微積分演習 第4回

$D_1 = D = \{ d \in \mathbb{R} \mid d^2 = 0 \}$

$d \in D$ & $\alpha \in \mathbb{R} \implies \alpha d \in D$

$$(\alpha d)^2 = \alpha^2 d^2 = 0$$

$d_1, d_2 \in D \implies d_1 + d_2 \in D_2 = \{ d \in \mathbb{R} \mid d^3 = 0 \}$

$$(d_1 + d_2)^2 = d_1^2 + 2d_1 d_2 + d_2^2 \neq 0$$

$$(d_1 + d_2)^3 = d_1^3 + 3d_1^2 d_2 + 3d_1 d_2^2 + d_2^3 = 0$$

$d_1, d_2, d_3 \in D$

$$(d_1 + d_2 + d_3)^3 = d_1^3 + d_2^3 + d_3^3 + 3d_1^2 d_2 + 3d_1^2 d_3 + 3d_1 d_2^2 + 3d_2^2 d_3 + 3d_1 d_3^2 + 3d_2 d_3^2 + 6d_1 d_2 d_3$$

$$(d_1 + d_2 + d_3)^4 = 0$$

$$\underbrace{d_1 d_2 d_3}, \underbrace{d_1 d_2 d_3}, \underbrace{d_1 d_2 d_3}, \underbrace{d_1 d_2 d_3}$$

同様に

$$(d_1 + d_2 + \cdots + d_n)^{n+1} = 0$$

$d_1, d_2, \cdots, d_n \in D \implies d_1 + d_2 + \cdots + d_n \in D_n$

$D_n = \{ d \in \mathbb{R} \mid d^{n+1} = 0 \}$

$$\boxed{d_1 \in D_n,\ d_2 \in D_m \Longrightarrow d_1 + d_2 \in D_{n+m}}$$

$(d_1 + d_2)^{n+m+1}$

$d_1^{\,i} d_2^{\,j} \qquad i+j = n+m+1$

0にならないためには

$i \leq n,\ j \leq m$ より $i+j \leq n+m$

$d_1, d_2, d_3 \in D$

$$d_1 + d_2 + d_3 = \underset{\in D_2}{(d_1 + d_2)} + \underset{\in D_1}{d_3} \in D_3$$

帰納法

$f : \mathbb{R} \to \mathbb{R} \qquad x \in \mathbb{R}$ 2回微分

$$f(x + d_1) = f(x) + f'(x) d_1$$

$f' : \mathbb{R} \to \mathbb{R}$

$d_1, d_2 \in D$

$$\begin{aligned} f(x + d_1 + d_2) &= f(x + d_1) + f'(x + d_1) d_2 \\ &= f(x) + f'(x) d_1 + \{ f'(x) + f''(x) d_1 \} d_2 \\ &= f(x) + f'(x)(d_1 + d_2) + f''(x) d_1 d_2 \end{aligned}$$

$d_1 d_2 = \dfrac{(d_1 + d_2)^2}{2}$ だから

$$f(x) + f'(x)(d_1 + d_2) + \frac{f''(x)}{2}(d_1 + d_2)^2$$

$d_1, d_2, d_3 \in D$

$$\begin{aligned} f(x + d_1 + d_2 + d_3) &= f(x + d_1 + d_2) + f'(x + d_1 + d_2) d_3 \\ &= f(x) + f'(x)(d_1 + d_2) + f''(x) d_1 d_2 \\ &\quad + \{ f'(x) + f''(x)(d_1 + d_2) + f'''(x) d_1 d_2 \} d_3 \\ &= f(x) + f'(x)(d_1 + d_2 + d_3) \\ &\quad + f''(x)(d_1 d_2 + d_2 d_3 + d_3 d_1) + f'''(x) d_1 d_2 d_3 \end{aligned}$$

