Introduction to Linear Algebra

Kazuaki TAIRA

Why do you study Mathematics ?

The Role of Mathematics
in
Natural Sciences

Mechanism of Mathematical Analysis

Natural Phenomenon

Mathematical Analysis
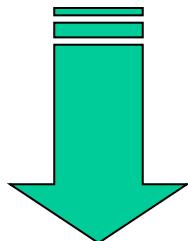
Mathematical Modeling
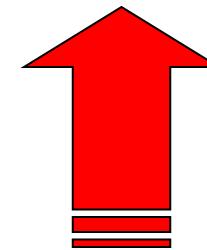
Differential Equations ⇒ Solution

# Weather Forecast

Mechanism of Weather Forecast

Weather

Weather Forecast
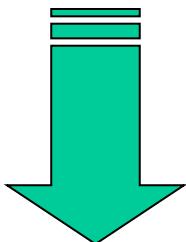
Mathematical Modeling
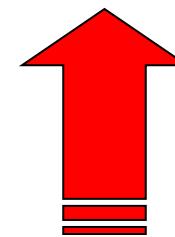
Navier - Stokes Equations $\Rightarrow$ **Approximation Solution**

Numerical Analysis

Navier-Stokes Equations in Fluid Dynamics

$$\rho \frac{D\mathbf{V}}{Dt}$$

$$= -\nabla p + \rho \mathbf{B} + \mu \Delta \mathbf{V} + \frac{1}{3}\mu \nabla \cdot \operatorname{div} \mathbf{V}$$

Inertia Force

= Pressure + Force + Viscosity + Stress

Bird's-Eye View of Analysis

Bird's- Eye View

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                          ┌─────────────┐
                          │  Analysis   │
                          └──────┬──────┘
              ┌──────────────────┴──────────────────┐
     ┌────────────────┐                   ┌────────────────────┐
     │ Vector Analysis│                   │    Differential    │
     │                │                   │    Equations       │
     └───────┬────────┘                   └──────────┬─────────┘
        ┌────┴─────┐               ┌─────────────────┼─────────────────┐
  ┌──────────┐ ┌──────────────┐ ┌───────────┐ ┌──────────┐ ┌──────────────┐
  │ Calculus │ │Linear Algebra│ │  Riemann  │ │  Series  │ │   Complex    │
  │          │ │              │ │  Integral │ │          │ │   Analysis   │
  └────┬─────┘ └──────────────┘ └─────┬─────┘ └────┬─────┘ └──────┬───────┘
   ┌───┴────┐                         │            │              │
┌──────────┐ ┌──────────┐      ┌────────────┐ ┌──────────┐ ┌──────────┐
│   Set    │ │ General  │      │ Set Theory │ │ Complex  │ │ Calculus │
│  Theory  │ │ Topology │      │  General   │ │ Numbers  │ │          │
│          │ │          │      │  Topology  │ │          │ └────┬─────┘
└──────────┘ └──────────┘      └────────────┘ └──────────┘      │
                                                          ┌──────────┐
                                                          │ Complex  │
                                                          │ Numbers  │
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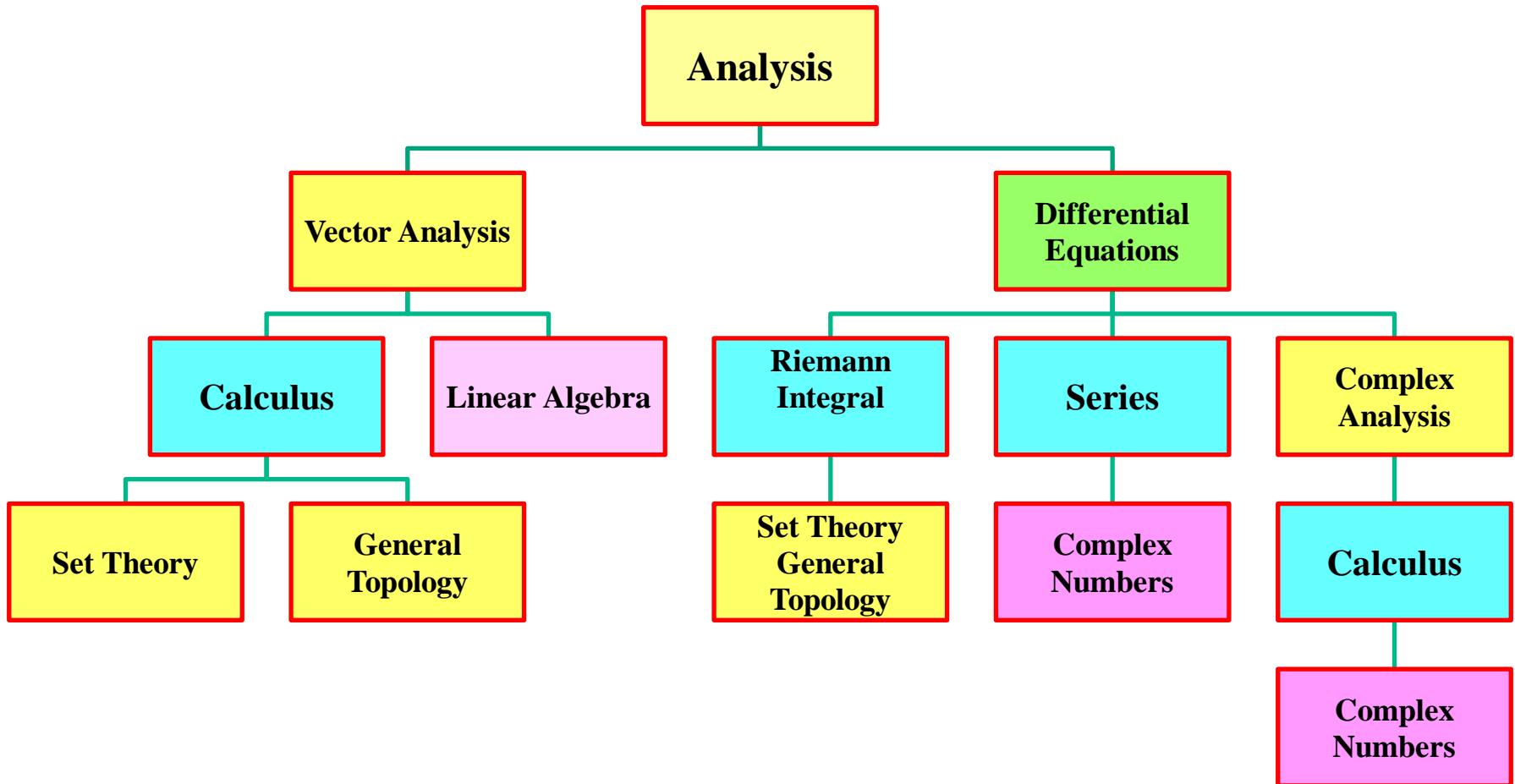
Mathematics versus Physics

Bird's-Eye View

Theme	Mathematics	Physics
Differential Equations	Ordinary Differential Equations	Newton's Equation of Motion
Infinite Series	Fourier Series	Eigenfunction Expansions (Principle of Superposition)
Vector Analysis	Calculus on Surfaces	Continuum Mechanics

Mathematical Thoughts

Mathematical Thoughts

（Ⅰ）　**Mathematical Reasoning**
（Ⅱ）　**Mathematical Ideas**
（Ⅲ）　**Mathematical Image**

Numerical Analysis

Role of Numerical Analysis

Mathematics	Analysis	Numerical Analysis
Physics	Theoretical Physics	Physical Experiments

Mathematical System
of
Numbers

Set	Algebra	Analysis
Complex Numbers	$+ - \times \div$	**Complete**
Real Numbers	$+ - \times \div$	**Complete**
Rational Numbers	$+ - \times \div$	
Integers	$+ - \times$	
Natural Numbers	$+ \times$	

Real Numbers

Real Numbers
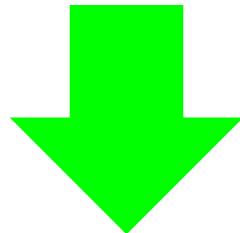
$\downarrow$

Sequences
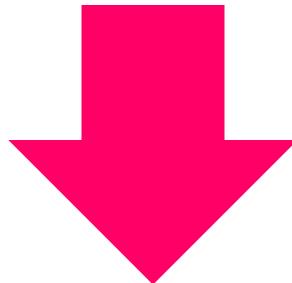
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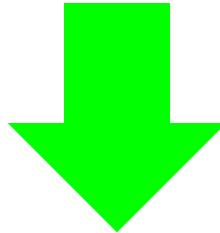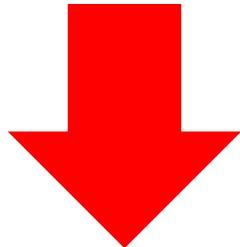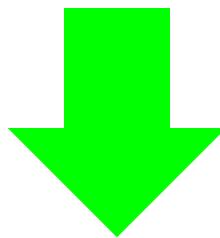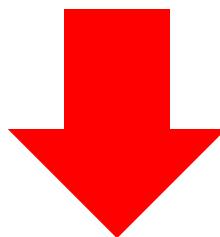
Series

