On Regularity Properties of Sets of Reals and Inaccessible Cardinals

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Submitted to the Graduate School of
Pure and Applied Sciences
in Partial Fulfillment of the Requirements
for the Degree of Master of Science

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February 2016

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Introduction

It is widely known that there exist Lebesgue non-measurable sets of reals under the Axiom of Choice. One such typical example is *Vitali set*:

Theorem 0.1 (Vitali). *Assume* AC*. Then a complete system of representatives of* $\mathbb{R}/\mathbb{Q}$ *is a Lebesgue non-measurable set of reals.*

If we analyze the proof of Vitali's Theorem, we can point out crucial ingredients:

(1) Axiom of Choice used for taking representative system for $\mathbb{R}/\mathbb{Q}$,

(2) Translation invariance of Lebesgue measure μ, and

(3) σ-additivity of μ.

So the question: if we take away one or more of these conditions, can every set of reals be measurable (in a suitable sense)?

In this thesis, we will focus on the first one: use of the Axiom of Choice. In particular, we will give a proof for the following results due to Solovay and Shelah:

Theorem 0.2 (Solovay [32] and Shelah [30]). *The following two statements are equiconsistent:*

(1) ZFC+ *"the existence of an inaccessible cardinal* (IC)*".*

(2) ZF + DC+ *"Every set of reals is Lebesgue measurable* (LM)*".*

Solovay [32] showed the direction (1) $\implies$ (2) and later Shelah [30] showed (2) $\implies$ (1).

Some glossaries:

ZF and ZFC stand for Zermelo–Fraenkel set theory and one with the Axiom of Choice.

Equiconsistent theories have the same consistency strength; if one can prove the consistency from the consistency of the other and vice versa, they are said to be *equiconsistent*.

DC stands for the *Axiom of Dependent Choice*, which is the weakened form of AC:

$$\forall A\, \forall R \subseteq A \times A\ [\forall x \in A\, \exists y \in A\, (y\ R\ x) \implies \exists \{x_n\}_{n=0}^{\infty} \subseteq A\, \forall n\, x_{n+1}\ R\ x_n] \,.$$

DC is known to be strong enough to establish basic results in measure theory and analysis.

Inaccessible cardinal is a kind of large cardinal, which is enough to prove the consistency of ZFC from its existence.

Precise definition will be given in what follows.

There are many other regularity properties of sets of reals, such as Baire Property. Such properties together with Lebesgue measurability can be unified as the concept of *I-regularity* due to Khomskii. In ZFC, we can prove the existence of sets without such properties, but Khomskii [15] showed that results similar to Solovay's Theorem also hold for some class of such regularity properties. We also step into this generalization as well.

This thesis is structured as follows. First, we review preliminary results in Chapter 1. It covers basics of mathematical logic, set theory, descriptive set theory (theory of sets of reals) and theory of forcing. Readers can once skip this chapter and come back later when unfamiliar results are found in main materials.

In Chapter 2, we will give a proof of the Solovay's theorem that says "The existence of an inaccessible cardinal implies the consistency of ZF+DC+LM". In fact, we will show the generalized form of Solovay's Theorem due to Khomskii [15] mentioned above. There we also discuss on complete Ramsey Property in detail, which is another example of I-regularity.

The converse result of Shelah is proven in Chapter 3. Actually, we give a proof of Shelah's Theorem roughly based on the well-known proof by Raisonnier and its variant of Todorcevic.

We briefly investigate, in Appendix A, the system indicated by Solovay's Theorem. In such system, some desirable properties contradicting with the usual mathematics within ZFC: every set of reals is Lebesgue measurable, every functional from Banach space is continuous, and so on. This system can be regarded as the alternative foundation to develop the theory of (functional) analysis.

1. Acknowledgements

Firstly, I'm really grateful for my supervisor Masahiro Shioya, who took time for me and gave me tremendous advice. Secondly, I thank Daiske Ikegami for giving me lecture about descriptive set theory, especially on regularity properties. Also, Tomoki Mihara and Keita Yamamoto suggested me some references.

At last, of course, I really thank my parents' support for my life and study.

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1. Preliminaries

In this chapter, we will briefly review the basic preliminary results, without proofs for most parts. Since the contents of this chapter is mostly classical and well-known, readers can skip this chapter and come back when additional knowledge is needed.

1.1. Basics of Mathematical Logic and Set Theory

1.1.1. Elementary Results in Basic Set Theory and Infinite Combinatorics

In this subsection, we briefly review the basic concepts and notations of infinite combinatorics. We refer readers to Levy [19] or Kunen [16] for more detail.

First, we introduce the notational conventions and basic definitions:

Definition 1.1.

- V stands for the *universe of sets*, that is, the (proper) class of all sets.
- A class x is *transitive* if $y \in z \in x$ implies $y \in x$.

  For any class x, the *transitive closure* $\operatorname{trcl}(x)$ of x is the smallest transitive class containing x as its subset. If x is a set, then so is $\operatorname{trcl}(x)$.
- A class $(A, <)$ is *well-ordered* if it is totally ordered and its every non-empty subset has the least element. We call well-ordered class $(A, <)$ is *set-like*, if for every $x \in A$, $A_x := \{\, y \in x \mid y \mathrel{R} x \,\}$ is always a set.
- **On** denotes the collection of all ordinals. We use $<$ for the canonical well-order, which coincides with $\in$, on **On**.
- Let $A \subseteq \mathbf{On}$ be a set. We write $\sup A := \min \{\, \alpha \in \mathbf{On} \mid \forall \beta \in A \, \beta \leq \alpha \,\}$ for the *supremum* of A. $\sup^+ A = \sup \{\, \alpha + 1 \mid \alpha \in A \,\}$ stands for the *strict supremum* of A.
- **Cd** is the collection of all (well-ordered) cardinals, that is, the collection of ordinals without any bijection from the ordinal from smaller ones. The cardinals form a set-like well-ordered class with respect to canonical order $<$. We denote α-th infinite well-ordered cardinal by $\aleph_\alpha$.
- If f is a function then we write $\operatorname{dom}(f)$ and $\operatorname{ran}(f)$ for its domain and range.

  For any class A, we denote image of A by f by $f[A]$ or $f \, '' \, A$.
- Let α be an ordinal. We write $\alpha + 1$ for the immediate successor ordinal of α. We write α^+ for the smallest cardinal larger than α.
- For any ordinal α and classes A and B,

$$[A]^{<\alpha} := \{\, a \subseteq A \mid |a| < \alpha \,\}, \quad [A]^{\alpha} := \{\, a \subseteq A \mid |a| = \alpha \,\}, \quad [A]^{\leq\alpha} := [A]^{<(\alpha+1)}$$

$${}^{A}B := \{\, f \mid f : A \to B \,\}, \quad {}^{<\alpha}B := \left\{ f \;\middle|\; \begin{array}{l} f : \text{function}, \\ \operatorname{dom}(d) < \alpha, \operatorname{ran} f \subseteq B \end{array} \right\}, \quad {}^{\leq\alpha}A := {}^{<(\alpha+1)}A.$$

For cardinals κ and λ,

$$\kappa^\lambda = \left|{}^\lambda\kappa\right|, \quad \kappa^{<\lambda} = \left|{}^{<\kappa}\lambda\right|, \quad [\kappa]^\lambda = \left|[\kappa]^\lambda\right|.$$

- We define the *beth numbers* as follows:

$$\beth_0 = \aleph_0 = \omega, \quad \beth_{\alpha+1} = 2^{\beth_\alpha}, \quad \beth_\gamma = \sup_{\alpha<\gamma} \beth_\gamma \ (\gamma : \text{limit ordinal}).$$

- *Continuum Hypothesis* (CH) is the following statement: $2^\omega = \omega_1$.

  Generalized Continuum Hypothesis (GCH) is the following statement: $\forall\kappa\, 2^\kappa = \kappa^+$.
- Let κ, λ and $\langle \kappa_\alpha \mid \alpha < \lambda \rangle$ be cardinals. We write $\kappa_\alpha \nearrow \kappa$ if $\langle \kappa_\alpha \mid \alpha < \lambda \rangle$ is a strictly increasing sequence with $\kappa_\alpha < \kappa$ for each $\alpha < \lambda$ and $\sup_\alpha \kappa_\alpha = \kappa$.
- For ordinal α, the *cofinality* $\operatorname{cf} \alpha$ of α is the smallest ordinal γ such that there is some strictly increasing sequence $\langle \alpha_\eta \mid \eta < \gamma \rangle$ such that $\alpha_\eta \nearrow \alpha$.

  A cardinal κ is called *regular* if $\operatorname{cf} \kappa = \kappa$, otherwise we say that κ is *singular*.
- A cardinal κ is *strongly limit* if $2^\lambda < \kappa$ for any cardinal $\lambda < \kappa$.
- A cardinal κ is *inaccessible* if κ is an uncountable regular strongly limit cardinal.
- The *pairing function* Γ on ω is the bijection on ω defined by $\Gamma(n, m) = (n+m)(n+m+1)/2+m$.
- $\mathcal{P}(A)$ stands for the powerset of A.
- We define the cumulative hierarchy of the universe as follows:

$$V_0 := \emptyset, \quad V_{\alpha+1} := \mathcal{P}(V_\alpha), \quad V_\gamma := \bigcup_{\alpha<\gamma} V_\alpha \ (\gamma : \text{limit}).$$

  Then we have $V = \bigcup_{\alpha\in\mathbf{On}} V_\alpha$.
- For any set x, we define the *rank* of x by $\operatorname{rank}(x) := \min\{\, \alpha \in \mathbf{On} \mid x \in V_{\alpha+1} \,\}$.
- For an infinite $\kappa \in \mathbf{Cd}$, we denote the set of *sets hereditarily of size* $< \kappa$ by $H(\kappa)$. That is, $H(\kappa) := \{\, a \mid |\operatorname{trcl}(\{\, a \,\})| < \kappa \,\}$.

The size of V_α and $H(\kappa)$ have the following characterization:

Fact 1.1. *For every infinite* $\alpha \in \mathbf{On}$, $|V_\alpha| = \beth_\alpha$. *For every infinite* $\kappa \in \mathbf{Cd}$, $|H(\kappa)| = 2^{<\kappa}$.

Here are combinatorial concepts which is used throughout this thesis:

Definition 1.2. Let X be a non-empty set.

- $\mathcal{A} \subseteq \mathcal{P}(X)$ is called a *σ-algebra on X* if it contains $\emptyset$ and X, and is closed under complements, countable unions and countable intersections.

Let $\mathcal{A}$ be a σ-algebra on some set X.

- For $\mathcal{F} \subseteq \mathcal{A}$, we call $\mathcal{F}^* := \{\, X \setminus A \mid X \in \mathcal{F} \,\}$ as the *dual* of $\mathcal{F}$.

- $\mathcal{I} \subseteq \mathcal{A}$ is called an *ideal over* $\mathcal{A}$ if $\emptyset \in \mathcal{I}$, $X \notin \mathcal{I}$, $\mathcal{I}$ is downward closed (i.e. $A \subseteq B \in \mathcal{I}$ implies $A \in \mathcal{I}$) and closed under finite unions (i.e. $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$). $\mathcal{I}$ is called an *ideal on* X if it's an ideal over $\mathcal{P}(X)$.

  $\mathcal{I}$ is σ*-complete* if it's closed under countable unions. We call σ-complete ideals as σ*-ideals.*

  We define $\mathcal{I}^+ := \{ A \in \mathcal{A} \mid A \notin \mathcal{I} \}$ and call each element of $\mathcal{I}^+$ as an $\mathcal{I}$*-positive set.*

  $\mathcal{I}$ is σ*-saturated* if for any uncountable $\mathcal{B} \subseteq \mathcal{I}^+$, there is distinct $A, B \in \mathcal{B}$ with $A \cap B \in \mathcal{I}^+$.

  An ideal $\mathcal{I}$ is called *prime* if $A \cap B \in \mathcal{I}$ implies either $A \in \mathcal{I}$ or $B \in \mathcal{I}$ for every $A, B \in \mathcal{A}$.

  Ideals on σ-algebra can be viewed as the collection of “small” or “null” set.

- $\mathcal{F} \subseteq \mathcal{A}$ is called a *filter over* $\mathcal{A}$ if $X \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$, and it's upward closed (i.e. $A \supseteq B \in \mathcal{F}$ implies $A \in \mathcal{F}$) and closed under finite intersections. $\mathcal{F}$ is called *filter on* X if it's a filter over $\mathcal{P}(X)$.

  $\mathcal{F}$ is σ*-complete* if it's closed under countable intersections as well.

  A filter $\mathcal{F}$ is *maximal*, or a *ultrafilter* if either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$ holds for each $A \in \mathcal{A}$.

  Elements of a filter can be viewed as “large” or “measure one” sets.

- We call the filter $F_0 := \{ a \subseteq \omega \mid |\omega \setminus c| < \aleph_0 \}$ on ω as *Fréchet filter.*

- $\mathcal{D} \subseteq \mathcal{P}(X)$ is a Δ*-system* if there is the set $r \subseteq X$ such that for any two distinct $x, y \in \mathcal{D}$ we have $x \cap y = r$. We call such r as the *root* of $\mathcal{D}$.

Notions of filter and ideal are dual each other, so we could formulate filters as “$\mathcal{F}^*$ forms an ideal over $\mathcal{A}$” and vice versa. In addition, “prime ideals” and “ultrafilters” are mutually dual concepts obviously.

Under the axiom of choice, we can always extend every non-principal filter to some ultrafilter:

Lemma 1.2 (Boolean Prime Ideal Lemma). *Every ideal* $\mathcal{I}$ *can be extended to some prime ideal. Every filter* $\mathcal{F}$ *can be extended to some ultrafilter.*

The following Delta System Lemma is often used:

Lemma 1.3 (Delta System Lemma). *Let* $\lambda < \kappa$ *be infinite regular cardinals and suppose we have* $\theta^{<\lambda} < \kappa$ *for any* $\theta < \kappa$*. If* $\mathcal{A}$ *is a family of sets of cardinality* κ *and we have* $|A| < \lambda$ *for all* $A \in \mathcal{A}$*, then there exists a delta system* $\mathcal{B} \in [\mathcal{A}]^\kappa$*.*

1.1.2. Elementary Mathematical Logic

We introduce the basic concepts and results of Mathematical Logic here. For more detailed discussion and proofs, we refer readers to Kunen [16, 18] or Jech [11].

First we introduce the formal concept of “formulae”.

Definition 1.3. The language $\mathcal{L}_{\mathrm{ZF}}$ of set theory is the language consists with one binary predicate symbol ϵ. More precisely, for any classes M and A, $\mathcal{L}_{\mathrm{ZF}}^A(M)$*-formula* is defined recursively as follows:

(1) Let a and b be any variable or parameters in M. Then “$a \mathrel{\varepsilon} b$” and “$\mathring{A}(a)$” is $\mathcal{L}_{\mathrm{ZF}}^A(M)$-formula.

  A formula of this form is called an *atomic formula.*

(2) If φ and ψ are $\mathcal{L}^A_{\mathrm{ZF}}(M)$-formulae, then so are "$\neg\varphi$" and "$\varphi \wedge \psi$".

(3) If x is a variable and φ is a $\mathcal{L}^A_{\mathrm{ZF}}(M)$-formula, so is "$\forall x\, \varphi$".

We use following abbreviations:

$$\varphi \vee \psi :\equiv \neg(\neg\varphi \wedge \neg\psi), \quad \varphi \implies \psi :\equiv \neg\varphi \vee \psi,$$
$$\exists x\, \varphi :\equiv \neg\forall x\, \neg\varphi, \quad \exists x \in a\, \varphi :\equiv \exists x\, (x \in a \wedge \varphi), \quad \forall x \in a\, \varphi :\equiv \forall x\, (x \in a \implies \varphi).$$

We write $\mathcal{L}_{\mathrm{ZF}}(M) = \mathcal{L}^{\emptyset}_{\mathrm{ZF}}(M)$ and $\mathcal{L}_{\mathrm{ZF}} = \mathcal{L}_{\mathrm{ZF}}(\emptyset)$.

We often write $\varphi(x_1, \dots, x_n)$ to indicate the formula φ has its variables in $x_1, \dots, x_n$. If $t_1, \dots, t_n$ are parameters, then $\varphi(t_1, \dots, t_n)$ is the formula obtained from φ by replacing each x_i by t_i.

Then we can define the *interpretation* of $\mathcal{L}_{\mathrm{ZF}}(M)$-formulae as follows:

Definition 1.4. Let $N \subseteq M$ be sets. We define the relation $M \models \varphi$ (pronounce: *M models φ* or *M satisfies φ*) between set M and $\mathcal{L}^A_{\mathrm{ZF}}(N)$-formula φ recursively:

(1) $M \models$ "$x \mathrel{\varepsilon} y$" if $x \in y$.

(2) $M \models$ "$\mathring{A}(x)$" if $x \in A \cap M$

(3) $M \models$ "$\varphi \wedge \psi$" if both $M \models \varphi$ and $M \models \psi$ hold.

(4) $M \models$ "$\neg\varphi$" if $M \not\models \varphi$.

(5) $M \models$ "$\forall x\, \varphi$" if $M \models \varphi(x)$ holds for all $x \in M$.

Let A be a class and M a set. We define the set $\mathrm{Def}^A(M)$ of *definable sets* in M with A as follows:

$$a \in \mathrm{Def}^A(M) \overset{\text{def}}{\iff} a = \{\, b \in B \mid M \models \varphi(b) \,\} \text{ for some } \varphi(x) \in \mathcal{L}^A_{\mathrm{ZF}}(M).$$

Above satisfaction relation for *sets* can be defined within the object-level. But we cannot use the same definition for *proper classes*. For that purpose, we use the meta-operation called *relativization*:

Definition 1.5. Let M and A be definable (possibly proper) classes and φ a formula. The *relativization* φ^M of φ to M is the formula obtained from φ by replacing all quantifiers $\forall x$ by $\forall x \in M$.

It is clear by definition that we have $M \models$ "φ" $\iff \varphi^M$ if M is a set. But, in set theory, we often use a proper class as a model of set theory hence we have to use relativization instead of the satisfaction relation. Next, we define the hierarchy of formulae:

Definition 1.6. We define the hierarchy $\Delta_n, \Sigma_n, \Pi_n$ of formulae in $\mathcal{L}^A_{\mathrm{ZF}}(M)$ as follows:

(1) φ is Δ_0, Σ_0 and Π_0 if every quantifier in φ in *bounded*; i.e. every quantifiers in φ is of form "$\forall x \in a$", with x and a variables.

(2) $\varphi(x)$ is Σ_{n+1} if $\varphi(x) \equiv \exists y_1 \cdots \exists y_n\, \psi(x, y_1, \dots, y_n)$ for some Π_n-formula $\psi(x, y_1, \dots, y_n)$.

(3) $\varphi(x)$ is Π_{n+1} if $\varphi(x) \equiv \forall y_1 \cdots \forall y_n\, \psi(x, y_1, \dots, y_n)$ for some Σ_n-formula $\psi(x, y_1, \dots, y_n)$.

(4) $\varphi(x)$ is Δ_n^Γ if there are Σ_n-formula $\psi(x)$ and Π_n-formula $\theta(x)$ such that

$$\Gamma \vdash \text{“}\forall x[\varphi(x) \iff \psi(x) \iff \theta(x)]\text{”}.$$

We often drop Γ if it's clear from the context.

We will treat the multiple models of set theory, so we have to consider what concepts or properties are preserved among many models.

Definition 1.7.
- M is a *transitive model* of a theory Γ under the ambient theory Σ if M is transitive class and for any $\varphi \in \Gamma$ we have $\Sigma \vdash \varphi^M$.
- Let $N \subseteq M$ and A be (possibly proper) classes.

  A $\mathcal{L}_{\mathrm{ZF}}^A(N)$-formula $\varphi(\vec{x})$ is *upward absolute between N and M* if $\varphi^N(\vec{a})$ then $\varphi^M(\vec{a})$ for all $\vec{a} \in N$. $\varphi(\vec{x})$ is *downward absolute between* N and M if $\varphi^M(\vec{a})$ then $\varphi^N(\vec{a})$ for all $\vec{a} \in N$.

  We say that φ is *absolute between N and M*, denoted by $N \prec_\varphi M$, if it is both upward and downward absolute

  A theory $\Sigma \subseteq \mathcal{L}_{\mathrm{ZF}}^A(N)$ is *absolute* between N and M (notation: $N \prec_\Sigma N$) if we have $N \prec_\varphi M$ for every $\varphi \in \Sigma$. We also say that N is Σ-*elementary submodel* of M or M is Σ-*elementary extension* of N in this situation. In particular, if $N \prec_\varphi M$ for any formula φ then we write just $N \prec M$ and call N as a *elementary submodel* of M and M a *elementary extension* of N.

  We call φ (or Σ) is *absolute for N* if φ (or Σ) is absolute between N and V.
- Absoluteness of predicates means the absoluteness of its defining formula.

  Function F defined by a formula $F(x) = y \iff \psi(x, y)$ is *absolute* between N and M if $\psi(x, y)$ defines functions F^N in N and F^M in M, and $F^N(a) = b \iff F^M(a) = b$ for all $a, b \in N$.

The following three lemmas are often used to establish absoluteness of many concepts:

Lemma 1.4. *Every Δ_0-concept is absolute between any transitive models.*

Lemma 1.5. *Let A be a definable class, R a definable, set-like well-founded binary relation, and G a definable binary relation. Let F be a function defined recursively by the following formula:*

$$\forall a \in A\, F(a) = G(a, F \restriction A_x).$$

Suppose that M is some transitive model of $\mathrm{ZF} - \mathrm{PowerSet}$*, R, A and G are all absolute for M, $(R : \textit{set-like})^M$ and $x \in M \cap A \implies A_x \subseteq M$. Then $F^M(a)$ is defined on $A \cap M$ and F is absolute for M.*

Lemma 1.6. *Every Σ_1-concept is upward absolute between any transitive models. Every Π_1-concept is downward absolute between any transitive models.*
In particular, Every Δ_1^Γ-concept is absolute between any transitive model of Γ.

Proof of Lemmas 1.4, 1.5 and 1.6. See Kunen [18, 16]. ☐

In particular, essentially “finite” objects are absolute for any transitive model:

Lemma 1.7. *As a unary predicate, ω and V_ω are absolute for any transitive model.*

With this in mind, we define the following concept:

Definition 1.8. $\mathcal{L}_{\mathrm{ZF}}^{A}$-formula $\varphi(\vec{x})$ is called *arithmetical* in A if all the quantifiers in φ are bounded to V_ω.

Corollary 1.8. *Every arithmetical concepts are absolute for any transitive model.*

Below, we list absolute concepts which is fairly often used:

Fact 1.9. *The following concepts are downward absolute:*

- *Predicate "κ is cardinal",*
- *Predicate "κ is regular".*

Concepts below are all absolute between transitive models:

- *f is a function,*
- *$f : x \to y$,*
- *f is a surjection,*
- *f is an injection,*
- *f is an bijection,*
- *$\alpha = \sup A$,*
- *$f(x) = a$,*
- *Function Def^A,*
- *$\alpha \in \mathbf{On}$,*
- *$x \in V_\alpha$,*
- *$\mathrm{rank}(x) = \alpha$.*

The following (meta-)theorem enables us operation in the opposite direction in some sense:

Theorem 1.10 (Reflection Theorem). *For each formula $\varphi(\vec{x})$ in the language of set theory, there exists $\alpha \in \mathbf{On}$ such that $V_\alpha \prec_\varphi V$.*

1.2. Measure Theory and Descriptive Set Theory

In this section, we will review the basic notions and facts in descriptive set theory (the theory of sets of reals). For detailed exposition, we refer readers to Kechris [14], Moschovakis [20] or Jech [11, Chapters 25 and 32].

We develop the theory in this section in the system ZF + DC, and we explicitly mention the system for results which needs AC or provable within weaker system than ZF + DC.

First, we will define the precise definition of "reals" and give the basic theory of sets of reals.

Definition 1.9 (Baire space and Cantor space).

- The *Baire space* ${}^{\omega}\omega$ is the collection of infinite sequences of natural numbers topologized with basic open sets of the following form:

$$[s] := \{\, a \in {}^{\omega}\omega \mid s \subseteq a \,\} \qquad (s \in {}^{<\omega}\omega).$$

  ${}^{\omega}\omega$ is homeomorphic to the space of irrational numbers with subset topology (via continued fractions).

- The *Cantor space* ${}^{\omega}2$ is the collection of infinite sequences of $\{0,1\}$ topologized with basic open sets $\langle [s] \mid s \in {}^{<\omega}2 \rangle$ of similar form as Baire space but with its range restricted to $2 = \{0,1\}$.

  Note that ${}^{\omega}2$ can be regarded as the famous Cantor set, or subspace of ${}^{\omega}\omega$. Note that ${}^{\omega}2$ is a compact space, which differs with Baire space.

We call each element $z \in {}^{\omega}2$ or $z \in {}^{\omega}\omega$ as a *real.* This terminology is justified because the collection of rational numbers is countable and we interested only in notions in which we can ignore countable exceptions.

Note that, already in ZF, both ${}^{\omega}\omega$ and ${}^{\omega}2$ are *Polish spaces*; i.e. separable and completely metrizable. Most of below can be generalized to arbitrary uncountable Polish space; see Moschovakis [20].

The following concept of Borel sets is central one:

Definition 1.10. For a topological space $(X, \mathscr{O})$, $\mathscr{B}(X, \mathscr{O})$ is the smallest σ-algebra containing all basic open sets of X. A set $B \in \mathscr{B}(X, \mathscr{O})$ is called a *Borel set.*

We often write $\mathscr{B}$ for $\mathscr{B}(X, \mathscr{O})$ if $(X, \mathscr{O})$ is clear from the context.

The following operations are often used to define classes of real sets:

Definition 1.11. For $A \subseteq X$, $\neg A := X \setminus A$. For $A \subseteq X \times Y$, we define

$$\exists^Y A := \{\, x \in X \mid \exists y \in Y \; (x,y) \in A \,\}, \qquad \forall^Y A := \neg \exists^Y \neg A.$$

We define, for $\Gamma \subseteq \mathcal{P}(X)$, $\neg \Gamma := \{\, \neg A \mid A \in \Gamma \,\}$. Similar for $\exists^Y \Gamma$ and $\forall^Y \Gamma$.

With operations above, we have the following characterization of Borel sets:

Fact 1.11 (ZF + CC)**.** *We define* $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Delta}^0_\alpha$ $(\alpha < \omega_1)$ *by the transfinite recursion as follows:*

$$\begin{aligned}
\mathbf{\Sigma}^0_1 &:= \{\, \textit{the collection of all open sets in } {}^{\omega}\omega \,\} \\
\mathbf{\Pi}^0_1 &:= \{\, \textit{the collection of all closed sets in } {}^{\omega}\omega \,\} = \neg \mathbf{\Sigma}^0_1 \\
\mathbf{\Sigma}^0_\alpha &:= \left\{ \bigcup_{n<\omega} A_n \;\middle|\; \{A_n\}_{n<\omega} \subseteq \bigcup_{\xi<\alpha} \mathbf{\Pi}^0_\xi \right\} \\
\mathbf{\Pi}^0_\alpha &:= \left\{ \bigcap_{n<\omega} A_n \;\middle|\; \{A_n\}_{n<\omega} \subseteq \bigcup_{\xi<\alpha} \mathbf{\Sigma}^0_\xi \right\} = \neg \mathbf{\Sigma}^0_\alpha.
\end{aligned}$$

Then $\mathscr{B} = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha$.

Above fact suggests the following way to *code* Borel sets by reals:

Definition 1.12.

