No. 1
Date '16.12.5

微積分第21回

複素数

$a+bi$ （a,bは実数）

(a,b)の組. 2次元の線型空間 $\mathbb{R}^2$
足し算、スカラー倍

掛け算は.

$$(a_1+b_1 i)(a_2+b_2 i) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$$

$\alpha\beta = \beta\alpha$ 可換.

ベクトル解析
古典的 —— 3次元に特化
$\left(\begin{array}{l}\text{ベクトル場} \\ \text{スカラー場}\end{array}\right)$

$f_i : \mathbb{R}^3 \to \mathbb{R}$. $e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$$\begin{tikzcd}
\text{スカラー場} \arrow[r, "\mathrm{grad}"] & \text{ベクトル場} \arrow[r, "\mathrm{rot}"] & \text{ベクトル場} \arrow[r, "\mathrm{div}"] & \text{スカラー場} \\
\varphi \arrow[d, leftrightarrow] & f_1e_1+f_2e_2+f_3e_3 \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & & \varphi : \mathbb{R}^3 \to \mathbb{R} \arrow[d, leftrightarrow] \\
\varphi & f_1dx+f_2dy+f_3dz & f_1dy\wedge dz + f_2dz\wedge dx + f_3dx\wedge dy & \varphi(dx\wedge dy\wedge dz) \\
\text{0次の微分形式} \arrow[r, "d"] & \text{1次の微分形式} \arrow[r, "d"] & \text{2次の微分形式} \arrow[r, "d"] & \text{3次の微分形式}
\end{tikzcd}$$

以下、複素数の場合を考えていく
2次元

$$f_i : \cancel{\mathbb{R}^2} \to \cancel{\mathbb{R}} \quad (\mathbb{C} \to \mathbb{C}) \qquad f : \mathbb{C} \to \mathbb{C}$$

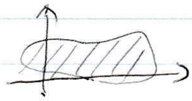
0次の微分形式 $\xrightarrow{d}$ 1次の微分形式 $\xrightarrow{d}$ 2次の微分形式

$\varphi : \cancel{\mathbb{R}^2} \to \cancel{\mathbb{R}}$ ($\mathbb{C} \to \mathbb{C}$) $\qquad \omega = f_1 dx + f_2 dy \qquad \omega = f\, dx \wedge dy$

線積分 $\gamma : [a, b] \to \mathbb{C}$

$$\int_\gamma \omega = \int_a^b \omega(\gamma(t))\, \gamma'(t)\, dt$$

面積分 領域Ω



$$\Sigma : \underset{s}{[a_1, b_1]} \times \underset{t}{[a_2, b_2]} \to \mathbb{C}$$

$$\int_\Omega \omega = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \omega(\Sigma(s,t)) \left(\frac{\partial \Sigma}{\partial s}(s,t), \frac{\partial \Sigma}{\partial t}(s,t) \right)$$

微分

$z = x + iy$

$\varphi : \mathbb{C} \to \mathbb{C}$ 関数

$$d\varphi = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy$$

$\boxed{\text{0次} \to \text{1次}}$

$f_i : \mathbb{C} \to \mathbb{C}$

$\omega = f_1 dx + f_2 dy$ (1次の微分形式)

$$d\omega = \left(\frac{\partial f_1}{\partial x} dx + \frac{\partial f_1}{\partial y} dy \right) \wedge dx + \left(\frac{\partial f_2}{\partial x} dx + \frac{\partial f_2}{\partial y} dy \right) \wedge dy$$

$$= \frac{\partial f_1}{\partial x} \underbrace{dx \wedge dx}_{0} + \frac{\partial f_1}{\partial y} \underbrace{dy \wedge dx}_{-dx \wedge dy} + \frac{\partial f_2}{\partial x} dx \wedge dy + \frac{\partial f_2}{\partial y} \underbrace{dy \wedge dy}_{0}$$

$$= \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx \wedge dy$$

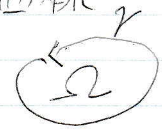
$\boxed{\text{1次} \to \text{2次}}$

積分定理は成り立つ.

曲線 $\gamma : [a, b] \to \mathbb{C}$

$$\underbrace{\int_\gamma d\varphi}_{線積分} = \varphi(\gamma(b)) - \varphi(\gamma(a))$$

閉曲線



w: 1次の微分形式

$$\int_\Omega dw = \int_\gamma w$$

微分.

$$\varphi : \mathbb{R} \to \mathbb{R}$$

$$d\varphi = \varphi' dx$$

$$(d\varphi)(x) = \varphi'(x)\,dx$$

複素数の場合は…

$$\varphi : \mathbb{C} \to \mathbb{C}$$

$$z = x + iy$$

$$dz = dx + i\,dy \quad (定義)$$

$$d\varphi = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy$$

$$= (a + bi)\,dz$$

イコールとなるとき実数と同じ形。そのための条件を調べよう.

$$= (a + bi)(dx + i\,dy)$$

$$= \underbrace{(a\,dx \boxed{-b}\,dy)}_{実部} + i\,\underbrace{(a\,dy + \boxed{b}\,dx)}_{虚部}$$

$\varphi : \mathbb{C} \to \mathbb{C}$

$\varphi = \varphi_1 + i\varphi_2$ $(\varphi_i : \mathbb{C} \to \mathbb{R})$ と書ける

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi_1}{\partial x} + i\frac{\partial \varphi_2}{\partial x}$$

$$\frac{\partial \varphi}{\partial y} = \frac{\partial \varphi_1}{\partial y} + i\frac{\partial \varphi_2}{\partial y}$$ であるから.

$$\begin{aligned} d\varphi &= \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy \\ &= \left(\frac{\partial \varphi_1}{\partial x} + i\frac{\partial \varphi_2}{\partial x}\right)dx + \left(\frac{\partial \varphi_1}{\partial y} + i\frac{\partial \varphi_2}{\partial y}\right)dy \\ &= \left(\frac{\partial \varphi_1}{\partial x}dx + \boxed{\frac{\partial \varphi_1}{\partial y}}dy\right) + i\left(\frac{\partial \varphi_2}{\partial y}dy + \boxed{\frac{\partial \varphi_2}{\partial x}}dx\right) \end{aligned}$$

したがって.

$$\boxed{\frac{\partial \varphi_1}{\partial x} = \frac{\partial \varphi_2}{\partial y}, \quad \frac{\partial \varphi_1}{\partial y} = -\frac{\partial \varphi_2}{\partial x}.}$$

Cauchy-Riemann の方程式 という.

Cauchy-Riemann の方程式が成り立つような関数を.
正則関数
解析関数 という.