Real Numbers and Decimal System

Real Numbers	Decimal System	Classification
Natural Numbers	Positive Integers	Rational
Integers	Integers	Rational
Fractional Numbers	Finite Decimal	Rational
Fractional Numbers	Recurring Decimal	Rational
Non-Fractional Numbers	Non-Recurring Decimal	Irrational

Finite Decimal (1)

$$\frac{1}{4} = 0.25$$

$$\frac{118}{25} = 4.72$$

Finite Decimal (2)

$$0.0625 = \frac{625}{10000}$$

$$= \frac{1}{16}$$

Recurring Decimal (1)

$$\frac{83}{74} = 1.1216216216\cdots$$

$$= 1.1\dot{2}1\dot{6}$$

$$\frac{89}{13} = 6.846153846153\cdots$$

$$= 6.\dot{8}46\dot{1}5\dot{3}$$

Recurring Decimal (2)

$$1.1\dot{2}\dot{1}\dot{6} = 1.1216216216\cdots$$

$$= 1.1 + 0.0216 + 0.0000216 + \cdots$$

$$= \frac{11}{10} + 216 \times \frac{1}{10^4} + 216 \times \frac{1}{10^7} + \cdots$$

$$= \frac{11}{10} + 216 \times \frac{1}{10^4}\left(1 + \frac{1}{10^3} + \cdots\right)$$

$$= \frac{11}{10} + 216 \times \frac{1}{10^4} \times \frac{1}{1 - \frac{1}{10^3}}$$

$$= \frac{11205}{9990} = \frac{83}{74}$$

Non-Recurring Decimal

$$\sqrt{2} = 1.41421356\cdots$$

$$e = 2.718281828459 04\cdots$$

The square root of a prime number is irrational (1)

Let p be a prime number.

Assume that $\sqrt{p}$ is rational.

(*) $\qquad \sqrt{p} = \dfrac{n}{m}$

Here the right$-$hand side is irreducible.

The square root of a prime number is irrational (2)

$(*) \implies$

$(**) \quad n^2 = pm^2$

p **is a prime number**

n^2 **is a multiple of** $p \iff$

n **is a multiple of** p

$n = pa + (**) \implies$

$pm^2 = n^2 = p^2a^2 \implies$

$m^2 = pa^2$

The square root of a prime number is irrational (3)

$$m^2 = pa^2$$

**implies that**

m **is a multiple of** p :

$$m = pb$$

$$\implies$$

$$\sqrt{p} = \frac{n}{m} = \frac{pa}{pb} = \frac{a}{b}$$

(contradiction)

Theory
of
Real Numbers

Main Theme

How do we characterize **irrational numbers** ?
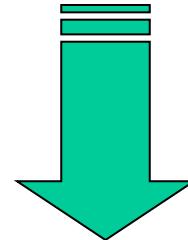
What is the **convergence** of sequences ?

Completeness

Convergence
of Sequences

Definition of Convergence

$\{a_n\}$ **sequence of real numbers**

$\{a_n\}$ **converges** **to** a

$$\overset{\text{def}}{\Longleftrightarrow}$$

$\forall \varepsilon > 0, \ \exists N = N(\varepsilon) \in \mathbf{N}$ **such that**

$\forall n \geq N \Rightarrow \left| a_n - a \right| < \varepsilon$

Cauchy's Test

Cauchy's Test

$$\{a_n\} \text{ converges}$$

$$\Longleftrightarrow$$

$$\lim_{n,m\to\infty} |a_n - a_m| = 0$$

Sequences

Sequences versus Functions

	Domain of Definition	Range
Sequence	**Natural Numbers**	**Real Numbers**
Functions	**Real Numbers**	**Real Numbers**

Definition

The sequnce $\{a_n\}$ **converges to** a

$$\overset{\text{def}}{\Longleftrightarrow}$$

$$\forall\, \varepsilon > 0,\ \exists N = N(\varepsilon) \in \mathbf{N} \text{ such that}$$

$$\forall n \geq N \Rightarrow |a_n - a| < \varepsilon$$

Notation $: \lim_{n \to \infty} a_n = a$

Fundamental Example

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

Examples (1)

$$(1)\ \lim_{n \to \infty} \frac{n+1}{n} = 1$$

$$(2)\ \lim_{n \to \infty} \frac{n+1}{n^2} = 0$$

$$(3)\ \lim_{n \to \infty} \left(\sqrt{n^2 + 1} - n \right) = 0$$

Example (2)

$$\lim_{n \to \infty} a^n = \begin{cases} 0 & \textbf{if} \quad 0 < a < 1 \\ 1 & \textbf{if} \quad a = 1 \\ +\infty & \textbf{if} \quad a > 1 \end{cases}$$

Examples (3)

$$(1)\ \lim_{n \to \infty} a^{\frac{1}{n}} = 1 \quad \textbf{for} \quad a > 0$$

$$(2)\ \lim_{n \to \infty} n^{\frac{1}{n}} = 1$$

Bounded Monotone Sequence

Fundamental Theorem

Every bounded, monotone increasing sequence itself converges.

$$a_n \leq \exists M \quad （\text{Bounded}）$$

$$a_n \leq a_{n+1} \quad （\text{Monotone increasing}）$$

Example (Napier's Number)

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

$$e = \overset{\exists}{\lim_{n \to \infty}} \left(1 + \frac{1}{n}\right)^n$$

Bounded Sequences

Fact

A convergent sequence is bounded.

Bolzano-Weierstrass Theorem

Bolzano (1781−1848)

Weierstrass (1815−1897)

Bolzano-Weierstrass Theorem

Every bounded sequence has a convergent subsequence.

Numerical Analysis

Newton's Method versus Bisection Method

Method	Newton's Method	Bisection Method
Hypotheses	Differentiability Monotonicity	Continuity
Merits Demerits	Strong Hypotheses Rapid Convergence	Weak Hypotheses Slow Convergence
Background	Convergence of Monotone Sequences	Intermediate Value Theorem

Newton's Approximation Method

Isaac Newton (1642-1727)
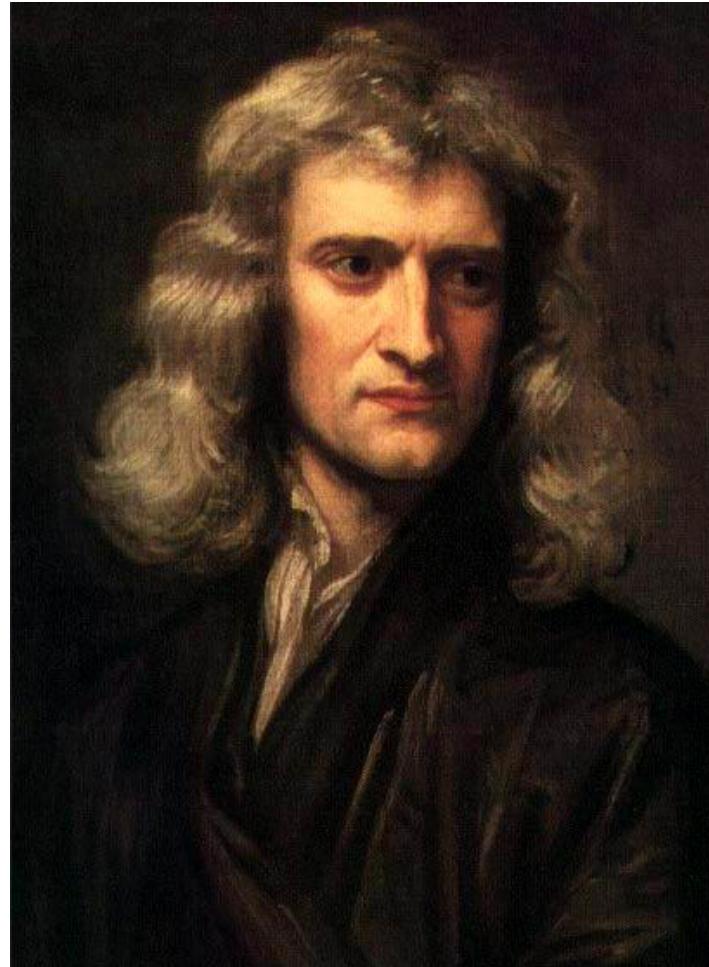
Newton's Approximation Method