- T is a *tree on* $\alpha_1 \times \cdots \times \alpha_n$ if $T \subseteq \bigcup_{k<\omega}(\alpha_1^k \times \cdots \times \alpha_n^k)$ and closed under initial segments i.e. $\forall t = (t_1, \ldots, t_n) \in T \, \forall k < \mathrm{lh}(t) \; t \restriction k = (t_1 \restriction k, \ldots t_n \restriction k) \in T$.

  We regard T as ordered by the reverse inclusion $\supseteq$.

- For tree T and $t \in T$,

  $$\begin{aligned}
  \mathrm{succ}_T(t) &:= \{\, s \in T \mid \exists k < \omega \; s = t ^\frown k \,\} \\
  \mathrm{term}(T) &:= \{\, s \in T \mid \nexists s' \in T \; s' \supsetneq s \,\}
  \end{aligned}$$

  That is, $\mathrm{succ}_T(t)$ is the collection of $\supset$-immediate predecessors of t in T, and $\mathrm{term}(T)$ is the collection of $\supset$-minimal elements of T.

- A tree T is called *well-founded* if T has no infinite strictly $\supset$-descending sequence.
- (T, σ) is called a *Borel code* if T is a well-founded tree on ω and $\sigma : \mathrm{term}(T) \to {}^{<\omega}\omega$. We denote the collection of Borel codes as $\mathbf{BC}$.
- For $c = (T, \sigma) \in \mathbf{BC}$ and $t \in T$, we define Borel set $B_{c,t}$ coded by σ by the recursion on t:

$$B_{c,t} := \begin{cases} [\sigma(t)] & (\mathrm{succ}_T(t) = \emptyset) \\ \neg B_{c,s} & (\mathrm{succ}_T(t) = \{ s \}) \\ \bigcup_{s \in \mathrm{succ}_T(t)} B_{c,s} & (\text{otherwise}). \end{cases}$$

  We write $B_c := B_{c,\emptyset}$.

Note that each Borel code can be regarded as a pair of functions from ω to ω, so we can identify each Borel code $c \in \mathbf{BC}$ with some real, via some suitable bijection.

By easy induction on α, we can show that every $\mathbf{\Sigma}^0_\alpha$-set can be coded by some Borel code, and vice versa:

Fact 1.12. *If $c \in \mathbf{BC}$ then B_c is a Borel set. In the system* ZF + CC*, the converse also holds: every Borel set $B \in \mathscr{B}$ can be expressed as $B = B_c$ for some $c \in \mathbf{BC}$.*

Now, we can define the Lebesgue measure on Borel sets as follows:

Definition 1.13.
- For each $s \in {}^{<\omega}\omega$, we define *measure* $\mu([s])$ on basic open set $[s]$ by induction on $\mathrm{lh}(s)$:

$$\mu([\emptyset]) := 1, \quad \mu([s ^\frown n]) := \frac{1}{2^{n+1}} \mu([s]).$$

- For each open set $U = \bigsqcup_{n<\omega} [s_n]$ (disjoint union), let

$$\mu(U) := \sum_{n<\omega} \mu([s_n]).$$

- For each closed set A, let $\mu(A) := 1 - \mu({}^\omega\omega \setminus A)$.
- Having defined μ on $\mathbf{\Pi}^0_\xi$ and $\mathbf{\Sigma}^0_\xi$ for all $\xi < \alpha$, if $A \in \mathbf{\Sigma}^1_\alpha$ and $A = \bigcup_n B_n$ for $\{ B_n \}_n \subseteq \bigcup_{\xi<\alpha} \mathbf{\Pi}^0_\xi$, then let

$$\mu(A) := \sup_{k<\omega} \mu\left(\bigcup_{n<k} B_n \right).$$

  Note that every $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^1_\alpha$ sets are closed under finite unions.
- If $A \in \mathbf{\Pi}^1_\alpha$, let $\mu(A) := 1 - \mu({}^\omega\omega \setminus A)$.

We define a measure on Cantor space ${}^\omega 2$ similarly, but with a slight modification to get the whole space have measure 1 as follows:

$$\mu([s ^\frown n]) = \frac{1}{2} \mu([s]).$$

Then we can define the Lebesgue measurability for general sets of reals:

Definition 1.14. For any $A \subseteq {}^{\omega}\omega$ or ${}^{\omega}2$, we define *outer measure* $\mu^*(A)$ *of* A as follows:

$$\mu^*(A) := \inf \{ \mu(B) \mid B \in \mathscr{B} \wedge A \subseteq B \} .$$

Then A is said to be *null* if $\mu^*(A) = 0$. We denote the collection of Lebesgue null sets by null. This forms an ideal on $\mathscr{B}$, and, in ZF + CC, it is σ-saturated σ-ideal.

A is *Lebesgue measurable* (or simply *measurable*) if there exists $B \in \mathscr{B}$ such that $A \triangle B$ is null. For any measurable A, we write $\mu(A) = \mu^*(A)$ and call it as *Lebesgue measure* of A.

This definition seems somewhat different from usual definition of Lebesgue measurability for subsets of $\mathbb{R}$. Carefully analyzing the above definition, we can see that they are certainly coincides, because two definitions of "null" coincides.

Next, we will discuss on some topological properties:

Definition 1.15. Let $(X, \mathscr{O})$ be a topological space.

- We denote the interior, closure and exterior of $S \subseteq X$ in the topology $\mathscr{O}$ by $\operatorname{int}_{\mathscr{O}} S$, $\operatorname{cl}_{\mathscr{O}} S$ and $\operatorname{ext}_{\mathscr{O}} S$ respectively. We often drop $\mathscr{O}$ if it's clear from the context.
- $E \subseteq X$ is *regular* open set if $\operatorname{int} \operatorname{cl} E = E$. We denote the set of regular open sets by $\operatorname{ro}(X, \mathscr{O})$.
- $A \subseteq X$ is *nowhere dense* if for any open $U \subseteq X$, there is some open set $V \subseteq A$ with $U \cap V = \emptyset$.
- $A \subseteq {}^{\omega}\omega$ is said to be *meager* (or, *of first category*) if it is a countable union of nowhere dense sets. A subset of X which is non-meager is called *of second category.* $A \subseteq X$ is *comeager* if $X \setminus A$ is meager.

  The collection of all meager Borel sets forms σ-saturated σ-ideal and we denote it by meager.
- $A \subseteq X$ has *Baire property* if there exists open set $B \in \mathscr{O}$ such that $A \triangle B$ is meager.

Next, we will define *projective sets* which play an important role:

Definition 1.16.

- For tree T on $\alpha_1 \times \cdots \times \alpha_n$, (*cofinal*) *path through* T is defined by:

$$[T] := \left\{ (y_1, \ldots, y_n) \in {}^{\omega}\omega \times \prod_{1 \leq i \leq n} {}^{\omega}\alpha_i \;\middle|\; \forall k < \omega \, (y_1 \restriction k, \ldots, y_n \restriction k) \in T \right\}.$$

  An (ℓ-th) *projection of* T is defined by:

$$\begin{aligned} p_\ell[T] &:= \exists^{{}^{\omega}\alpha_{\ell+1} \times \cdots \times {}^{\omega}\alpha_n} [T] \\ &= \{ (y_1, \ldots, y_\ell) \mid \exists \vec{y} \, (y_1, \ldots, y_\ell, \vec{y}) \in [T] \} . \end{aligned}$$

  We write $p[T]$ for the first projection $p_1[T]$ of T.
- $A \subseteq ({}^{\omega}\omega)^k$ is said to be α-*Suslin* if there exists some tree T on $\omega^k \times \alpha$ such that $A = p_k[T]$. An ω-Suslin set is called *analytic* and a complement of analytic set is called *coanalytic.*

- For any $k < \omega$,

$$\begin{aligned}
\mathbf{\Sigma}^1_1(({}^{\omega}\omega)^k) &:= \left\{ A \subseteq ({}^{\omega}\omega)^k \;\middle|\; A : \text{analytic} \right\} \\
\mathbf{\Pi}^1_1(({}^{\omega}\omega)^k) &:= \left\{ A \subseteq ({}^{\omega}\omega)^k \;\middle|\; A : \text{coanalytic} \right\} = \neg\, \mathbf{\Sigma}^1_1(({}^{\omega}\omega)^k) \\
\mathbf{\Sigma}^1_{n+1}(({}^{\omega}\omega)^k) &:= \exists^{{}^{\omega}\omega}\, \mathbf{\Pi}^1_n(({}^{\omega}\omega)^{k+1}), \quad \mathbf{\Pi}^1_{n+1}(({}^{\omega}\omega)^k) := \neg\, \mathbf{\Sigma}^1_{n+1}(({}^{\omega}\omega)^k) \\
\mathbf{\Delta}^1_n(({}^{\omega}\omega)^k) &:= \mathbf{\Sigma}^1_n(({}^{\omega}\omega)^k) \cap \mathbf{\Pi}^1_n(({}^{\omega}\omega)^k).
\end{aligned}$$

We say $A \subseteq ({}^{\omega}\omega)^k$ is *projective* if it belongs to some $\mathbf{\Delta}^1_n(({}^{\omega}\omega)^k)$.
We define projective sets in Cantor space ${}^{\omega}2$ in the similar way.

In particular, trees on ω corresponds to closed subset of ${}^{\omega}\omega$:

Lemma 1.13 (ZF)**.** *If T is a tree on ω, then $[T] \subseteq {}^{\omega}\omega$ is closed subset of ${}^{\omega}\omega$. Conversely, if $F \subseteq {}^{\omega}\omega$ is closed subset then the tree $T(F) := \{ f \restriction n \mid f \in F, n < \omega \}$ on ω satisfies $F = [T(F)]$.*
Similar result also holds for Cantor space ${}^{\omega}2$.

The following fact states the structure of projective hierarchy:

Fact 1.14. *In any uncountable Polish space, the following (proper) inclusion holds*

$$\begin{tikzcd}
 & \mathbf{\Sigma}^1_1 \arrow[dr, phantom, sloped, "\subsetneq"] & & \mathbf{\Sigma}^1_2 \arrow[dr, phantom, sloped, "\subsetneq"] & & \mathbf{\Sigma}^1_3 & \ddots \\
\mathbf{\Delta}^1_1 \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & & \mathbf{\Delta}^1_2 \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & & \mathbf{\Delta}^1_3 \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & & \\
 & \mathbf{\Pi}^1_1 \arrow[ur, phantom, sloped, "\subsetneq"] & & \mathbf{\Pi}^1_2 \arrow[ur, phantom, sloped, "\subsetneq"] & & \mathbf{\Pi}^1_3 & \iddots
\end{tikzcd}$$

The notion of projective sets subsumes Borel sets; in fact, Borel sets are exactly the same as $\mathbf{\Delta}^1_1$-sets:

Fact 1.15. $\mathscr{B} = \mathbf{\Delta}^1_1$.

Analytic sets have following characterization:

Fact 1.16. *Following are equivalent:*

(1) *$A \subseteq ({}^{\omega}\omega)^k$ is analytic.*

(2) *A is a continuous image of some Borel set in some $({}^{\omega}\omega)^{\ell}$.*

(3) *A is a projection of some Borel set in some $({}^{\omega}\omega)^k \times ({}^{\omega}\omega)^{\ell}$.*

(4) *A is a projection of some closed set in some $({}^{\omega}\omega)^k \times ({}^{\omega}\omega)^{\ell}$.*

By this fact, we can adopt (2) above as the definition of analytic sets for general Polish spaces, and define projective hierarchy starting from that.

Note that above $\mathbf{\Sigma}, \mathbf{\Pi}, \mathbf{\Delta}$ are all typeset in *boldface*. There are another *lightface* version $\Sigma^1_n, \Pi^1_n, \Delta^1_n$ of *effective projective hierarchy*:

Definition 1.17.
- For $x \in {}^{\omega}\omega$ and $A \subseteq ({}^{\omega}\omega)^k$, A is $\Sigma^1_1(x)$ if A is a projection of some tree T on $\omega^k \times \omega$ which is definable over the structure $(V_\omega, \in, x)$.

- $\Pi^1_n(x) := \neg\Sigma^1_n(x), \Sigma^1_{n+1} := \exists^{{}^\omega\omega}\, \Pi^1_n(x), \Delta^1_n(x) := \Sigma^1_n(x) \cap \Pi^1_n(x)$.
- $\Sigma^1_n := \Sigma^1_n(0), \Pi^1_n := \Pi^1_n(0), \Delta^1_n = \Delta^1_n(0)$.

The following fact illustrates how effective and non-effective projective sets relate:

Fact 1.17. *$A \in \boldsymbol{\Sigma}^1_n \iff A \in \Sigma^1_n(x)$ for some $x \in {}^\omega\omega$. Similar facts hold for Π^1_n and Δ^1_n.*

There is also a handy characterization of lightface projective sets:

Fact 1.18. *The following are equivalent:*

(1) *$A \in \Sigma^1_n(x)$.*

(2) *There exists a formula $\varphi[a, b, \vec{c}]$, where a, b and $\vec{c}$ are all new predicate symbols, such that*

$$A = \left\{ y \in ({}^\omega\omega)^k \;\middle|\; \exists z_1 \in {}^\omega\omega\, \forall z_2 \in {}^\omega\omega \ldots \mathsf{Q}_n z_n \in {}^\omega\omega\, (V_\omega, \in, x, y, \vec{z}) \models \varphi[x, y, \vec{z}] \right\},$$

where $\mathsf{Q}_k = \begin{cases} \forall & (k : \textit{even}) \\ \exists & (k : \textit{odd}) \end{cases}$

Similar fact holds for $\Pi^1_n(x)$ with the order of quantifiers interleaved.

This fact suggests the following definition of the class of formula:

Definition 1.18. We call formulae of the following form as *projective formula*:

$$\mathsf{Q}_1 x_1 \in {}^\omega\omega \;\cdots\; \mathsf{Q}_n x_n \in {}^\omega\omega\; \varphi,$$

where, φ is some arithmetical formula and each Q_i is either $\forall$ or $\exists$. Notions defined by some projective formula is also called *projective* as well.

In the context of descriptive set theory, the following absoluteness result concerning projective hierarchy due to Mostwski is useful:

Lemma 1.19 (Mostwski Absoluteness). *Every $\boldsymbol{\Pi}^1_1$-concepts (and hence $\boldsymbol{\Sigma}^1_1$-concepts) are absolute for all sufficiently large transitive models.*

Proof. See Jech [11]. ☐

In the lemma above, "sufficiently large" means that it satisfies enough axioms of ZFC to prove the $\boldsymbol{\Pi}^1_1$-ness.

1.2.1. More on Measure and Category in Cantor Space ${}^\omega 2$

Here, we will see classical results on measure and category mainly in Cantor space ${}^\omega 2$. We refer readers to Oxtoby [21] for more detailed discussion on the duality of measure and category.

The following theorems relate regularity properties in product spaces and ones in its factor space:

Theorem 1.20 (Fubini Theorem for Measure). *Let X, Y be Polish space and $A \subseteq X \times Y$. A is null if and only if $A^y = \{\, x \in X \mid (x, y) \in A \,\}$ is null for all $y \in Y$ but measure zero set of Y, if and only if $A_x = \{\, y \in Y \mid (x, y) \in A \,\}$ is null for all $x \in X$ but measure zero set of X.*

Theorem 1.21 (Fubini Theorem for Category). *Let X, Y be Polish spaces and $A \subseteq X \times Y$. Then A is meager if and only if A^y is meager for all $y \in Y$ but meager set of Y, if and only if $A_x = \{ y \in Y \mid (x, y) \in A \}$ is meager for all $x \in X$ but meager set of X.*

Note that, both of above theorems can be established already in ZF + CC, if we restrict X and Y to ${}^{\omega}2$, ${}^{\omega}\omega$ or finite discrete spaces.

Next, we define the concept of *tail set.*

Definition 1.19. $A \subseteq {}^{\omega}\omega$ is *tail* if for any $n < \omega$, there exists $A_n \subseteq {}^{\omega}\omega$ such that $A = {}^{n}2 \times A_n$. In other words, A is a tail set if memberships of A are determined only by the *tail* segments.

The following two results are classical facts stating that tail sets have "Zero-One" property in some senses:

Lemma 1.22 (Zero-One Law for Measure, ZF + CC**).** *If $A \subseteq {}^{\omega}\omega$ is a measurable tail set, then either $\mu(A) = 1$ or $\mu(A) = 0$.*

Proof. Since A is a tail set, we can fix $B_n \subseteq {}^{\omega}2$ such that $A = {}^{n}2 \times B_n$ for each $n < \omega$. Note that, by definition, we have $\mu(B_n) = \mu({}^{n}2 \times B_n) = \mu(E)$ for each n.

It suffices to show that $\mu(E) = \mu(E) \times \mu(E)$, which follows from the following more general statement:

$$\forall A \subseteq {}^{\omega}2 : \text{measurable}\, \mu(E \cap A) = \mu(E)\mu(A)$$

By σ-additivity of the Lebesgue measure, which follows from CC, we may assume that A to be some basic open set $[s]$ for some $(s \in {}^{<\omega}2)$. Well, letting $s \in {}^{n}2$ we have

$$\mu(E \cap [s]) = \mu\left(({}^{n}2 \times B_n) \cap (\{ s \} \times {}^{\omega \setminus n}2)\right) = \mu(\{ s \} \times B_n) = \mu([s]) \times \mu(E).$$

□

Lemma 1.23 (Zero-One Law for Category). *If $A \subseteq {}^{\omega}\omega$ has a tail set with Baire Property, then A is either meager or comeager.*

Proof. As above, we fix $B_n \subseteq {}^{\omega}2$ such that $A = {}^{n}2 \times B_n$ for each $n < \omega$.

We suppose that A is not comeager and show A is meager. In this case, since A and ${}^{\omega}2 \setminus A$ has Baire property, there is some non-empty open set $G \subseteq {}^{\omega}2$ with $M := G \mathbin{\triangle} A \in \mathsf{meager}$. In particular, we can pick some $s \in {}^{<\omega}2$ with $[s] \subseteq G$. Letting $n := \mathrm{lh}(s)$, we have $A = {}^{n}2 \times B_n$ and hence $[s] \cap A = \{ s \} \times B_n$. Then we have $\{ s \} \times B_n \subseteq G \subseteq ({}^{\omega}2 \setminus E) \cup M$ and hence $\{ s \} \times B_n \subseteq M$, which means $\{ s \} \times B_n$ is meager. Therefore, by Fubini Theorem 1.21 for Category, at least one of $\{ s \}$ or B_n must be meager. But, since $\{ s \}$ is a point in a discrete space ${}^{n}2$, it's not meager and B_n must be meager. Then, again by Theorem 1.21, $A = {}^{n}2 \times B_n$ should be meager, which was what we wanted. □

Here is an immediate result on measurability of filters, which follows from above two lemma:

Lemma 1.24 (Sierpinski, ZF + CC**).** *Every measurable filter $\mathcal{F}$ extending Fréchet filter is null.*

Proof. First note that $\mathcal{F}$ is a tail set. Hence, by Zero-One Law for Measure (Fact 1.22), we must have $\mu(\mathcal{F}) = 0$ or 1. But the latter case does not hold. Let $T : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ by $h(a) := \omega \setminus a$. Clearly T preserves measurability and measure. In particular, $\mu(T(\mathcal{F})) = \mu(\mathcal{F})$. But we have $T(\mathcal{F}) \cap \mathcal{F} = \emptyset$, and hence

$$1 \geq \mu(T(\mathcal{F})) + \mu(\mathcal{F}) = 2\mu(\mathcal{F}).$$

It follows that $\mathcal{F}$ must be null. ☐

Under the Axiom of Choice, it's well-known that there exists a set without Baire property and non-measurable:

Theorem 1.25 (Vitali). *Under* AC*, there exists a set of real which is non-measurable and without Baire property.*

Proof. We give somewhat different proof of this theorem, specialized to Cantor space ${}^{\omega}2$ rather than the real line $\mathbb{R}$.

Since we assume AC, we can take some non-principle ultrafilter $\mathcal{U} \subsetneq \mathcal{P}(\omega)$ extending Fréchet filter by Prime Ideal Lemma 1.2.

First suppose $\mathcal{U}$ is Lebesgue measurable. Then, by Sierpinski's Lemma 1.24, we have $\mu(\mathcal{U}) = 0$. But, since the mapping T used above is measure preserving map and $\mathcal{U}$ is an ultrafilter, we have $\mu({}^{\omega}2) = \mu(\mathcal{U}) + \mu(T[\mathcal{U}]) = 0$, which is contradiction.

Next, we show that $\mathcal{U}$ is without property of Baire. Well, if $\mathcal{U}$ has Baire Property, it should be either meager or comeager by Zero-One Law for Category (Fact 1.23). Since T is clearly a homeomorphism so we have that $\mathcal{U}$ is (co)meager if and only if $T[\mathcal{U}]$ is (co)meager. But ${}^{\omega}2 = \mathcal{U} \cup T[\mathcal{U}]$, this cannot be happen! ☐

Also, well-known Baire's Category Theorem can be proven in ZF + DC with the usual proof:

Theorem 1.26 (Baire's Category Theorem). *Every complete metric space is of second category, i.e. not meager. In other words, intersections of countable open dense set of a complete metric space remains to be dense.*

Actually, Baire's Category Theorem restricted to ${}^{\omega}2$, ${}^{\omega}\omega$ and their closed subspaces can be established within ZF:

Proof of Baire's Category Theorem for closed subsets of ${}^{\omega}\omega$ or ${}^{\omega}2$, in ZF. Let $K \neq \emptyset$ be closed subspace of ${}^{\omega}2$ or ${}^{\omega}\omega$ and $U_n \cap K$ be dense open subsets in K.

Fix arbitrary $z \in K$ and $\ell < \omega$. We pick $s_n \in {}^{<\omega}\omega$ such that $z \upharpoonright \ell \subseteq s_n \subsetneq s_{n+1}$ and $\emptyset \neq [s_n] \cap K \subseteq U_n$ holds for each $n < \omega$. We take such s_n's by induction.

For $n = 0$, we can just pick $z \upharpoonright \ell \subseteq s_0 \in {}^{<\omega}\omega$ so that $\emptyset \neq [s_0] \cap K \subseteq U_0 \cap K$, because $U_0 \cap K$ is open dense in K. In particular, we can pick s_0 to be the least element w.r.t. the canonical well-ordering on ${}^{<\omega}\omega$. Having s_n defined, we pick s_{n+1}. Since $U_{n+1} \cap K$ is dense in K, we have $[s_n] \cap U_{n+1} \cap K \neq \emptyset$. Hence there is some $s_{n+1} \supsetneq s_n$ with $[s_{n+1}] \cap K \subseteq U_{n+1} \cap K$. Again, we can pick s_{n+1} as the least such s w.r.t. the canonical well-ordering on ${}^{<\omega}\omega$.

Since each s_n's taken in the above construction can be uniquely determined, we can do this within ZF. So let $x := \bigcup_n s_n$. By construction, x is a real and belongs to each $[s_n] \cap U_n$. In addition, because we have $[x \upharpoonright n] \cap K \neq \emptyset$ for each n, it must be the case that $x \in K$ since K is closed. So we have $x \in K \cap \bigcap_n U_n$. In particular, $x \in [z \upharpoonright \ell]$ and hence $[z \upharpoonright \ell] \cap \bigcap_n U_n \cap K \neq \emptyset$.

This shows that every basic neighbourhood of z intersects with $\bigcap_n U_n \cap K$ and hence it's dense. ☐

Definition 1.20. Let $A \subseteq {}^{\omega}2$.

- For $s \in {}^{<\omega}2$, $A_{\lfloor s \rfloor} := \{ x \in {}^{\omega}2 \mid s^\frown x \in A \}$. Note that, by the definition of the standard measure of ${}^{\omega}2$, we have
$$\mu(A \cap [s]) = \mu([s])\mu(A_{\lfloor s \rfloor})$$
- We say A has *density* $d_A(x)$ at x if the following limit exists:
$$d_A(x) := \lim_{n\to\infty} \mu(A_{\lfloor x\restriction n \rfloor}) = \lim_{n\to\infty} \frac{\mu([x \restriction n] \cap B)}{\mu([x \restriction n])}.$$

Theorem 1.27 (Lebesgue Density Theorem). *If B is measurable, then B has density 1 at almost everywhere in ${}^{\omega}2$. More precisely, if we let $\Phi(B) := \{ x \in {}^{\omega}2 \mid d_B(x) = 1 \}$ then $\mu(B \triangle \Phi(B)) = 0$.*

Proof. It suffices to show that $E \setminus \Phi(E)$ is null. Let
$$A_\varepsilon := \left\{ x \in E \;\middle|\; \liminf_{n\to\infty} \frac{\mu(E \cap [x \restriction n])}{\mu([x \restriction n])} < 1 - \varepsilon \right\} \quad (\varepsilon \in (0,1)).$$

Then we can write
$$E \setminus \Phi(E) = \bigcup_{\varepsilon \in \mathbb{Q}_{>0}} A_\varepsilon.$$

With these in mind, we will show this theorem by contradiction: suppose there is $A \subseteq {}^{\omega}2$ with $A = A_\varepsilon$ for some $\varepsilon \in \mathbb{Q}^+$ and $\mu^*(A) > 0$.

Then, for such an A, by the definition of outer measure, there must be open set $G \supseteq A$ such that
$$\mu(G) \leq \frac{\mu^*(A)}{1-\varepsilon}.$$

Let us define
$$\mathscr{E} := \{ [s] \mid [s] \subseteq G, \mu(E \cap [s]) \leq (1-\varepsilon)\mu([s]) \}.$$

Then $\mathscr{E}$ satisfies following properties:

(i) $\mathscr{E}$ has as an element every sufficiently small open neighbourhoods of each point of A.

(ii) For any pairwise disjoint family $\{ [s_n] \mid n < \omega \} \subseteq \mathscr{E}$, we have $\mu^*(A \setminus \bigcup_n [s_n]) > 0$.

Condition (i) immediately follows. To see (ii):
$$\begin{aligned}\mu^*\left(A \cap \bigcup_n [s_n]\right) &\leq \sum_{n<\omega} \mu(E \cap [s_n]) \\ &\leq (1-\varepsilon) \sum_{n<\omega} \mu([s_n]) \leq (1-\varepsilon)\mu(G) \\ &\leq \mu^*(A).\end{aligned}$$

Using these conditions, we will take disjoint $[s_n]$'s from $\mathscr{E}$ inductively. Suppose $[s_k]$'s are defined for all $k < n$. Let $\mathscr{E}_n := \{ I \in \mathscr{E} \mid \forall k < n\, I \cap [s_k] = \emptyset \}$. Note that conditions (i) and (ii) ensures $\mathscr{E}_n \neq \emptyset$. Letting $d_n := \sup \{ \mu(I) \mid I \in \mathscr{E}_n \}$, we pick $[s_n] \in \mathscr{E}_n$ so that $\mu([s_n]) > d_n/4$.

Put $B := A \setminus \bigcup_n [s_n]$. Then by (ii), we have $\mu^*(B) > 0$. Hence, taking sufficiently large $N < \omega$ we have:

$$(\star) \qquad \sum_{N<n<\omega} \mu([s_n]) < \frac{\mu^*(B)}{8}.$$

For such $n > N$, enlarging $[s_n]$'s to $[t_n] \supseteq [s_n]$ so that $\mu([t_n]) = 8\mu([s_n])$. Then we have, by $(\star)$,

$$\sum_{N<n<\omega} \mu([t_n]) = 8 \sum_{N<n<\omega} \mu([s_n]) < \mu^*(B).$$

So we can fix $z \in B \setminus \bigcup_{n>N}[t_n]$. Since we also have that $x \in A \setminus \bigcup_{n\leq N}[s_n]$, there must be some $\ell < \omega$ with $[x \restriction \ell] \in \mathscr{E}_N$. Then we have some $n > N$ such that $[s_n] \cap [x \restriction \ell] \neq \emptyset$. Otherwise, by the definition of $\mathscr{E}_n$, we have $[x \restriction \ell] \in \mathscr{E}_n$ for all $n > N$. Then we have $\mu([x \restriction \ell]) \leq d_n < 4\mu([s_n])$, but

$$1 \geq \frac{\mu^*(B)}{8} > \sum_{N<n<\omega} \mu([s_n]) > \frac{1}{4} \sum_{N<n<\omega} \mu([x \restriction \ell]) = \infty.$$

So take minimum such n. Then we have $n > N$ and $\mu([x \restriction \ell]) \leq d_n < 4\mu([s_n])$. But in this case we have $x \in [x \restriction \ell] \subseteq J_n$. Contradiction! □

1.3. Basic Inner Model Theory

In this chapter, we will introduce basic concepts in the area of *inner model theory*. Inner model theory is a branch of set theory about definable proper class models of set theory.

