A Review

Hirokazu Nishimura
Institute of Mathematics, University of Tsukuba

November 9, 2015

Abstract

Quantum Field Theory (usually called QFT by its acronym) is a splendid hybrid of three major themes of modern physics, namely, the quantum theory, the field concept and special relativity. Although many mathematicians are quite afraid of it, the central concepts and technicalities of QFT are easy to grasp. In particular, many daily tools of working quantum field theorists such as Feynman diagrams and renormalization group flows are even picturesque. We are very happy to live in a universe where low energy physics is decoupled from high energy physics, the former being described surely by quantum field theories and the latter being depicted possibly by string theories or the like. Contemporary mathematicians are no longer allowed to boast of their ignorance of QFT as bliss. Recall, by way of example, how Witten's supersymmetric QFT of $U(1)$ gauge symmetry to be called the Seiberg-Witten theory replaced Donaldson's topological QFT of $SU(2)$ gauge symmetry, which could be called a revolution. This book is by no means a textbook on QFT. Technical results are usually presented without proofs, for which the readers are referred to appropriate references. Even technical terms are often introduced without palpable definitions. It is very difficult to imagine a novice in category theory, when reading the book, to help being swallowed by a flood of the jargon of higher category theory. The book should be regarded as a rough design of the author's grandiose approach to QFT. If you want a good textbook on QFT for contemporary mathematicians, Zeidler's ongoing all-inclusive six-volume-to-be textbook [Quantum Field Theory, Springer] is most recommendable. To our great fortunes, its first three volumes [Zbl 1124.81002, Zbl 1155.81005, Zbl 1228.81005] are already available. Its fourth volume is supposed to deal with quantum mathematics in general, while its sixth volume is intended for quantum gravity and string theory. Every mathematician and every physicist are waiting for the remaining three volumes with a lot of enthusiasm. If the author wants to write a textbook on QFT on his lines, it should be at least as voluminous as Zeidler's. The main hulk of the book is divided into three parts, namely, Part 1 (Mathematical Preliminaries) ranging from Chapter 2 through Chapter 13 and taking about 280 pages, Part 2 (Classical Trajectories and Fields) ranging from Chapter 14 through Chapter 16

and taking about 40 pages, and Part 3 (Quantum Trajectories and Fields) ranging from Chapter 17 through Chapter 24 and taking about 100 pages. The first chapter is an introduction taking about 30 pages. The remarkably unique contents covered in Part 1, as mathematical tools found in books oriented towards QFT, are Chapter 2 (A Categorical Toolbox), Chapter 3 (Parametrized and Functional Differential Geometry), Chapter 9 (Homotopical Algebra), Chapter 10 (A Glimpse at Homotopical Geometry) and Chapter 13 (Gauge Theories and Their Homotopical Poisson Reduction).

Chapter 2 consists of five sections, finding its sequel in Chapter 9. The first section §2.1 (Higher Categories, Doctrines and Theories) introduces higher categories, doctrines as $(n+1)$-categories, theories and models as objects of the same doctrine. Adjunctions, (left and right) Kan extensions, limits and colimits (as special cases of right and left Kan extensions) are dealt with from a standpoint of higher categories. The Yoneda dual n-categories $\mathcal{C}^\vee$ and $\mathcal{C}^\wedge$ of an n-category $\mathcal{C}$ are defined, and the higher Yoneda lemma is stated as a hypothesis. Even Ehresmann's sketches, as a natural generalization of algebraic theories in the sense of Lawvere [Zbl 1062.18004], are touched from a standpoint of higher categories. This book chooses the presentation upon the setting of doctrines and higher categorical logic. To the advantage of having a direct homotopical generalization, §2.2 (Monoidal Categories) gives monoidal categories in the doctrinal guise, while symmetric monoidal categories are introduced in §2.3 (Symmetric Monoidal Categories) in an equivalent way that the traditional one is introduced by generalizing Segal's theory of Γ-spaces. §2.4 (Grothendieck Topologies) is a succinct review. The final section §2.5 (Categorical Infinitesimal Calculus) gives Quillen's tangent category [Zbl 0234.18010] for a genuinely categorical treatment of differential calculus, to the advantage of allowing one to deal simultaneously with all geometric structures in the book and of generalizing directly to the higher categorical setting of doctrines and theories. Grothendieck connections introduced in the section will play a unifying role in Chapter 7 (Connections and Curvature).

Chapter 3, consisting of three sections, gives appropriate tools for differential geomety on spaces of fields, which are directly susceptible of homotopical generalizations in Chapter 10. Roughly speaking, there are two approaches, namely, parametrized and functional ones. The parametrized approach is adept at treating differential forms, while the functional one is so at dealing with vector fields. The parametrized approach can be seen in [Zbl 0118.36206] and [Zbl 0237.00012] in the arena of algebraic geometry and in [Zbl 1269.53003] in the terrain of differential geometry. Synthetic differential geometry (cf. [Zbl 1091.51002]) is an adequate mixture of both approaches. After §3.1 (Parametrized Geometry) and §3.2 (Functional Geometry), §3.3 is devoted to showing how categorical infinitesimal calculus depicted in §2.5 is to be applied to differential geometry.