$$r > 0, \ a_0 > 0$$

$$a_{n+1} := \frac{1}{2}\left(a_n + \frac{r}{a_n}\right), \quad n = 0, 1, 2, \ldots$$

$$\Longrightarrow$$

$$a_n \downarrow \sqrt{r} \quad (n \to \infty)$$

Example (Square root of 2)

$$a_0 = 2, \qquad a_1 = \frac{3}{2}$$

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n} \right)$$

$$\implies$$

$$\lim_{n \to \infty} = \sqrt{2}$$

Newton's Method (1)

$$y = x^2 - 2$$

a_2 a_1 a_0

Newton's Method (2)

$$y = x^2 - 2$$

$(a_n, a_n^2 - 2)$

a_{n+1}
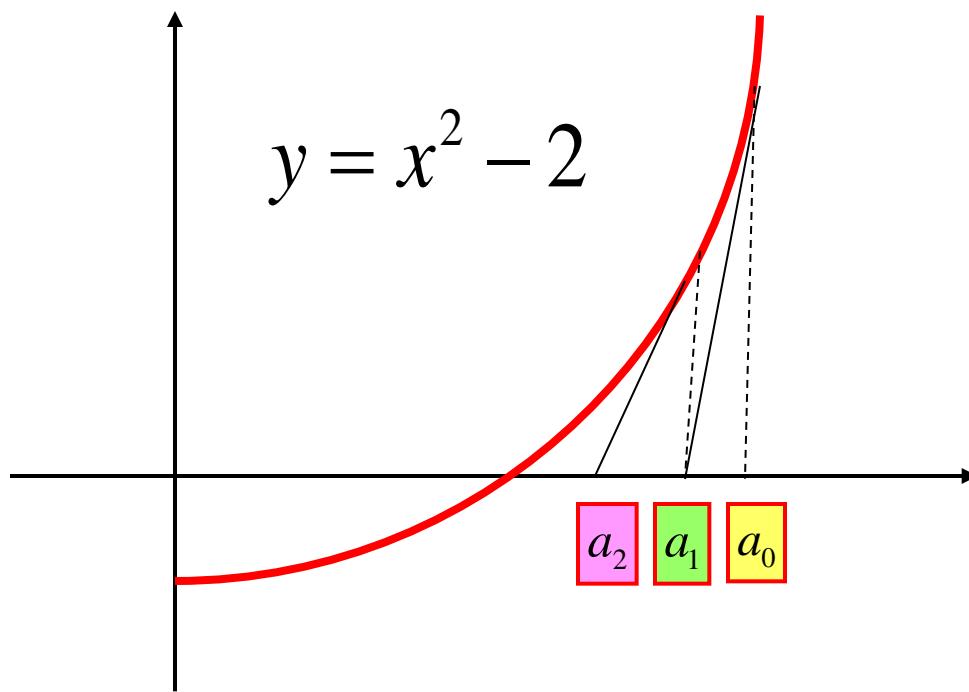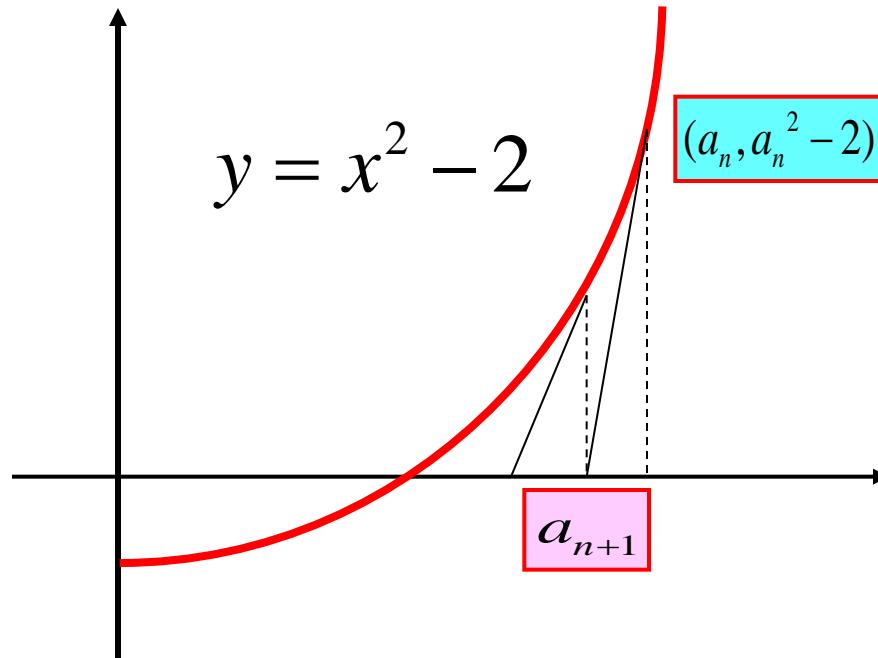
Tangent Line **at** $\left(a_n, a_n^2 - 2\right):$

$$y = 2a_n(x - a_n) + a_n^2 - 2 = 2a_n x - a_n^2 - 2$$

Bisection Method

Principle of Successive Subdivision

Cantor (1845−1918)

Cantor's Nested-Interval Property

$\{I_n\}$ **Sequence of closed intervals**

(1) $I_{n+1} \subset I_n$

(2) $|I_n| \to 0$

$\Rightarrow$

$$\bigcap_{n=1}^{\infty} I_n = \{\text{One Point}\}$$

Sequence Version

$$(1) \quad a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots \leq b_{n+1} \leq b_n \leq b_2 \leq b_1$$

$$(2) \quad b_n - a_n \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

$$\Rightarrow$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n$$

Bisection Method（1）

$$y = x^2 - 2$$

a_1

a_2
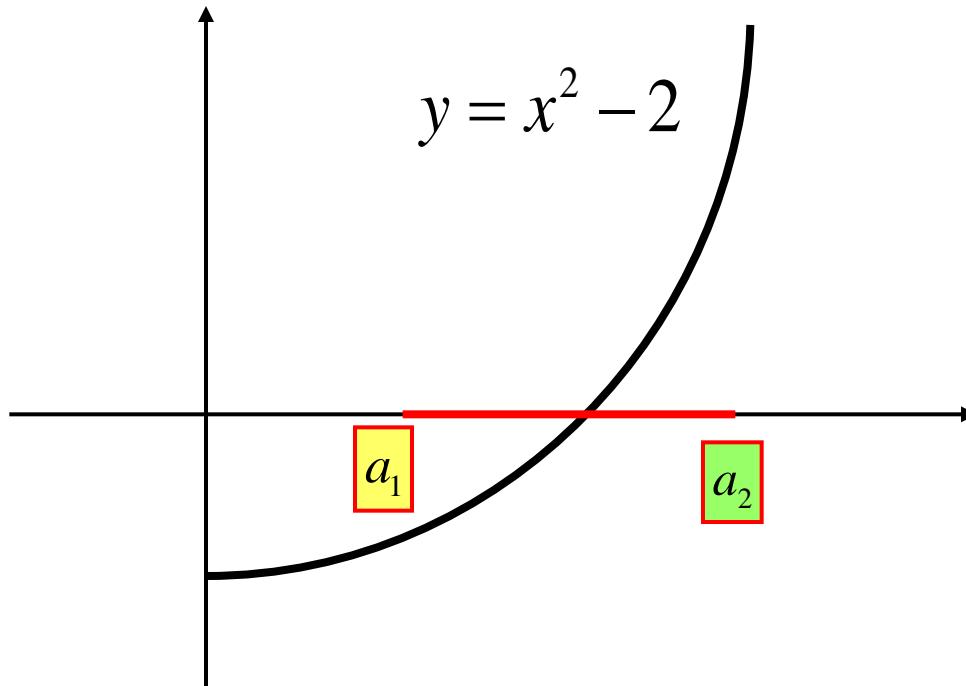
$$\sqrt{2} : \textbf{Square Root of } 2$$

