No. 1

Date '16. 6. 20

微積分 第8回

$f: \mathbb{R}^2 \to \mathbb{R}$ 2変数の関数

点 $x \in \mathbb{R}^2$ で微分

$f'(x): \mathbb{R}^2 \to \mathbb{R}$ への線型写像

1×2 行列で表示される

$$[a, b]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto ax_1 + bx_2$$

$$f'(x) = \left[\frac{\partial f}{\partial x_1}(x) \quad \frac{\partial f}{\partial x_2}(x)\right]$$ 横ベクトル

$f': \mathbb{R}^2 \to$ 1×2 行列全体

2次元の横ベクトル全体

x で微分

$f''(x): \mathbb{R}^2 \to$ (2次元の横ベクトル全体) への線型写像

$$(y_1, y_2) \longmapsto \underset{1\times 2}{(y_1, y_2)} \underset{2\times 2}{\begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(x) \end{bmatrix}}$$

$$= \left[y_1 \frac{\partial^2 f}{\partial x_1 \partial x_1} + y_2 \frac{\partial^2 f}{\partial x_2 \partial x_1} \quad y_1 \frac{\partial^2 f}{\partial x_1 \partial x_2} + y_2 \frac{\partial^2 f}{\partial x_2 \partial x_2} \right]$$

1×2 行列

$f''(x)(a)(b) \in \mathbb{R}$

$$\underset{1\times 2}{(a_1\ a_2)} \underset{2\times 2}{\begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}} = \underset{1\times 1}{\bigcirc}$$

極値

$f : \mathbb{R}^2 \to \mathbb{R}$

$g_a : t \in \mathbb{R} \longmapsto f(x+ta) \in \mathbb{R}$

g_a を考える.

$$\left.\begin{array}{l} t \longmapsto x+ta \\ \quad y \longmapsto f(y) \end{array}\right\} \text{合成}$$

$t=0$ で微分

$$f'(x)(a)$$

$$\left[\frac{\partial f}{\partial x_1}(x) \quad \frac{\partial f}{\partial x_2}(x)\right]\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$\boxed{\frac{\partial f}{\partial x_1}(x) = \frac{\partial f}{\partial x_2}(x) = 0}$$

必要条件

t での g_a の微分

$t \longmapsto f'(x+ta)(a)$ をもう一度 t で微分.

$$\begin{array}{l} t \longmapsto x+ta \\ \quad y \longmapsto f'(y) \\ \qquad \bigcirc \longmapsto \_\,(a) \quad \text{線型} \end{array}$$

$f''(x+ta)(a)(a)$

a に関わらず >0 のとき 極小値
<0 のとき 極大値

No. 3

Date

$$(a_1\ a_2)\begin{array}{c} \alpha \qquad\qquad \gamma \\ \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1 \partial x_1} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2 \partial x_2} \end{bmatrix} \\ \gamma \qquad\qquad \beta \end{array} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$= [\alpha a_1 + \gamma a_2,\ a_1\gamma + a_2\beta]\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$= \alpha a_1^2 + \gamma a_1 a_2 + a_1 a_2 \gamma + a_2^2 \beta$$

$$= \alpha a_1^2 + 2\gamma a_1 a_2 + \beta a_2^2$$

>0 なら極小
<0 なら極大

$a_2 \neq 0$ とする
a_2^2 でわる

$$\alpha\left(\frac{a_1}{a_2}\right)^2 + 2\gamma\frac{a_1}{a_2} + \beta > 0 \text{ となるには}$$

$$\alpha > 0,\ D = \gamma^2 - \alpha\beta < 0$$

$$\alpha\left(\frac{a_1}{a_2}\right)^2 + 2\gamma\frac{a_1}{a_2} + \beta < 0 \text{ となるには}$$

$$\alpha < 0,\ D = \gamma^2 - \alpha\beta < 0$$

問題

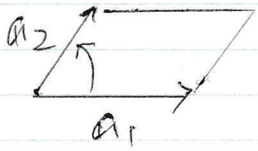
次の関数の極値を求めよ

(1) $f(x, y) = 2 - 3(x+y) + x^3 + y^3$

(2) $f(x, y) = x^3 + y^3 - 9xy + 27$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$a_1 = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}, \quad a_2 = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$



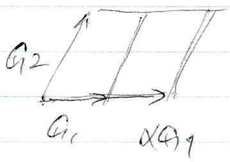
で張られる平行四辺形の
符号のついた面積を $S(a_1, a_2)$ とする.

$a_1 \to a_2$
反時計回り $+$
時計回り $-$

① $S(a_1, a_2) = -S(a_2, a_1)$

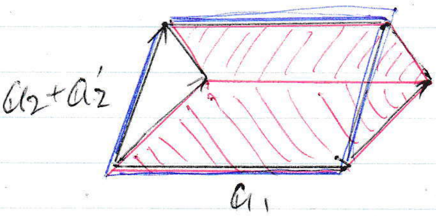
$a_1 = a_2$ とすると $S(a_1, a_1) = -S(a_1, a_1)$
$2S(a_1, a_1) = 0$
$S(a_1, a_1) = 0$

② $S(\alpha a_1, a_2) = \alpha S(a_1, a_2)$



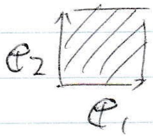
$S(a_1, \alpha a_2) = \alpha S(a_1, a_2)$
も成り立つ.

③ $S(a_1, a_2 + a_2') = S(a_1, a_2) + S(a_1, a_2')$



$$S(\mathbf{e}_1, \mathbf{e}_2) = 1$$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



$$S(\mathbf{e}_2, \mathbf{e}_1) = -1$$
$$S(\mathbf{e}_1, \mathbf{e}_1) = 0$$
$$S(\mathbf{e}_2, \mathbf{e}_2) = 0$$

$$a_1 = a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2$$
$$a_2 = a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2$$
と表す.

$$\begin{aligned} S(a_1, a_2) &= S(a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2,\ a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2) \\ &= a_{11}a_{12}\underbrace{S(\mathbf{e}_1, \mathbf{e}_1)}_{0} + a_{11}a_{22}\underbrace{S(\mathbf{e}_1, \mathbf{e}_2)}_{1} \\ &\quad + a_{21}a_{12}\underbrace{S(\mathbf{e}_2, \mathbf{e}_1)}_{-1} + a_{21}a_{22}\underbrace{S(\mathbf{e}_2, \mathbf{e}_2)}_{0} \\ &= a_{11}a_{22} - a_{21}a_{12} \end{aligned}$$

平行四辺形の面積 $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$\left(\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} \right)$$

$$a_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}, \quad a_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}, \quad a_3 = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}$$

a_1, a_2, a_3 で張られる平行六面体の
符号のついた体積を $V(a_1, a_2, a_3)$ とする.

$a_1 \to a_2$ 右ねじの進む方向に a_3 なら $+$
逆 なら $-$
} あるいは 右手系 $+$
左手系 $-$