Here is the precise definition:

Definition 1.21. Let Γ and Σ be some theories of set theory. A definable class M is an *inner model* for Γ under the ambient theory Σ if

- M is transitive class definable in Σ,
- M contains all ordinals, and
- M satisfies all axiom of Γ; i.e., we have $\Sigma \models \varphi^M$ for each $\varphi \in \Gamma$.

We often omit Γ and Σ if they are clear.

Note that the concept of "inner model" is not defined in ZFC but in the metatheory.

The following lemma is useful to check if a given class is inner-model of ZF:

Definition 1.22. A class M is *almost universal* if for any sub*set* $a \subset M$ there is $x \in M$ with $a \subseteq x$.

Lemma 1.28. *If transitive class M is almost universal and satisfies the Axiom of Separation, then M is a model for* ZF.

In this thesis, we will use two typical inner models: $L[z]$ and $\mathrm{HOD}(A)$, which we will define in the succeeding subsections.

1.3.1. Constructibility

For more detail on constructibility, see Devlin [5].

The *constructible universe* L is introduced by Gödel and widely used in the modern set theory. Intuitively, constructible universe is the collection of sets which can be effectively constructed in a bottom-up manner:

Definition 1.23. Let A be some (possibly proper) class. We define the *constructible universe* $L[A]$ *relative* to A as follows:

$$\begin{aligned} L_0[A] &:= \emptyset, \quad L_{\alpha+1} := \mathrm{Def}^A(L_\alpha[A]),\\ L_\gamma[A] &:= \bigcup_{\alpha<\gamma} L_\alpha[A] \ (\gamma : \text{limit ordinal}),\\ L[A] &:= \bigcup_{\alpha\in\mathbf{On}} L_\alpha[A]. \end{aligned}$$

In particular, we put $L_\alpha := L_\alpha[\emptyset]$ and $L := L[\emptyset]$, and we call L as the *constructible universe.*

This much look like the definition of the cumulative hierarchy of the universe V, but use Def^A instead of $\mathcal{P}$. Below we list the prominent properties of $L[A]$:

Theorem 1.29 (Gödel). *There is the global set-like well-ordering $<_{L[A]}$ on the entire $L[A]$, i.e. $<_{L[A]}$ is a total order on $L[A]$, $\left\{ x \in L[A] \,\middle|\, x <_{L[a]} a \right\}$ forms always a set and every non-empty subclass of $L[A]$ has the $<_{L[A]}$-least element.*

We call $<_{L[A]}$ as the canonical well-ordering of $L[A]$.

Theorem 1.30 (Gödel). *$L[A]$ is the least inner model of* ZF *with $A \subseteq L[A]$. In particular, L is the least inner model of* ZF*.*

Moreover, notions $x \in L[A]$ is absolute.

Lemma 1.31. *If $a \subseteq \kappa$ then $L[a] \models \forall\lambda \geq \kappa\, [2^\lambda = \lambda^+]$. In particular, if $a \subseteq \omega$ then $L[a] \models$* GCH*.*

Lemma 1.32 (Gödel). *For every $z \in {}^\omega\omega$, $L[z]$ and canonical well-ordering $<_{L[z]}$ of $L[z]$ are both $\Sigma^1_2(z)$.*

Lemma 1.33. *If $A \subseteq \omega$ then $z \in {}^\omega\omega \cap L[A] \implies \left|\left\{ x \in {}^\omega\omega \,\middle|\, x <_{L[A]} z \right\}\right| < \aleph_1$. Therefore, the height of canonical well-ordering of $L[A]$ on ${}^\omega\omega \cap L[A]$ is ω_1.*

1.3.2. Ordinal Definability

We now introduce the concept of *ordinal definable sets* which slightly seems similar to constructible sets.

Definition 1.24. Let A be a set and z be a (probably proper) class.

- A is *ordinal definable with parameters in z* (denoted by $A \in \mathrm{OD}(z)$) if there is a $\mathcal{L}_{\mathrm{ZF}}(z \cup \mathbf{On})$-formula $\varphi(x)$ such that $A = \{\, a \mid \varphi(a) \,\}$.
- A is *hereditary ordinal definable with parameters in z* (denoted by $A \in \mathrm{HOD}(z)$) if $\mathrm{trcl}(\{\, A \,\}) \in \mathrm{OD}(z)$. That is, A is in $\mathrm{HOD}(z)$ if it's entirely made of $\mathrm{OD}(z)$-sets only.

At first look, the notion of ordinal definability seems indefinable in ZF. But by Reflection Theorem 1.10, we can write down the defining formula of $\mathrm{OD}(z)$ and $\mathrm{HOD}(z)$.

Lemma 1.34. $\mathrm{OD}(z)$ *and* $\mathrm{HOD}(z)$ *is definable over* ZF*.*

Proof. Let φ, $\vec{\alpha}$ and $\vec{b} \in z$ be witness of $A \in \mathrm{OD}(z)$. By Reflection Theorem 1.10, we can take γ with $A, \vec{b}, \vec{\alpha} \in V_\gamma$ and φ is absolute between V_γ and V. So in this case:

$$A = \left\{ x \in V_\gamma \,\middle|\, V_\gamma \models \varphi(x, \vec{\alpha}, \vec{b}) \right\}.$$

On the other hand, sets of the above form are obviously in $\mathrm{OD}(z)$. So we have

$$A \in \mathrm{OD}(z) \iff \exists \gamma \, \exists \varphi \, \exists \vec{\alpha} \in \mathbf{On} \cap V_\gamma \, \exists \vec{b} \in z \cap V_\gamma \, \left[A = \left\{ x \in V_\gamma \,\middle|\, V_\gamma \models \varphi(x, \vec{\alpha}, \vec{b}) \right\} \right].$$

Similar for $\mathrm{HOD}(z)$. ☐

$\mathrm{HOD}(z)$ is often used as an inner model for ZF without AC.

Lemma 1.35. *For any class z,* $\mathrm{HOD}(z)$ *is an inner model of* ZF*.*

Proof. We use Lemma 1.28. It is clear that $\mathrm{HOD}(z)$ is transitive. Since $\mathrm{HOD}(z)$ is definable by Lemma 1.34, it is also obvious that $\mathrm{HOD}(z)$ satisfies the Axiom of Separation.

So it suffices to check $\mathrm{HOD}(z)$ is almost universal. But $V_\alpha \cap \mathrm{HOD}(z) \in \mathrm{HOD}(z)$ by the absoluteness of rank function (Fact 1.9), and hence this also holds. ☐

$\mathrm{HOD}(z)$ is not necessarily satisfy the Axiom of Choice; but for certain form of z, we can prove that the Axiom of Dependent Choice holds in $\mathrm{HOD}(z)$:

Lemma 1.36. *Assume* DC*. Then* DC *holds in* $\mathrm{HOD}({}^\omega z)$ *for any class* $z \supseteq \omega$*.*

Proof. Let $A, R \in \mathrm{HOD}({}^\omega z)$ be arbitrary set with $R \subseteq A^2$ and $\forall x \in A \, \exists y \in A \; y \; R \; x$. Since DC holds in the outer model V, there exists $f \in {}^\omega A \cap V$ such that $f(n+1) \; R \; f(n)$ for all $n < \omega$. Hence, it suffices to show that $\mathrm{HOD}({}^\omega z)$ is closed under ω-sequence, namely to show ${}^\omega(\mathrm{HOD}({}^\omega z)) \subseteq \mathrm{HOD}({}^\omega z)$.

To that end, fix an arbitrary $f : \omega \to \mathrm{HOD}({}^\omega z)$. We show $f \in \mathrm{OD}({}^\omega z)$. All we have to do is to find some formula and parameters which f can be defined by. First observe that the following definition gives a definable surjection from $\mathbf{On} \times {}^\omega z$ to $\mathrm{OD}({}^\omega z)$:

$$F(\alpha, z) := \begin{cases} \{ \, x \in V_\gamma \mid V_\gamma \models \varphi(x, s, z') \, \} & (\gamma \frown s = h(\alpha), \; z = \ulcorner \varphi \urcorner \frown z') \\ \emptyset & (\text{otherwise}) \end{cases}$$

where $\ulcorner \varphi \urcorner$ stands for Gödel number of formula φ, and $h : \mathbf{On} \overset{\sim}{\to} {}^{<\omega}\mathbf{On}$ is a bijection induced by the canonical weighted lexicographical order.

Because $f(n) \in \mathrm{HOD}({}^\omega z)$ for all $n < \omega$ and F is a surjection, there exists $s_n \in {}^\omega z$ and $\alpha_n \in \mathbf{On}$, for each $n < \omega$, with $f(n) = F(\alpha_n, s_n)$. Note that here we need the Axiom of Countable Choice, which DC implies. Then let $u(\Gamma(n,m)) := s_n(m)$ for $n, m < \omega$ (recall that $\Gamma : \omega \times \omega \to \omega$ is the canonical paring function on naturals). Then we can define f from the ω-sequence u of ordinals as follows:

$$f := \{ \, \langle n, x \rangle \mid F(u(\Gamma(n+1, -)), u(0)(n)) = x \, \}.$$

This shows $f \in \mathrm{OD}({}^\omega z)$. We can easily see that $\mathrm{trcl}(\{ f \}) \subseteq \mathrm{OD}({}^\omega z)$ hence $f \in \mathrm{HOD}({}^\omega z)$. ☐

By the proof above, we have especially that $({}^{\omega}\omega)^V \in \mathrm{HOD}({}^{\omega}z)$ for $z \supseteq \omega$. Hence, by the absoluteness of Δ_0-concepts, Lemma 1.4, all projective concepts are absolute between $\mathrm{HOD}({}^{\omega}z)$ and V:

Theorem 1.37. *Let $z \supset \omega$ be any class. If the formula φ is projective then* $\mathrm{HOD}({}^{\omega}z) \prec_{\varphi} V$*. Hence, all the projective concepts are absolute for* $\mathrm{HOD}({}^{\omega}z)$*.*

Corollary 1.38. *The concept of "x is countable" is absolute for* $\mathrm{HOD}({}^{\omega}\omega)$ *or* $\mathrm{HOD}({}^{\omega}\mathbf{On})$*.*

1.4. The Basic Theory of Forcing

The method of forcing is an inevitable tool in modern set theory to construct various model of set theories to establish consistency results. Intuitively, forcing method is a way to adjoin "generic" element(s) with desired properties to the existing universe. Of course, this cannot be done because such an element doesn't exist in the universe strictly speaking. Rather, we treat *approximations* of such generic elements instead. Kunen [17, 16] is the standard detailed textbook of the forcing method.

Precisely, we deal with ordered sets of approximating conditions for such generic elements.

Definition 1.25. $\mathbb{P} = (\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$ is a *pseudo-ordered set* (*poset* for short) if $\leq_{\mathbb{P}}$ is a transitive and reflexive binary relation on $\mathbb{P}$ and $\mathbb{1}_{\mathbb{P}} \in \mathbb{P}$ is *a* largest element of $\mathbb{P}$ with regard to $\leq_{\mathbb{P}}$. We also call a poset as a *forcing notion* or *notion of forcing.* An element $p \in \mathbb{P}$ is often called a *forcing condition.* We say that q *extends* p if $q \leq p$.

We often drop the subscript $\mathbb{P}$ and denote $\leq$, $\mathbb{1}$ if $\mathbb{P}$ is clear from the context.

Note that we don't require posets to be *antisymmetric*; i.e., although if $p \leq q$ and $q \leq p$, it's not necessarily the case that $p = q$. Hence, $\mathbb{1}_{\mathbb{P}}$ is "a" largest element; there might be multiple distinct largest element in $\mathbb{P}$.

We call an antisymmetric poset as a (*partially*) *ordered set.*

Also, the definition of posets is readily Δ_0-property and hence absolute between transitive models by Lemma 1.4.

As noted above, every forcing condition $p \in \mathbb{P}$ can be seen as an approximation accessible from the current universe of a generic object. Such approximating conditions are considered to be ordered by a degree of freedom. We are now at the point of defining a *generic filters* which can be regarded as approximated by forcing conditions:

Definition 1.26. Let $\mathbb{P}$ be a poset.

- p and q are *compatible* (denoted by $p \parallel q$) if there is $r \in \mathbb{P}$ with $r \leq p$ and $r \leq q$.

  If p and q don't have any common extension, then they are said to be *incompatible* and we write $p \perp q$ in such a case.

- A poset $\mathbb{P}$ is *separative* if for every $r \leq p$ we have $r \parallel q$, then $p \leq q$.

- $p \in \mathbb{P}$ is an *atom* if for any $q, r \leq p$ we have $q \parallel r$. Posets without atom are said to be *atomless.*

- $U \subseteq \mathbb{P}$ is called *open* if U is downward closed; i.e., $q \leq p \in U \implies q \in U$. We define the *poset topology* $\mathscr{O}_{\mathbb{P}}$ of $\mathbb{P}$ by this definition of open sets.

- $D \subseteq \mathbb{P}$ is *predense below* p if every $q \leq p$ has a common extension with at least one element of D, i.e., $\forall q \leq p\, \exists r \in D\, q \parallel r$. D is *predense* in $\mathbb{P}$ if it is predense below $\mathbb{1}$.
- $D \subseteq \mathbb{P}$ is *dense* below p if every $q \leq_{\mathbb{P}} p$ has an extension in D, i.e., $\forall q \leq p\, \exists r \in D\, r \leq p$. D is *dense* in $\mathbb{P}$ if D is dense below $\mathbb{1}$.

  Note that D is dense in the sense of order if and only if it's dense in the poset topology.
- $A \subseteq \mathbb{P}$ is an *antichain* in $\mathbb{P}$ if A is pairwise incompatible, i.e., $p, q \in A \implies p = q \vee p \perp q$.
- $F \subseteq \mathbb{P}$ is a *filter* if F is upward closed (i.e. $q \geq p \in F \implies q \in F$) and $p, q \in F$ then they have common extension $r \in F$.
- Let $\mathcal{D} \subseteq \mathcal{P}(\mathbb{P})$ be some family of dense sets of $\mathbb{P}$. A filter $G \subseteq \mathbb{P}$ is $\mathcal{D}$*-generic* if $D \cap G \neq \emptyset$ for every dense subset of $\mathbb{P}$ with $D \in \mathcal{D}$.

  If M is some transitive model with $\mathbb{P} \in M$, then $\{\, D \in M \mid D \subseteq \mathbb{P} : \text{dense in } \mathbb{P} \,\}$-generic filter is called $\mathbb{P}$*-generic over* M.

All of dense sets, predense sets and maximal antichains can be viewed as properties which "generically hold". Hence, we can consider a generic filter as an ideal object which satisfies all such properties.

We have the following natural characterization of genericity:

Fact 1.39. *Let M be a transitive model, $\mathbb{P} \in M$ a poset, and $G \subseteq \mathbb{P}$ a filter. The following are equivalent:*

(1) *G is a $\mathbb{P}$-generic over M,*

(2) *G meets every* open dense *subset $D \subseteq \mathbb{P}$ with $D \in M$,*

(3) *G meets every* predense *subset $D \subseteq \mathbb{P}$ with $D \in M$,*

(4) *G meets every* maximal antichain *$A \subseteq \mathbb{P}$ with $A \in M$.*

If G is generic over M, then G does not necessarily belong to M. In fact, in all cases which we are going to treat, G isn't in M, which is natural because we want to adjoin "new" elements to existing universe of sets.

So we need a way to refer to elements in the new universe with G; we use $\mathbb{P}$-*name* for such a purpose.

Definition 1.27. Let M be a transitive model and $\mathbb{P} \in M$ be a poset. We define the collection $M^{\mathbb{P}}$ of $\mathbb{P}$-*names in* M by transfinite recursion as follows:

$$\begin{gathered} M^{\mathbb{P}}_0 := \emptyset, \quad M^{\mathbb{P}}_{\alpha+1} := \mathcal{P}^M(M^{\mathbb{P}}_\alpha \times \mathbb{P}) \\ M^{\mathbb{P}}_\gamma := \bigcup_{\alpha<\gamma} M^{\mathbb{P}}_\alpha \quad (\text{if } \gamma : \text{limit}) \\ M^{\mathbb{P}} := \bigcup_{\alpha \in \mathbf{On} \cap M} M^{\mathbb{P}}_\alpha. \end{gathered}$$

Intuitively, $\mathbb{P}$-names can be viewed as a "$\mathbb{P}$-valued probabilistic set". We often use greek letters $\sigma, \tau, \vartheta, \ldots$ and dotted roman alphabets $\dot{a}, \dot{x}, \dot{y}, \dot{z}, \ldots$ for variables ranging over $\mathbb{P}$-names.

If G is $\mathbb{P}$-generic filter over M, then we define the *evaluation* τ^G of $\mathbb{P}$-name τ by recursion:

$$\mathrm{val}(\tau, G) := \tau^G := \left\{ \sigma^G \;\middle|\; \langle \sigma, p \rangle \in \tau, p \in G \right\}.$$

We call $M[G] := \left\{ \tau^G \;\middle|\; \tau \in M^{\mathbb{P}} \right\}$ as a $\mathbb{P}$-*generic extension* of M by G. By convension, we sometimes write $\dot{x}$ for some $\mathbb{P}$-name of a particular $x \in M[G]$.

For $x \in M$, we define $\check{x} \in M^{\mathbb{P}}$ by $\check{x} := \{ \langle \check{y}, \mathbb{1} \rangle \mid y \in x \}$. Note that $\check{x}^G = x$ for all $x \in M$ and hence we have $M \subseteq M[G]$. Abusing notation, we also write $\check{M} := \{ \langle \check{x}, \mathbb{1} \rangle \mid x \in M \}$.

We call $\dot{G} := \{ \langle \check{q}, q \rangle \mid q \in \mathbb{P} \}$ as *the canonical name of* a $\mathbb{P}$-generic filter.

We call $\mathcal{L}^{\check{M}}_{\mathrm{ZF}}(M^{\mathbb{P}})$ as *the forcing language of* $\mathbb{P}$ in M and denote it by $\mathcal{FL}_{\mathbb{P}}$.

Theorem 1.40 (Generic Extension Theorem). *Let M be a transitive model of (a sufficiently large finite fragment of) set theory, $\mathbb{P} \in M$ a poset, and $G \subseteq \mathbb{P}$ a $\mathbb{P}$-generic filter over M. Then $M[G]$ is the smallest transitive model with $M[G] \supseteq M$ and $G = \dot{G}^G \in M[G]$.*

In addition, if M satisfies (some large finite fragment of) ZF*, then so does $M[G]$. Furthermore, if $M \models$ AC, then $M[G] \models$ AC.*

If M is a countable transitive model (*ctm*) of a sufficiently large finite fragment of ZF, we can always take a generic filter over M clearly.

So, what if we don't have access to a generic filter over M, for example, in case of $M = V$? In such a case, in theory, we have to give up to treat G directly and do with $\mathbb{P}$-names and approximations of generic objects. *The Forcing Relation* enables us to do such an argument:

Definition 1.28. Let φ be a $\mathcal{FL}_{\mathbb{P}}$-formula and $p \in \mathbb{P}$. We define the relation $p \Vdash^M_{\mathbb{P}} \varphi$, *$p$ forces φ in M*, by recursion on the structure of φ:

$$\begin{aligned}
p \Vdash^M_{\mathbb{P}} \tau = \vartheta &\overset{\text{def}}{\iff} \forall \sigma \in \mathrm{dom}(\tau) \cup \mathrm{dom}(\vartheta)\ \forall q \leq p\, [q \Vdash \sigma \in \tau \iff q \Vdash \sigma \in \vartheta] \\
p \Vdash^M_{\mathbb{P}} \sigma \in \tau &\overset{\text{def}}{\iff} \forall r \leq p\, \exists q \leq r\, \exists \langle \vartheta, s \rangle \in \tau\ (q \leq s \wedge q \Vdash^M_{\mathbb{P}} \vartheta = \sigma) \\
p \Vdash^M_{\mathbb{P}} \psi \wedge \varphi &\overset{\text{def}}{\iff} (p \Vdash^M_{\mathbb{P}} \psi) \wedge (q \Vdash^M_{\mathbb{P}} \varphi) \\
p \Vdash^M_{\mathbb{P}} \neg\varphi &\overset{\text{def}}{\iff} \forall q \leq p\ q \nVdash^M_{\mathbb{P}} \varphi \\
p \Vdash^M_{\mathbb{P}} \forall x\, \varphi(x) &\overset{\text{def}}{\iff} \forall \tau \in M^{\mathbb{P}}\ p \Vdash^M_{\mathbb{P}} \varphi(\tau).
\end{aligned}$$

We write $\Vdash^M_{\mathbb{P}} \varphi$ if $\mathbb{1}_{\mathbb{P}} \Vdash^M_{\mathbb{P}} \varphi$. We often drop $\mathbb{P}$ and M if it's clear.

Note that, the recursive definition above takes place in *metatheory*; for example, in the case of $M = V$, "$p \Vdash \forall x\, \varphi(x)$" involves quantification over proper class. But, for each fixed φ, the formula representing $p \Vdash \varphi$ can be written in M.

Here are basic properties of the forcing relation, which directly follows from the definition of the forcing relation:

Lemma 1.41. (1) *$p \Vdash \varphi, q \leq p \implies q \Vdash \varphi$.*

(2) *$p \Vdash \varphi \iff \{ q \in \mathbb{P} \mid q \Vdash \varphi \}$: dense below p.*

(3) *No $p \in \mathbb{P}$ forces both φ and $\neg\varphi$.*

(4) *For any $p \in \mathbb{P}$ and $\varphi \in \mathcal{FL}_{\mathbb{P}}$, there must be some $q \leq p$ with $q \Vdash \varphi$ or $q \Vdash \neg\varphi$.*

(5) *For* $D \subseteq \mathbb{P}$, $p \Vdash \check{D} \cap \dot{G} \neq \emptyset$ *if and only if* D *is predense below* p.

With these in mind, we have following fundamental theorem of Forcing:

Theorem 1.42 (Forcing Theorems). (1) (Truth Lemma) *Suppose* G *is a* $\mathbb{P}$*-generic filter over* M*. For formula* $\varphi(a_1, \ldots, a_n)$ *and* $\dot{x}_1, \ldots, \dot{x}_n \in M^{\mathbb{P}}$*, we have*

$$M[G] \models \varphi(\dot{x}_1^G, \ldots, \dot{x}_n^G) \iff \exists p \in G\, p \Vdash_{\mathbb{P}}^M \varphi(\dot{x}_1, \ldots, \dot{x}_n).$$

Furthermore, if $p \Vdash_{\mathbb{P}}^M \varphi$ *and* G *is a* $\mathbb{P}$*-generic filter over* M *with* $p \in G$*, then* $M[G] \models \varphi$.

(2) $\mathbb{1} \Vdash \varphi$ *for any tautology* φ *of the First Order Logic.* $M \models \mathrm{ZF}$ *then* $\mathbb{1} \Vdash^M \mathrm{ZF}$*. If* $M \models \mathrm{AC}$*, then* $\mathbb{1} \Vdash^M \mathrm{AC}$.

(3) *Conversely, we have* $\mathbb{1} \Vdash$ *"*$\check{V}$ *: an inner model"*

(4) $\Vdash \dot{G}$ *: generic filter over* $\check{V}$.

(5) (Δ_1-absoluteness) *if* φ *is* Δ_1*-sentence then* $\mathbb{1} \Vdash^M \varphi$.

In practice, Forcing Theorems enables us to think of $V^{\mathbb{P}}$ as some generic extension $V[G]$ by generic filter G over V, although there might not be such a G. So, in what follows, we will abuse a notation $M[G]$ and think as if there actually exists a generic filter G over M, even if M is uncountable or a proper class, for example $M = V$.

Following definition and lemma provide a way to check the preservation of cardinal:

Definition 1.29.
- $\mathbb{P}$ has *κ-chain condition* (*κ-c.c.* for short) $\overset{\text{def}}{\iff}$ every antichain of $\mathbb{P}$ has cardinality $< \kappa$. $\mathbb{P}$ has *countable chain condition* (*c.c.c.* for short) if $\mathbb{P}$ has ω_1-c.c.

  In particular, if $\mathcal{I}$ is some σ-ideal over σ-algebra $\mathcal{B}$, then c.c.c. of $\mathcal{I}^+$ is equivalent to σ-saturation of $\mathcal{I}$.
- $\mathbb{P}$ is *κ-closed* $\overset{\text{def}}{\iff}$ for any decreasing sequence of length $< \kappa$ has lower bound in $\mathbb{P}$, i.e. for any $\xi < \kappa$ and $\{\, p_\alpha \mid \alpha < \xi \,\} \subseteq \mathbb{P}$, if $p_\alpha \leq p_\beta$ for all $\beta < \alpha < \xi$ then there exists $p^* \in \mathbb{P}$ such that $p^* \leq p_\alpha$ for all $\alpha < \xi$. $\mathbb{P}$ is *countably closed* if it is ω_1-closed.

Lemma 1.43. *Let* κ *be regular cardinal.*

(1) *If* $\mathbb{P}$ *has* κ*-c.c. then* $\mathbb{P}$ *preserves cardinals* $\geq \kappa$.

(2) *If* $\mathbb{P}$ *is* κ*-closed then* $\mathbb{P}$ *preserves cardinals* $\leq \kappa$.

The following example illustrates the typical usage of c.c. and closure properties:

Lemma 1.44. *Let* $\mu < \kappa$ *be regular infinite cardinals. We regard* ${}^{<\mu}\kappa$ *as poset ordered with reverse inclusions, i.e.,* $p \leq q \iff p \supseteq q$*. Then* ${}^{<\mu}\kappa$ *preserves cardinals* $\leq \mu$ *and* $|\kappa| = \mu$ *in generic extension. Moreover, if* $\kappa^{<\mu} = \kappa$ *then it also preserves cardinals* $> \kappa$.

Proof. Clearly ${}^{<\mu}\kappa$ is μ-closed by definition, so it preserves cardinals $\leq \mu$. If $\kappa^{<\mu} = \kappa$, then $|{}^{<\mu}\kappa| = \kappa^{<\mu} = \kappa$, so ${}^{<\mu}\kappa$ has κ^+-c.c. and hence it preserves cardinals $> \kappa$.