Bisection Method（2）

$$y = x^2 - 2$$
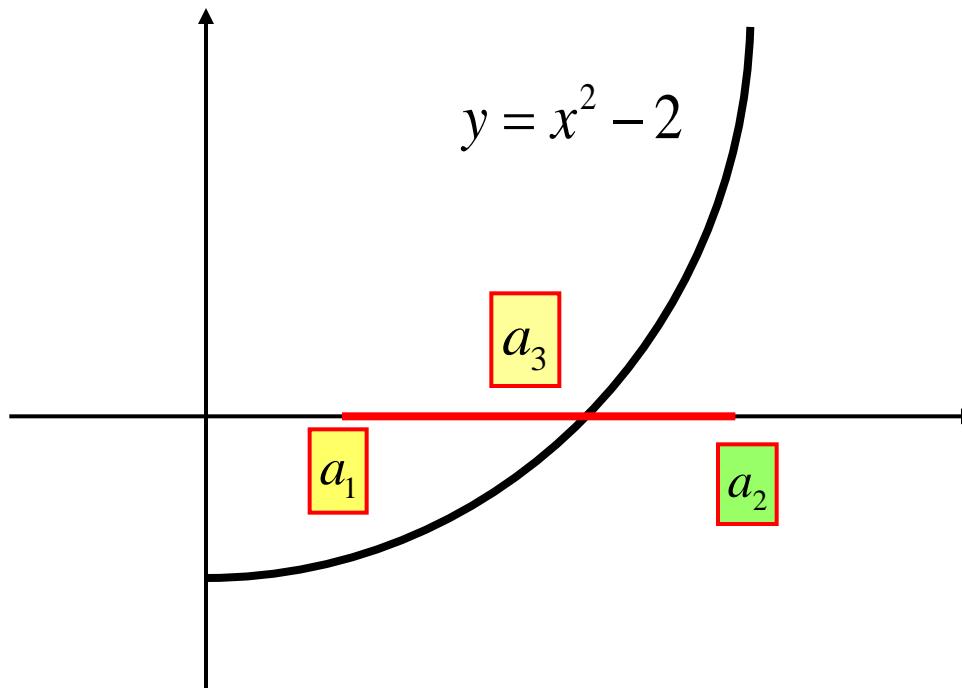
Bisection Method（3）

$$y = x^2 - 2$$

a_3 a_2

a_4
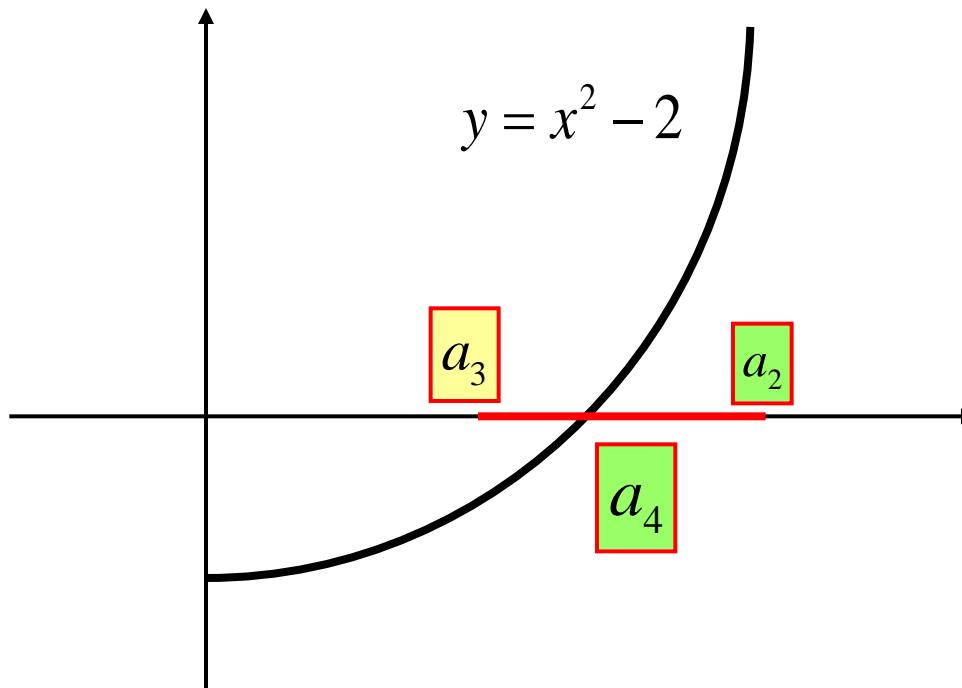
Square Root of 2 （1）

(1) $1^2 < 2 < 2^2 \Rightarrow 1 < \sqrt{2} < 2$

$\sqrt{2} \in I_1 = [1, 2]$

(2) $(1.4)^2 = 1.96 < 2 < (1.5)^2 = 2.25$

$\Rightarrow 1.4 < \sqrt{2} < 1.5$

$\sqrt{2} \in I_2 = [1.4, 1.5]$

(3) $(1.41)^2 = 1.9881 < 2 < (1.42)^2 = 2.0164$

$\Rightarrow 1.41 < \sqrt{2} < 1.42$

$\sqrt{2} \in I_3 = [1.41, 1.42]$

Square Root of 2 （2）

(n) $a_n{}^2 < 2 < b_n{}^2 \Rightarrow a_n < \sqrt{2} < b_n$

$$b_n - a_n = \frac{1}{10^n}$$

$\sqrt{2} \in I_n = [a_n, b_n]$

$\Rightarrow$

$\begin{cases} a_n \uparrow \alpha \\ b_n \downarrow \alpha \end{cases}$

$\alpha = \sqrt{2}$