ここで、

$$\left[\begin{array}{l} d_1d_2+d_1d_3+d_2d_3 = \dfrac{(d_1+d_2+d_3)^2}{2} \\ d_1d_2d_3 = \dfrac{(d_1+d_2+d_3)^3}{3!} \end{array}\right]$$

だから。

$$f(x)+f'(x)(d_1+d_2+d_3)+\frac{f''(x)}{2}(d_1+d_2+d_3)^2 + \frac{f'''(x)}{3!}(d_1+d_2+d_3)^3$$

$d_1, d_2, d_3, d_4 \in D$

$$f(x+d_1+d_2+d_3+d_4)$$

$$= f(x+d_1+d_2+d_3) + f'(x+d_1+d_2+d_3)\,d_4$$

$$= f(x)+f'(x)(d_1+d_2+d_3)+f''(x)(d_1d_2+d_1d_3+d_2d_3) + f'''(x)\,d_1d_2d_3$$
$$+ \left\{ f'(x)+f''(x)(d_1+d_2+d_3) + f'''(x)(d_1d_2+d_1d_3+d_2d_3) + f''''(x)\,d_1d_2d_3 \right\} d_4$$

$$= f(x)+f'(x)(d_1+d_2+d_3+d_4)$$
$$+ f''(x)\underbrace{(d_1d_2+d_1d_3+d_1d_4+d_2d_3+d_2d_4+d_3d_4)}_{\frac{(d_1+\cdots+d_4)^2}{2}}$$
$$+ f'''(x)\underbrace{(d_1d_2d_3+d_1d_2d_4+d_1d_3d_4+d_2d_3d_4)}_{\frac{(d_1+\cdots+d_4)^3}{3!}}$$
$$+ f''''(x)\underbrace{d_1d_2d_3d_4}_{\frac{(d_1+\cdots+d_4)^4}{4!}}$$

一般形

$d_1, \cdots, d_n \in D$

$$f(x+d_1+\cdots+d_n) = f(x) + f'(x)(d_1+\cdots+d_n) + \frac{f''(x)}{2}(d_1+\cdots+d_n)^2 + \cdots + \frac{f^{(i)}(x)}{i!}(d_1+\cdots+d_n)^i + \cdots + \frac{f^{(n)}(x)}{n!}(d_1+\cdots+d_n)^n$$

$f : \mathbb{R} \to \mathbb{R}$

f：多項式と仮定（m次）

$$f(x+y) = a_0 + a_1 y + a_2 y^2 + \cdots + a_m y^m$$

x：固定

（a_0を求めるには $y=0$ とおく。）

$$a_0 = f(x)$$
$$a_1 = f'(x)$$
$$a_2 = \frac{f''(x)}{2}$$
$$a_3 = \frac{f^{(3)}(x)}{3!}$$
$$\vdots$$
$$a_m = \frac{f^{(m)}(x)}{m!}$$

Taylor展開

任意の関数
無限小のレベルでは
多項式

数II　多項式関数の微積

数III　sin, cos, e^x …の微積

ここで無限次の多項式で書けることを認める。

$$f(x) = e^x = a_0 + a_1 x + a_2 x^2 + \cdots\cdots$$

$$a_0 = f(0) = 1$$

$$a_1 = f'(0) = 1 \qquad (e^x)' = e^x$$

$$a_2 = \frac{f''(0)}{2} = \frac{1}{2}$$

$$a_3 = \frac{f^{(3)}(0)}{3!} = \frac{1}{3!}$$

$$\vdots$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!} + \cdots$$

$$f(x) = \sin x = a_0 + a_1 x + a_2 x^2 + \cdots\cdots$$

$$a_0 = f(0) = 0$$

$$a_1 = f'(0) = 1 \qquad (\sin x)' = \cos x$$

$$a_2 = \frac{f''(0)}{2} = 0 \qquad (\sin x)'' = -\sin x$$

$$a_3 = \frac{f^{(3)}(0)}{3!}$$

$$\vdots$$