To complete the proof, let G be ${}^{<\mu}\kappa$-generic and $f := \bigcup G$. We show that f is a surjection from μ onto κ. Let $D_\alpha := \{\, p \in {}^{<\mu}\kappa \mid \alpha \in \operatorname{ran}(p) \,\}$ for $\alpha < \kappa$. Every D_α is dense in $\mathbb{P}$ so we have $D_\alpha \cap G \neq \emptyset$ for every $\alpha < \kappa$. This implies that f is surjective. $\square$

Definition 1.30. Let $\langle(\mathbb{P}_i, \leq_i, \mathbb{1}_i) \mid i \in I\rangle$ be posets and λ be infinite cardinal.
The λ-*support product* of $\langle P_i \mid i \in I\rangle$, $(\prod_{i\in I}^{<\lambda} \mathbb{P}_i, \leq, \mathbb{1})$, is defined as follows:

$$\prod_{i\in I}^{<\lambda} \mathbb{P}_i := \left\{ p \subseteq I \times \bigcup_{i\in I} \mathbb{P}_i \;\middle|\; p \text{ is a function} \wedge \forall i \in I\, [p(i) \in \mathbb{P}_i] \wedge |\operatorname{dom}(p)| < \lambda \right\},$$

$$p \leq q \overset{\text{def}}{\iff} \operatorname{dom}(p) \supseteq \operatorname{dom}(q) \wedge \forall i \in \operatorname{dom}(q)\, [p(i) \leq_i q(i)], \qquad \mathbb{1} := \emptyset.$$

$\prod_{i\in I}^{<\lambda} \mathbb{P}_i$ subsumes the all effects of $\mathbb{P}_i$ as a forcing notion. To make this fact precise, we first need to define the concept of *projections*:

Definition 1.31. Let $\mathbb{P}$ and $\mathbb{Q}$ be posets. $f : \mathbb{P} \to \mathbb{Q}$ is a *projection* if

(1) f is order preserving,

(2) $\forall p \in \mathbb{P}\, \forall q \in \mathbb{Q}\, [q \leq f(p) \implies \exists p' \leq_{\mathbb{P}} p\, f(p') \leq q]$.

Lemma 1.45. *Let G be a $\mathbb{P}$-generic filter over V and $f : \mathbb{P} \to \mathbb{Q}$ be a projection. Then $H := \{ q \in \mathbb{Q} \mid \exists p \in G\, [f(p) \leq q] \}$ is $\mathbb{Q}$-generic over V.*

Proof. By Lemma 1.39, it is enough to show that if $D \subseteq \mathbb{Q}$ is open dense in $\mathbb{Q}$ then $D \cap H \neq \emptyset$.
Let $E := \{ p \in P \mid \exists q \in D\, [f(p) \leq q] \} \subseteq \mathbb{P}$. Then E is dense in $\mathbb{P}$ since D is dense and f is a projection. Hence, we have some $p \in G \cap E \neq \emptyset$ by the genericity of G, and, by the definition of E, there exists $q \in D$ with $f(p) \leq q$. Then $f(p) \in D \cap H$ since D is open and by definition of H. ☐

Corollary 1.46. *Let $\mathbb{P} := \prod_{i\in I}^{<\lambda} \mathbb{P}_i$. If G is $\mathbb{P}$-generic over V, $G_i := \{ p \in \mathbb{P}_i \mid \exists s \in G\, s(i) \leq p \}$ is $\mathbb{P}_i$-generic over V.*

Proof. Clearly, $\mathbb{P} \ni p \mapsto p(i) \in \mathbb{P}_i$ is a projection. Hence, the statement follows immediately. ☐

One useful application of projections is the following *Product Lemma*:

Lemma 1.47 (Product Lemma). *Let $\mathbb{P}, \mathbb{Q}$ be posets and K be $(\mathbb{P} \times \mathbb{Q})$-generic filter over V. Set $G := \{ p \in \mathbb{P} \mid \exists q \in \mathbb{Q}\ \langle p, q\rangle \in K \}$ and $H := \{ q \in \mathbb{Q} \mid \exists p \in \mathbb{P}\ \langle p, q\rangle \in K \}$. Then the followings holds:*

(1) *G is $\mathbb{P}$-generic over V and H is $\mathbb{Q}$-generic over $V[G]$ with $V[K] = V[G][H]$*

(2) *H is $\mathbb{Q}$-generic over V and G is $\mathbb{Q}$-generic over $V[H]$ with $V[K] = V[H][G]$.*

Conversely, if at least one of the above two conditions holds true, then $G \times H$ is $(\mathbb{P} \times \mathbb{Q})$-generic filter over V and $V[G \times H] = V[G][H] = V[H][G]$.

Furthermore, in either case the following holds:

$$(p, q) \Vdash_{\mathbb{P}\times\mathbb{Q}} \varphi \iff p \Vdash_{\mathbb{P}} \text{“}\check{q} \Vdash_{\check{\mathbb{Q}}} \varphi\text{”} \iff q \Vdash_{\mathbb{Q}} \text{“}\check{p} \Vdash_{\check{\mathbb{P}}} \varphi\text{”}$$

Proof. See [16]. ☐

1.4.1. Boolean Completion and Forcing Equivalence

In this subsection, we will discuss on the *forcing equivalence* and Boolean completion.

Definition 1.32 (Dense embedding). Let $\mathbb{P}$ and $\mathbb{Q}$ be posets. $e : \mathbb{P} \to \mathbb{Q}$ is called *dense embedding*, denoted by $e : \mathbb{P} \xrightarrow[d]{} \mathbb{Q}$, if

(1) e is monotone: $p \leq_{\mathbb{P}} p'$ then $e(p) \leq_{\mathbb{Q}} e(p')$ for any $p, p' \in \mathbb{P}$.

(2) e preserves incompatibility: $p \perp_{\mathbb{P}} p' \iff e(p) \perp_{\mathbb{Q}} e(p')$.

(3) $e[\mathbb{P}]$ is dense in $\mathbb{Q}$: for any $q \in \mathbb{Q}$, there is some $p \in \mathbb{P}$ with $e(p) \leq_{\mathbb{Q}} q$.

Note that converse direction is not required in condition (1).
Dense embeddings assures some kind of "forcing equivalence".

Fact 1.48. *If $e : \mathbb{P} \xrightarrow[d]{} \mathbb{Q}$ then $\mathbb{P}$ and $\mathbb{Q}$ are forcing equivalent; more precisely, the followings holds:*

(1) *If G is $\mathbb{P}$-generic over V then $H := \{ q \in \mathbb{Q} \mid \exists p \in \mathbb{P}\, e(p) \leq q \}$ is $\mathbb{Q}$-generic over V and $V[G] = V[H]$.*

(2) *If H is $\mathbb{Q}$-generic over V then $H := e^{-1}[G]$ is $\mathbb{P}$-generic over V and $V[H] = V[G]$.*

(3) *Define $e_* : V^{\mathbb{P}} \to V^{\mathbb{Q}}$ by $e_*(\tau) = \{ \langle e_*(\dot{x}), e(p) \rangle \mid \langle \dot{x}, p \rangle \in \tau \}$. Then,*

$$p \Vdash_{\mathbb{P}} \varphi(\vec{\tau}) \iff e(p) \Vdash_{\mathbb{Q}} \varphi(e_*(\vec{\tau})).$$

To analyze the notion of being "forcing equivalent" more systematically, we introduce the notion of *complete Boolean algebra*:

Definition 1.33. $\mathbb{B} = (\mathbb{B}, \leq, -, +, \cdot, 0, 1)$ called a *Boolean algebra* if the following conditions met:

(1) $(\mathbb{B}, \leq, 0, 1)$ forms partially ordered set with 0 and 1 least and largest element.

(2) For $b, c \in \mathbb{B}$, $b + c$ gives the $\leq$-least upper bound of b and c.

(3) For $b, c \in \mathbb{B}$, $b \cdot c$ gives the $\leq$-largest lower bound of b and c.

(4) For $b \in \mathbb{B}$, $b + (-b) = 1$.

A Boolean algebra $\mathbb{B}$ is said to be *complete* if any subset $A \subseteq \mathbb{B}$ has the least upper bound $\Sigma_{\mathbb{B}} A$ and the largest lower bound $\Pi_{\mathbb{B}} A$ in $\mathbb{B}$.

By the following definition and lemma, we see that complete Boolean algebras capture all the properties related to the forcing:

Definition 1.34. The *Boolean completion* $\mathbb{B}(\mathbb{P})$ of a forcing poset $\mathbb{P}$ is the complete Boolean algebra $\mathbb{B}(\mathbb{P})$ together with some dense embedding $i : \mathbb{P} \xrightarrow[d]{} \mathbb{B}(\mathbb{P}) \setminus \{ 0 \}$.

Fact 1.49. *Every poset $\mathbb{P}$ has the unique Boolean completion $\mathbb{B}(\mathbb{P})$. In particular, $\mathbb{B}(\mathbb{P})$ is isomorphic to the collection $\mathrm{ro}(\mathbb{P}, \mathscr{O}_{\mathbb{P}})$ of regular open sets in the poset topology of $\mathbb{P}$ ordered by inclusion. Then $\mathbb{P}$ can be densely embedded into $\mathrm{ro}(\mathbb{P})$ and $\mathrm{ro}(\mathbb{P})$ forms a complete Boolean algebra.*
Furthermore, if $j : \mathbb{P} \xrightarrow[d]{} \mathbb{Q}$, then $\mathbb{B}(\mathbb{P}) \simeq \mathbb{B}(\mathbb{Q})$.

Proof. See Jech [11]. □

By the following lemma, as for forcing, we can always assume separativity of posets without loss of generality:

Lemma 1.50. *Every Boolean algebra is separative.*

By convention, by "forcing by complete Boolean algebra $\mathbb{B}$" we mean "forcing by $\mathbb{B} \setminus \{ 0 \}$ regarded as a poset". In the theory of forcing by Boolean algebra, we can compute the "truth value" of formula φ as follows:

Definition 1.35. The *truth value* $[\![\varphi]\!]^M_{\mathbb{B}} \in \mathbb{B}$ of φ is defined as follows:

$$[\![\dot{x} \in \dot{y}]\!]^M_{\mathbb{B}} := \sum_{(\dot{z},q)\in\dot{y}} q \cdot [\![\dot{x} = \dot{z}]\!]^M_{\mathbb{B}}$$

$$[\![\dot{x} = \dot{y}]\!]^M_{\mathbb{B}} := \left(\prod_{(\dot{z},q)\in\dot{x}} (-q + [\![\dot{z} \in \dot{y}]\!]_{\mathbb{B}})^M \right) \cdot \left(\prod_{(\dot{z},q)\in\dot{y}} (-q + [\![\dot{z} \in \dot{x}]\!]^M_{\mathbb{B}}) \right)$$

$$[\![\varphi \wedge \psi]\!]^M_{\mathbb{B}} := [\![\varphi]\!]^M_{\mathbb{B}} \cdot [\![\psi]\!]^M_{\mathbb{B}} \qquad [\![\neg\varphi]\!]^M_{\mathbb{B}} := -\, [\![\varphi]\!]^M_{\mathbb{B}}$$

$$[\![\forall x\, \varphi(x)]\!]^M_{\mathbb{B}} := \prod_{\dot{x}\in M^{\mathbb{B}}} [\![\varphi(\dot{x})]\!]^M_{\mathbb{B}}$$

We often drop M and $\mathbb{B}$ if they are clear.

Fact 1.51. $p \leq [\![\varphi]\!]$ *if and only if* $p \Vdash \varphi$.

Hence, we can identify every forcing formula φ with some regular open set in the poset topology of $\mathbb{P}$. The following fact illustrates this situation:

Fact 1.52. *Let* $\mathbb{P}$ *be a forcing poset and* $E \subseteq \mathbb{P}$ *a regular open set. Then* $p \Vdash \check{E} \cap \dot{G} \neq \emptyset$ *if and only if* E *is dense below* p.

Consequently, $p \Vdash \varphi$ *if and only if a regular open set* $[\![\varphi]\!]_{\mathbb{B}(\mathbb{P})} \subseteq \mathbb{P}$ *is dense below* p.

Another application of dense embeddings is the following Zero-One law:

Definition 1.36. Poset $\mathbb{P}$ is *homogeneous* if for any $p, q \in \mathbb{P}$ there exists dense endomorphism $\pi : \mathbb{P} \to \mathbb{P}$ such that $p \parallel \pi(q)$.

Lemma 1.53 (Zero-One Law). *If* $\mathbb{P}$ *is homogeneous and* φ *is a forcing formula with parameters in* V*, i.e.* $\varphi \equiv \varphi(\check{x}_1, \dots \check{x}_n)$ *with* $x_i \in V$*, then either* $\mathbb{1} \Vdash \varphi$ *or* $\mathbb{1} \Vdash \neg\varphi$ *holds.*

Proof. See [28]. □

Definition 1.37. Let $\mathbb{P}$ be a poset, $\theta > 2^{|\mathbb{P}|}$ and $M \prec H(\theta)$ a countable elementary submodel with $\mathbb{P} \in M$. $p \in \mathbb{P}$ is called $(M, \mathbb{P})$-*master condition* if for any dense subset $D \in M$ of $\mathbb{P}$, $D \cap M$ is predense below p in $\mathbb{P}$.

A poset $\mathbb{P}$ is called *proper* if for any regular $\theta > 2^{|\mathbb{P}|}$ and countable elementary submodel $M \prec H(\theta)$ with $\mathbb{P} \in M$, if $p \in \mathbb{P} \cap M$ then there are $(M, \mathbb{P})$-*master condition* $q \in \mathbb{P}$ below p.

Properness of posets are introduced by S. Shelah. Shelah [31] is the basic reference for proper forcings. Clearly, properness of poset is preserved by dense embedding.

The following lemma is convenient to test if the given poset is proper:

Lemma 1.54. *If* $\mathbb{P}$ *has c.c.c. or is countably closed, then it's proper.*

2. Solovay's Model: Inaccessible Cardinal to Measurability

In this chapter, we will establish the following theorem due to Solovay:

Theorem 2.1. *Let κ be an inaccessible cardinal and G be $\mathrm{Col}(\omega, {<}\kappa)$-generic over V. Then in $N := \mathrm{HOD}({}^{\omega}\mathbf{On})^{V[G]}$, every set of reals is Lebesgue measurable and has Baire property.*

2.1. Borel Codes, Solovay Sets and Regularity Properties

One important ingredient of Solovay's Theorem is the concept of *Solovay* sets. This need the theory of Borel codes developed in Section 1.2.

First, it is obvious that the notion of being "Borel code" is absolute because it is a Π^1_1-concept.

Lemma 2.2. *The notion of "Borel code" is absolute between transitive models for* ZF*. That is, if M is some transitive model for (a large fragment of)* ZF*, then $\mathbf{BC}^M \subseteq \mathbf{BC}$.*

Proof. $\sigma = \langle t, c \rangle \in \mathbf{BC}$ can be write as follows:

$$c : {}^{<\omega}\omega \to {}^{<\omega}\omega, t : \text{well-founded tree on } \omega.$$

The former condition is clearly arithmetical. The latter condition that t is well-founded can be expressed as follows:

$$\nexists \sigma \in {}^{\omega}({}^{<\omega}\omega)\, \forall n < \omega\, [\sigma(n) \in t \wedge \sigma(n+1) \supsetneq \sigma(n)].$$

And this formula is clearly Π^1_1. Hence, by the Mostwski Absoluteness of Π^1_1-concepts (Lemma 1.19), the relation $c \in \mathbf{BC}$ is absolute. ☐

Next, we will establish some sort of absoluteness:

Lemma 2.3. *Relation $x \in B_{c,t}$ is absolute for transitive models.*

Proof. $B_{c,t}$ is defined by induction from absolute functions, hence so is $B_{c,t}$ by Lemma 1.5. ☐

Lemma 2.4. *Relation $B_c = \emptyset$ is absolute.*

Proof. Note that the relation "$x \in B_c$" between x and c is arithmetical and hence $B_c = \emptyset$ is $\mathbf{\Pi}^1_1$. Hence, by the Mostwski absoluteness of $\mathbf{\Pi}^1_1$-relations (Lemma 1.19), $B_c = \emptyset$ is absolute. ☐

Lemma 2.5. *Let M be some transitive model and $c, d \in \mathbf{BC}^M$. Then there are some $-c, c{\cdot}d, c{-}d \in \mathbf{BC}^M$ such that following holds in any transitive model including M:*

$$B_{-c} = \neg B_c \qquad B_{c \cdot d} = B_c \cap B_d \qquad B_{c-d} = B_c \setminus B_d.$$

Proof. Let $c = (T, \sigma_0), d = (S, \tau_1)$. Obviously the following definitions work:

$$\begin{aligned} -c &:= (\{\, 0 ^\frown t \mid t \in T \,\}, 0 ^\frown s \mapsto \sigma_0(s)) \\ c \cdot d &:= (\{\, \langle 0 \rangle \,\} \cup \{\, \langle 0,0 \rangle ^\frown t \mid t \in T \,\} \cup \{\, \langle 0,1 \rangle ^\frown s \mid s \in S \,\}, (0 ^\frown i ^\frown s) \mapsto \sigma_i(s)) \\ c - d &:= c \cdot -d. \end{aligned}$$

□

Corollary 2.6. *Let M be some transitive model and $c, d \in \mathbf{BC}^M$. Then, $(B_c = B_d)^M$ if and only if $B_c = B_d$.*

Proof. "If" direction is obvious, so it's enough to show "only if" part. We show the contrapositive form: suppose $B_c \setminus B_d \neq \emptyset$ and show $(B_c \setminus B_d \neq \emptyset)^M$. But $B_c \setminus B_d = B_{c-d}$, hence we have $B_{c-d} \neq \emptyset$. By Lemma 2.4, we have $(B_{c-d} \neq \emptyset)^M$ and, by Lemma 2.5, we have $(B_c \setminus B_d \neq \emptyset)^M$. □

Hence, if B is a Borel set in some transitive model M, then we can compute the natural "extension" of B in larger transitive models.

Notation. *Let $M \subseteq N$ be some transitive models, $B \in \mathscr{B}^M$ and $c \in \mathbf{BC}^M$ some code for B. Then we write $B^N := (B_c)^N$. If G is some generic filter over M and $N = M[G]$, then $B^G := B^{M[G]}$. We will also write B^* if N is obvious from context.*

We use the above notation to "transfer" Borel sets between multiple transitive models.

The following facts are easy to verify and useful to "compute" the value of Borel sets:

Fact 2.7. *Let $M \subseteq N$ be some inner models, $s \in {}^{<\omega}\omega$ and $A, A_i \in M$ be Borel sets.*

- $([s]^M)^N = [s]^N$
- $({}^\omega\omega \cap M \setminus A)^N = {}^\omega\omega \cap N \setminus A^N$
- $(\bigcup_{n<\omega} A_n)^M = \bigcup_n A_n^M$
- $(\bigcap_{n<\omega} A_n)^M = \bigcap_n A_n^M$

Furthermore, by well-founded recursion in t of $c = (t, \sigma)$, we can easily show the following:

Lemma 2.8. *The function $c \mapsto \mu(B_c)$ on $\mathbf{BC}$ is absolute for transitive models.*

We are now ready to define the central concept of *Solovay sets*:

Definition 2.1. Let M be a transitive model of ZFC.

- $x \in {}^\omega\omega$ is said to be *generic over M* if there exists some notion of forcing $\mathbb{P} \in M$ and $\mathbb{P}$-generic filter G over M such that $M[G]$ is the $\subseteq$-least transitive model with $M \cup \{\, x \,\} \subseteq M[G]$. We write $M[x]$ for such (unique) $M[G]$.
- $A \subseteq {}^\omega\omega$ is called *Solovay over M* if there exists some formula $\varphi(x, y_1, \dots, y_n)$ in the language of set theory and parameters $a_1, \dots, a_n \in M$ such that for any generic $z \in {}^\omega\omega$:

$$z \in A \iff M[z] \models \varphi(z, a_1, \dots, a_n).$$

Note that the above definition of "Solovayness" is a *metadefinition*; by Forcing Theorems 1.42, we can write the formula expressing $M[z] \models \varphi$ for each individual formula φ, but these are scheme in metatheory. Thus, the assertion "A is Solovay over M" means that we can write down some specific, concrete formula φ and can prove φ catches every generic reals in A.

So, the following key lemma, referring to Solovay sets, is also a metatheorem:

Lemma 2.9. *If M is transitive model with $(2^{\aleph_0})^M < \aleph_1$ and $A \subseteq {}^\omega\omega$ is Solovay over M, then A is Lebesgue measurable and has Baire property.*

In fact, Solovay sets satisfies more general notion of *I-regularity*, due to Khomskii [15]:

Definition 2.2. Let I be a σ-ideal over $\mathscr{B}$.

- $I^+ := \{\, B \in \mathscr{B} \mid B \notin I \,\}$ be a poset, ordered by ordinal inclusion.
- $A \subseteq {}^\omega\omega$ is *I-regular* if $\forall B \in I^+ \, \exists C \in I^+ \, [C \leq B \land (C \subseteq A \lor C \cap A = \emptyset)]$.
- For σ-ideal I, we write $\mathrm{Reg}(I)$ for the statement "every set of reals has I-regularity".
- PReg is short for the statement "every set of reals has I-regularity for any projective σ-ideal I with I^+ proper as a forcing notion".

Examples 2.1.

- *A is Lebesgue measurable iff A is null-regular.*
- *A has Baire property iff A is meager-regular.*

We will analyze one more example of a regularity property in section 2.4. For more examples see Khomskii [15].

Then, Lemma 2.9 follows from the following theorem

Theorem 2.10. *Let M be transitive model with $(2^{2^{\aleph_0}})^M < \aleph_1$ and I be σ-ideal with I^+ proper as a poset. If $A \subseteq {}^\omega\omega$ is Solovay over M, then A has an I-regularity.*

To that end, we define *I-genericity* for each I:

Definition 2.3. Let I be a σ-ideal and M a transitive model.

- $I \restriction M := \left\{ B \in \mathscr{B}^M \,\middle|\, B^* \in I \right\}, \quad I^+ \restriction M := (I \restriction M)^+ = \mathscr{B}^M \setminus (I \restriction M)$.
- $x \in {}^\omega\omega$ is said to be *I-generic* over M if $G_x := \left\{ B \in \mathscr{B}^M \,\middle|\, x \in B^* \right\}$ forms an $(I^+ \restriction M)$-generic filter over M.
- We say that A is *random* if A is null-generic, and *Cohen* if meager-generic.

Next, we show that if I^+ is proper, there is the unique I-generic real. To that end, we need the following Lemma:

Lemma 2.11. *Let I be a σ-ideal on $\mathscr{B}$ and G an $(I^+ \restriction M)$-generic filter over M. Then G satisfies following:*

- *$I^* \subseteq G$,*

- *G is a $\mathscr{B}^M$-ultrafilter, that is, $A \notin G$ implies $({}^\omega\omega)^M \setminus A \in G$, and*
- *G is σ-M-complete; i.e. if $\langle A_n \in G \,|\, n < \omega\rangle \in M$ is an ω-sequence* in M, *then $\bigcap_n A_n \in G$.*

Proof. If $A \in (I \restriction M)^*$ then $\{ B \in (I^+ \restriction M) \mid B \subseteq A \} \in M$ is dense in $I^+ \restriction M$. Then there is some $B \in G$ with $B \subseteq A$ and hence we have $A \in G$ because G is upward closed. To see the maximality of G, first note that $D_A := \{ B \in I^+ \restriction M \mid B \subseteq A \vee B \cap A \in I \} \in M$ is clearly dense in $I^+ \restriction M$ for each A and we can pick $B \in G \cap D_A$. If $B \subseteq A$, then we have $A \in G$. If $A \cap B \in I$ then, by preceding arguments, we have $B, A^c \cup B^c \in G$ and hence $A^c \supseteq A^c \cap B \in G$ and $A^c \in G$.

To see σ-M-completeness, it's enough to fix a sequence $\langle A_n \,|\, n < \omega\rangle \in M$ with $\bigcup_n A_n \in G$, and show $A_n \in G$ for at least one $n < \omega$. For a while, we discuss in M and we confuse I, I^+ and I^* with $I \restriction M$, $I^+ \restriction M$ and $I^* \restriction M$ respectively. Let $A_\omega := {}^\omega\omega \setminus \bigcup_n A_n$ and $D := \{ B \in I^+ \mid \exists n < \omega\, [B \subseteq A_n] \vee B \subseteq A_\omega \}$. Note that D is dense set in I^+: if there is some $B \in I^+$ with $B \cap A_n \in I$ for each n and $B \cap A_\omega \in I$, then, by σ-completeness of I, $B = B \cap \bigcup_n A_n \cup B \cap A_\omega \in I$, which is a contradiction.

Stepping outside M, since G is generic over M, we can pick $B \in G \cap D$. Then we have either $B \subseteq A_n$ for some n or $B \subseteq A_\omega$. But, in the latter case, we have $A_\omega = ({}^\omega\omega)^M \setminus \bigcup_n A_n \in G$, which is impossible since $\bigcup_n A_n \in G$. So we have $B \subseteq A_n$ for some n, and, since G is a filter, we have $A_n \in G$ for such n. ☐

Lemma 2.12. *Let G be $(I^+ \restriction M)$-generic filter over M. There exists the unique I-generic real $x_G \in {}^\omega\omega \cap M[G]$:*

$$(\star) \qquad \forall B \in \mathscr{B}^M\, [x_G \in B^V \iff B \in G].$$

Proof. From now on, we fix a generic filter G over M. Uniqueness can be easily verified: if x and y are as desired, then for any $s \in {}^{<\omega}\omega$ we have $x \in [s] \iff [s] \in G \iff y \in [s]$. Hence, they coincide on every initial segment and should be equal.

So it is sufficient to construct such an x. First, observe that $[s] \in G$ holds for exactly one $s \in {}^n\omega$ for each n. With this in mind, our strategy is as follows:

(1) Construct s_n's such that $s_n : n \to \omega$ and $[s_n] \in G$ for all n.

(2) Let $x = \bigcup_n s_n$. Then $(\star)$ holds for every basic open sets.

(3) Show that the collection $\mathscr{C}$ of sets of reals with $(\star)$ forms a σ-M-algebra.

So let's construct such s_n's. As $[\emptyset] = {}^\omega\omega$, we can simply let $s_0 = \emptyset$. If s_n had been constructed as desired, then following D is dense below $[s_n]$:

$$D := \{ C \in I^+ \mid \exists k\, C \leq [s_n {}^\frown k] \}.$$

Suppose not; let $B \leq [s_n]$ be a counterexample. Then $B \cap [s_n {}^\frown k] \in I$ and $B \cap \bigcup_k [s_n {}^\frown k] \in I$ by σ-completeness of I. On the other hand, we have $B = B \cap \bigcup_k [s_n {}^\frown k] \in I^+$, which is contradiction.

Thus, we have defined $x := \bigcup_n s_n$ as desired. By definition, it is clear that $\mathscr{C}$ is closed under complement and contains all basic open sets, so it suffices to see that $\mathscr{C}$ is closed under countable intersections in M. So suppose $\{ B_n \mid n < \omega \} \in \mathcal{P}(\mathscr{C}) \cap M$. Then we have:

$$\begin{aligned} x \in \left(\bigcap_{n<\omega} B_n\right)^* = \bigcap_n B_n^* &\iff \forall n < \omega\, x \in B_n^* && \text{(Fact 2.7)} \\ &\iff \forall n < \omega\, B_n \in G && \text{(by } B_n \in \mathscr{C}\text{)} \\ &\iff \bigcap_n B_n \in G. && (G : \sigma\text{-complete by Lemma 2.11}) \end{aligned}$$

So $\mathscr{C}$ forms σ-algebra containing all basic open sets and contains all Borel sets in M. $\square$

Now, we will see that reals are "almost everywhere" I-generic if I^+ is proper notion of forcing. To state this precisely, we must first introduce suitable ideal extending I:

Definition 2.4. Let σ-ideal I on $\mathscr{B}$.

$$N_I := \{\, A \subseteq {}^\omega\omega \mid \forall B \in I^+ \, \exists C \leq B \, [C \cap A = \emptyset] \,\}.$$

To see that N_I is a σ-ideal extending I, we need the following characterization of properness for ideals:

Lemma 2.13. *Let I be σ-ideal. Then the followings are equivalent:*

(1) *I^+: proper*

(2) $\forall \theta > 2^{\left|I^+\right|} \, \forall M \prec H(\theta) \, (I^+ \in M) \, \forall B \in M \cap I^+$
$$C := \{\, x \in B \mid x : M\text{-}generic \,\} \in I^+.$$
*Here, $x \in {}^\omega\omega$ is M-*generic *for elementary submodel M if $\{\, A \in I^+ \restriction M \mid x \in A \,\}$ meets every dense set of I^+ in M.*

Proof. First observe that C is Borel set and

$$C = B \cap \bigcap \left\{ \bigcup (D \cap M) \,\middle|\, D \in M : \text{dense in } I^+ \right\}.$$

For (1) $\implies$ (2): Fix $B \in M \cap I^+$. By properness, we can take (M, I^+)-master condition $B' \leq B$. If $D \in M$ is dense then $D \cap M$ is predense below B', and hence $B' \setminus \bigcup(D \cap M) \in I$. There are only countably many dense set in M hence we have $\bigcup \{\, B' \setminus \bigcup(D \cap M) \mid D \in M : \text{dense} \,\} \in I$ by σ-completeness. Then we have

$$B' \cap \bigcap \left\{ \bigcup (D \cap M) \,\middle|\, D \in M : \text{dense} \right\} \in I^+,$$

and this contains C above so $C \in I^+$.