Complex Numbers

Carl Friedrich Gauss

Gauss

◆ **Carl Friedrich Gauss (1777-1855)**
**German Mathematician and Physicist**

Complex Number

$$a + ib = c + id$$

$$\Longleftrightarrow$$

$$a = c, b = d$$

$$i = \sqrt{-1}$$

Complex Plane

$$z = a + ib$$

b

0

a
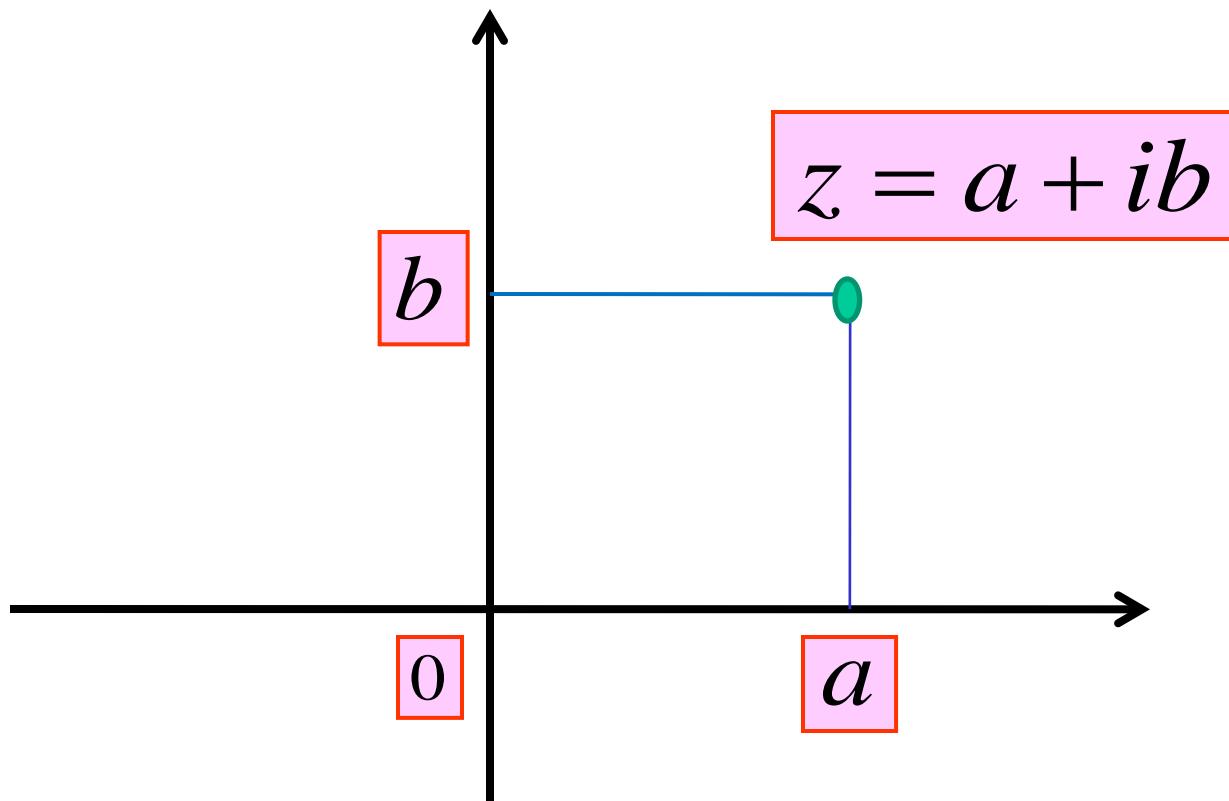
Conjugate of a Complex Number

$$z = a + ib$$

$$\Longrightarrow$$

$$\bar{z} = a + i(-b) = a - ib$$

$$z = a + ib$$

$$b$$

$$a$$

$$0$$

$$-b$$
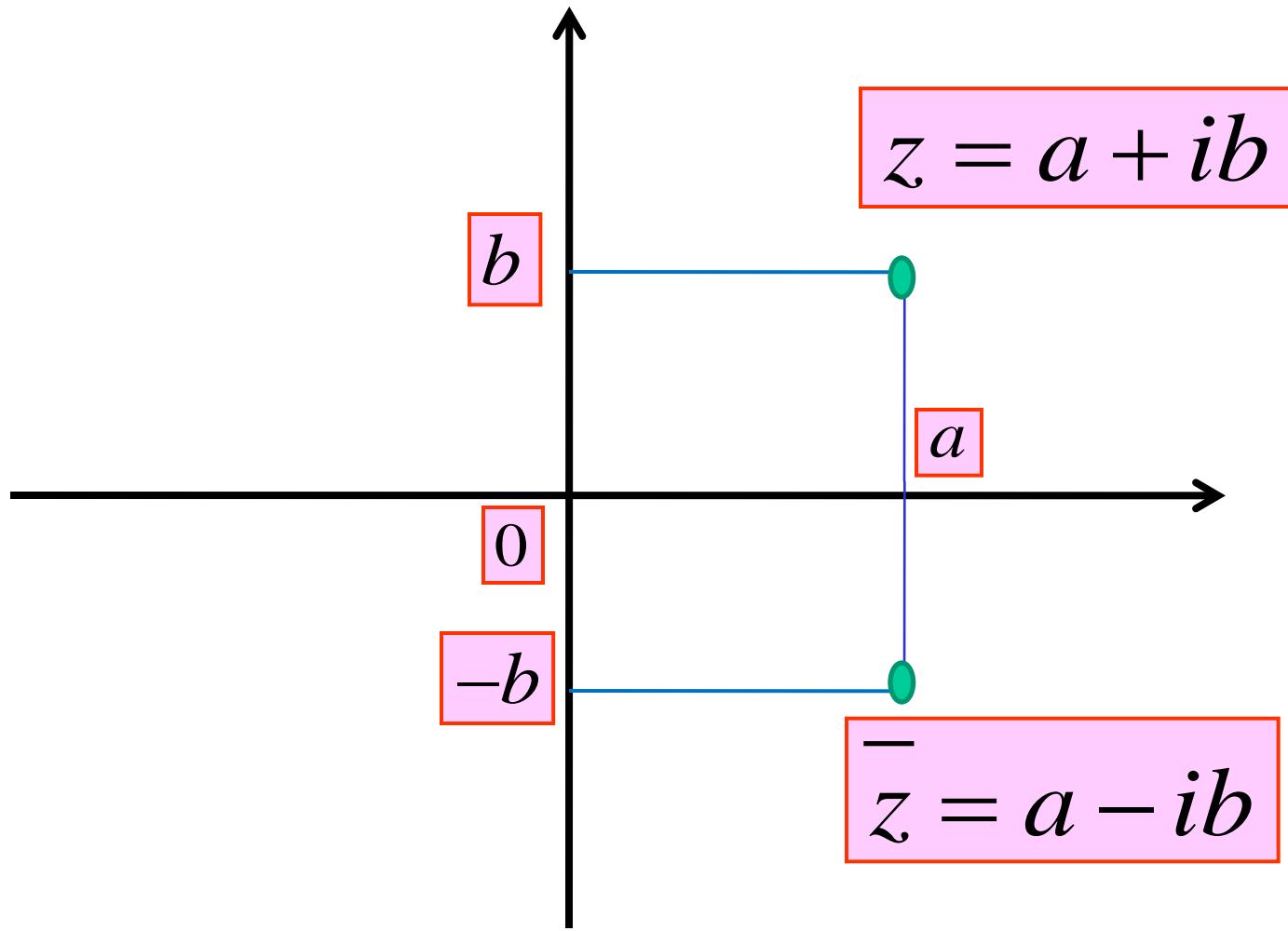
$$\overline{z} = a - ib$$

Absolute Value of a Complex Number

$$z = a + ib$$

$$\Rightarrow$$

$$|z| = |a + ib| = \sqrt{a^2 + b^2}$$

$$b$$

$$z = a + ib$$

$$|z|$$

$$0$$

$$a$$
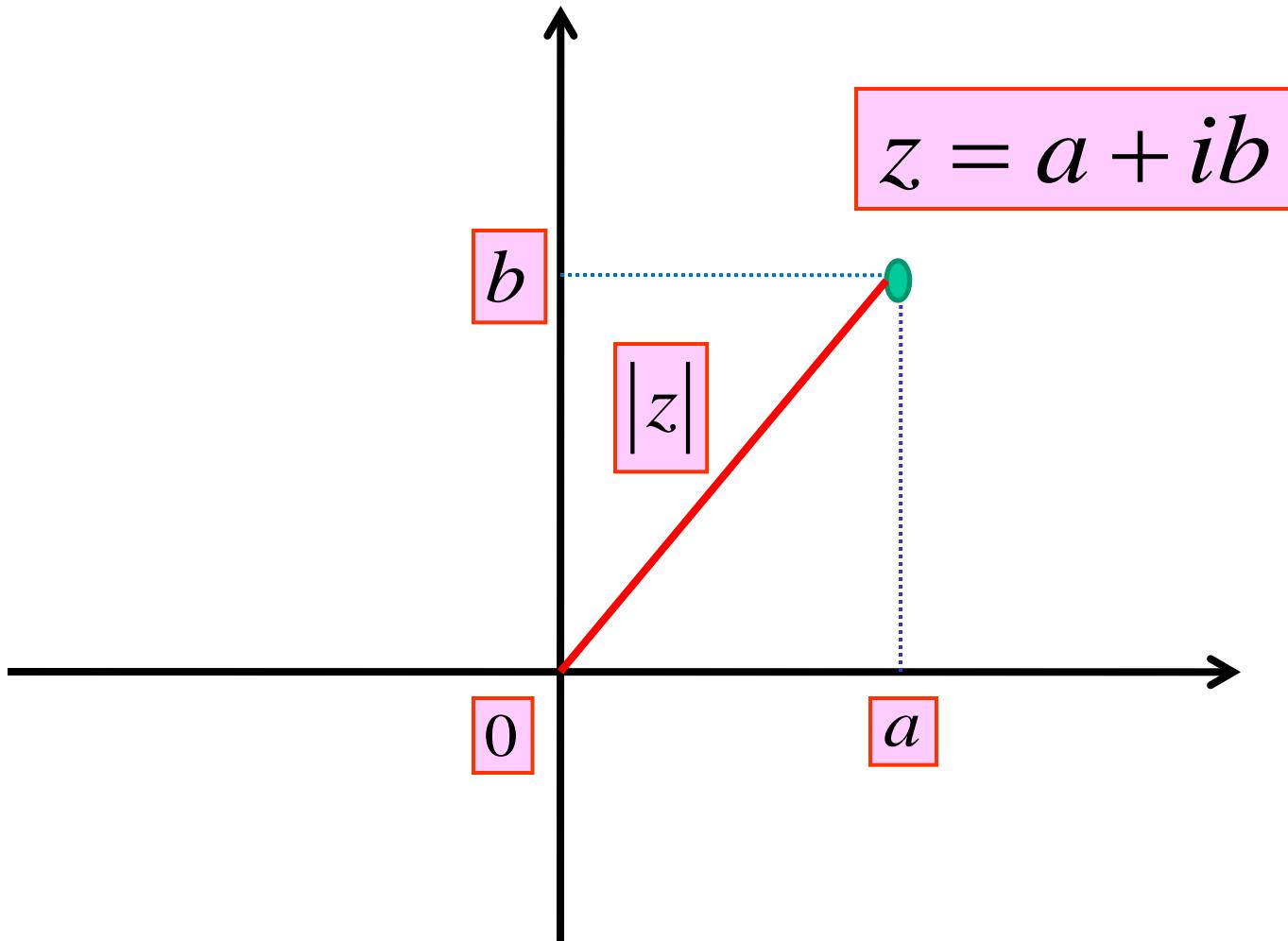
Polar Coordinates of a Complex Number

