微積分演習 第13回

微分方程式

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x' = a_{11}x + a_{12}y$$
$$y' = a_{21}x + a_{22}y$$

Aの固有値
異なる2実解 α, β

$$A = P^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}P$$

$$e^{tA} = e^{t P^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}P}$$
$$= e^{P^{-1} t\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}P}$$
$$= P^{-1}e^{t\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}}P$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{tA}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$ c_1, c_2 は定数
初期条件

$$\begin{cases} x = c_{11}e^{t\alpha} + c_{12}e^{t\beta} \\ y = c_{21}e^{t\alpha} + c_{22}e^{t\beta} \end{cases}$$

元の微分方程式をみたすこと，初期条件を考慮して解ける。

$$\forall t \in \mathbb{R}.\ c_1 e^{t\alpha} + c_2 e^{t\beta} = 0$$
$$\Rightarrow c_1 = c_2 = 0$$

固有方程式
複素解 $a \pm bi$ $(b \neq 0)$ をもつとき

$$A = P^{-1}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}P$$

$$\begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} \underset{対応}{\longleftrightarrow} a_1 + b_1 i$$

$$\begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} \longleftrightarrow a_2 + b_2 i$$

$$\begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} + \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} \longleftrightarrow (a_1 + b_1 i) + (a_2 + b_2 i)$$

$$\begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} \longleftrightarrow (a_1 + b_1 i)(a_2 + b_2 i)$$

$$e^{\underbrace{t\begin{pmatrix} a & -b \\ b & a \end{pmatrix}}_{=B}} \longleftrightarrow e^{t(a+bi)}$$

$$= E + tB + \frac{1}{2}t^2B^2 + \cdots\cdots$$

$$= e^{ta+tbi}$$

$$= e^{ta}\ e^{tbi}$$

$$= e^{ta}(\cos tb + i \sin tb)$$

$$= e^{ta}\cos tb + i e^{ta}\sin tb$$

$$\begin{pmatrix} e^{ta}\cos bt & -e^{ta}\sin tb \\ e^{ta}\sin bt & e^{ta}\cos tb \end{pmatrix}$$

$$A = P^{-1}\begin{pmatrix} a & -b \\ b & a \end{pmatrix} P$$

解は $P^{-1}\begin{pmatrix} e^{ta}\cos bt & -e^{ta}\sin bt \\ e^{ta}\sin bt & e^{ta}\cos bt \end{pmatrix} P \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ 未定定数

$$x = c_{11}e^{ta}\cos bt + c_{12}e^{t\beta}\sin bt$$
$$y = c_{21}e^{ta}\cos bt + c_{22}e^{t\beta}\sin bt$$ の形

微分して. 前と同じ要領

$$(\forall t \in \mathbb{R})\ c_1 e^{t\alpha}\cos bt + c_2 e^{t\beta}\sin bt = 0$$
$$\Longrightarrow c_1 = c_2 = 0$$

を利用

例 $A = \begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix}$

$$|tE - A| = \begin{vmatrix} t & 2 \\ -1 & t-2 \end{vmatrix} = t(t-2)+2$$
$$= t^2 - 2t + 2 = 0$$
$$t = 1 \pm i$$

$$\begin{cases} x = c_{11} e^t \cos bt + c_{12} e^t \sin bt \\ y = c_{21} e^t \cos bt + c_{22} e^t \sin bt \end{cases}$$

$$x' = c_{11}(e^t \cos bt - e^t \sin bt) + c_{12}(e^t \sin bt + e^t \cos bt) = -2y$$
$$y' = c_{21}(e^t \cos bt - e^t \sin bt) + c_{22}(e^t \sin bt + e^t \cos bt) = x + 2y$$

$$\begin{cases} c_{11} + c_{12} = -2c_{21} \\ -c_{11} + c_{12} = -2c_{22} \\ c_{21} + c_{22} = c_{11} + 2c_{21} \\ -c_{21} + c_{22} = c_{12} + 2c_{22} \end{cases}$$

文字は2つ消える。初期条件を考慮.

例
$$\begin{cases} x' = y \\ y' = -b^2 x \quad (b > 0) \end{cases} \qquad A = \begin{pmatrix} 0 & 1 \\ -b^2 & 0 \end{pmatrix}$$

$$|tE - A| = \begin{vmatrix} t & -1 \\ b^2 & t \end{vmatrix} = t^2 + b^2 = 0$$
$$t = \pm bi$$

$$\begin{cases} x = c_{11} \cos bt + c_{12} \sin bt \\ y = c_{21} \cos bt + c_{22} \sin bt \end{cases}$$

$$x = c_{11} \cos bt + c_{12} \sin bt$$
$$y = bc_{12} \cos bt - bc_{11} \sin bt$$

$\boxed{x'' = -b^2 x}$ $\xrightarrow[\text{とおく}]{x' = y}$ $\boxed{\begin{cases} x' = y \\ y' = -b^2 x \end{cases}}$

2階方程式

1階連立方程式

単振動.