Let's show opposite direction. If $C \in I^+$ then $C \Vdash \dot{x}_G \in (\check{C})^{\dot{G}}$ and hence by the definition of C, $C \Vdash \forall D \in M : \text{open dense}\, \dot{G} \cap M \cap D \neq \emptyset$, so C is desired $(M, \mathbb{P})$-master condition. $\square$

Lemma 2.14. *Let I be a σ-ideal. N_I coincides with I on Borel sets. Furthermore, if I^+ is a proper forcing notion, then N_I is σ-complete.*

Proof. It is clear that N_I is an ideal and coincides with I on $\mathscr{B}$. It remains to show the σ-completeness of N_I for proper I^+.

To that end, fix $A_n \in N_I$ $(n < \omega)$ and we shall prove that $A := \bigcup_n A_n \in N_I$. It is enough to fix $B \in I^+$ and find $C \subseteq B$ with $C \in I^+, C \cap A = \emptyset$.

Note that, by the definition of N_I, $D_n := \{\, B \in I^+ \mid B \cap A_n = \emptyset \,\}$ is dense subset of I^+ for each $n < \omega$. So let's take countable $M \prec H(\theta)$ with $I^+, B, D_n \in M$ for sufficiently large θ. By Lemma 2.13, $C := \{\, x \in B \mid x : M\text{-generic} \,\}$ is I-positive and $C \subseteq B$. Hence, every $D_n \in M$ is a dense subset of I^+, we have $x \in \bigcup(D_n \cap M)$ for every n and $x \in C$. Then by definitions of D_n's, $C \cap A = \emptyset$. $\square$

With the lemma above, we can now easily show:

Lemma 2.15. *Let M be a transitive model with $(2^{\aleph_0})^M < \aleph_1$, I be σ-ideal with I^+ proper notion of forcing. Assume that either $(2^{2^{\aleph_0}})^M < \aleph_1$ or I is σ-saturated.*

Then $A := \{\, x \in {}^\omega\omega \mid x : \text{not } I\text{-generic over } M \,\} \in N_I$.

Proof. We first show the case of $(2^{2^{\aleph_0}})^M < \aleph_1$. By assumption there are only countably many dense subsets of $I^+ \restriction M$ in M, because there are only $2^{\aleph_0}$ Borel sets. So we can enumerate such dense subsets as $\{\, D_n \mid n < \omega \,\}$. Then $A_n := {}^\omega\omega \setminus \bigcup (D_n)^V \in N_I$ by Lemma 2.14. If $x \in {}^\omega\omega$ is not I-generic over M, then $x \notin \bigcup (D_n)^V$ for some $n < \omega$. Hence, $x \in \bigcup_n A_n$ and we get $A \subseteq \bigcup_n A_n \in N_I$.

The case where I is σ-saturated is much easier. In this case, I^+ has c.c.c. as a notion of forcing by definition. By σ-saturation, every maximal antichain of I^+ in M must be countable. Since there are only $2^{\aleph_0}$-many Borel sets and $(2^{\aleph_0})^M < \aleph_1$, so we can list maximal antichains of $I^+ \restriction M$ in M as $\{\, D_n \mid n < \omega \,\}$ in V. The remaining argument is similar to the discussion above. ☐

We need one more lemma to prove Theorem 2.10 of Khomskii:

Lemma 2.16. *Let M be some transitive model and I a σ-ideal with I^+ proper as a poset. Furthermore, assume that either $(2^{\aleph_0})^M < \aleph_1$ or I is σ-saturated. If $A \subseteq {}^\omega\omega$ is Solovay over M, then there exists $B \in \mathscr{B}$ such that $\forall x : I\text{-generic}\,[x \in A \iff x \in B]$.*

Proof. Fix $\vec{a} \in M$ and a formula φ witnessing Solovayness of A. That is, $x \in A \iff M[x] \models \varphi(x, \vec{a})$ for arbitrary generic $x \in {}^\omega\omega$.

Let $\dot{x}_G$ be the canonical name for the generic real corresponding to the generic filter $\dot{G}$. Inside M, take a maximal antichain $E \subseteq I^+ \restriction M$ consisting of conditions $p \in I^+ \restriction M$ with $p \Vdash \varphi(\dot{x}, \vec{a})$.

Since we assume that either $(2^{\aleph_0})^M < \aleph_1$ or I is σ-saturated, we can list E as $E = \{\, p_n \mid n < \omega \,\}$ in the ground model V. Now we put $B := \bigcup_{n<\omega} (p_n)^V$. It is clear that $B \in \mathscr{B}$, so it suffices to show that B catches every I-generic reals in A.

Let $x \in {}^\omega\omega$ be an I-generic real over M and G the corresponding $I^+ \restriction M$-generic filter. Note that every I-generic real is a generic real. Then, as A is Solovay over M, we have

$$\begin{aligned} x \in A &\iff M[G] \models \varphi(x, \vec{a}) && (x : \text{generic and } A : \text{Solovay}) \\ &\iff \bigcup_n p_n \in G && (\text{By maximality of } E \text{ and Lemma 2.11}) \\ &\iff x \in \bigcup_n p_n^V = B, && (\text{By Lemma 2.12}) \end{aligned}$$

which was what we wanted. ☐

Corollary 2.17. *Let M be transitive model and I a σ-ideal with I^+ proper as a poset. Also assume that $\{\, x \in {}^\omega\omega \mid x : \text{not } I\text{-generic over } M \,\} \in N_I$ and either $(2^{\aleph_0})^M < \aleph_1$ or I is σ-saturated. If $A \subseteq {}^\omega\omega$ is Solovay over M, then A has an I-regularity.*

Proof. Fix arbitrary $B \in I^+$ and we will find $C \leq_{I^+} B$ with $C \subseteq A$ or $C \cap A = \emptyset$.

By the assumption, the element of A is I-generic N_I-almost everywhere. This means that, by the definition of N_I, we can shrink B not to contain any non I-generic reals beforehand. Then by the Lemma 2.16 above, we may also assume that $A \subseteq {}^\omega\omega$ is Borel. Now, if $B \cap A \in I^+$ then we let $C := B \cap A$ and otherwise $C := B \setminus A$. ☐

Proof of Theorem 2.10. It follows immediately from Lemmas 2.15, 2.16 and Corollary 2.17. ☐

2.2. Levy Collapse and its Basic Properties

We finally introduce the concept of *Levy collapse* which plays a crucial role in Solovay's theorem:

Definition 2.5. Let $\omega \leq \mu < \kappa$ be regular cardinals and κ inaccessible.

We define *Levy collapse* collapsing κ to μ^+, denoted by $\mathrm{Col}(\mu, {<}\kappa)$, as follows:

$$\mathrm{Col}(\mu, {<}\kappa) := \prod_{\mu < \alpha < \kappa}^{<\mu} ({}^{<\mu}\alpha).$$

We put $\mathrm{supt}(p) := \{ \langle \alpha, \beta \rangle \mid \alpha \in \mathrm{dom}(p) \wedge \beta \in \mathrm{dom}(p(\alpha)) \}$ for $p \in \mathrm{Col}(\mu, {<}\kappa)$.

First, we show the combinatorial properties of $\mathrm{Col}(\mu, {<}\kappa)$:

Lemma 2.18. $\mathrm{Col}(\mu, {<}\kappa)$ *has κ-c.c. and is μ-closed.*

Proof. To prove κ-c.c, apply Delta System Lemma 1.3 twice. Note that inaccessibility is needed to apply Delta System Lemma.

μ-closure follows from the regularity of μ and μ-support condition. ☐

Corollary 2.19. $\mathrm{Col}(\mu, {<}\kappa)$ *preserves cardinals $\leq \mu$ and $\geq \kappa$.*

Corollary 2.20. $\mathrm{Col}(\mu, {<}\kappa) \Vdash \check{\kappa} = \check{\mu}^+$.

Proof. By Lemma 1.44 and Corollary 1.46, $\mathrm{Col}(\mu, {<}\kappa)$ adds surjections from μ to α for every $\mu < \alpha < \kappa$ and hence every cardinal $\mu < \alpha < \kappa$ cease to be a cardinal. Since κ remains a cardinal in $\mathrm{Col}(\mu, {<}\kappa)$-generic extension by Corollary 2.19, we have $\mathrm{Col}(\mu, < \kappa) \Vdash \check{\kappa} = \check{\mu}^+$. ☐

Lemma 2.21. $\mathrm{Col}(\mu, {<}\kappa)$ *is homogeneous.*

Proof. Let $p, q \in \mathrm{Col}(\mu, {<}\kappa)$ be arbitrary. We define automorphism $\pi : \mathrm{Col}(\mu, {<}\kappa) \to \mathrm{Col}(\mu, {<}\kappa)$ as follows:

$$\pi(r)(\alpha)(\beta) := \begin{cases} q(\alpha)(\beta) & ((\alpha, \beta) \in \mathrm{supt}(p) \cap \mathrm{supt}(q) \wedge r(\alpha)(\beta) = q(\alpha)(\beta)) \\ p(\alpha)(\beta) & ((\alpha, \beta) \in \mathrm{supt}(p) \cap \mathrm{supt}(q) \wedge r(\alpha)(\beta) = q(\alpha)(\beta)) \\ r(\alpha)(\beta) & (\text{otherwise}) \end{cases}$$

It is easy to check that π is really a dense automorphism. ☐

By the lemma above and Lemma 1.53, we have the following corollary to be used in the proof of Solovay's Theorem:

Corollary 2.22. *If φ is a $\mathcal{FL}_{\mathbb{P}}$-formula, then either $\mathbb{1} \Vdash_{\mathrm{Col}(\mu, {<}\kappa)} \varphi$ or $\mathbb{1} \Vdash_{\mathrm{Col}(\mu, {<}\kappa)} \neg\varphi$.*

The next lemma states that "short sequences" in generic extensions by $\mathrm{Col}(\mu, {<}\kappa)$ comes from some "intermediate stage" of collapsing:

Lemma 2.23. *Let G be $\mathrm{Col}(\mu, {<}\kappa)$-generic over V. For any $f \in {}^{<\kappa}\mathbf{On} \cap V[G]$, there exists some cardinal $\lambda < \kappa$ with $f \in V[G \upharpoonright \lambda]$.*

Proof. Take arbitrary $f \in {}^{<\kappa}\mathbf{On} \cap V[G]$ and fix some nice $\mathrm{Col}(\mu, {<}\kappa)$-name $\dot{f}$ for f; namely, there exists maximal antichains A_n consisting of conditions deciding the value of $\dot{f}(\check{n})$ and $\dot{f}$ can be expressed as $\dot{f} := \left\{ \left\langle \widecheck{\langle n, \alpha \rangle}, p \right\rangle \,\middle|\, p \in A_n \wedge p \Vdash \dot{f}(\check{n}) = \check{\alpha} \right\}$.

Since $\mathrm{Col}(\mu, {<}\kappa)$ has κ-c.c., by Lemma 2.18, and κ is regular, there exists some $\lambda < \kappa$ with $A_n \subseteq \mathrm{Col}(\mu, {<}\lambda)$ for each $n < \lambda$. Then $\dot{f}$ is actually a $\mathrm{Col}(\mu, {<}\lambda)$-name hence $f = \dot{f}^{G \restriction \lambda} \in V[G \restriction \lambda]$.□

Corollary 2.24. *For any $\mu < \kappa$ and $f \in {}^{<\kappa}V[G \restriction \mu] \cap V[G]$, we can find some cardinal $\lambda \in [\mu, \kappa)$ with $f \in V[G \restriction \lambda]$.*

Proof. Since κ remains inaccessible in $V[G \restriction \mu]$ for $\lambda < \kappa$, we can assume without loss of generality that $\lambda = 0$ and $f \in {}^{<\kappa}V$.

Although it is not neccessarily hold that $\mathrm{ran}(f) \in V$, but we can find some $\gamma \in \mathbf{On}$ with $\mathrm{ran}(f) \subseteq V_\gamma$ and we can regard $f \in {}^{<\kappa}V_\gamma$. In V, one can find suitable bijection h between $\beth_\gamma \in \mathbf{On}$ and V_γ by Fact 1.1. Hence, we can code f with $f^* \in {}^{<\kappa}\beth_\gamma$ through this bijection $h \in V$ and we can find some $\lambda < \kappa$ such that $f^* \in V[G \restriction \lambda]$. Since $h \in V \subseteq V[G \restriction \lambda]$, we can decode f^* to f using h in $V[G \restriction \lambda]$ and we have $f \in V[G \restriction \lambda]$. □

Note that the argument above cannot apply to $f \in {}^{<\kappa}V[G]$, since we cannot take a sequence of ordinals coding f in ground model in general.

From now on, we will concentrate on the case of $\mu = \omega$ and let G be $\mathrm{Col}(\omega, {<}\kappa)$-generic filter over V. We prove the final ingredient of Solovay's Theorem, which states some kind of "universality" of Levy collapse $\mathrm{Col}(\omega, {<}\kappa)$:

Lemma 2.25 (Factor Lemma). *Let G be $\mathrm{Col}(\omega, {<}\kappa)$-generic over V, $\lambda < \kappa$ and $\mathbb{P} \in H(\kappa)^{V[G \restriction \lambda]}$. If $s \in V[G]$ is a $\mathbb{P}$-generic filter over $V[G \restriction \lambda]$, then there is some $\mathrm{Col}(\omega, {<}\kappa)$-generic filter $H \in V[G]$ over $V[G \restriction \lambda][s]$ such that $V[G] = V[G \restriction \lambda][s][H]$.*

Carefully analyzing the proof below, we can assume that $\lambda = 0$ without loss of generality.

The following two lemmas are keys to prove the lemma above:

Lemma 2.26. *Let μ be an infinite cardinal and $\mathbb{P}$ an atomless poset and $\mathbb{1}_{\mathbb{P}} \Vdash |\check{\mu}| = \aleph_0$. Then for any $p \in \mathbb{P}$, there exists an antichain A below p with $|A| = \mu$.*

Proof. Induct on μ; the case $\mu = \omega$ is trivial, so we consider the case in which $\mu > \omega$.

First, consider the subcase that $\mathrm{cf}\,\mu = \omega < \mu$. In such a case, we can fix some strictly increasing ω-sequence $\langle \mu_n \mid n < \omega \rangle$ of regular uncountable cardinals with $\mu_n \nearrow \mu$. Since $\mu_n < \mu$ and $\mathbb{1} \Vdash |\check{\mu}| = \aleph_0$, we also have that $\mathbb{1} \Vdash |\check{\mu_n}| = \aleph_0$. By the induction hypothesis, we can take $p_n \leq p$ and some antichain $A_n \subseteq$ below p_n with following properties for each $n < \omega$:

- $p_n \leq p$,
- $p_{n+1} \in A_n$, and
- A_n is an antichain of size μ_n.

Then, if we put $A := \bigcup_n (A_n \setminus \{ p_{n+1} \})$, we have that A is an antichain below p and $|A| = \sup_n |A_n| = \mu$.

Last, consider the case $\mathrm{cf}\,\mu > \omega$. Let $\dot{h}$ be a nice $\mathbb{P}$-name for the surjection from ω to μ and A_n maximal antichain deciding the value of $\dot{h}(n)$ for each $n < \omega$. We define the function $f : (\omega \times \bigcup_n A_n) \to \kappa$ as follows:

$$f(n,p) := \begin{cases} \xi & (\text{if } p \Vdash \dot{h}(\check{n}) = \check{\xi}) \\ 0 & (\text{otherwise}) \end{cases}$$

Since $\dot{h}$ is a name for a surjection, it follows that f is also a surjection onto κ. But then, since $\mathrm{cf}\,\mu > \omega$, there must exists some $n < \omega$ with $|A_n| \geq \kappa$. □

Lemma 2.27. *Let $\mu \geq \omega$ and $\mathbb{P}$ be a separative poset with $|\mathbb{P}| = \mu$. If $\mathbb{1}_{\mathbb{P}} \Vdash |\check{\mu}| = \aleph_0$ then there exists dense embedding $\pi : {}^{<\omega}\mu \xrightarrow[d]{} \mathbb{P}$.*

Proof. First observe that $\Vdash_{\mathbb{P}} |\dot{G}| = \aleph_0$ since $\Vdash \dot{G} \subseteq \check{\mathbb{P}}$ and $|\mathbb{P}| = \mu$. So we can fix a nice $\mathbb{P}$-name $\dot{f}$ for a surjection from ω to $\dot{G}$.

We define the value of $\pi(p)$ by the induction on the length of $p \in {}^{<\omega}\mu$.

For $\mathrm{lh}(p) = 0$, let $\pi(\emptyset) = \mathbb{1}_{\mathbb{P}}$.

Suppose we have defined the value of $\pi(p)$ for $\mathrm{lh}(p) = n$. We have to define the value for $\pi(p \frown \langle i \rangle)$ for each $i < \mu$. By Lemma 2.26 above, we can fix some antichain A below $\pi(p)$ of size μ. In addition, we can refine each element of A to decide the value of $\dot{f}(n)$. Then, we enumerate A as $A = \{\, q_i \mid i < \mu \,\}$ and put $\pi(p \frown \langle i \rangle) := q_i$.

It is easy to check that π defined above is monotone and preserves incompatibility. It remains to show that $\pi[{}^{<\omega}\mu]$ is dense in $\mathbb{P}$.

Let $A_n := \{\, \pi(p) \mid p \in {}^{n}\mu \,\}$. Observe that each A_n is an antichain maximal amongst ones deciding the value of $\dot{f}$ below n.

To prove the density of π, we fix an arbitrary $r \in \mathbb{P}$ and find some $p \in {}^{<\omega}\mu$ with $\pi(p) \leq r$. Specifically, since $\mathbb{P}$ is separative by our assumption, it suffices to find some p with $\pi(p) \Vdash \check{r} \in \dot{G}$.

Since $r \Vdash \check{r} \in \dot{G}$, we can take $s \leq r$ and $n < \omega$ such that $s \Vdash \dot{f}(\check{n}) = \check{r}$. We can also assume that s decides each value of $\dot{f}$ upto n, without any loss of generality. Then, by the observation on A_n above, we can fix $p \in {}^{<\omega}\mu$ such that $\pi(p) \parallel s$ and $\pi(p) \in A_{n+1}$. Since both s and $\pi(p)$ decides the value of $\dot{f}$ upto n and are compatible with each other, we have $\pi(p) \Vdash \dot{f}(\check{n}) = \check{r}$, which implies that $\pi(p) \Vdash \check{r} \in \dot{G}$. □

With lemmas above, we can prove Factor Lemma 2.25 for Levy Collapse can be showed as follows:

Proof of Factor Lemma 2.25. First, by Corollary 2.20, $\mathbb{P}$ is countable in $V[G]$ and so is s hence they can be coded by ω-sequences of elements of $V[G]$. Therefore, s sits in $V[G \upharpoonright \nu]$ for some $\nu < \kappa$ by Lemma 2.23, so we fix a $\mathrm{Col}(\omega, {<}\nu)$-name $\dot{s}$ for s with $\dot{s}_{V[G\upharpoonright\nu]} = s$, by Lemma 2.23. Furthermore, taking $\nu < \kappa$ sufficiently large, we may also assume that $|\mathbb{P}| < \nu$, since κ is inaccessible. If we let $G_0 := G \upharpoonright (\nu + 1), G_1 := G \upharpoonright \{\, \nu + 1 \,\}, G_2 := \mathrm{Col}(\omega, [\nu + 2, \kappa))$, we have $V[G] = V[G_0][G_1][G_2]$ by Product Lemma 1.47.

We will “approximate” $\mathrm{Col}(\omega, {<}\nu + 1)$ inside $V[s]$ by some smaller poset $\mathbb{Q}$ so that G_0 is $\mathbb{Q}$-generic over $V[s]$. Namely, we will show the following claim:

Claim 1. *There is $\mathbb{Q} \in V[s]$ with $G_0 \subseteq \mathbb{Q} \subseteq \mathrm{Col}(\omega, {<}\nu + 1)$ such that G_0 is $\mathbb{Q}$-generic over $V[s]$.*

First, let’s assume the claim above and show our goal. Note that, by the minimality of a generic extension, we have $V[G_0] \subseteq V[s][G_0] \subseteq V[G_0]$ and hence $V[s][G_0] = V[G_0]$.

Since $\mathbb{Q} \times \mathrm{Col}(\omega, \{\, \nu+1 \,\})$ collapses ν to ω and has cardinality ν, there exists dense embedding $j : {}^{<\omega}\nu \xrightarrow[d]{} \mathbb{Q} \times \mathrm{Col}(\omega, \{\, \nu+1 \,\})$. There is also another dense embedding $i : {}^{<\omega}\nu \xrightarrow[d]{} \mathrm{Col}(\omega, {<}\nu+2)$, so let

$$H := \{\, p \in \mathrm{Col}(\omega, {<}\nu+1) \mid \exists q \in {}^{<\omega}\nu\, [i(q) \leq p \wedge j(q) \in G_0 \times G_1] \,\}$$

Then H is $\mathrm{Col}(\omega, {<}\nu+2)$-generic over $V[s]$, hence we have:

$$\begin{gathered} V[s][H] = V[s][G_0 \times G_1] = V[s][G_0][G_1] = V[G_0][G_1] \\ \therefore V[G] = V[G_0][G_1][G_2] = V[s][H \times G_2]. \end{gathered}$$

Here, by Product Lemma 1.47, $H \times G_2$ is $\mathrm{Col}(\omega, {<}\kappa)$-generic over $V[s]$, which was what we wanted.

It's left to show that there actually exists such $\mathbb{Q} \in V[s]$. In $V[s]$, we inductively define $\mathbb{Q}_\delta$ as follows:

$$\mathbb{Q}_0 := \left\{ p \in \mathrm{Col}(\omega, {<}\nu+1) \;\middle|\; \forall r \in \mathbb{P} \begin{bmatrix} p \Vdash \text{“}\check{r} \in \dot{s}\text{”} \implies r \in s \\ p \Vdash \text{“}\check{r} \notin \dot{s}\text{”} \implies r \notin s \end{bmatrix} \right\}, \tag{2.1}$$

$$\mathbb{Q}_{\alpha+1} := \left\{ p \in \mathbb{Q}_\alpha \;\middle|\; \begin{gathered} \forall D \in V[s] : \text{open dense in}\, \mathrm{Col}(\omega, {<}\nu+1) \\ \exists p' \leq p\, [p' \in D \cap \mathbb{Q}_\alpha] \end{gathered} \right\}, \tag{2.2}$$

$$\mathbb{Q}_\gamma := \bigcap_{\beta<\gamma} \mathbb{Q}_\beta \qquad (\gamma : \text{limit}). \tag{2.3}$$

Then there must be some δ with $\mathbb{Q}_\delta = \mathbb{Q}_{\delta+1}$; we let $\mathbb{Q} := \mathbb{Q}_\delta$ for the least such δ.

First we show that $G_0 \subseteq \mathbb{Q}$. We prove this by contradiction; suppose $G_0 \setminus G \neq \emptyset$. Then $\beta = \alpha+1$ for some α by definition. Fix $p \in G_0 \setminus \mathbb{Q}_{\alpha+1}$. Then there exists some dense $D \subseteq \mathrm{Col}(\omega, {<}\nu+1)$ with $p' \in D \implies p' \notin \mathbb{Q}_\alpha$ for all $p' \leq p$. But such D is also dense in $\mathbb{Q}_\alpha$ so there must be some $p' \leq p$ with $p \in D \cap \mathbb{Q}_\alpha$. Contradiction!

Finally, we will show that G_0 is $\mathbb{Q}$-generic over $V[s]$. Assume contrary: fix some $D \in V[s]$ dense in $\mathbb{Q}$ with $D \cap G_0 = \emptyset$. By Forcing Theorems 1.42, We can take $\mathrm{Col}(\omega, {<}\nu+1)$-names $\dot{D}$ and $\dot{\mathbb{Q}}$, and $p \in G_0$ such that $\dot{D}_{G_0} = D, \dot{\mathbb{Q}}_{G_0} = \mathbb{Q}$ and

$$p \Vdash \text{“}\dot{D} \cap \dot{G}_0 = \emptyset \wedge \dot{G}_0 \subseteq \dot{\mathbb{Q}} \wedge \dot{D} : \text{dense in } \dot{\mathbb{Q}}\text{”}.$$

Furthermore, $\dot{D}$ can be defined in terms of $\mathrm{Col}(\omega, {<}\nu+1)$-name $\dot{s}$ and some $\mathbb{P}$-name $\dot{D}'$. Since $p \in G_0 \subseteq \mathbb{Q}$ and D is dense in $\mathbb{Q}$, we have $V[s] \models \exists q \leq p\, [q \in D]$. So fix such a $q \leq p$.

By condition (2.2) and $\mathbb{Q} = \mathbb{Q}_\delta = \mathbb{Q}_{\delta+1}$, every $\mathrm{Col}(\omega, {<}\nu+1)$-dense set is also dense in $\mathbb{Q}$. Furthermore, $\mathbb{Q}$ has only countably many dense sets viewed from $V[G]$, so we can choose another $\mathrm{Col}(\omega, {<}\nu+1)$-generic filter G_0' over $V[s]$ with $q \in G_0 \subseteq \mathbb{Q}$. Moreover, by condition (2.1), we have that $\dot{s}_{G_0'} = s$ for such a G_0'. Since $\dot{D}$ can be defined from $\dot{s}$, we also have $\dot{D}_{G_0'} = D$. Because $q \leq p$, we have $q \Vdash \dot{G}_0 \cap \dot{D} = \emptyset$ and hence $V[G_0'] \models G_0' \cap D = \emptyset$. On the other hand, by the choice of q, we have $V[G_0'] \models q \in D \cap G_0'$. Contradiction! ☐

2.3. Proof of Solovay's Theorem

Putting all the ingredients introduced above together, we are now ready to prove the following Solovay–Khomskii Theorem:

Theorem 2.28 (Khomskii). *Let κ be inaccessible and I a projective σ-ideal on $\mathscr{B}$ with I^+ proper as forcing notion. If G is $\mathrm{Col}(\omega, {<}\kappa)$-generic over V, then every set of reals is I-regular in $(\mathrm{HOD}({}^{\omega}z))^{V[G]}$ for any infinite class $z \subseteq V[G \restriction \mu]$ and $\mu < \kappa$.*

Proof. Let $M := (\mathrm{HOD}({}^{\omega}z))^{V[G]}$ and I a σ-ideal as above. We fix a set $A \in M$ of reals and show that A has I-regularity.

Because I is projective, notion of being "I-regular" is absolute between $V[G]$ and M by Theorem 1.37. Hence, by Theorem 2.10, it suffices to show that A is Solovay over some inner model N of $V[G]$ with $(2^{2^{\aleph_0}})^N < \aleph_1^{V[G]}$. So we have to search such an inner model N.