$$z = a + ib$$

$$= r(\cos\theta + i\sin\theta)$$

b

$r = |z|$

θ

0

a
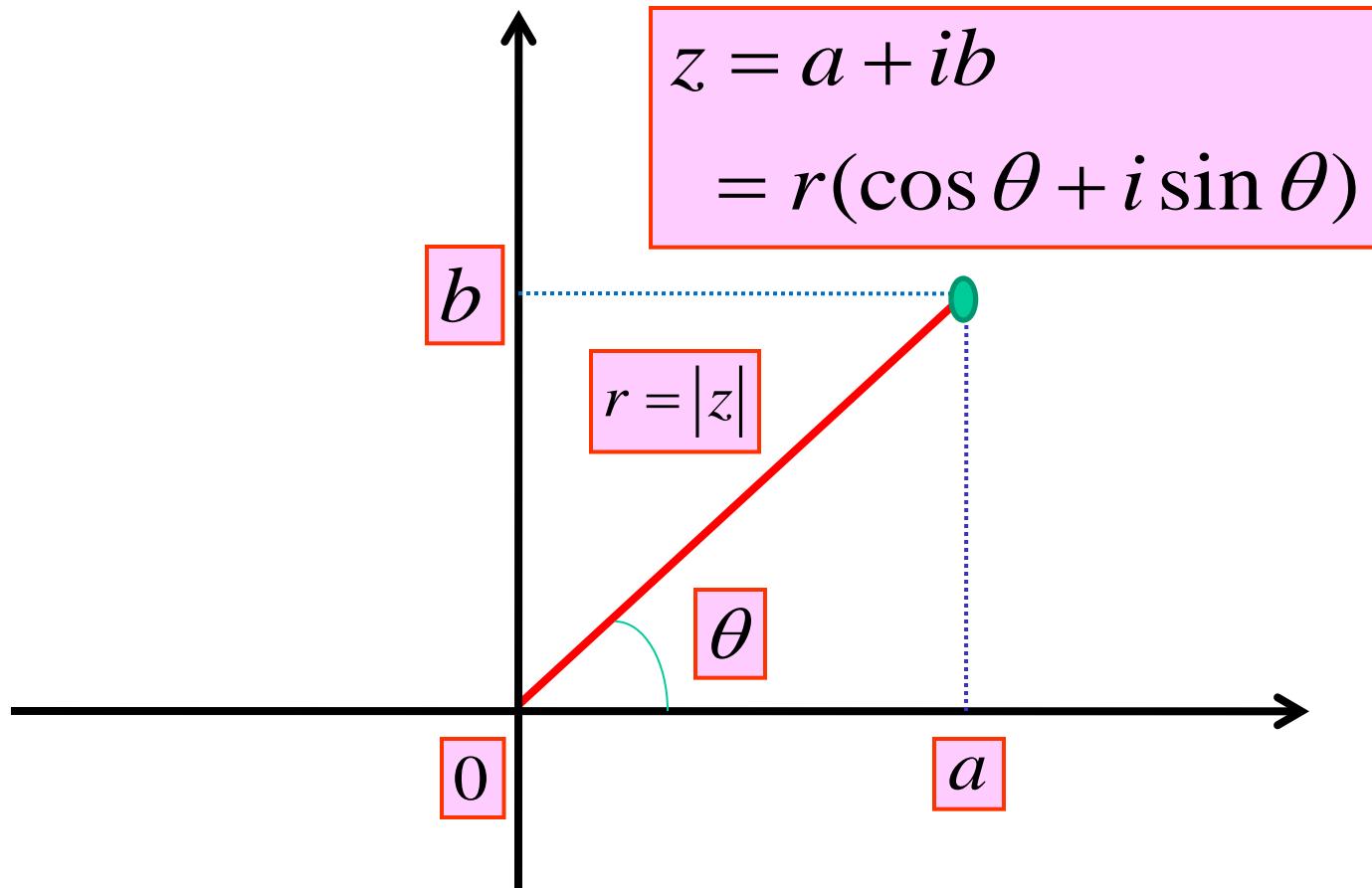
Sum of Complex Numbers

$$z = a + ib, \quad w = c + id$$

$$\Longrightarrow$$
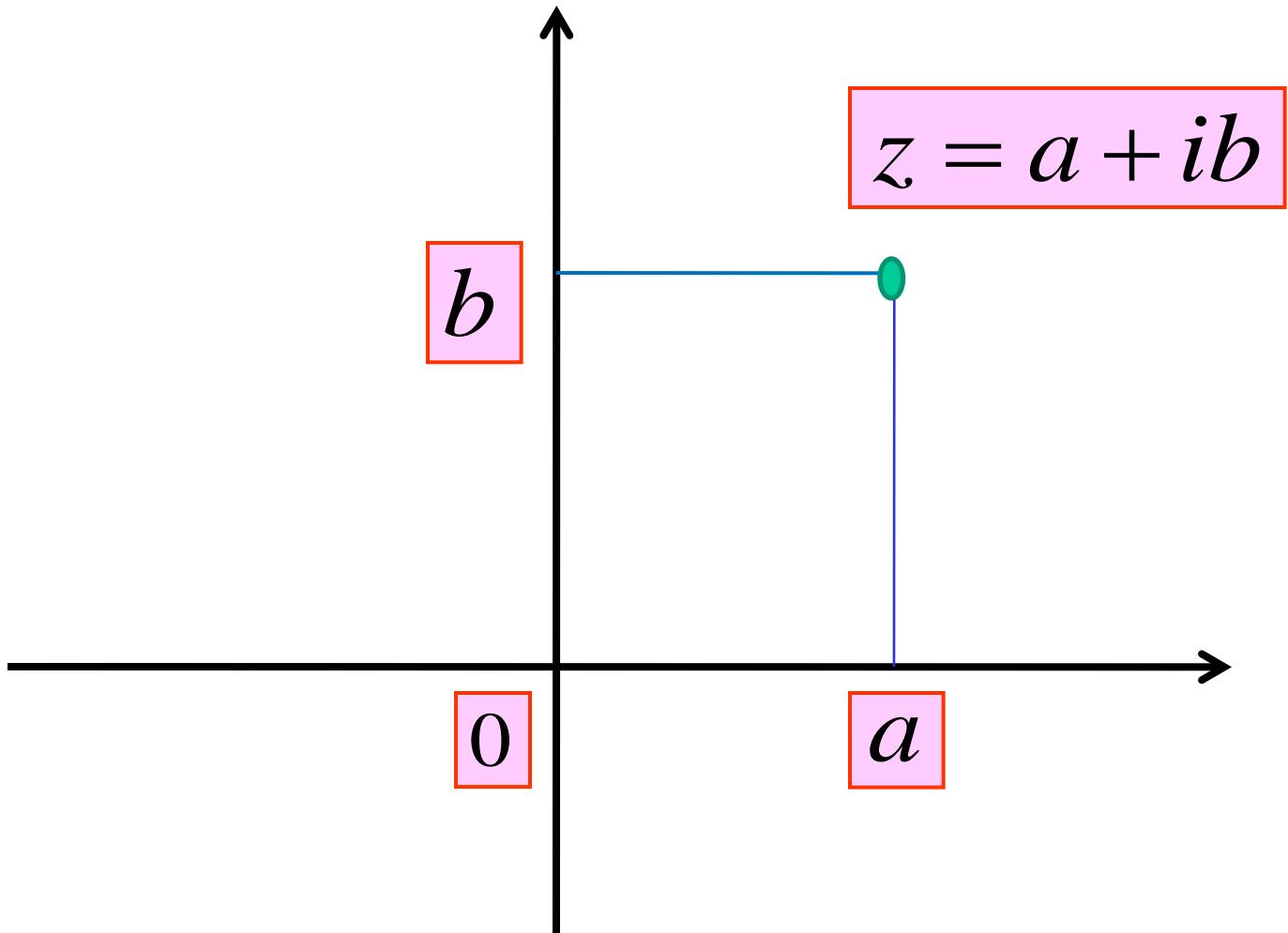
$$z + w = (a + c) + i(b + d)$$

