第2回

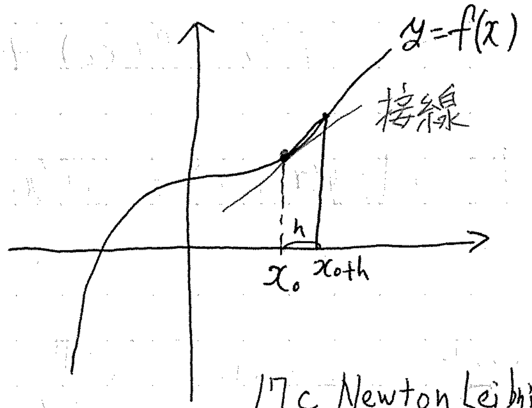
数理科学III B
10/11(火)

Synthetic differential geometry
総合的
within homotopy type theory

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

平均の変化率

極限 ⇒ 微分
19c ε-δ



17c Newton Leibniz
18c Euler, Lagrage

グラフ
曲がっている

$$h^2 = 0 \Rightarrow \underset{\text{十分小さい}h}{h=0}$$

$$D = \{ d \in \mathbb{R} \mid d^2 = 0 \}$$

$$f(x_0+d) - f(x_0) = \underset{\exists ! a \in \mathbb{R}}{a} d \quad (\forall d \in D)$$

$$f(x_0+d) = f(x_0) + a d$$

a ― $f'(x_0)$

$$(f+g)' = f' + g'$$

$$\lim_{h \to 0} \frac{f(x_0+h) + g(x_0+h) - \{f(x_0)+g(x_0)\}}{h} = \left(\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}\right)\left(\lim_{h \to 0}\frac{g(x_0+h)-g(x_0)}{h}\right)$$

$$f(x_0+d) + g(x_0+d) = f(x_0) + f'(x_0)d + g(x_0) + g(x_0)d$$
$$= f(x_0) + g(x_0) + \{f'(x_0) + g'(x_0)\}d$$

$$(fg)' = f'g + fg'$$

$$\lim_{h \to 0} \frac{f(x_0+h)g(x_0+h) - f(x_0)g(x_0)}{h} = \frac{f(x_0+h)g(x_0+h) - f(x_0)g(x_0+h) + f(x_0)g(x_0+h) - f(x_0)g(x_0)}{h}$$

Leibniz が発見　　冪零無限小

$$= \lim_{h\to 0} \frac{\{f(x_0+h)-f(x_0)\}g(x_0+h) + f(x_0)\{g(x_0+h)-g(x_0)\}}{h}$$

$$= f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

$$f(x_0+d)\,g(x_0+d) = \{f(x_0)+f'(x_0)d\}\{g(x_0)+g'(x_0)d\}$$

$$= f(x_0)g(x_0) + \underline{\{f'(x_0)g(x_0)+f(x_0)g'(x_0)\}}d + f'(x_0)g'(x_0)\underbrace{d^2}_{=0}$$

Kock − Lawvere（ラヴィアー）の公理 → topos

$\varphi: D \to \mathbb{R}$　　$\exists!\, a \in \mathbb{R}$

$\forall d \in D$　　$\varphi(d) = \varphi(0) + ad$.

$\boxed{\text{微分可能な関数}}$

$f: \mathbb{R} \to \mathbb{R}$　　固定 $x_0 \in \mathbb{R}$

$\varphi(d) = f(x_0+d)$　　$\varphi(0) = f(x_0)$

$$f(x_0+d) = \varphi(d) = f(x_0) + ad$$

$$\mathbb{R}[x] \xrightarrow{\mathbb{R}\text{代数}} \mathbb{R}$$

多項式環

$\mathbb{R}$代数としての準同型 → $\mathbb{R}$と同一視可能

$$\mathrm{Spec}_{\mathbb{R}}\,\mathbb{R}[x] = \mathbb{R}$$

$$\mathbb{R}[x]/(x^2) \to \mathbb{R}$$

$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n$

$$\mathrm{Spec}_{\mathbb{R}}\,\mathbb{R}[x]/x^2 = D$$

$$D_1 = D = \{d \in \mathbb{R} \mid d^2 = 0\}$$

$$\left.\begin{array}{l} d \in D \\ \alpha \in \mathbb{R} \end{array}\right\} \Rightarrow (\alpha d)^2 = \alpha^2 \underbrace{d^2}_{=0} = 0$$