Well, since $A \in \mathrm{HOD}({}^{\omega}z)$, there exists some formula $\varphi(x, y, w)$, $\alpha \in \mathbf{On}$ and $s \in {}^{\omega}z$ with

$$A = \{\, x \in {}^{\omega}\omega \mid \varphi(x, s, \alpha) \,\}.$$

Then, by Corollary 2.24, we can fix $\lambda < \kappa$ such that $f \in V[G \restriction \lambda]$. So let $N := V[G \restriction \lambda]$. By Product Lemma 1.47, $V[G \restriction \lambda]$ can be viewed as an inner model of $V[G]$ and $(2^{2^{\aleph_0}})^N < \aleph_1^{V[G]}$ because $V[G]$ collapses all uncountable cardinals $< \kappa$ to be countable. So it suffices to show the following Claim:

Claim 1. *A is Solovay over N.*

All we have to do is to find some formula $\tilde{\varphi}$ such that $x \in A \iff N[x] \models \tilde{\varphi}(x)$ holds for any generic real $x \in {}^{\omega}\omega$ over N. So let us fix a generic real x over N. Note that such an x can be added by some "small" forcing notion. So, applying the Factor Lemma 2.25, there is some $\mathrm{Col}(\omega, {<}\kappa)$-generic filter $H \in V[G]$ over $N[x]$ such that $V[G] = V[G \restriction \lambda][x][H]$. Then,

$$\begin{aligned} x \in A &\iff N[x][H] = V[G] \models \varphi(x, s, \alpha) \\ &\iff \exists p \in H\ N[x] \models \text{“}p \Vdash_{\mathrm{Col}(\omega,<\kappa)} \varphi(\check{x}, \check{s}, \check{\alpha})\text{”} \end{aligned}$$

$\mathrm{Col}(\omega, {<}\kappa)$ is homogeneous by Lemma 2.21, and by the Zero-One Law 1.53 we have:

$$\iff N[x] \models \text{“}\mathbb{1} \Vdash_{\mathrm{Col}(\omega,<\kappa)} \varphi(\dot{x}, \check{s}, \check{\alpha})\text{”}.$$

By Forcing Theorem 1.42, the formula "$\mathbb{1} \Vdash_{\mathrm{Col}(\omega,<\kappa)} \varphi(\check{x}, \check{s}, \check{\alpha})$" can be stated in $N[x]$. So letting $\tilde{\varphi}(x) :\equiv$ "$\mathbb{1} \Vdash_{\mathrm{Col}(\omega,<\kappa)} \varphi(\check{x}, \check{s}, \check{\alpha})$" we have:

$$x \in A \iff N[x] \models \tilde{\varphi}(x).$$

This shows that A is Solovay over N. ☐

Corollary 2.29 (Solovay [32]). *Under the same assumption on κ and z as above, every set of reals has Baire property and Lebesgue measurable in* $(\mathrm{HOD}({}^{\omega}z))^{V[G]}$.

Proof. This corollary is almost obvious, but note that null and meager are clearly projective σ-ideal and their positive sets is proper as a forcing, since they are σ-saturated and hence proper by Lemma 1.54. ☐

2.4. Ramsey Property in Solovay's Model

Another example of regularity property is the following notion of (*complete*) *Ramsey Property*:

Definition 2.6.

- $A \subseteq [\omega]^{\omega}$ has *Ramsey Property* if $\exists x \in [\omega]^{\omega}\ [[x]^{\omega} \cap B = \emptyset \vee [x]^{\omega} \subseteq B]$.
- For $s \in [\omega]^{<\aleph_0}$ and $A \in [\omega]^{\omega}$, $[s, A]^{\omega} := \{\, x \in [\omega]^{\omega} \mid s \subset x \wedge x \setminus s \subset A \setminus \sup^{+} s \,\}$. That is, $[s, A]^{\omega}$ is the collection of $x \in [\omega]$ such that its initial segment coincides with s and its tail part is subsumed by A.

- *Mathias forcing* $(\mathbb{M}, \leq, \emptyset)$ is the forcing notion defined as below:

$$\mathbb{M} := \left\{ (s, A) \;\middle|\; s \in [\omega]^{<\aleph_0}, A \in [\omega]^\omega, s \cap A = \emptyset \right\},$$
$$(t, B) \leq (s, A) \overset{\text{def}}{\iff} s \subseteq_{\text{end}} t \wedge t \setminus s \subseteq A, B \subseteq A.$$

  Each condition $(s, A) \in \mathbb{M}$ approximates some $x \in [s, A]^\omega$.

- $A \subseteq [\omega]^\omega$ is *completely Ramsey*, or has *complete Ramsey Property* if

$$\forall (s, F) \in \mathbb{M}\, \exists H \in [F]^\omega \;\; [s, H]^\omega \cap A = \emptyset \vee [s, H]^\omega \subseteq A.$$

- $x \subseteq [\omega]^\omega$ is called *Ramsey null* if $\forall (s, A)\, \exists B \in [A]^\omega \; x \cap [s, B]^\omega = \emptyset$. We denote the collection of all Ramsey null Borel sets as I_{RN}. $x \subseteq [\omega]^\omega$ is called *Ramsey positive* if it's completely Ramsey but not Ramsey null.

It is clear by definition that complete Ramsey property implies Ramsey property.

Note that $[\omega]^\omega$ can be viewed as ${}^\omega\omega$ via increasing enumeration; that is, we can identify $A \in [\omega]^\omega$ with strictly increasing enumeration $e_A : \omega \to A$ of elements of A. Complete Ramsey property can be thought as another example of our regularity properties in this sense. It is immediate that Ramsey null sets are completely Ramsey.

Under AC, there is a set without Ramsey Property:

Lemma 2.30. *Under* AC *(especially existence of well-order on ${}^\omega 2$), there exists a set X of reals without Ramsey property.*

Proof. Enumerate $[\omega]^\omega$ as $\{ A_\xi \mid \xi < 2^\omega \}$ and we recursively take x_ξ and y_ξ so that

(1) $x_\xi \in [A_\xi]^\omega \setminus (\{ x_\gamma \mid \gamma < \xi \} \cup \{ x_\gamma \mid \gamma < \xi \})$,

(2) $y_\xi \in [A_\xi]^\omega \setminus \{ x_\gamma \mid \gamma \leq \xi \}$.

These x_ξ's and y_ξ's can be taken, because we have a well-order on $[\omega]^\omega$ and $|[A_\xi]^\omega| = 2^\omega$ for each $\xi < 2^\omega$.

Once this is done, clearly $X := \{ x_\xi \mid \xi < 2^\omega \}$ doesn't have Ramsey property: if $A_\xi \in [\omega]^\omega$ then $x_\xi \in [A_\xi]^\omega \cap A$ and $y_\xi \in [A_\xi]^\omega \setminus A$. ☐

In what follows, we prove that "every set of reals is completely Ramsey" holds in Solovay's model via Khomskii Theorem 2.28. We have to establish the following results:

(I) Every Borel set has complete Ramsey property and I_{RN} is a σ-ideal on $\mathscr{B}$.

(II) Notions of being "Completely Ramsey" and "I_{RN}-regular" coincides.

(III) ${I_{\text{RN}}}^+$ is proper as a notion of forcing.

The following *Ellentuck topology* on $[\omega]^\omega$ plays a central role in each step above:

Definition 2.7. The *Ellentuck Topology* on $[\omega]^\omega$ is the topology generated by basic open sets of the form $[s, A]^\omega$ for $(s, A) \in \mathbb{M}$. We denote the collection of Ellentuck open sets by $\mathscr{E}$.

In this section, we prefix "E-" with topological notions in Ellentuck topology to avoid confusions.

Observe that, if $s \in {}^{<\omega}\omega$, we have $[s] = [s, \omega]^\omega$ and hence above Ellentuck topology is finer than the relative topology on $[\omega]^\omega$ induced by the usual topology of ${}^\omega\omega$.

First, we show (I): all Borel sets are completely Ramsey and Ramsey nullality is preserved by a countable union.

To that end, we introduce several temprary terms for convenience: we say $A \in [\omega]^\omega$ *accepts* $s \in [\omega]^{<\omega}$ *under* $S \subseteq [\omega]^\omega$ if $[s, A]^\omega \subseteq S$. $A \in [\omega]^\omega$ *rejects* $s \in [\omega]^{<\omega}$ *under* $S \subseteq [\omega]^\omega$ if $[s, B]^\omega \nsubseteq S$ for all $B \in [A]^\omega$. If A either accepts or rejects s under S, then A is said to *judge* s *under* S. We sometimes omit "under S" if S is clear from the context.

Lemma 2.31. *Let $S \subseteq [\omega]^\omega$ be any subset. Then, there exists $A \in [\omega]^\omega$ which judges every $s \in [A]^\omega$ under S.*

Proof. We will take $a_n < \omega$ and $A_n \in [\omega]^\omega$ recursively such that:

(a) $a_n := \min(A_n \setminus \{ a_k \mid k < n \})$,

(b) $A_{n+1} \subseteq A_n \setminus \{ a_k \mid k \leq n \}$, and

(c) A_n judges each $s \subseteq \{ a_k \mid k < n \}$ under S.

For $n = 0$, if there is some $C \subseteq \omega$ such that $[\emptyset, C]^\omega \subseteq S$, then $A_0 := C$ accepts $\emptyset$. If there are no such C, then $A_0 := \omega$ itself rejects $\emptyset$. In either case, required conditions clearly hold.

Suppose a_n and A_n have been defined. Since $\mathcal{P}(\{ a_k \mid k \leq n \})$ is finite, we can list them as $\{ s^n_k \mid k < 2^n \}$ and choose A^n_k so that each s^n_k is judged by A^n_k and $A^n_{k+1} \subseteq A^n_k \subseteq A_n$, by similar arguments in the case of $n = 0$. Then if we let $A_{n+1} := A^n_{2^n}$, all conditions are clearly met.

Then $A := \{ a_k \mid k < \omega \}$ is as desired. In fact, if $s \in [A]^\omega$ then there must be some $n < \omega$ such that $s \subseteq \{ a_n \mid n < \omega \}$ and A_n judges s by condition (c). If A_n accepts s, then we have $[s, A]^\omega \subseteq [s, A_n]^\omega \subseteq S$ since $A \subseteq A_n$. In case A_n rejects s, we have $[s, B]^\omega \setminus S \neq \emptyset$ for each $B \subseteq A \subseteq A_n$. ☐

Lemma 2.32. *Every E-open set has the Ramsey Property.*

Proof. Let S be an arbitrary E-open set and take A as in Lemma 2.31. If A accepts $\emptyset$ under S, then we have $[A]^\omega = [\emptyset, A]^\omega \subseteq S$ and hence S is Ramsey.

So let's suppose A rejects $\emptyset$ and find some $B \subseteq A$ with $[B]^\omega \cap S = \emptyset$.

Claim 1. *$\exists B \in [A]^\omega\ \forall s \in [B]^{<\omega}$ B rejects s under S.*

Proof. We inductively take $B_n := \{ b_k \mid k < n \}$ so that $A \setminus B_n$ rejects each subset of B_n. $B_0 = \emptyset$ clearly satisfies this requirement because A rejects $\emptyset$ by our assumption. So let's assume B_n has been taken as desired and we take b_n. First, observe that $Z_s := \{ z < \omega \mid A \setminus B_n \text{ accepts } s \cup \{ z \} \}$ is finite for each $s \subseteq B_n$. Otherwise, $Z_s \subseteq X \setminus B_n$ is an infinite subset accepting s which contradicts with the induction hypothesis. So we can put $b_n := \min \left(A \setminus \bigcup_{s \subseteq B_n} Z_s \right)$ and this clearly satisfies our requirement. ☐ (Claim 1)

It remains to show that $[B]^\omega \cap S = \emptyset$. Suppose otherwise; if $Z \in [B]^\omega \cap S$, then, since S is E-open, there must be some $s \subseteq_{\text{end}} Z$ with $[s, Z \setminus s]^\omega \subseteq S$. But, by the claim above, B must reject all of its finite subsets, in particular, it must be the case that $[s, Z]^\omega \nsubseteq S$. Contradiction! ☐

Lemma 2.33. *Every E-open set is completely Ramsey.*

Proof. Let S be an E-open set and fix $(s, A) \in \mathbb{M}$ arbitrarily. Since A is infinite, we can pick the strictly increasing enumeration $f : \omega \to A$ of A. If we define $f^* : [\omega]^\omega \to [\omega]^\omega$ by $f^*(X) := s \cup f[X]$, then f^* is continuous; indeed, if $X \in f^{*-1} \text{ ''} \, [t, B]^\omega$, then we have $X \in [f^{-1}[t], X \setminus f^{-1}[t]] \subseteq f^{*-1} \text{ ''} \, [t, B]^\omega$.

Let $T := f^{*-1}[S]$. Then T is E-open as well, and T has Ramsey Property by Lemma 2.32. So take $K \in [\omega]^\omega$ such that either $[K]^\omega \subseteq T$ or $[K]^\omega \cap T = \emptyset$ holds. Then, putting $H := f[K] \subseteq A$, we have $f^*[[K]^\omega] = [s, H]^\omega$. Hence, either $[s, H]^\omega \subseteq S$ or $[s, H]^\omega \cap S = \emptyset$ must hold. ☐

Corollary 2.34. *Every E-closed, open and closed set is completely Ramsey.*

Lemma 2.35. *If S is completely Ramsey, then $N := S \setminus \operatorname{int}_{\mathscr{E}} S$ is Ramsey null.*

Proof. Fix $(s, A) \in \mathbb{M}$. Since S is completely Ramsey, there must be $B \in [A]^\omega$ with $[s, B]^\omega \subseteq S$ or $[s, B]^\omega \cap S = \emptyset$. In the former case, since $[s, B]^\omega$ is E-open, we have $[s, B]^\omega \subseteq \operatorname{int}_{\mathscr{E}} S$ and we have $N \cap [s, B]^\omega = \emptyset$. In the latter case, it's clear that $[s, B]^\omega \cap N \subseteq [s, B]^\omega \cap S = \emptyset$. ☐

Lemma 2.36. *N is E-nowhere dense if and only if it is Ramsey null.*

Proof. Since "if" direction follows immediately from definition, we show "only if" direction. Let N be E-nowhere dense. By Lemma 2.35, we may also assume N to be E-closed. So fix $(s, A) \in \mathbb{M}$ arbitrarily. Since N is closed, it has complete Ramsey Property by Corollary 2.34 and hence there is $B \in [A]^\omega$ so that either $[s, B]^\omega N$ or $[s, B]^\omega \cap N = \emptyset$ holds. But, in the former case, $[s, B]^\omega$ should contain open set disjoint from N, since N is E-nowhere dense. Hence, we have $[s, B]^\omega \cap N = \emptyset$. ☐

Lemma 2.37. *The collection of Ramsey null sets is closed under countable union. Hence, I_{RN} forms σ-ideal on $\mathscr{B}$.*

Proof. Let $\langle N_k \mid k < \omega \rangle$ be a countable family of Ramsey null sets and we show that $N := \bigcup_k N_k$ is also Ramsey null.

To that end, fix $(s, A) \in M$ and pick $a_n < \omega$ and X_n so that

(1) $X_{n+1} \subseteq X_n \setminus \{ a_k \mid k \leq n \}$,

(2) $s \subseteq t \subseteq s \cup \{ a_k \mid k < n \} \implies [t, X_n]^\omega \cap N_n = \emptyset$,

(3) $a_n := \min X_n$.

Clearly, such a_n and X_n can be taken since each N_n is Ramsey null.

Then, it easily follows that $H := \{ a_k \mid k < \omega \} \subseteq A$ witnesses that N is Ramsey null. ☐

At this point, we can stablish the following characterization of complete Ramsey property and Ramsey nullarity in terms of Ellentuck topology:

Theorem 2.38. (1) *S is completely Ramsey if and only if it has E-Baire Property.*

(2) *S is Ramsey null, if and only if it is E-nowhere dense, if and only if it is E-meager.*

Proof. (1): Immediate from Lemma 2.35. (2): By Lemmas 2.36 and 2.37. ☐

Corollary 2.39. *Every Borel set has complete Ramsey Property.*

Proof. Let $\mathscr{R}$ be the collection of completely Ramsey sets. Clearly, $\mathscr{R}$ is closed under complement, and, by Corollary 2.34, $\mathscr{R}$ contains all open sets. So it's enough to show that it's also closed under countable unions.

Let $\langle S_n \mid n < \omega \rangle$ be a countable family of completely Ramsey sets. By Theorem 2.38 and Corollary 2.34, every S_n has E-Baire Property and can be written as $S_n = G_n \cup N_n$ for some E-open set G_n and Ramsey null N_n. Then $S := \bigcup_n S_n = \bigcup_n G_n \cup \bigcup_n N_n$ and it clearly has E-Baire Property. Hence, S is completely Ramsey again by Theorem 2.38. ☐

This complets step (I). We can now easily establish (II), namely, the equivalence between I_{RN}-regularity and complete Ramsey Property.

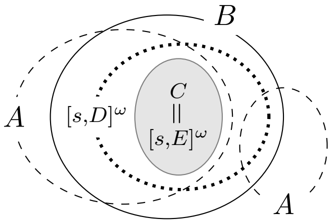
Lemma 2.40. *If* $B \in {I_{\mathrm{RN}}}^{+}$ *then there is* $(s, A) \in \mathbb{M}$ *such that* $[s, A]^{\omega} \subseteq B$.

Proof. Because B is not Ramsey null, there must be some $(s, C) \in \mathbb{M}$ with $[s, H] \nsubseteq B$ for every $H \in [C]^{\omega}$. On the other hand, since $B \in \mathscr{B}$, B is completely Ramsey and therefore there is some $B \in [C]^{\omega}$ with $[s, B]^{\omega} \subseteq B$. ☐

Lemma 2.41. $A \subseteq [\omega]^{\omega}$ *has complete Ramsey Property if and only if it's* I_{RN}*-regular.*

Proof. First, suppose A is completely Ramsey and we show A is I_{RN}-regular.

We arbitrarily choose $B \in {I_{\mathrm{RN}}}^{+}$ and find $C \leq_{{I_{\mathrm{RN}}}^{+}} B$ such that $C \cap A = \emptyset$ or $C \subseteq A$. By Lemma 2.40, one can take $(s, D) \in \mathbb{M}$ with $[s, D]^{\omega} \subseteq B$. Since A is completely Ramsey, we can take $E \in [D]^{\omega}$ so that either $[s, E]^{\omega} \subseteq A$ or $[s, E]^{\omega} \cap A = \emptyset$. So $C := [s, E]^{\omega}$ is as desired.



Let's show the converse: let S be I_{RN}-regular. Fixing $(s, A) \in \mathbb{M}$, we find $B \subseteq A$ so that either $[s, B]^{\omega} \subseteq S$ or $[s, B]^{\omega} \cap S = \emptyset$ holds. In particular, since $[s, B]^{\omega}$ is E-open, these conditions are equivalent to $[s, B]^{\omega} \subseteq \mathrm{int}_{\mathscr{E}}\, S$ and $[s, B]^{\omega} \subseteq \mathrm{ext}_{\mathscr{E}}\, S$ respectively.

Let $U = \mathrm{int}_{\mathscr{E}}\, S \cup \mathrm{ext}_{\mathscr{E}}\, S$. Since U is E-open, it has complete Ramsey Property by Lemma 2.33. Therefore, we can pick $B_0 \in [A]^{\omega}$ so that either $[s, B_0]^{\omega} \subseteq U$ or $[s, B_0]^{\omega} \cap U = \emptyset$ holds. Furthermore, by the I_{RN}-regularity of S, there must be $(t, X) \leq (s, B_0)$ such that $[t, X]^{\omega} \subseteq \mathrm{int}_{\mathscr{E}}\, S$ or $[t, X]^{\omega} \subseteq \mathrm{ext}_{\mathscr{E}}\, S$. Hence, we have $[s, B]^{\omega} \cap U \neq \emptyset$. Therefore, it's the case $[s, B]^{\omega} \subseteq \mathrm{int}_{\mathscr{E}}\, S \cup \mathrm{ext}_{\mathscr{E}}\, S$ and, by the complete Ramsey Property of $\mathrm{int}_{\mathscr{E}}\, S$ and $\mathrm{ext}_{\mathscr{E}}\, S$, one can take $C \subseteq B$ so that either $[s, C]^{\omega} \subseteq \mathrm{int}_{\mathscr{E}}\, S$ or $[s, C]^{\omega} \subseteq \mathrm{ext}_{\mathscr{E}}\, S$ holds, which is what we wanted. ☐

Finally, we will show (III) to complete this section. This step is divided to two parts:

(III-a) ${I_{\mathrm{RN}}}^{+}$ is forcing equivalent to Mathias forcing $\mathbb{M}$.

(III-b) $\mathbb{M}$ is proper as a notion of forcing.

For (III-a):

Lemma 2.42. ${I_{\mathrm{RN}}}^{+}$*, Mathias forcing* $\mathbb{M}$ *and* $\mathrm{ro}([\omega]^{\omega}, \mathscr{E})$ *are all forcing equivalent.*

Proof. Consider the following maps:

$$\begin{tikzcd}
h : \mathbb{M} \arrow[r] & {I_{\mathrm{RN}}}^{+} & i : \mathbb{M} \arrow[r] & \mathrm{ro}([\omega]^\omega, \mathscr{E}) & e : \mathscr{O}_{\mathbb{M}} \arrow[r] & \mathscr{E} \\
(s,A) \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & {[s,A]^\omega} \arrow[u, phantom, sloped, "\in"] & (s,A) \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & {[s,A]^\omega} \arrow[u, phantom, sloped, "\in"] & U \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \bigcup_{(s,A)\in U} [s,A]^\omega . \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

Then h, i and e are well-defined because $[s,A]^\omega$ is clearly clopen and hence regular open and is Ramsey positive by Lemma 2.33. In particular, it is easy to see that i, h are both dense embedding and follows that $\mathbb{M}$, ${I_{\mathrm{RN}}}^{+}$ and $\mathrm{ro}([\omega]^\omega, \mathscr{E})$ are all forcing equivalent by Fact 1.48. ☐

So, since properness is preserved by dense embeddings, it suffices to show (III-b) properness of $\mathbb{M}$.

Unfortunately, $\mathbb{M}$ is neither c.c.c. nor σ-closure, hence we cannot use Lemma 1.54 to establish its properness:

Example 2.2.

- *A descending chain* $\{ (n, \omega \setminus (n+1)) \mid n < \omega \} \subseteq \mathbb{M}$ *doesn't have any common extension in* $\mathbb{M}$.
- *For each* $f \in {}^{\omega}2$, *we define* $X_f := \{ f \restriction n \mid n < \omega \}$. *Then family* $\mathcal{A} := \{ X_f \mid f \in {}^{\omega}2 \}$ *is almost disjoint, i.e.,* $a \cap b$ *is finite for any distinct pair* $a, b \in A$. *Then, if* $h : {}^{<\omega}2 \xrightarrow{\sim} \omega$ *is the canonical bijection,* $\{ \langle \emptyset, h[X_f] \rangle \mid f \in {}^{\omega}2 \}$ *is clearly an uncountable antichain of* $\mathbb{M}$.

So we have to prove the properness of $\mathbb{M}$ directly. In particular, we shall prove this via the following lemma:

Lemma 2.43 (Prikry Lemma for Mathias Forcing). *Let* φ *be an* $\mathcal{FL}_{\mathbb{M}}$*-formula and* $(s,A) \in \mathbb{M}$. *Then there is some* $B \in [A]^\omega$ *such that* $(s,B) \Vdash \varphi$ *or* $(s,B) \Vdash \neg\varphi$.

Proof. In fact, we have already proven this lemma in effect. Actually, this lemma essentially follows from Lemma 2.33 of complete Ramsey Property of E-open sets.

To see that, fix $(s,A) \in M$. Recall that $[\![\varphi]\!]_{\mathbb{B}(\mathbb{P})}$ is regular open subset of $\mathbb{P}$. It immediately follows that $e([\![\varphi]\!]) \subseteq [\omega]^\omega$ is E-regular open as well. Applying Lemma 2.33, we can find $B \in [A]^\omega$ such that one of $[s,B]^\omega \subseteq e([\![\varphi]\!])$ or $[s,B]^\omega \cap e([\![\varphi]\!]) = \emptyset$ holds.

In the former case, since $e([\![\varphi]\!])$ is regular open, we have $[s,B]^\omega \subseteq \mathrm{int}_{\mathscr{E}}\, e([\![\varphi]\!])$, and, by definition, $[\![\varphi]\!]$ is dense below (s,B). Then, by Fact 1.52, we have $(s,B) \Vdash \varphi$.

In the latter case, $[\![\neg\varphi]\!]$ is dense below (s,B), and we have $(s,B) \Vdash \neg\varphi$. ☐

We need one more additional property:

Definition 2.8. Let N be some elementary submodel with $\mathbb{M} \in N$.

$m \in [\omega]^\omega$ is a *Mathias real* over N if $G_m := \{ (s,A) \in \mathbb{M} \mid s \subset_{\text{end}} m, m \setminus s \subseteq A \}$ meets every dense set in N.

For an $\mathbb{M}$-generic filter G over N, we call $m_G := \bigcup \{ s \mid (s,A) \in G \}$ as the *Mathias real* corresponding to G.

By construction, it is easy to see that $G = G_{m_G}$ and $m = m_{G_m}$. A property of being Mathias is inherited by its subset:

Lemma 2.44. *If* $m' \in [m]^\omega$ *and* m *is Mathias over* N, *then so is* m'.

Proof. Let $D \subseteq \mathbb{M}$ with $D \in N$. We show $G_{m'} \cap D \neq \emptyset$, that is, we find $(s, A) \in D$ with $m' \in [s, A]^\omega$. Using a term in Lemma 2.42, it is equivalent to $m' \in e(D)$. With this in mind, we let

$$D' := \{ (s, A) \in \mathbb{M} \mid \forall t \subseteq s\ [t, A]^\omega \subseteq e(D) \},$$

and show D' is a dense open set in $\mathbb{M}$. If we once have shown that, since m is Mathias property, we can pick $(s, X) \in D'$ with $s \subset_{\text{end}} m \subseteq s \cup X$. Letting $t := m' \cap s$, we have, by the definition of D', $m' \in [t, X]^\omega \subseteq e(D)$, which was what we wanted.

So it remains to show that D' is dense. Fix $(s, A) \in \mathbb{M}$ and enumerate $\mathcal{P}(s)$ as $\left\{ t_\ell \;\middle|\; \ell < 2^{|s|} \right\}$. By Prikry Lemma, we can take $A_{\ell+1} \subseteq A_\ell \subseteq A$, such that $[t_\ell, A_\ell]^\omega \subseteq e(D)$. This can be done because D is dense below (s, A). Let $B := A_{2^{|s|}-1}$. Then by our construction, $[t, B]^\omega \subseteq e(D)$ for all $t \subseteq s$ and hence we have $(s, B) \in D'$. This shows D' is dense in $\mathbb{P}$. ☐

We are now ready to prove the properness of $\mathbb{M}$:

Lemma 2.45. *$\mathbb{M}$ is a proper notion of forcing.*

Proof. Let $N \prec H(\theta)$ be a countable elementary submodel with $\mathbb{P} \in N$. We fix $(s, X) \in N \cap \mathbb{M}$ and find $(N, \mathbb{M})$-master condition $(t, Y) \leq (s, X)$. Using Mathias real condition, it is enough to find $(t, Y) \leq (s, X)$ with every $m \in [t, Y]^\omega$ is a Mathias real.