Difference of Complex Numbers

$$z = a + ib, \quad w = c + id$$

$$\Longrightarrow$$

$$z - w = (a - c) + i(b - d)$$

Product of Complex Numbers

$$z = a + ib, \quad w = c + id$$

$$\Rightarrow$$

$$zw = (ac - bd) + i(ad + bc)$$

$$i = \sqrt{-1} \Rightarrow i^2 = -1$$

$$z = a + ib$$

$$= r(\cos\theta + i\sin\theta)$$

b

$r = |z|$

θ

0

a
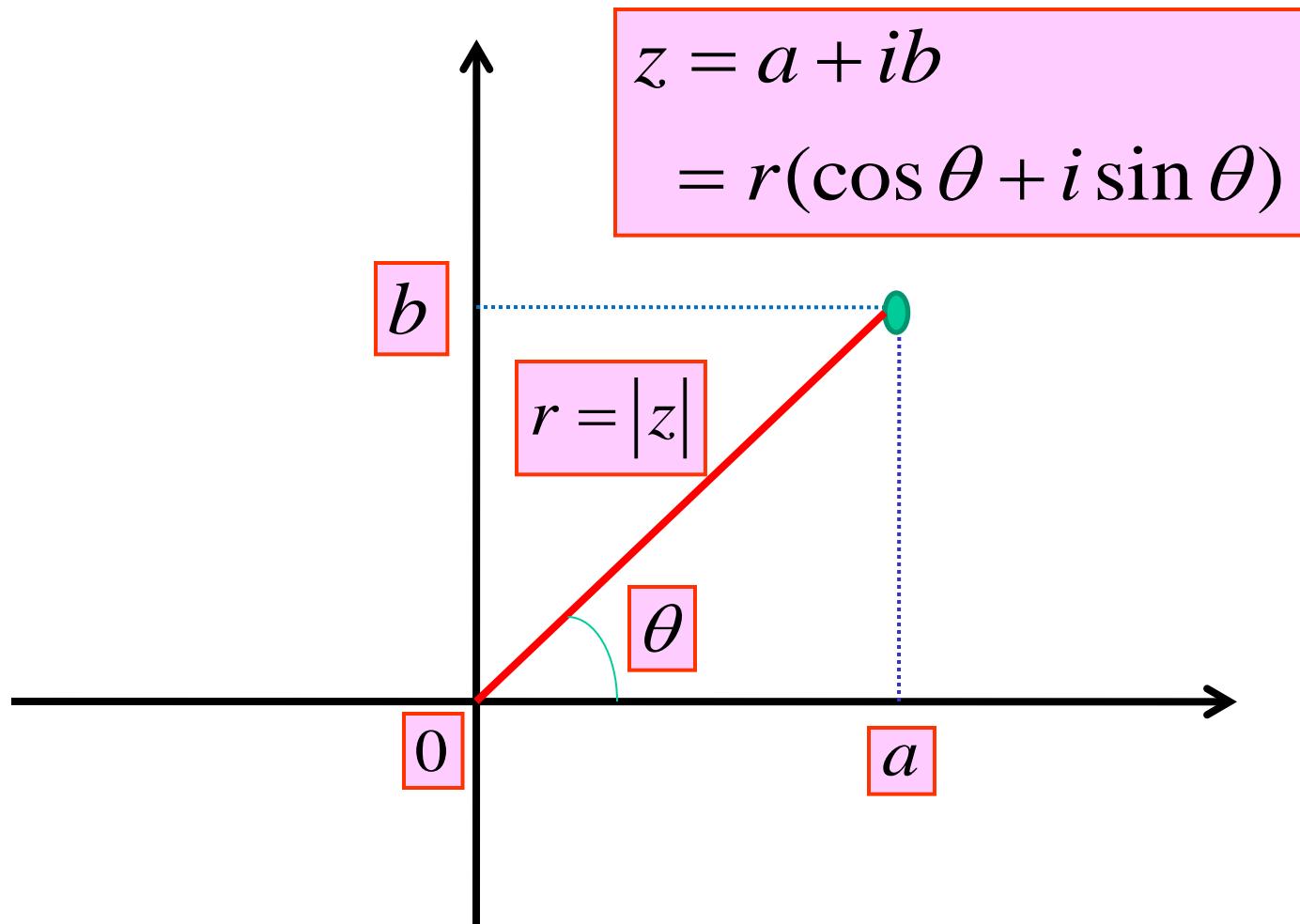
Product of Complex Numbers

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$w = s(\cos\omega + i\sin\omega) = se^{i\omega}$$

$$\Rightarrow$$

$$zw = rs(\cos(\theta + \omega) + i\sin(\theta + \omega))$$

$$= rse^{i(\theta + \omega)}$$

De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$\forall n \in \mathbf{Z}$$

Leonhard Euler (1707-1783)

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{i\pi} = \cos\pi + i\sin\pi = -1$$

Euler + De Moivre

$$(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n$$
$$= \cos n\theta + i\sin n\theta$$
$$= e^{in\theta} \quad (\forall n \in \mathbf{Z})$$

Algebraic Equation

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0$$

$$a_i \in \mathbf{C}$$

Fundamental Theorem of Algebra (Gauss)

Every algebraic equation

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0, \, a_0 \neq 0$$

has n roots in C counted with multiplicity.

Example (1)

$$ax + b = 0, \ a \neq 0$$

$$\implies$$

$$x = -\frac{b}{a}$$

Example (2)

$$ax^2 + bx + c = 0, \qquad a \neq 0$$

$$\Longrightarrow$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Imaginary Number

$$x^2 + 1 = 0$$
$$\implies$$
$$x = \pm\sqrt{-1}$$

Theory of Matrices

General Form of a Matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nm} \end{pmatrix}$$

Row of a Matrix

$$\left(a_{i1} \quad a_{i2} \quad \cdot \quad \cdot \quad a_{im} \right)$$

Colum of a Matrix

$$\begin{pmatrix} a_{1k} \\ a_{2k} \\ \cdot \\ \cdot \\ a_{nk} \end{pmatrix}$$
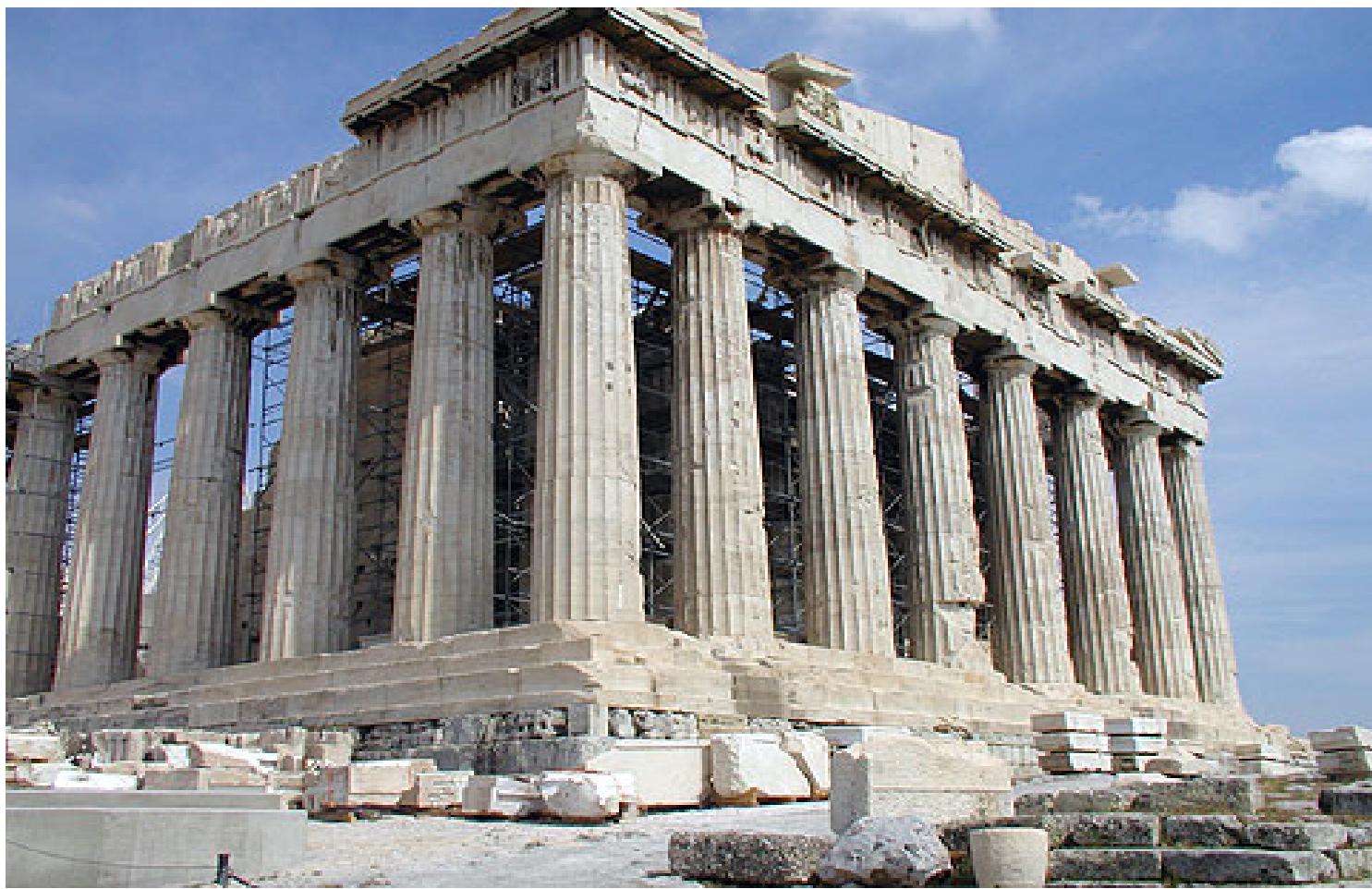
Operations of Matrices