Most problems one encounters in physics and mathematics take obstructions with them, and Chapter 9, consisting of 11 sections, gives main tools for general obstruction theory. All the obstructions considered in the book can be defined as some kind of higher Kan extensions of models of theories within a given doctrine. Nobody would expect a full treatment of homotopical algebra in only 40 pages or so, and one is referred to [Zbl 1017.55001], [Zbl 1072.18012], [Zbl 06376898] [Zbl 0949.55001] and [Zbl 0909.55001] for a full treatment of axiomatic homotopy theory and its intimately related topics. The principal objective in homotopical algebra is localizations of categories. §9.6 (∞-Categories) gives an elementary presentation of ∞-categories, while §9.10 (Higher Categories) gives their advanced presentation based upon Rezk [Zbl 1203.18015, Zbl 1203.18016]. §9.11 (Theories up to Homotopy and the Doctrine Machine) is concerned with homotopical doctrines and homotopical theories based upon the homotopical higher categoriies discussed in §9.10.

Homotopical geometry, a scrupulous device intended for studying obstruction-theoretic problems in geometry from a genuinely geometric standpoint, started with [Zbl 0091.03701], [Zbl 0224.13014], [Zbl 0238.13017] [Zbl 0234.18010] and others in the latter half of the previous century. Roughly speaking, it is obtained from the parametrized and functional geometry disucussed in Chapter 3 by simply replacing the category **Sets** of sets and maps by the category **sSets** of simplicial sets and simplicial maps endowed with the standard model category structure or the homotopical ∞-category $^{\infty}\mathbf{GRPD}$ of ∞-groupoids in the author's terminology (Theorem 9.5.1 and Definition 9.5.5). We have to produce a kind of differential calculus up to homotopy, carrying a well-behaved notion of higher stack in formalization of quotients and moduli spaces in covariant gauge theory. In Yang-Mills theory, we have to deal with a variable principal G-bundle P over a given spacetime M, which can be reformulated as a map $P : M \to BG$ with BG being the smooth classfying space for principal G-bundles. It is the derived geometry that provides the proper setting for differential calculus on spaces like BG. It is interesting to note that, as is often the case, physicists have encountered similar mathematical structures, independently of mathematicians, in endeavors for the BRST-BV formalism. Quantization can be regarded as a kind of deformation, and the modern theory of deformation has a great deal to do with homotopical geometry.

Homotopical algebraic geometry is now a well-established branch of mathematics, for which the reader is referred to [Zbl 1175.18001], [Zbl 1120.14012] and [Zbl 114514003]. Homotopical differential geometry is by no means settled, or rather, is left in a mess. For the first attempts at homotopical differential geometry the reader is referred to [Zbl 05717812] and [MR3221297]. This stark contrast between homotopical algebraic geometry and homotopical differential geometry comes simply from the fact that algebraic geometry as such is completely axiomatized or conceptually purifed due to Grothendieck's revolution in the arena in the middle of the previous century, while differential geometry is not yet. For the burgeoning axiomatic approach to differential geometry inspired by synthetic differential geometry, the reader is referred to [Zbl 1285.51009, Zbl 1285.51010, Zbl 1281.51013]. In Chapter 10, the author gives a systematic construction of homotopical spaces based upon the doctoral approach to categorical logic depicted in Chapter 2. Homotopical or derived geometry gives a natural setting for non-abelian cohomology. Following §10.3 (Non-abelian Cohomology) and §10.4 (Differential Cohomology), §10.5 (Geometric Stacks) gives a short account of geometric higher stacks after [arXiv:math.AG/9807409]. §10.6 (Homotopical Infinitesimal Calculus) explains how to adapt the methods in §2.5 to the ∞-categorical setting. §10.7 (Derived Symplectic Structures) sketches the notion of a closed form and a derived symplectic form on a derived stack. A striking difference between the classical and the derived settings is that a differential form's being closed is not an intrinsic property but an exotic structure in the latter setting. The final section §10.8 (Deformation Theory and Formal Geometry) is devoted to the derived deformation theory program due to Deligne, Drinfeld and Kontsevich.

Chapter 13 is an expansion of the author's [Zbl 06011053], defining gauge theories and investigating their classical aspects. The problems considered in the chapter are called local variational problems, and their equations of motion expressed by Euler-Lagrange equations are studied in use of non-linear algebraic analysis depicted in Chapter 12 (Algebraic Analysis of Non-Linear Partial Differential Equations).

Chapter 4 (Functional Analysis) is mainly concerned with the existence of solutions of linear partial differential equations. The presentation is well adapted to immediate generalizations to the super or graded setting, accommodating homotopical geometric methods discussed in Chapter 10 readily. The book [Zbl 0889.58001] in use of convenient locally convex topological vector spaces is conceptually in the proximity of the author's treatment. This chapter is succeeded by the coordinate-free treatment in Chapter 11 (Algebraic Analysis of Linear Partial Differential Equations) and Chapter 12 (Algebraic Analysis of Non-linear Partial Differential Equations), the former of which is based on [Zbl 1058.58011, Zbl 1213.18008, arXiv:1206.1435, arXiv:math/0702459], while the latter of which is based on.[Zbl 1222.83160].