Since N is countable, we can list dense set of $\mathbb{M}$ in N as $\{ D_n \mid n < \omega \}$. One can take $(s_n, A_n) \in \mathbb{M}$ with $(s_{n+1}, A_{n+1}) \leq (s_n, A_n) \leq (s, X)$ and $(s_n, A_n) \in D_n$ for each n. Then, $m := \bigcup_n s_n$ is clearly a Mathias real over N. By Lemma 2.44, every $m' \in [s \cap m, m \setminus s]^\omega$ is Mathias, which was what we wanted. ☐

This completes the following main theorem of this section:

Theorem 2.46. *In Solovay's Model, every set of reals is completely Ramsey.*

Proof. Since the properness is preserved by dense embedding, ${I_{\mathrm{RN}}}^+$ is a proper notion of forcing by Lemmas 2.42 and 2.45. Clearly I_{RN} is a projective σ-ideal, hence every set of reals is I_{RN}-regular in Solovay's Model by Khomskii's Theorem 2.28. By Lemma 2.41, I_{RN}-regularity is equivalent to complete Ramsey Property, so the Theorem follows. ☐

3. Shelah's Result and Raisonnier Filters: Measurability to Inaccessible Cardinal

In this chapter, we will prove the Shelah's Theorem, which is converse of Solovay's Theorem 2.1:

Theorem 3.1 (Shelah [30]). *Assume* $\mathrm{ZF}+\mathrm{CC}$*. If every set of reals is Lebesgue measurable, then* ω_1^V *is inaccessible in* $L[z]$ *for every* $z \in {}^{\omega}\omega$*.*
Hence, $\mathrm{Con}(\mathrm{ZF}+\mathrm{DC}+\mathrm{LM})$ *implies* $\mathrm{Con}(\mathrm{ZFC}+\mathrm{IC})$*.*

Together with Solovay's Theorem 2.1, we have the following consistency strength result:

Theorem 3.2 (Solovay [32] and Shelah [30]). *The following theories are equiconsistent:*

- $\mathrm{ZFC}+$ *"There exists an inaccessible cardinal".*
- $\mathrm{ZF}+\mathrm{CC}+$ *"Every set of reals is Lebesgue measurable".*
- $\mathrm{ZF}+\mathrm{DC}+$ *"Every set of reals is Lebesgue measurable".*

On the other hand, Shelah [30] also shows that the statement "every set of reals has Baire Property" does not require any inaccessibles. We don't step into this direction; see Shelah [30] or Bartoszyński–Judah [2].

To prove Shelah's Theorem, all the arguments in this chapter will be done in ZF, and we indicate the place using some sort of Choice Principles like this. ⋆CC

3.1. Projective Measurability and Inaccessibility to the Reals

More precise statement of Shelah's Theorem is as follows:

Theorem 3.3 (Shelah [30], CC). *If every* $\boldsymbol{\Sigma}^1_3$*-set of reals is Lebesgue measurable, then* $\aleph_1^V$ *is an inaccessible cardinal in* $L[z]$ *for every* $z \in {}^{\omega}\omega$*.*

One might wonder why the above doesn't state just "$\aleph_1^V$ is inaccessible in L", which is already enough to establish the consistency strength. There are several reasons for this.

One reason is that the statement $\forall z\, L[z] \models$ "ω_1^V is inaccessible" is known to be itself very strong regularity property in the projective hierarchy. For example, this statement is equivalent to the statement "Every $\boldsymbol{\Pi}^1_1$-set of reals has perfect set property" and many known regularity properties in $\boldsymbol{\Sigma}^1_2$ follows from it; we refer readers to Khomskii [15] and Brendle–Löwe [4] for this topic.

Secondly, although the statements "ω_1^V is inaccessible in L" and "ω_1^V is inaccessible in $L[z]$ for every $z \in {}^{\omega}\omega$" have the same consistency strength (use Levy collapse in L to get the consistency of the latter from the former), they are not equivalent in ZFC (take $z \in {}^{\omega}\omega$ coding a surjection $f : \omega \to \omega_1^L$ in V and think in $L[z]$).

The final, and main reason is that we actually show the following statement to establish Theorem 3.3:

Theorem 3.4 (Shelah [30], CC**).** *Assume that every* $\mathbf{\Sigma}^1_2$*-set of reals is Lebesgue measurable and there exists a real* $z \in {}^\omega\omega$ *with* $\aleph_1^V = \aleph_1^{L[z]}$*. Then there is a* $\Sigma^1_3(x)$*-set of reals which is not Lebesgue measurable.*

To see how this statement relates to Theorem 3.3, we need the following definition and characterization:

Definition 3.1. ω_1 *is inaccessible to the reals* if $\aleph_1^{L[z]} < \aleph_1^V$ for every $z \in {}^\omega\omega$.

Lemma 3.5 (CC)**.** *The following are equivalent:*

(1) ω_1 *is inaccessible to the reals; i.e.* $\forall z \in {}^\omega\omega\ \aleph_1^{L[z]} < \aleph_1^V$.

(2) $\forall z \in {}^\omega\omega,\ L[z] \models$ *"*ω_1^V *: inaccessible".*

$\star$CC

Proof. The direction (2) $\implies$ (1) is obvious, so let's show the opposite direction. First, note that the regularity of ω_1^V follows from CC. Because the notion of being regular is Π_1-concept, ω_1^V is regular also in $L[z]$ for every $z \in {}^\omega\omega$ by Lemma 1.6.

So all we have to show is that ω_1^V remains strongly limit cardinal in $L[z]$. Because $L[z] \models$ GCH for every $z \in {}^\omega\omega$, by Lemma 1.31, it is enough to show that $L[z] \models \omega_1^V$: "limit cardinal" for every real z.

We argue by contradiction; suppose otherwise. Then there is some $x \in {}^\omega\omega$ and α with $L[x] \models \omega_1^V = \alpha^+$. Since α is countable in V, we can take some bijection $f : \omega \to \alpha$. We will define $z \in {}^\omega\omega$ coding information of both x and f as follows:

$$z(n) := \begin{cases} z(\frac{n}{2}) & (n : \text{even}) \\ 1 & (n = 3^k \cdot 5^\ell, f(k) < f(\ell)) \\ 0 & (\text{otherwise}). \end{cases}$$

We can define x and f in terms of z hence we have $x, f \in L[z]$. In particular, α is countable in $L[z]$ and we have $\alpha^{+L[z]} = \omega_1^{L[z]}$. We also have that $L[x] \subseteq L[z]$ hence $\alpha^{+L[x]} \leq \alpha^{+L[z]}$. But then we have:

$$\omega_1^V = \alpha^{+L[x]} \leq \alpha^{+L[z]} = \omega_1^{L[z]} < \omega_1^V.$$

Contradiction! □

By Lemma 3.5, we can get Theorem 3.3 by taking the contrapositive form of Theorem 3.4, and hence we state Theorem 3.3 for every $L[z]$ rather than just in L.

3.2. Lebesgue (Non-)Measurability of Filters

We will prove Shelah's Theorem 3.4 following the alternative proof due to Raisonnier [23]. In fact, our proof is roughly based on Todorcevic's variant of Raisonnier's proof which is described in Bekkali [3], but modified to explicitly use the notion of *rapid filters* a la Raisonnier.

Definition 3.2. A filter $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is *rapid* if $\mathcal{F}$ extends Fréchet filter and following holds:

$$\forall f : \omega \to \omega \text{ monotone } \exists a \in \mathcal{F}\ \forall n < \omega\ |f(n) \cap a| \leq n.$$

Intuitively, rapid filter uniformly bounds divergence speed of any increasing sequences. The following lemma makes this intuition precise:

Lemma 3.6. *A filter $\mathcal{F}$ on ω extending Fréchet filter is rapid if and only if there exists strictly increasing $g : \omega \to \omega$ such that for any monotone $f : \omega \to \omega$ there exists $a \in \mathcal{F}$ such that $|a \cap f(n)| \leq g(n)$.*

Proof. Since "only if" direction is clear, so assume there exists uniformly bounding strictly increasing function $g : \omega \to \omega$.

So let $f : \omega \to \omega$ be any monotone function. If we put $f^*(n) := f(g(n+1))$, then f^* is also monotonic. Then by assumption, there must be $a \in \mathcal{F}$ with $|f(g(n+1))| \leq g(n)$ for all n. If $k \in [g(n), g(n+1))$ then we have $|f(k) \cap a| \leq |f(g(n+1)) \cap a| \leq g(n) \leq n$. So a is a witness for "almost rapidness". Using the fact that $\mathcal{F}$ contains Fréchet filter, if we shrink a to $a^* := a \setminus f(g(0))$, a^* is still in $\mathcal{F}$. Then we have, for $k < g(n)$, $|a^* \cap f(k)| = 0 \leq k$ and for $k \geq g(0)$ we have $|a^* \cap f(k)| \leq |a \cap f(k)| \leq k$. This shows that $\mathcal{F}$ is rapid. ☐

Next we will consider the measurability of filters regarded as a set of reals. Note that identifying $a \subseteq \omega$ with its characteristic function $\chi_a : \omega \to 2$, we can view filter $\mathcal{F} \subseteq \mathcal{P}(\omega)$ as a set of reals.

Mokobodzki shows that no rapid filter is measurable, which plays an essential role in our proof of Shelah's Theorem:

Lemma 3.7 (Mokobodzki). *Rapid filters are not Lebesgue measurable.*

Proof of Lemma 3.7. We prove by contradiction. Assume that a rapid filter $\mathcal{F}$ is Lebesgue measurable.

Since $\mathcal{F}$ contains Fréchet filter, $\mathcal{F}$ must be null by Sierpinski's Lemma 1.24. So it's enough to show that $\mathcal{F}$ has positive outer measure to get contradiction. In particular, it suffices to show that $\mathcal{F}$ meets every closed positive sets.

Fix closed $B \subseteq {}^{\omega}2$ with $\mu(B) > 0$. For convenience, we denote $\mathring{S} := \bigcup \{ [s] \mid s \in S \}$ for $S \subseteq {}^{<\omega}2$.

We inductively take $T_i \subseteq {}^{<\omega}2$ and $n_i := \sup \{ \mathrm{lh}(s) \mid s \in T_i \}$ so that:

(a) $s \in T_i, t \in T_{i+1} \implies \mathrm{lh}(s) < \mathrm{lh}(t)$,

(b) $\mu(B \setminus \mathring{T}_{i+1}) < 2^{-(n_i+i+2)}$,

(c) $s \in T_i \implies \mu(B_{\lfloor s \rfloor}) > 1 - 2^{-(i+2)}$.

We put $T_{-1} := \emptyset$ for convenience. Having T_i and n_i defined, we will define T_{i+1}. Well, to get conditions (a) and (c) met, we define

$$S := \left\{ s \in {}^{<\omega}2 \;\middle|\; \mathrm{lh}(s) > n_i, \mu(B_{\lfloor s \rfloor}) > 1 - \frac{1}{2^{(i+2)}} \right\},$$

and take T_{i+1} as a subset of S. We enumerate S as $\{ t_k \mid k < \omega \}$; using the canonical well-ordering on ${}^{<\omega}2$, this can be done within ZF. We take the minimal $m > 0$ such that

$$\mu\left(\mathring{S} \setminus \bigcup_{k<m} [t_k] \right) < \frac{1}{2^{n_i+i+2}}.$$

So let $T_{i+1} := \{ t_k \mid k < m \}$. By Lebesgue Density Theorem 1.27, we have $\mu(A \setminus \mathring{S}) = 0$. Then,

$$\mu(A \setminus \mathring{T}_{i+1}) = \mu(A \setminus \mathring{S}) + \mu(\mathring{S} \setminus \mathring{T}_{i+1}) < \frac{1}{2^{n_i+i+2}},$$

and hence T_{i+1} satisfies condition (b).

Note that the above construction doesn't require any Choice Principles, since T_{i+1} and n_{i+1} are uniquely determined by the previous T_i and n_i.

Next we get $b \in \mathcal{F} \cap A$. Fix $a \in \mathcal{F}$ so that $|n_{i+1} \cap a| \leq i$ by rapidness of $\mathcal{F}$. We recursively take $s_i \in T_i$ and $x_i \in {}^\omega 2$ so that following conditions hold:

(i) $s_i \subsetneq s_{i+1}$,

(ii) $x_i \in [s_i] \cap B \cap \mathring{T}_i$,

(iii) $\forall n \in (a \cap n_i)\ s_i(n) = 1$.

We only need information about s_{i-1} to define s_i and x_i, so let $s_{-1} := \emptyset$ for convenience. Having s_i, x_i defined as desired, we define s_{n+1}, x_{n+1} as follows. To make condition (iii) hold, let $H^i := \{\, x \in {}^\omega 2 \mid \forall n \in [\mathrm{lh}(s_i), n_{i+1}]\ x(n) = 1 \,\}$ and we will choose x_i from it. Note that we have $\mu(H) \geq \frac{1}{2^{i+2}}$ because $|a \cap n_{i+1}| \leq i + 2$ by our choice of $a \in \mathcal{F}$.

Claim 1. $\mu(H^i \cap [s_i] \cap B) > 2^{-(n_i+i+2)}$

Proof. First note that, since the defining conditions of H^i and $[s_i]$ doesn't overlap, they are independent: $\mu(H^i \cap [s_i]) = \mu(H^i)\mu([s_i])$. Then we have:

$$\begin{aligned}
\mu(H^i \cap [s_i] \cap B) &= \mu(H^i \cap [s_i]) - \mu(H^i \cap [s_i] \setminus B) \\
&\geq \mu(H^i \cap [s_i]) - \mu([s_i] \setminus B) \\
&> \mu(H^i)\mu([s_i]) - 2^{-(i+2)}\mu([s_i]) \qquad \text{(by } s_i \in T_i \text{ and (c))} \\
&\geq \frac{1}{2^{n_i+i+2}},
\end{aligned}$$

which was what we wanted. ☐ (Claim 1)

In adittion, we have $H^i \cap [s_i] \cap B \cap \mathring{T}_{i+1} \neq \emptyset$. Otherwise, by Claim 1 and (b), we have

$$\frac{1}{2^{n_i+i+2}} < \mu(H^i \cap [s_i] \cap B) \leq \mu(B \setminus \mathring{T}_{i+1}) < \frac{1}{2^{n_i+i+2}},$$

which is a contradiction.

So we can pick $x_{i+1} \in B \cap [s_i] \cap H^i \cap \mathring{T}_{i+1}$ and we can choose $s_{i+1} \in T_{i+1}$ with $s_{i+1} \subsetneq x_{i+1}$ by definition.

So we defined s_i's and x_i's as desired. Let $b := \bigcup_{i<\omega} s_i$. Then by condition (iii), $a(n) = 1$ implies $b(n) = 1$ and hence $b \in \mathcal{F}$. We also have $b \in B$ because $x_i \to b$ in ${}^\omega 2$ by condition (ii) and B is closed. This shows that $b \in \mathcal{F} \cap B$ which was what we wanted. ☐

3.3. Raisonnier Filter and Its Rapidness

So, to prove Shelah's Theorem 3.4, it suffices to find some rapid filter which is $\boldsymbol{\Sigma}^1_3$ regarded as a set of reals.

Under the assumption of Shelah's Theorem 3.4, we will prove the following *Raisonnier filter* is our desired rapid filter:

Definition 3.3.

- For $f, g \in {}^\omega 2$ with $f \neq g$, we define $h(f, g) := \min\{\, n \mid f \restriction n \neq g \restriction n \,\}$.

- If $A \subseteq {}^{\omega}2$ then $H(A) := \{\, h(f,g) \mid f,g \in A, f \neq g \,\}$.
- *Raisonnier Filter* $\mathcal{F}(x)$ associated with x is defined as follows:

$$a \in \mathcal{F}(x) \iff \exists \langle F_i \subseteq {}^{\omega}2 \,|\, i < \omega\rangle \left[\bigcup_i H(F_i) \subseteq a \wedge {}^{\omega}2 \cap L[x] \subseteq \bigcup_i F_i\right].$$

  In other words, $\mathcal{F}(x)$ is a filter generated by coverings of ${}^{\omega}2 \cap L[x]$.

First, we show the Raisonnier filter $\mathcal{F}(x)$ is actually a filter:

Lemma 3.8 (CC)**.** *Assume $\omega_1^V = \omega_1^{L[x]}$. Then $\mathcal{F}(x)$ is filter on ω extending Fréchet filter.*

Proof. First, we will see $\mathcal{F}(x)$ is indeed a filter. By definition, $\mathcal{F}(x)$ is clearly upward closed.

So take $a,b \in \mathcal{F}(x)$ and witnessing coverings $\left\langle F_n^a, F_n^b \,\middle|\, n < \omega\right\rangle$. Take $\Gamma : \omega \times \omega \to \omega$ be pairing bijection on ω and let $F_{\Gamma(n,m)} := F_n^a \cap F_n^b$. Then it is clear that $L[x] \cap {}^{\omega}2 \subseteq \bigcup_k F_k$ and $\bigcup_k F_k \subseteq a \cup b$.

Next we show $\emptyset \notin \mathcal{F}(x)$, where we need the condition $\aleph_1^V = \aleph_1^{L[x]}$. Fix $a \in \mathcal{F}(x)$ arbitrary and let $\langle F_n \,|\, n < \omega\rangle$ be some covering of $L[x] \cap {}^{\omega}2$ witnessing $a \in \mathcal{F}(x)$. By Lemma 1.31 and our assumption $\aleph_1^V = \aleph_1^{L[x]}$, we have $|\bigcup_n F_n| \geq |{}^{\omega}2 \cap L[x]| = (2^{\omega})^{L[x]} = \aleph_1^{L[x]} = \aleph_1^V$. Hence, by the regularity of $\aleph_1$, ⋆CC
at least one F_n is uncountable; then we can pick $f,g \in F_n, f \neq g$ and hence $h(f,g) \in H(F_n) \subseteq a$.

It remains to see that $\mathcal{F}(x)$ extends Fréchet filter. Fixing $n < \omega$ arbitrary, we show $\omega \setminus n \in \mathcal{F}(x)$. But it is easy to see that $\langle [s] \,|\, s \in {}^{n}2\rangle$ witnesses $\omega \setminus n \in \mathcal{F}(x)$. □

We next compute the complexity of $\mathcal{F}(x)$:

Lemma 3.9. *The complexity of $\mathcal{F}(x)$ is $\Sigma_3^1(x)$ as a set of reals.*

Proof. First note that a covering $\langle F_n \,|\, n \in \omega\rangle$ witnessing $a \in \mathcal{F}(x)$ can be taken as closed subsets of ${}^{\omega}2$. More precisely, we see that $H(\mathrm{cl}(F_n)) = H(F_n)$ for each n. It is clear that $H(F_n) \subseteq H(\mathrm{cl}(F_n))$ so we show the converse inclusion. To that end, fix $f,g \in \mathrm{cl}(F_n), f \neq g$ and let $k := h(x,y) \in H(\mathrm{cl}(F_n))$. Since they belong to $\mathrm{cl}(F_n)$, there are $f',g' \in F_n$ with $f \restriction k = f' \restriction k, g \restriction k = g' \restriction k$ respectively. Then we clearly have $h(f,g) = h(f',g') \in H(F_n)$.

With the above in mind, we can define $\mathcal{F}(x)$ by the following formula using tree representation (see Lemma 1.13):

$$a \in \mathcal{F}(x) \iff \exists \langle T_n : n < \omega\rangle : \text{trees on } \omega \left[{}^{\omega}2 \cap L[z] \subseteq \bigcup_n [T_n] \wedge \bigcup_n H([T_n]) \subseteq a\right].$$

Note that, through canonical bijection ${}^{<\omega}2 \overset{\sim}{\to} \omega$ and ${}^{\omega}({}^{\omega}2) \overset{\sim}{\to} {}^{\omega}2$, we can view a sequence of trees on ω as a single real. Hence, by Fact 1.18, it suffices to show that formula inside brakets can be written in $\Pi_2^1(x)$-manner.

Well, we can write:

$$\begin{aligned} {}^{\omega}2 \cap L[z] \subseteq \bigcup_n [T_n] &\iff \forall x \in {}^{\omega}2\, [x \notin L[z] \vee \exists n < \omega\, \forall k < \omega\; x \restriction k \in T_n], \\ \bigcup_n H([T_n]) \subseteq a &\iff \forall x,y \in {}^{\omega}\omega \left[\begin{array}{r} x \neq y \wedge \forall m < \omega\, (x \restriction m, y \restriction m \in T_n) \\ \implies h(x,y) \in a \end{array}\right]. \end{aligned}$$

By Gödel's Lemma 1.32 on complexity, it is easy to see that both are indeed $\Pi_2^1(x)$-formula. □

To exploit our assumption of the $\mathbf{\Sigma}^1_2$-measurability, we need the following definition and lemma:

Definition 3.4. We define $A(x) \subseteq {}^{\omega}2 \times {}^{\omega}2$ as follows:

$$\begin{aligned}&(u,v) \in A(x)\\ &\quad \overset{\text{def}}{\iff} \exists \alpha < \omega_1\, \exists z \in L_\alpha[x] \begin{bmatrix} z \text{ codes null } G_\delta \text{ set } S_z \text{ with } u \in S_z \\ \forall y <_{L_\alpha[x]} z\, [S_y \text{ doesn't code null } G_\delta \text{ null set with } v \in S_y] \end{bmatrix}.\end{aligned}$$

Note that $A(x)$ is clearly a $\Sigma^1_2(x)$-subset of ${}^{\omega}2 \times {}^{\omega}2$.

Lemma 3.10 (CC)**.** *The following are equivalent:*

(1) *$A(x)$ is Lebesgue measurable.*

(2) *Union of all null Borel sets coded in $L[x]$ is null in V.*

Proof. We need the direction (1) $\implies$ (2) only, so we just prove this. See Bekkali [3] and Theorem 25.19 in Jech [11] for the converse implication.

So let $A(x)$ be measurable. Put $\alpha := \omega_1^{L[x]}$ and let $\langle z_\xi \mid \xi < \alpha \rangle$ be Borel codes for G_δ-sets in $L[x]$ well-ordered by canonical order $<_{L[x]}$ of $L[x]$ (i.e. we require $\xi < \eta \implies z_\xi <_{L[x]} z_\eta$) and $\langle G_\xi \mid \xi < \alpha \rangle$ be corresponding G_δ-set in V. We define the "uniquification" of G_ξ's and their union:

$$\tilde{G}_\xi := G_\xi \setminus \bigcup_{\eta < \xi} G_\eta, \qquad G := \bigcup_{\xi < \alpha} G_\xi = \bigcup_{\xi < \alpha} \tilde{G}_\xi.$$

By definition we have $A(x) \subseteq G \times G$ and

$$(u,v) \in A(x) \iff \exists \xi < \eta < \delta\, [u \in \tilde{G}_\xi \wedge v \in \tilde{G}_\eta].$$

By the definition of outer measure, every Lebesgue measurable sets can be approximiated from outer by G_δ-subset, so it suffices to show that G is null.

For each $u \in G$, we let $\xi(u) < \alpha$ be the unique $\xi < \alpha$ with $u \in \tilde{G}_\xi$. Note that we can write $A(x) = \{\, (u,v) \in G \times G \mid \xi(u) < \xi(v) \,\}$. Then we have, for each $v \in {}^{\omega}2$,

$$A^v(x) = \{\, u \in G \mid \xi(u) < \xi(v) \,\} = \bigcup_{\xi < \xi(v)} \tilde{G}_\xi.$$

$\star$CC By Lemma 1.33 on the height of $<_{L[x]}$, $A^v(x)$ is a countable union of null sets and hence itself null.
$\star$CC Then, by Fubini Theorem 1.20 for Measure, the set $\{\, u \in G \mid A_u(x) : \text{not null} \,\}$ is null.

We consider two cases: if $A_u(x)$ is non-null for all $u \in G$ then G must be null by discussion above. If there are $u \in G$ with $A_u(x)$ is null then:

$$G = A^u(x) \cup \tilde{G}_{\xi(u)} \cup A_u(x),$$

which is a finite union of null sets, and hence itself null. ☐

The last ingredient is combinatorial principle associating functions and small open sets:

Lemma 3.11. *There are a sequence $\langle N_f \mid f \in {}^\omega\omega \rangle$ of null G_δ-subsets of ${}^\omega 2$, and a sequence $\langle \varphi_U \subseteq {}^\omega 2 \mid U :$ open set with $\mu(U) < 1 \rangle$ of functions with $\varphi_U \in \prod_{n<\omega} [\omega]^{\leq 2^{n+1}}$, with the following property:*

$$\forall U : \text{open set } (\mu(U) < 1)\, \forall f : \omega \to \omega\, [N_f \subseteq U \implies \forall^\infty n < \omega\, f(n) \in \varphi_U(n)].$$

Namely, $N_f \subseteq U$ implies φ_U "almost captures" f.

Furthermore, if M is some transitive model and $f \in M$ then N_f is coded in M.

Proof. We define open sets $B_{n,m} := \left\{ x \in {}^\omega 2 \,\middle|\, \forall k \leq n\, x(2^n \cdot 3^m \cdot 5^k) = 0 \right\}$ for each $n, m < \omega$. Then $\langle B_{n,m} \mid n, m < \omega \rangle$ obviously satisfies the following properties:

(i) $\mu(B_{n,m}) = 1/2^{n+1}$,

(ii) $\langle B_{n,m} \mid n, m < \omega \rangle$ is independent; namely, for any finite $I \subsetneq \omega \times \omega$ we have:

$$\mu\left(\bigcap_{(n,m)\in I} B_{n,m} \right) = \prod_{(n,m)\in I} \mu(B_{n,m}).$$

Similarly $\langle {}^\omega 2 \setminus B_{n,m} \mid n, m < \omega \rangle$ is also an independet sequence.

The last property holds because defining condition of $B_{n,m}$'s doesn't overlap each other if $(n, m) \neq (n', m')$.

First, for each $f : \omega \to \omega$, we define N_f as follows:

$$B^f_m := \bigcup_{n \geq m} B_{n,f(n)}, \qquad N_f := \bigcap_{m<\omega} B^f_m$$

Note that N_f is null G_δ-set since $\mu(B^f_m) \leq \frac{1}{2^m}$.

Next, we define φ_U fixing open set $U \subseteq {}^\omega 2$ with $\mu(U) < 1$. Let $K := {}^\omega 2 \setminus U$. We can assume without loss of generaility that $[s] \cap K \implies \mu([s] \cap K) > 0$ for each $s \in {}^{<\omega}2$, because there are at most only countably many s with $\mu([s] \cap K) = 0$ and hence we can safely remove them from K. Note that after removing such $[s]$ from K, the following properties remains true:

(1) $K \cap U = \emptyset$,

(2) K is a closed subset of ${}^\omega 2$ and hence itself Polish space, and

(3) All basic open sets $[s] \cap K$ in K has positive measure.

Only these properties for K will be used in what follows.

We put

$$T_K := \{ s \in {}^{<\omega}2 \mid [s] \cap K \neq \emptyset \}, \qquad A_s(n) := \{ m \mid K \cap [s] \cap B_{n,m} = \emptyset \}$$
$$\varphi_U(n) := \bigcup_{\substack{s \in T_K \\ n(s) \leq n}} A_s(n).$$

Let's show this φ_U satisfies our requirement.

First we show that φ_U has desired upper bound. To that end, we fix canonical bijection $i : {}^{<\omega}2 \overset{\sim}{\to} \omega$.

Claim 1. $\exists \langle n(s) < \omega \mid s \in T_K \rangle \ \forall n \geq n(s) \ \left[\frac{|A_s(n)|}{2^{n+1}} \leq \frac{1}{2^{i(s)}+1} \right]$

First, note that $|A_s(n)| < \aleph_0$ for each s. To see that, first observe that if $m \in A_s(n)$ then $K \cap [s] \subseteq B^c_{n,m}$. Then, by conditions (i) and (ii), there are only finitely many $m \in A_s(n)$ because otherwise we have $\mu(K \cap [s]) = 0$ which cotradicts with (3).

