第4回

type theory

集合論　集合入門　原理的　$P: A \to U$ universe

types　演算　$\sum_{x:A} P(x)$　$P(x) = B$　$A \times B$ 直積

$\prod_{x:A} P(x)$　$P(x) = B$　$A \to B$

dependent type theory 1.12

n-type　n<

equality type 1　indentity type 1

A: type

$Id_A : A \to A \to U$

$A \times A \to U$

$A \to A \to U$

$f(x, y)$　$x \to f(x, -)$

$Id_A(a, b)$　$a, b : A$

$a =_A b$ type　$a = b$

$p, q : a =_A b$　path $p : a = b$　pはaとbの indentification witness (証人)

$p =_{a =_A b} q$

A 2.10

$$\frac{\Gamma \vdash A : U \quad \Gamma \vdash a : A \quad \Gamma \vdash b : A}{\Gamma \vdash a =_A b : U} = \text{-formation}$$

$$\frac{\Gamma \vdash A : U \qquad \Gamma \vdash a : A}{\Gamma \vdash \underline{\mathrm{refl}_a} : a =_A a}$$ $=$ - introduction

reflection a A

$$\frac{\begin{array}{c}\Gamma, x:A,\ y:A,\ p: x =_A y \vdash C : U \\ \Gamma, z:A \vdash c : C[z,z,\mathrm{refl}_z/x,y,p] \quad \Gamma \vdash a:A \\ \Gamma \vdash b:A \quad \Gamma \vdash p' : a =_A b\end{array}}{\Gamma \vdash \mathrm{ind}_{=_A}(x,y,p.C,\ z.c,\ a,\ b,\ p') : C[a,b,p'/x,y,p]}$$ $=$ -elimiation

induction primple ∈ one of the most subtle of typetheory and crucial to the homotopy type theory.

$$\frac{\Gamma, x:A,\ y:A,\ p: x =_A y \vdash C : U \quad \Gamma, z:A \vdash c : C[z,z,\mathrm{refl}_z/x,y,p] \quad \Gamma \vdash a : A}{\Gamma \vdash \mathrm{ind}_{=_A}(x,y,p.C,\ z.c,\ a,\ a,\ \mathrm{refl}_a) \equiv c[a/z] : C[a,a,\mathrm{refl}_a/x,y,p]}$$ $=$ -computation

1.12.1 path induction

$$\left.\begin{array}{l} C : \prod_{x,y:A} (x =_A y) \to U \\ c : \prod C(x,x,\mathrm{refl}_x) \end{array}\right\} \Longrightarrow$$

dependent function $\exists f : \prod_{x,y:A} \prod_{p:x=_A y} C(x,y,p)$ such that $f(x,x,\mathrm{refl}_x) :\equiv c(x)$

$$\mathrm{ind}_{=_A} : \prod_{C : \prod_{x,y:A} (x =_A y) \to U} \Big(\prod_{x:A} C(x,x,\mathrm{refl}_x)\Big) \to \prod_{x,y:A} \prod_{p:x=_A y} C(x,y,p)$$

$$\frac{\mathrm{ind}_A}{J}(C, c, x, x, \mathrm{refl}_x) :\equiv c(x)$$

Based path induction
基点のある.

$a : A$: fixed

$C : \prod_{x:A} (a =_A x) \longrightarrow U$
$c : C(a, \mathrm{refl}_a)$
$\Rightarrow$ $\exists f : \prod_{x:A} \prod_{p : a = x} C(x, p)$ such that $f(a, \mathrm{refl}_a) :\equiv c$

equivalent Yoneda lemma

object a of A

$\mathrm{Hom}(\mathbb{A}(a, -), F) = F(a)$

1.12.2 Equivalence of path induction and based path induction

easier based path induction $\Rightarrow$ path induction

path 前提

$C : \prod_{x, y : A} (x =_A y) \longrightarrow U$
$c : \prod_{x:A} C(x, x, \mathrm{refl}_x)$

given $x : A$

$C' : \prod_{y:A} (x =_A y) \longrightarrow U$
$C' :\equiv C(x)$
$c' : C'(x, \mathrm{refl}_x)$
$c' :\equiv c(x)$

C', c' は based path induction の前提

$g : \prod_{y:A} \prod_{p:x=y} C'(y,p)$ discharge $x:A$

with $g(x, \mathrm{refl}_x) :\equiv c'$

gの codomain は $C(x,y,p)$

$f : \prod_{x,y:A} \prod_{p:x=_A y} C(x,y,p)$ with $f(x,x,\mathrm{refl}_x) \equiv g(x,\mathrm{refl}_x) :\equiv c' \equiv c(x)$

path ind $\Longrightarrow$ based path ind.

$D : \prod_{x,y:A} (x =_A y) \to U$

$D(x,y,p) :\equiv \prod_{C : \prod_{z:A}(x =_A z) \to U} C(x, \mathrm{refl}_x) \to C(y,p)$

$d : \prod_{x:A} D(x,x,\mathrm{refl}_x)$

$d :\equiv \lambda x. \lambda C. \lambda(c : C(x,\mathrm{refl}_x)). c$

path induction を用いて

$f : \prod_{x,y:A} \prod_{p:x=_A y} D(x,y,p)$ with $f(x,x,\mathrm{refl}_x) :\equiv d(x)$

Dの定義をほどいてやって.
(unfold)

$f : \prod_{x,y:A} \prod_{p:x=_A y} \prod_{C:\prod_{z:A}(x=_A z) \to c} C(x,\mathrm{refl}_x) \to C(y,p)$

given $x:A$ $p : a =_A x$

$f(a,x,p,C,c) : C(x,p)$