微積分演習 第6回

$$f(x+d_1+\cdots+d_n) = f(x) + f'(x)\lambda_n^1(d_1,\cdots,d_n)$$
$$+ \cdots + f^{(n)}(x)\lambda_n^n(d_1,\cdots,d_n)$$

の証明.

n に関する帰納法

$n=1$ のとき, $f(x+d_1) = f(x) + f'(x)d$ 成立.

$n=k$ のとき, $f(x+d_1+\cdots+d_k) = f(x) + f'(x)\lambda_k^1(d_1,\cdots,d_k)$
$$+ \cdots + f^{(k)}(x)\lambda_k^k(d_1,\cdots,d_k)$$

と仮定.

$n=k+1$ のとき, $f(x+d_1+\cdots+d_{k+1})$

$$= f(x+d_{k+1}) + f'(x+d_{k+1})\lambda_k^1(d_1,\cdots,d_k)$$
$$+ f''(x+d_{k+1})\lambda_k^2(d_1,\cdots,d_k) + \cdots + f^{(k)}(x+d_{k+1})\lambda_k^k(d_1,\cdots,d_k)$$

$$= f(x) + f'(x)d_{k+1} + \left(f'(x) + f''(x)d_{k+1}\right)\lambda_k^1(d_1,\cdots,d_k)$$
$$+ \left(f''(x) + f'''(x)d_{k+1}\right)\lambda_k^2(d_1,\cdots,d_k)$$
$$+ \cdots + \left(f^{(k)}(x) + f^{(k+1)}(x)d_{k+1}\right)\lambda_k^k(d_1,\cdots,d_k)$$

$$= f(x) + f'(x)d_{k+1} + f'(x)\lambda_k^1(d_1,\cdots,d_k)$$
$$+ f''(x)d_{k+1}\lambda_k^1(d_1,\cdots,d_k) + f''(x)\lambda_k^2(d_1,\cdots,d_k)$$
$$+ f'''(x)d_{k+1}\lambda_k^2(d_1,\cdots,d_k) + f'''(x)\lambda_k^3(d_1,\cdots,d_k)$$
$$+ \cdots$$

$f^{(i)}(x)$ の係数を見て.

$$d_{k+1}\lambda_k^{i-1}(d_1,\cdots,d_k) + \lambda_k^i(d_1,\cdots,d_k) = \lambda_{k+1}^i(d_1,\cdots,d_{k+1})$$

したがって $n=k+1$ で成立.

帰納法により示された.

$x = x(t)$ 時刻の関数

$x' = ax$ 微分方程式

$$\xrightarrow{d_1} \xrightarrow{d_1+d_2} d_1+d_2+d_3$$

$x(0) = C$ (定数)

$$x(d_1) = x(0) + x'(0) d_1$$
$$= C + aC d_1$$
$$= C(1+ad_1)$$

$$x(d_1+d_2) = x(d_1) + x'(d_1) d_2$$
$$= x(d_1) + ax(d_1) d_2$$
$$= x(d_1)(1+ad_2)$$
$$= C(1+ad_1)(1+ad_2)$$

$$x(d_1+d_2+d_3) = x(d_1+d_2) + x'(d_1+d_2) d_3$$
$$= x(d_1+d_2) + ax(d_1+d_2) d_3$$
$$= x(d_1+d_2)(1+ad_3)$$
$$= C(1+ad_1)(1+ad_2)(1+ad_3)$$

$$x(d_1+ \cdots + d_n) = C(1+ad_1) \cdots (1+ad_n)$$

$$= C\left\{1 + \underbrace{a(d_1+\cdots+d_n)}_{\lambda_n^1(d_1,\cdots,d_n)} + \underbrace{a^2(d_1d_2+\cdots\cdots)}_{\lambda_n^2(d_1,\cdots,d_n)} + \cdots\cdots\right\}$$

$$\lambda_n^1(d_1,\cdots,d_n) \qquad \lambda_n^2(d_1,\cdots,d_n)$$
$$\parallel \qquad\qquad\qquad \parallel$$
$$d_1+\cdots+d_n \qquad \frac{(d_1+\cdots+d_n)^2}{2}$$

$$= C\left\{1 + a(d_1+\cdots+d_n) + a^2\frac{(d_1+\cdots+d_n)^2}{2} + \cdots + a^n\frac{(d_1+\cdots+d_n)^n}{n!}\right\}$$

$$C \cdot e^{at}$$

$x = \sin t$ とする.

$x' = \cos t$

$x'' = -\sin t$ なので $x'' = -x$ をみたしている.

$\boxed{x'' = -x}$ 解く.

$x(0) = C_1$
$x'(0) = C_2$ とする.

$x(0) = C_1$
$x(d_1) = x(0) + x'(0) d_1$
$\quad = C_1 + C_2 d_1$

$x(d_1 + d_2) = x(d_1) + x'(d_1) d_2$
$\quad = (C_1 + C_2 d_1) + \{ C_2 + (-C_1) d_1 \} d_2$
$\quad = C_1 (1 - d_1 d_2) + C_2 (d_1 + d_2)$

$x(d_1 + d_2 + d_3) = x(d_1 + d_2) + x'(d_1 + d_2) d_3$
$\quad = C_1 (1 - d_1 d_2) + C_2 (d_1 + d_2)$
$\qquad + (\underbrace{x'(d_1)}_{C_2 - C_1 d_1} + \underbrace{x''(d_1) d_2}_{\substack{-x(d_1) d_2 \\ = -(C_1 + C_2 d_1) d_2}}) d_3$

$\quad = C_1 \underline{(1 - d_1 d_2 - d_1 d_3 - d_2 d_3)} + C_2 \underline{(d_1 + d_2 + d_3 - d_1 d_2 d_3)}$
$\qquad\quad {}^{''} 1 - \dfrac{(d_1 + d_2 + d_3)^2}{2} \qquad\qquad \dfrac{(d_1 + d_2 + d_3)^3}{3!}$

$\cos t = 1 - \dfrac{t^2}{2} + \cdots$
$\sin t = t - \dfrac{t^3}{3!} + \cdots$

宿題
　$x(d_1 + \cdots + d_4)$ を計算せよ.