Furthermore, series $\sum_{n<\omega} \frac{|A_s(n)|}{2^{n+1}}$ converges. In fact, by the independence of ${}^\omega 2 \setminus B_{n,m}$'s we have:

$$\begin{aligned} 0 < \mu([s] \cap K) &\leq \mu \left(\bigcap_{n<\omega} \bigcap_{m \in A_s(n)} ({}^\omega 2 \setminus B_{n,m}) \right) \\ &= \prod_{\substack{n<\omega \\ m \in A_s(n)}} \mu({}^\omega 2 \setminus B_{n,m}) = \prod_{n<\omega} \left(1 - \frac{1}{2^{n+1}}\right)^{|A_s(n)|} \\ &\leq e^{-\sum_{n<\omega} \frac{|A_s(n)|}{2^{n+1}}}, \end{aligned}$$

where the last inequality hold because $1 - x \leq e^{-x}$ for any $x \in \mathbb{R}$. This shows $\sum_{n<\omega} \frac{|A_s(n)|}{2^{n+1}} < \infty$ and hence $\frac{|A_s(n)|}{2^{n+1}}$ converges to 0. Hence, we can take the minimal $n(s)$ such that $n \geq n(s)$ implies $\frac{|A_s(n)|}{2^{n+1}} \leq \frac{1}{2^{i(s)+1}}$.

Above Claim gives our intended upper bound:

Claim 2. $|\varphi_U(n)| \leq 2^{n+1}$.

$$\frac{|\varphi_U(n)|}{2^{n+1}} \leq \sum_{\substack{s \in T_K \\ n(s) \leq n}} \frac{|A_s(n)|}{2^{n+1}} \leq \sum_{\substack{s \in T_K \\ n(s) \leq n}} \frac{1}{2^{i(s)+1}} \leq \sum_{k<\omega} \frac{1}{2^{k+1}} = 1.$$

Next, we will see that this φ_U almost captures f.

Claim 3. $N_f \subseteq U \implies \forall^\infty n < \omega \, f(n) \in \varphi_U(n)$

So let's assume $N_f \subseteq U$. Note that, since we have $U \cap K = \emptyset$, $K \cap N_f = K \cap \bigcap_m B^f_m = \emptyset$. Then, there must be some $s \in T_K$ and $m < \omega$ with $[s] \cap K \cap B^f_m = \emptyset$. To see that, suppose we have $[s] \cap (K \cap B^f_m) \neq \emptyset$ for each s, m. Then each $K \cap B^f_m$ meets every basic open sets in the closed subspace K and hence dense open subsets in K. We have then, by Baire's Category Theorem 1.26, $\bigcap_{m<\omega} B^f_m \cap K = N_f \cap K$ is also a dense subset of K. But this contradicts with $N_f \cap K = \emptyset$. Note that, since K is a closed subspace of ${}^\omega 2$, we don't need any choice principles here.

So take the minimal such s and m and let $\ell := \max \{ m, n(s) \}$. We prove that the value of f above ℓ are all captured by φ_U. Well, if $n \geq m, n(s)$ then $B_{n,f(n)} \subseteq B^f_n \subseteq B^f_m$. Hence, we have $K \cap [s] \cap B_{n,f(n)} = \emptyset$ and therefore $f(n) \in A_s(n) \subseteq \varphi_U(n)$ by definition of $A_s(n)$ and $n(s)$. □

With tools above, we can finally prove the rapiedness of Raisonnier Filter $\mathcal{F}(x)$:

Theorem 3.12 (CC)**.** *If $\omega_1^V = \omega_1^{L[x]}$ and $A(x)$ is measurable then $\mathcal{F}(x)$ is rapid.*

⋆CC *Proof.* Fix an increasing sequence $\langle n_i \mid i < \omega \rangle$. By Lemma 3.8 and 3.6, it suffices to find $a \in \mathcal{F}(x)$ with $|a \cap n_i| \leq 2^{i+2} - 2$.

Let $M := L[x]$. For each $f \in M \cap {}^{\omega}2$, we define $\bar{f} : \omega \to {}^{<\omega}2$ by $\bar{f}(i) := f \restriction n_i$. Clearly $\bar{f}$ is in
M. Confusing ${}^{<\omega}2$ with ω, there are null G_δ-set $N_{\bar{f}}$ in V coded in M by Lemma 3.11. Then, by
$\star$CC Lemma 3.10, $M := \bigcup_{f \in M \cap {}^{\omega}2} N_f$ is null. Hence, we can pick open set $U \supseteq M$ with $\mu(U) < 1$ and
$\star$CC again by Lemma 3.10, we have for each $f \in M \cap {}^{\omega}2$

$$(\star) \qquad \forall^{\infty} i < \omega \ \bar{f}(i) = f \restriction n_i \in \varphi_U(i).$$

With this in mind, we define $a \subseteq \omega$ as follows:

$$n \in a \overset{\text{def}}{\iff} [i = \min \{\, i \mid n \leq n_i \,\} \implies \exists s, t \in \varphi_U(i) \ h(s,t) = n].$$

Claim 1. $a \in \mathcal{F}(x)$

We will find $\langle F_s \subseteq {}^{\omega}2 \,|\, i < \omega, s \in {}^{n_i}2 \rangle$ such that $H(F_s) \subseteq a$ and F_s covers ${}^{\omega}2 \cap M$. We define F_s as follows:

$$F_s := \{\, f \in {}^{\omega}2 \cap M \mid s \subsetneq f, \forall i \geq \mathrm{lh}(s) \ [\bar{f}(i) = f \restriction n_i \in \varphi_U(i)] \,\}.$$

It immediately follows from $(\star)$ that F_s's covers $M \cap {}^{\omega}2$. To prove $H(F_s) \subseteq a$ for each s, fix $f, g \in F_s$ with $f \neq g$ and show $h(f,g) \in a$. Let $i := \min \{\, j < \omega \mid f \restriction n_j \neq g \restriction n_j \,\}$ and $t := f \restriction n_i, u := g \restriction n_i$. Since $s \subsetneq t, u$, we have $t, u \in \varphi_U(i)$. Then we have $h(f,g) = h(t,u) \leq n_i$ and hence $h(f,g) \in a$ as desired.

It remains to show that $|a \cap n_i| \leq 2^{i+2} - 2$. Well, by Lemma 3.11, we have $|a \cap (n_{i-1}, n_i]| \leq |\varphi_U(i)| \leq 2^{i+1}$ for each i. Hence,

$$|a \cap n_i| = \sum_{1 \leq j \leq i} |a \cap (n_{j-1}, n_j]| \leq \sum_{1 \leq j \leq i} 2^{j+1} = 2^{i+2} - 2.$$

This completes the proof of rapidness of $\mathcal{F}(x)$. ☐

Then Shelah's Theorems 3.4 and 3.3 immediately follows:

Proof of Theorem 3.4 of $\Sigma^1_3(x)$*-nonmeasurability.* Asuume every $\mathbf{\Sigma}^1_2$-set is measurable and $\aleph_1^V = \aleph_1^{L[x]}$ for some $x \in {}^{\omega}\omega$. Since every $\mathbf{\Sigma}^1_2$-set is measurable, so is $A(x)$. Then by Theorem 3.12, $\mathcal{F}(x)$ is a $\Sigma^1_3(x)$-rapid filter. Therefore, by Mokobodzki's Lemma 3.7, $\mathcal{F}(x)$ is a non-measurable $\Sigma^1_3(x)$-set. ☐

Proof of Theorem 3.3. Assuming $\mathbf{\Sigma}^1_3$-measurability we show that ω_1 is inaccessible to the reals.

We prove this by contradiction. So suppose ω_1 is not inaccessible to the reals. Then by Lemma 3.5, there must be some $z \in {}^{\omega}\omega$ with $\omega_1^V = \omega_1^{L[x]}$. Since every $\mathbf{\Sigma}^1_2$-set is measurable as well, by Theorem 3.4, Raisonnier filter $\mathcal{F}(x)$ is Lebegue non-measurable $\Sigma^1_3(x)$-set. This contradicts with our assumption of $\mathbf{\Sigma}^1_3$-measurability. ☐

A. Alternative Analysis: Some Applications of Theorems of Solovay and Shelah to Analysis

In this chapter, we will see how does Mathematics in Solovay's model differ from "ordinary" Mathematics and can be used for an alternative foundation for (mainly functional) analysis.

In what follows, we denote Solovay's model as $\mathcal{M} := (\mathrm{HOD}({}^{\omega}\mathbf{On}))^{V[G]}$. For more wide and comprehensive survey, we refer readers to Howard–Rubin [10, Section 1 of Part III] and it's support page [26], where Solovay's model is denoted as $\mathcal{M}5(\aleph)$ there.

By the argument above, ZF+DC+PReg holds true in Solovay's model. So the theorems derivable from this system holds also true in Solovay's Model. Interestingly, many of results can be derived just from ZF + DC + BP. It is worth mentioning that the consistency for this system does NOT require inaccessibles; see Shelah's original paper [30] or Bartoszyński–Judah [2].

Although, regularity properties proven in Chapter 2.3 is for Cantor space ${}^{\omega}2$ or Baire space ${}^{\omega}\omega$. In general, they cannot necessarily be extended to other general Polish spaces, but Baire Property and measurability can easily be generalized to such cases. So, we can use ZF + DC + BP and ZF + DC + LM not only for Baire and Cantor spaces, but for general Polish spaces, for example separable Banach spaces.

A.1. Automatic Continuity of Linear Functionals

One powerful example of theorems derivable from ZF + DC + BP is the following theorem due to Wright [35], which ensures some strong form of automatic continuity:

Theorem A.1 (Wright [35]). *Let V be a Banach space and W a second countable vector space (in particular, separable metrizable vector space). If $T : V \to W$ is linear then T is continuous.*

To that end, we first have to prove the following lemma, which states every map from Polish space is "continuous almost everywhere":

Lemma A.2 (Wright [35]). *Let X be a separable complete metric space and Y a second countable space. For any map $f : X \to Y$, there exists some meager subset N of X such that $f \restriction (X \setminus N)$ is continuous in the relative topology of $X \setminus N$.*

Proof. Let $\{ U_n \mid n < \omega \}$ be a countable open base of Y. It suffices to find some meager set N of X such that $(f \restriction X \setminus N)^{-1}[U_n]$ is open in $X \setminus N$ for each $n < \omega$.

Well, let $A_n := f^{-1}[U_n]$. By BP and DC (especially the Countable Choice), there exists a family $\langle G_n \,|\, n < \omega \rangle$ of open sets of X such that every $N_n := G_n \vartriangle A_n$ is meager in X. In particular, $A_n \setminus N_n = G_n \setminus N_n$ and hence if we put $N := \bigcup_n N_n$ then $A_n \setminus N = G_n \setminus N$ for each n. So, every $A_n \setminus N$ is open in the relative topology of $X \setminus N$ for each n. But $A_n \setminus N = (f \restriction X \setminus N)^{-1}[U_n]$, this shows that every preimage of basic open sets in Y become open in $X \setminus N$. ☐

Then it follows that every sublinear functional from Banach space is continuous:

Lemma A.3 (Wright [36]). *If V is Banach space and $p : V \to \mathbb{R}$ is a sublinear functional, then p is continuous.*

Proof. First, we show that p is continuous at origin.

Fix arbitrary sequence $\boldsymbol{x}_n \to \boldsymbol{0}$ in V. We exploit Lemma A.2 to show $p(\boldsymbol{x}_n) \to 0$ avoiding "discontinuous" part of p. Note that V is not assumed to be separable so we cannot directly apply Lemma A.2. Hence, we instead consider the closed subspace $W := \operatorname{span}_n \boldsymbol{x}_n$, which is separable because the collection of $\mathbb{Q}$-linear combinations of $\boldsymbol{x}_n$'s is a countable dense subset for W.

So, by Lemma A.2, p is continuous outside of some meager set $N \subseteq W$. We put

$$M := \bigcup_{k<\omega} \bigcup_{n<\omega} 2^k(N - \boldsymbol{x}_n) \cup \bigcup_{k<\omega} 2^k N.$$

M is a countable union of meager subsets, hence M is meager. By Baire's Category Theorem, we can pick $\boldsymbol{z} \notin M$ so that $\boldsymbol{x}_n + \frac{\boldsymbol{z}}{2^k}, \frac{\boldsymbol{z}}{2^k} \notin N$.

Then by the continuity of p on $W \setminus N$, we have for each k,

$$p\left(\boldsymbol{x}_n + \frac{1}{2^k}\boldsymbol{z}\right) \xrightarrow{n\to\infty} p\left(\frac{1}{2^k}\boldsymbol{z}\right).$$

Then, by subadditivity we have:

$$\begin{tikzcd}
p\left(\boldsymbol{x}_n + \frac{1}{2^k}\boldsymbol{z}\right) - \frac{1}{2^k}p(\boldsymbol{z}) \arrow[d, "n\to\infty"'] & \leq & p(\boldsymbol{x}_n) & \leq & p\left(\boldsymbol{x}_n + \frac{1}{2^k}\boldsymbol{z}\right) + \frac{1}{2^k}p(-\boldsymbol{z}) \arrow[dd, "n\to\infty"] & \\
0 = \frac{1}{2^k}p(\boldsymbol{z}) - \frac{1}{2^k}p(\boldsymbol{z}) & \leq & \liminf\limits_{n\to\infty} p(\boldsymbol{x}_n) & & & \\
 & \leq & \limsup\limits_{n\to\infty} p(\boldsymbol{x}_n) & \leq & \frac{1}{2^k}p(\boldsymbol{z}) + \frac{1}{2^k}p(-\boldsymbol{z}) \arrow[r, "k\to\infty"] & 0.
\end{tikzcd}$$

Hence $p(\boldsymbol{x}_n) \to 0$.

The argument above shows the continuity of p at origin on entire space V. Finally, let $\boldsymbol{x}_n \to \boldsymbol{x}$ in V. Note that since $\boldsymbol{x}_n - \boldsymbol{x} \to \boldsymbol{0}$ it follows that $p(\boldsymbol{x}_n - \boldsymbol{x}) \to 0$. Then we have:

$$\begin{tikzcd}
p(\boldsymbol{x}) - p(\boldsymbol{x} - \boldsymbol{x}_n) \arrow[d, "n \to \infty"'] & \leq & p(\boldsymbol{x}_n) & \leq & p(\boldsymbol{x}_n - \boldsymbol{x}) + p(\boldsymbol{x}) \arrow[d, "n \to \infty"] \\
p(\boldsymbol{x}) & & & & p(\boldsymbol{x})
\end{tikzcd}$$

Hence $p(\boldsymbol{x}_n)$ converges to $p(\boldsymbol{x})$. ☐

Corollary A.4. *Every linear functional $f : V \to \mathbb{R}$ on Banach space V is continuous under* $\mathrm{ZF} + \mathrm{DC} + \mathrm{BP}$.

The proof above depends on the order structure of $\mathbb{R}$. The similar but slight simpler argument shows the following more general Lemma which is essentially equivalent to Theorem A.1:

Lemma A.5. *Let X be a separable complete metric topological group and Y a second countable topological group. If $h : X \to Y$ is a group homomorphism, then h is continuous.*

Sketch of Proof. Let $\boldsymbol{x}_n \to \boldsymbol{x}$ and take $\boldsymbol{z}$ so that $h(\boldsymbol{x}_n\boldsymbol{z}) \to h(\boldsymbol{x}\boldsymbol{z})$ as in the above proof of Lemma A.3. Then we have $h(\boldsymbol{x}_n) = h(\boldsymbol{x}_n\boldsymbol{z})h(\boldsymbol{z}^{-1}) \to h(\boldsymbol{x}\boldsymbol{z})h(\boldsymbol{z}^{-1}) = h(\boldsymbol{x})$ because h is a homomorphism.☐

Proof of Theorem A.1. The proof is almost the same as Lemma A.3, but we Lemma A.5 to get the continuous part around $\nvdash$. ☐

So, in the Solovay Model, we don't need the classical Closed Graph Theorem in some cases.

A.2. Hahn–Banach Theorems in Solovay's Model

On the other hand, there is the case that we cannot assume our standard results always hold in Solovay's Model. One such example is classical *Hahn–Banach Theorem*.

More precisely, Hahn–Banach Theorem refutes the axiom BP and LM:

Theorem A.6 (Solovay). *Under* ZF+DC*, "Hahn–Banach Theorem for separable normed space",* **SNHB** *for short, implies the existence of a set of reals without Baire property and measurability.*

Theorem A.6 above follows from following two facts:

Lemma A.7 (Solovay). *Under* ZF + DC + **SNHB***, there exists non-principal finitely-additive probabilistic measure m on ω.*

Proof. We define the norm $\|-\|$ on the space $\ell^\infty(\mathbb{R})$ of bounded sequences as follows:

$$\|\boldsymbol{x}\| = \sum_{n<\omega} \frac{|x_n|}{2^{2n}}, \quad \text{where } \boldsymbol{x} = \langle x_n \mid n < \omega \rangle \in \ell^\infty(\mathbb{R}).$$

Then the space $(\ell^\infty, \|-\|)$ is clearly separable, since finite sequences form a countable dense subset of ℓ^∞. We also define a sublinear functional $p : \ell^\infty(\mathbb{R}) \to \mathbb{R}$ by $p(\boldsymbol{x}) := \limsup_{n\to\infty} x_n$.

Then, applying **SNHB** to $0 \subseteq \ell^\infty(\mathbb{R})$ and $p(-)$, we can get linear functional $f : \ell^\infty(\mathbb{R}) \to \mathbb{R}$ with $f(\boldsymbol{x}) \leq \limsup_{n\to\infty} x_n$.

If we identify $s \subseteq \omega$ with its characteristic function $\chi_s : \omega \to 2$, then χ_s is trivially in $\ell^\infty(\mathbb{R})$. So we define $m(s) := f(\chi_s)$ and we see that this m is as desired.

Well, $m(\omega) = f(\mathbf{1}) \leq p(\mathbf{1}) = 1$ and $m(\omega) = f(\mathbf{1}) = -f(\mathbf{-1}) \geq -p(\mathbf{-1}) = 1$ so we have $m(\omega) = 1$. For additivity, if $x \cap y = \emptyset$ then

$$m(x \cup y) = f(\chi_x + \chi_y) = f(\chi_x) + f(\chi_y).$$

It remains to show that f is non-principal. But, if $x \subseteq \omega$ is finite, then clearly $\limsup \chi_x = 0$ holds and hence we have

$$0 = -p(-\chi_x) \leq -f(-\chi_x) = m(x) = f(\chi_x) \leq p(\chi_x) = 0.$$

☐

Lemma A.8 (Morillon). *Under* ZF+DC*, If there is a non-principal finitely additive probabilistic measure m on ω, then there exist sets of reals without Baire Property or measurability.*

Proof. We will identify $\mathcal{P}(\omega)$ and ${}^{\omega}2$ through characteristic functions. Let

$$A^{+} := \left\{ x \in {}^{\omega}2 \,\middle|\, m(x) > \frac{1}{2} \right\}, \quad A^{-} := \left\{ x \in {}^{\omega}2 \,\middle|\, m(x) \leq \frac{1}{2} \right\}.$$

We show that neither A^{+} or A^{-} have Baire Property. Since we have ${}^{\omega}2 = A^{+} \sqcup A^{-}$, A^{+} has Baire Property if and only if so does A^{-}. So let's assume that both A^{+} and A^{-} have Baire Property and get contradiction.

Since m is finitely additive, value of m cannot be changed by finite changes. That is, if $x \backslash n = y \backslash n$ for some n, then $m(x) = m(y)$. In particular, A^{+} and A^{-} are tail sets in ${}^{\omega}2$. Then, by Zero-One Law for Category (Fact 1.23), A^{+} and A^{-} are respectively meager or comeager. Note that since $A^{+} = {}^{\omega}2 \setminus A^{-}$, they cannot be both (co)meager at one time.

Consider the homeomorphism $h : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ defined by $h(x) = \omega \setminus x$. Since the notion of being (co)meager is topological, h preserves (co)meagerness. But, since $A^{+} = h(A^{-})$, we have A^{+} is meager if and only if A^{-} is meager. This is a contradiction!

Using Zero-One Law for Measure (Fact 1.22) and the fact that measure on basic open sets of ${}^{\omega}2$ only depends only on the length of its defining finite sequence, exactly the same argument shows that neither A^{+} or A^{-} is measurable. ☐

Corollary A.9. *In Solovay's model, Hahn–Banach Theorem (even restricted to separable normed spaces) is false.*

On the other hand, since we can use DC in $\mathcal{M}$, the following restricted form of Hahn–Banach Theorem derivable from ZF + DC can be used in $\mathcal{M}$:

Theorem A.10 (Separable Continuous Hahn–Banach Theorem, SCHB). *Let V be separable normed space and $p : V \to \mathbb{R}$ be sublinear (i.e. $p(\lambda \boldsymbol{x}) = \lambda p(\boldsymbol{x})$ and $p(\boldsymbol{x} + \boldsymbol{y}) \leq p(\boldsymbol{x}) + p(\boldsymbol{y})$ for $\boldsymbol{x}, \boldsymbol{y} \in V$ and $\lambda \in \mathbb{R}_{+}$) functional* continuous at origin. *Let $W \subseteq V$ be subspace of V and $f : W \to \mathbb{R}$ be linear function with $f(\boldsymbol{x}) \leq p(\boldsymbol{x})$ for any $\boldsymbol{x} \in W$.*

Then there exists linear function $\bar{f} : V \to \mathbb{R}$ extending f such that $f(\boldsymbol{x}) \leq p(\boldsymbol{x})$ for any $\boldsymbol{x} \in V$.

This Theorem can be proven by combining two lemmas:

Lemma A.11. *Let V be separable normed space and $p : V \to \mathbb{R}$ be sublinear functional* continuous at origin. *If $W \subseteq V$ is dense subspace of W and $f : W \to \mathbb{R}$ is linear functional with $f(\boldsymbol{x}) \leq p(\boldsymbol{x})$ for all $\boldsymbol{x} \in W$, then there exists linear functional $\bar{f} : V \to \mathbb{R}$ extending f such that $f(\boldsymbol{x}) \leq p(\boldsymbol{x})$ for any $\boldsymbol{x} \in V$.*

Proof. Since $W \subseteq V$ is dense in V, every $\boldsymbol{x} \in V$ can be written as $\boldsymbol{x}_n \to \boldsymbol{x} \quad (n \to \infty)$ for some $\langle \boldsymbol{x}_n \in W \mid n < \omega \rangle$. So, for such $\boldsymbol{x}_n \to \boldsymbol{x}$, define $\bar{f} : V \to \mathbb{R}$ by $\bar{f}(\boldsymbol{x}) = \lim_{n \to \infty} f(\boldsymbol{x})$. The continuity of p at origin ensures that $\langle f(\boldsymbol{x}_n) \mid n < \omega \rangle$ is Cauchy sequence:

$$f(\boldsymbol{x}_n - \boldsymbol{x}_m) \leq p(\boldsymbol{x}_n - \boldsymbol{x}_m) \to 0 \quad (\text{as } n, m \to \infty).$$

Hence it converges to the real number $\lim_{n \to \infty} f(\boldsymbol{x}_n)$. To check the well-definedness, for $\boldsymbol{x}_n \to \boldsymbol{x}, \boldsymbol{y}_n \to \boldsymbol{x}$:

$$f(\boldsymbol{x}_n) - f(\boldsymbol{y}_n) = f(\boldsymbol{x}_n - \boldsymbol{x}) + f(\boldsymbol{x} - \boldsymbol{y}_n) \leq p(\boldsymbol{x}_n - \boldsymbol{x}) + p(\boldsymbol{x} - \boldsymbol{y}_n) \to 0 \quad (\text{as } n \to \infty).$$

Then the linearity and domination condition easily follow. ☐

Lemma A.12. *Let V be separable normed space, $p : V \to \mathbb{R}$ be sublinear functional, $W \subseteq V$ a proper subspace of W, and $f : W \to \mathbb{R}$ is linear functional with $f(\boldsymbol{x}) \leq p(\boldsymbol{x})$ for all $\boldsymbol{x} \in W$. If $\boldsymbol{w} \in W \setminus V$, then there exists linear functional $\bar{f} : W + \mathbb{R}\boldsymbol{w} \to \mathbb{R}$ extending f such that $f(\boldsymbol{x}) \leq p(\boldsymbol{x})$ for any $\boldsymbol{x} \in W + \mathbb{R}\boldsymbol{w}$.*

Proof. Note that every $\boldsymbol{x} \in W + \mathbb{R}\boldsymbol{w}$ can be expressed as $\boldsymbol{x} = \boldsymbol{v} + \alpha\boldsymbol{w}$ for the unique $\boldsymbol{v} \in V$ and $\alpha \in \mathbb{R}$. It is also easy to check that $f_c(\boldsymbol{v} + \alpha\boldsymbol{w}) := f(\boldsymbol{v}) + \alpha c$ is a linear functional extending f for any choice of $c \in \mathbb{R}$.

So it remains to show that we can pick $c \in \mathbb{R}$ so that $\bar{f} = f_c$ satisfies the boundedness condition.

Well, we have, for any $\boldsymbol{y}, \boldsymbol{y}' \in W$,

$$f(\boldsymbol{y}) + f(\boldsymbol{y}') = f(\boldsymbol{y} + \boldsymbol{y}') \leq p(\boldsymbol{y} + \boldsymbol{y}') = p(\boldsymbol{y} - \boldsymbol{z} + \boldsymbol{y}' + \boldsymbol{z}) \leq p(\boldsymbol{y} - \boldsymbol{z}) + p(\boldsymbol{y}' + \boldsymbol{z})$$
$$\therefore f(\boldsymbol{y}) - p(\boldsymbol{y} - \boldsymbol{z}) \leq p(\boldsymbol{y}' + \boldsymbol{z}) - f(\boldsymbol{y}')$$

Hence, if we let $\beta_1 := \sup_{\boldsymbol{y} \in W} [f(\boldsymbol{y}) - p(\boldsymbol{y} - \boldsymbol{z})]$ and $\beta_2 := \inf_{\boldsymbol{y}' \in W} [p(\boldsymbol{y}' + \boldsymbol{z}) - f(\boldsymbol{y}')]$, we can pick $c \in [\beta_1, \beta_2]$. Then it is each to check that this c satisfies our requirement. ☐

Proof of Theorem A.10. First, apply Lemma A.12 repeatedly to extend the domain of f to the countable dense subset D of V. Then use Lemma A.11 to extend it to entire space V. ☐

Together with Theorem A.10 and Corollary A.9, it follows that the continuity condition on dominating sublinear functional in Separable Continuous Hahn–Banach Theorem A.10 cannot be dropped. So, the statement of **SCHB** is one of optimal variants of Hahn–Banach Theorems hold in Solovay's Model.

On the other hand, since $\ell^\infty(\mathbb{R})$ is not a Banach space (consider finite sequences converging to increasing sequence), one can consider the following variant of Hahn–Banach Theorem as another optimal variant of Hahn–Banach Theorem:

Theorem A.13 (Hahn–Banach Theorem for Separable Banach Spaces, SBHB). *Let V be separable* Banach *space and $p : V \to \mathbb{R}$ be sublinear functional.*

Let $W \subseteq V$ be subspace of V and $f : W \to \mathbb{R}$ be linear function with $f(\boldsymbol{x}) \leq p(\boldsymbol{x})$ for any $\boldsymbol{x} \in W$.

Then there exists linear function $\bar{f} : V \to \mathbb{R}$ extending f such that $f(\boldsymbol{x}) \leq p(\boldsymbol{x})$ for any $\boldsymbol{x} \in V$.

Although Solovay [32] states that the above "Hahn–Banach theorem for separable Banach spaces follows readily from DC", we cannot make out how to prove this in ZF + DC, because we cannot assume that such spaces have countable bases.

By the way, we've already shown, under the ZF + DC + BP, the automatic continuity of sublinear functional on Banach spaces in Lemma A.3. Hence, it follows that Theorem A.13 holds true in Solovay's Model, and it's at least consistent with ZF + DC!

Proof of Theorem A.13 under ZF + DC + BP. Since every sublinear functional is continuous at origin by Lemma A.3, it immediately follows from Theorem A.10. ☐

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