No. 1

Date '16 · 12 · 12

微積分 第22回

$f: \mathbb{C} \to \mathbb{C}$

$z = x + iy$

$dz = dx + idy$

$\left(\begin{array}{ll} f: \mathbb{R} \to \mathbb{R} & \text{を微分すると} \\ f' dx & \text{1次の微分係式} \end{array} \right)$

$$f' = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$
$$= \bigcirc \, dz$$

Cauchy-Riemann の方程式

コーシー リーマン

$f_i: \mathbb{C} \to \mathbb{R}, \quad f = f_1 + f_2 i$

$$\boxed{\begin{aligned} \frac{\partial f_1}{\partial x} &= \frac{\partial f_2}{\partial y} \\ \frac{\partial f_1}{\partial y} &= -\frac{\partial f_2}{\partial x} \end{aligned}}$$

$f: \mathbb{C} \to \mathbb{C}$

$$\begin{aligned} d(f\,dz) &= \frac{\partial f}{\partial x} dx \wedge dz + \frac{\partial f}{\partial y} dy \wedge dz \\ &= \frac{\partial f}{\partial x} dx \wedge (dx + idy) + \frac{\partial f}{\partial y} dy \wedge (dx + idy) \\ &= \frac{\partial f}{\partial x} i\, dx \wedge dy + \frac{\partial f}{\partial y} dy \wedge dx \\ &= \left(i \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy \end{aligned}$$

$f = f_1 + if_2$

$$\left(i \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \right) = i\left(\frac{\partial f_1}{\partial x} + i \frac{\partial f_2}{\partial x} \right) - \left(\frac{\partial f_1}{\partial y} + i \frac{\partial f_2}{\partial y} \right)$$

$$= \underbrace{-\left(\frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial x} \right)}_{\text{実部}} + i \underbrace{\left(\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y} \right)}_{\text{虚部}}$$

Cauchy-Riemannの方程式をみたすとき
$d(f\,dz)=0$ となる.

$f: \mathbb{C} \to \mathbb{C}$

$df = g\,dz$ と書ける
$\Updownarrow$
Cauchy-Riemannの方程式を満たす
(前回)

$d(f\,dz)=0$
$\Updownarrow$
Cauchy-Riemannの方程式を満たす
(今回)

- Cauchy-Riemannの方程式が成り立つ関数を正則関数, 解析関数といい, 対象にする.

$f(z)=c$ (定数関数) は正則

$f(z)=z$ (恒等関数) は正則

$z=x+iy$
$f_1(z)=x$
$f_2(z)=y$

$$\frac{\partial f_1}{\partial x}=1, \quad \frac{\partial f_2}{\partial y}=1$$
$$\frac{\partial f_1}{\partial y}=0 \quad \frac{\partial f_2}{\partial x}=0$$

$df = 1\,dz$.

f, g: 正則 $\Rightarrow$ $f+g$: 正則

$$d(f+g) = df + dg = f'dz + g'dz = (f'+g')dz$$

α：複素数

$$d(\alpha f) = \alpha\, df = \alpha f' dz$$

$$\begin{aligned} d(fg) &= (df)g + f(dg) \\ &= f'g\, dz + f\, g' dz \\ &= (f'g + fg')\, dz \end{aligned}$$

多項式の関数は正則

$$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_0$$

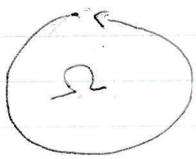
sin, cos も正則（無限次の多項式）

f：正則 $\Rightarrow df = f' dz$

f：正則 $\Rightarrow f'$ も正則

積分定理（2次元でも考えられる）

閉曲線 γ　　　　ω 1次の微分形式

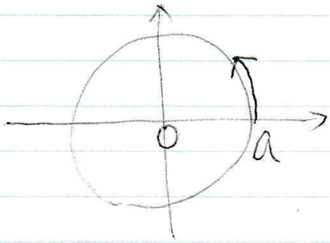


$$\boxed{\int_\gamma \omega = \int_\Omega d\omega}$$

$$\int_\gamma f\, dz = \int_\Omega \underbrace{d(f\, dz)}_{f：正則なら\ 0}$$

f：正則
$\Rightarrow$ 閉曲線に沿った
線積分は0

$\int_\gamma \frac{1}{z} dz$



原点中心
半径 a の円

$\theta \in [0, 2\pi] \mapsto a(\cos\theta + i\sin\theta)$

$\int_0^{2\pi} \frac{1}{a(\cos\theta + i\sin\theta)} a(\cos\theta + i\sin\theta)' d\theta$

$= \int_0^{2\pi} (\cos\theta - i\sin\theta)(-\sin\theta + i\cos\theta) d\theta$

$= \int_0^{2\pi} i \, d\theta$

$= [i\theta]_0^{2\pi}$

$= 2\pi i$