$$\alpha d \in D$$

$$d_1, d_2 \in D \Rightarrow (d_1+d_2)^2 = \underbrace{d_1^2}_{=0} + 2d_1d_2 + \underbrace{d_2^2}_{=0}$$

$$= 2d_1d_2$$

$$\underline{d_1+d_2 \notin D}$$

$$(d_1+d_2)^3 = \underbrace{d_1^3}_{=0} + \underbrace{3d_1^2d_2}_{=0} + \underbrace{3d_1d_2^2}_{=0} + \underbrace{d_2^3}_{=0} = 0$$

$D_n = \{d \in \mathbb{R} \mid d^{n+1} = 0\}$　　D_1 : 1次の無限小

$d_1, d_2 \in D_1 \Rightarrow d_1 + d_2 \in D_2$　　D_2 = 2次の ______

Lemma

$\left.\begin{array}{l} d_m \in D_m \\ d_n \in D_n \end{array}\right\} \Rightarrow d_m + d_n \in D_{m+n}$ を示せ

$d_m^{m+1} = 0$

$d_n^{n+1} = 0$

$$(d_m+d_n)^{m+n+1} = \sum_{k=0}^{m+n+1} {}_{m+n+1}C_k\,(d_m)^{(m+n+1-k)}(d_n)^k$$

$$= d_m^{m+n+1} + {}_{m+n+1}C_1\,(d_m)^{m+n}d_n + \cdots + d_n^{m+n+1}$$

$$= 0$$

(i) $\underline{k \leq n}$　$m+n+1-k \leq m$

(ii) $\underline{n+1 \leq k}$

$\therefore\ d_m + d_n \in D_{m+n}$

((i), (ii) の条件を同時に満たすことは出来ないので)

Corollary

$d_1, \cdots, d_n \in D = D_1 \Rightarrow \underbrace{d_1 + d_2 + \cdots + d_n}_{\in D_n} \in D_n$

証明 by induction on n

$d_1 + \cdots + d_{n-1} \in D_{n-1}, \quad d_n \in D = D_1$

(復習)

$f(x_0 + d_1) = f(x_0) + f'(x_0) d_1$

$$f(\underline{x_0 + (d_1 + d_2)}) = f(\underline{x_0} + d_1) + f'(x_0 + d_1) d_2$$

$$= (f(x_0) + f'(x_0) d_1) + \{ f'(x_0) + f''(x_0) d_1 \} d_2$$

$$= f(x_0) + f'(x_0)(d_1 + d_2) + f''(x_0) d_1 d_2$$

$$= f(x_0) + f'(x_0)\underline{(d_1 + d_2)} + \frac{f''(x_0)}{2} \underline{(d_1 + d_2)^2}$$

$$f(\underline{x_0 + d_1 + d_2} + d_3) = f(x_0 + d_1 + d_2) + f'(x_0 + d_1 + d_2) d_3$$

$$= f(x_0) + f'(x_0)(d_1 + d_2) + \frac{f''(x_0)}{2}(d_1 + d_2)^2$$

$$+ \left\{ f'(x_0) + f''(x_0)(d_1 + d_2) + \frac{f'''(x_0)}{2} \overset{d_1 d_2}{\cancel{(d_1 + d_2)^2}} \right\} d_3$$

$$= f(x_0) + f'(x_0)(d_1 + d_2 + d_3) + f''(x_0)(\underline{d_1 d_2 + d_1 d_3 + d_2 d_3}$$

$$+ f'''(x_0) \underline{d_1 d_2 d_3}$$