Chapter 8 (Lagrangian and Hamiltonian Systems), consisting of 8 sections, deals with Lagrangian methods in preference to Hamiltonian methods, because the covariance of the former is automatic in relativistic field theories and the former is more suitable to the conceptual treatment of symmetries than the latter. §8.6 (Hamilton-Jacobi Equations) replaces second-order Euler-Lagrange equations by first-order ones whose formal solution spaces can be investigated with simple geometric tools. §8.7 (Poisson Reduction) states the main theorem of Marsden and Weinstein's symplectic reduction in [Zbl 0327.58005]. §8.8 (The Finite Dimensional BRST Formalism) is based upon [Zbl 0642.17003], claiming the main theorem of homotopical perturbation theory.

The Lagrangian methods discussed in Chapter 8 are applied to classical physics such as Newtonian mechanics, Maxwell's theory of electromagnetism and Einstein's special and general relativity in Chapter 14 (Variational Problems of Experimental Classical Physics). Chapter 15 (Variational Problems of Experimental Quantum Physics) deals with the Klein-Gordon Lagrangian, the Dirac Lagrangian, classical Yang-Mills theory, the Yang-Mills theory with the fermionic matter Lagrangian, and the standard model Lagrangian as a combination of a classical Yang-Mills Lagrangian with matter and a Higgs/Yukawa component. Chapter 16 (Variational Problems of Theoretical Physics) deals with the Kaluza-Klein theory, the bosonic gauge theory, the Poisson sigma model, Chern-Simon field theory and its higher variant, supersymmetric particles and so on. The reviewer feels that the title of Part 2 should have been "Variational Problems in Theoretical and Experimental Physics".

Part 3 begins with Chapter 17 (Quantum Mechanics) and concludes with Chapter 24 (Factorization Spaces and Quantization). The author believes firmly that functional integrals (in particular, the nonperturbative normalization group and Dyson-Schwinger equations) give the right description of interacting quantum fields that is presently the easiest entry for general mathematicians into modern applications of QFT methods in mathematics. The basic idea of perturbative methods in QFT is to adopt explicit computations of Gaussian integrals in finite dimensions as definitions of the corresponding Gaussian integrals in QFT, resulting in the Dyson-Schwinger equation, which is appropriate for the definition of fermionic integral. This is the main topic of Chapter 18 (Mathematical Difficulties of Perturbative Functional Integrals). Chapter 19 (The Connes-Kreimer-van Suijlekom View of Renormalization) is an overview of [Zbl 1194.81165] and [Zbl 1207.81083], following the Hopf algebra/Riemann-Hilbert description of renormalization in [Zbl 1032.81026] and [Zbl 1042.81059]. After Chapter 20 (Nonperturbative Quantum Field Theory), Chapter 21 (Perturbative Renormalization à la Wilson) gives the results of [Zbl 1221.81004], which is based axiomatically upon the Lagrangian formulation making use of Feynman's sum over histories in place of the familiar Hamiltonian formalism. Costello's ideas stem from [Leo P. Kadanoff, Scaling laws for Ising models near t_c, Physics 2 (1966), 263-272], [K. G. Wilson, Renormalization group and critical phenomena, I, Physical Review B **4** (1971), 3174-3183], [J. Polchinski, Renormalization and effective Lagrangians, Nuclear Physics, B (1984), 231-269] and others. Chapter 22 (Causal Perturbative Quantum Field Theory) presents a curved-spacetime version ([Zbl 1184.81099]) of the coordinate-free presentation of causal perturbative QFT in the spirit of the Bogoliubov-Epstein-Glaser renormalization method ([Zbl 1216.81075]). The method was extended to gauge theories in [Zbl 1263.81245]. Chapter 23 (Topological Deformation Quantization), concerned with deformation quantization of topological theories, is a sequel to §10.5 (Deformation Theory and Formal Geometry), the main idea behind the author's geometric interpretation being that deforming categories of representations of algebras is easier than deformation algebras, as in deformations of Hopf algebras discussed in §6.2 (Twists and Deformations) of Chapter 6 (Hopf Algebras), so that "the problem is to prove that a given $(k-1)$-monoidal dg-category is the category of modules over an E_k-algebra." Chapter 24 (Factorization Spaces and Quantizations) turns to non-topological theories and is concerned with the formalism of $\mathcal{D}$-geometry over the Ran space and factorization spaces, where n-monoidal dg-categories are replaced by chiral categories in deformation quantization. The basic idea is to exploit deformations of the stable ∞-category of modules over a given factorization algebra. Topological deformation quantizations discussed in Chapter 23 can be regarded as categorical factorization quantizations by looking on n-monoidal dg-categories as locally constant factorization categories over $\mathbb{R}^n$.