Sum of Matrices

$$A + B = \left(a_{ij} + b_{ij} \right)$$

$$= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdot & \cdot & a_{1m} + b_{1m} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdot & \cdot & a_{2m} + b_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \cdot & \cdot & a_{nm} + b_{nm} \end{pmatrix}$$

Difference of Matrices

$$A - B = \left(a_{ij} - b_{ij} \right)$$

$$= \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \cdot & \cdot & a_{1m} - b_{1m} \\ a_{21} - b_{21} & a_{22} - b_{22} & \cdot & \cdot & a_{2m} - b_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} - b_{n1} & a_{n2} - b_{n2} & \cdot & \cdot & a_{nm} - b_{nm} \end{pmatrix}$$

Scalar Multiple of a Matrix

$$\alpha A = \left(\alpha a_{ij} \right)$$

$$= \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdot & \cdot & \alpha a_{1m} \\ \alpha a_{21} & \alpha a_{22} & \cdot & \cdot & \alpha a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha a_{n1} & \alpha a_{n2} & \cdot & \cdot & \alpha a_{nm} \end{pmatrix}$$

Product of Matrices (1)

$$\mathbf{Ab}$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nm} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_m \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_1 + a_{12}b_2 + \cdot\cdot + a_{1m}b_m \\ a_{21}b_1 + a_{22}b_2 + \cdot\cdot + a_{2m}b_m \\ \cdot \\ \cdot \\ a_{n1}b_1 + a_{n2}b_2 + \cdot\cdot + a_{nm}b_m \end{pmatrix}$$

Motivation

$$y_1 = 2x_1 + 5x_2$$

$$y_2 = -x_2$$

$$y_3 = -x_1 + 4x_2$$

$$\Longrightarrow$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 0 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Example

$$\mathbf{Ab} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

Product of Matrices (2)

$$AB = \left(\sum_{k=1}^{m} a_{ik} b_{kj} \right)$$

Zero Matrix

$$O = \begin{pmatrix} 0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 \end{pmatrix}$$

$$A + O = O + A = A$$

Unit Matrix

$$E_n = \left(\textcolor{red}{\delta_{ij}} \right) = \begin{pmatrix} 1 & 0 & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 1 \end{pmatrix}$$

$$E_m A = A, \quad A E_n = A$$

Kronecker's Delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$E = \left(\delta_{ij} \right)$$

Inverse Matrix

$$A = \left(a_{ij}\right)_{1 \le i, j \le n}, \quad B = \left(a_{ij}\right)_{1 \le i, j \le n}$$

$$AB = BA = E_n$$

$$\Longleftrightarrow$$

$$B = A^{-1} : \text{inverse matrix of } A$$

Uniqueness of an Inverse Matrix

$$A B_1 = B_1 A = E_n$$

$$A B_2 = B_2 A = E_n$$

$$\Longrightarrow$$

$$B_1 = B_1 E$$

$$= B_1 \left(A B_2 \right) = \left(B_1 A \right) B_2$$

$$= E_n B_2 = B_2$$

Transposed Matrix

$$A = (a_{ij})$$

$$\Longrightarrow$$

$$^{t}A = (a_{ji})$$

Example

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 4 & 1 & 0 \\ -5 & 1 & 2 \end{pmatrix}$$

$${}^{t}A = \begin{pmatrix} 3 & 4 & -5 \\ 0 & 1 & 1 \\ -1 & 0 & 2 \end{pmatrix}$$

Symmetric Matrix

$$A = (a_{ij})$$

$${}^{t}A = A$$

$$\Longleftrightarrow$$
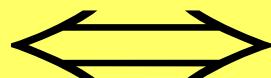
$$a_{ij} = a_{ji}$$

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 3 \end{pmatrix}$$

Alternating Matrix

$$A = (a_{ij})$$

$${}^t A = -A$$

$$\Longleftrightarrow$$
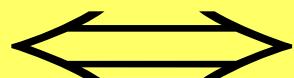
$$a_{ij} = -a_{ji}$$

Example

$$\begin{pmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{pmatrix}$$

Addition Theorem of Trigonometric Functions

Rotation (1)

$$A(\theta)\mathbf{e}_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

θ

0

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Rotation (2)

$$A(\theta)\mathbf{e}_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

θ

0
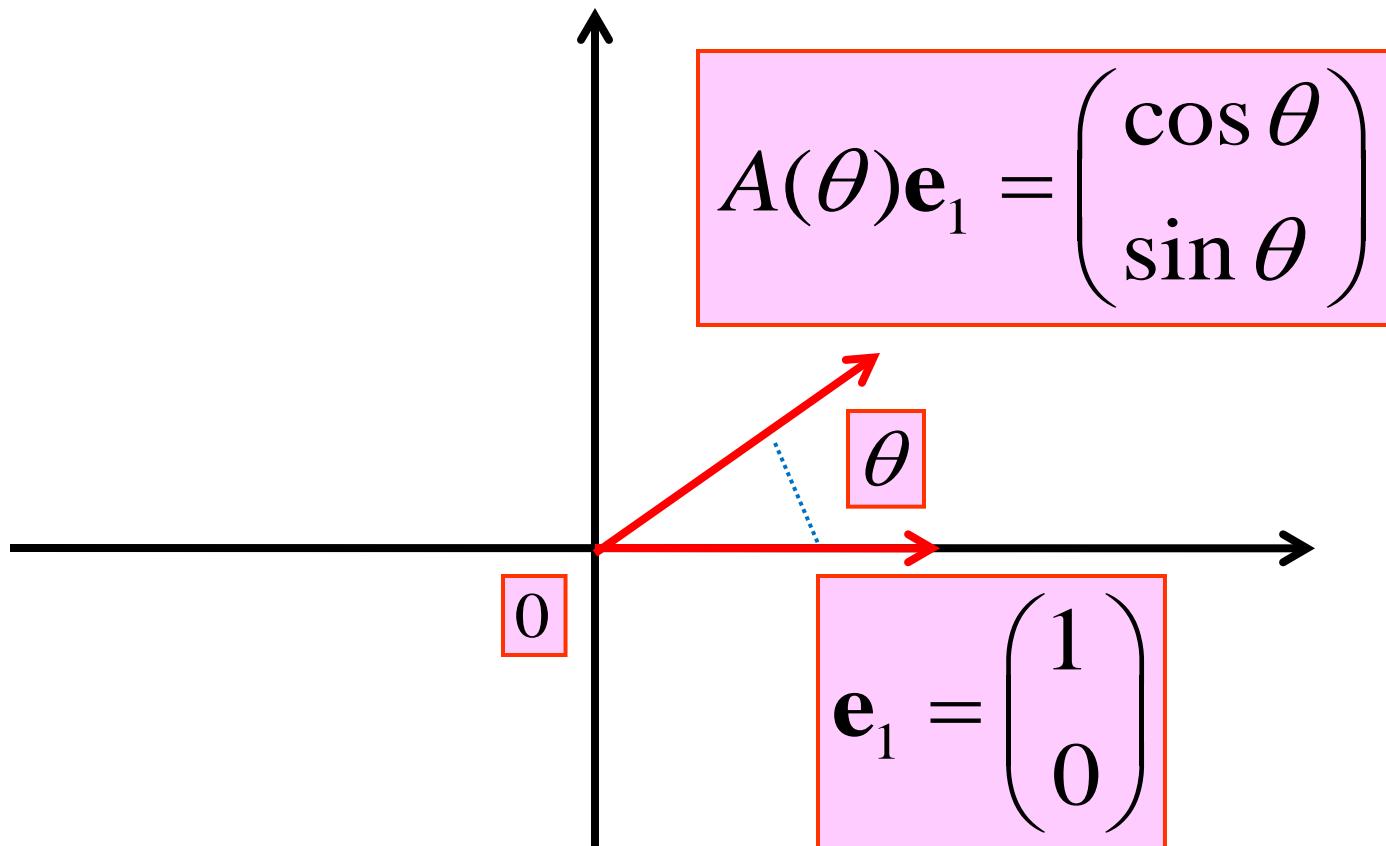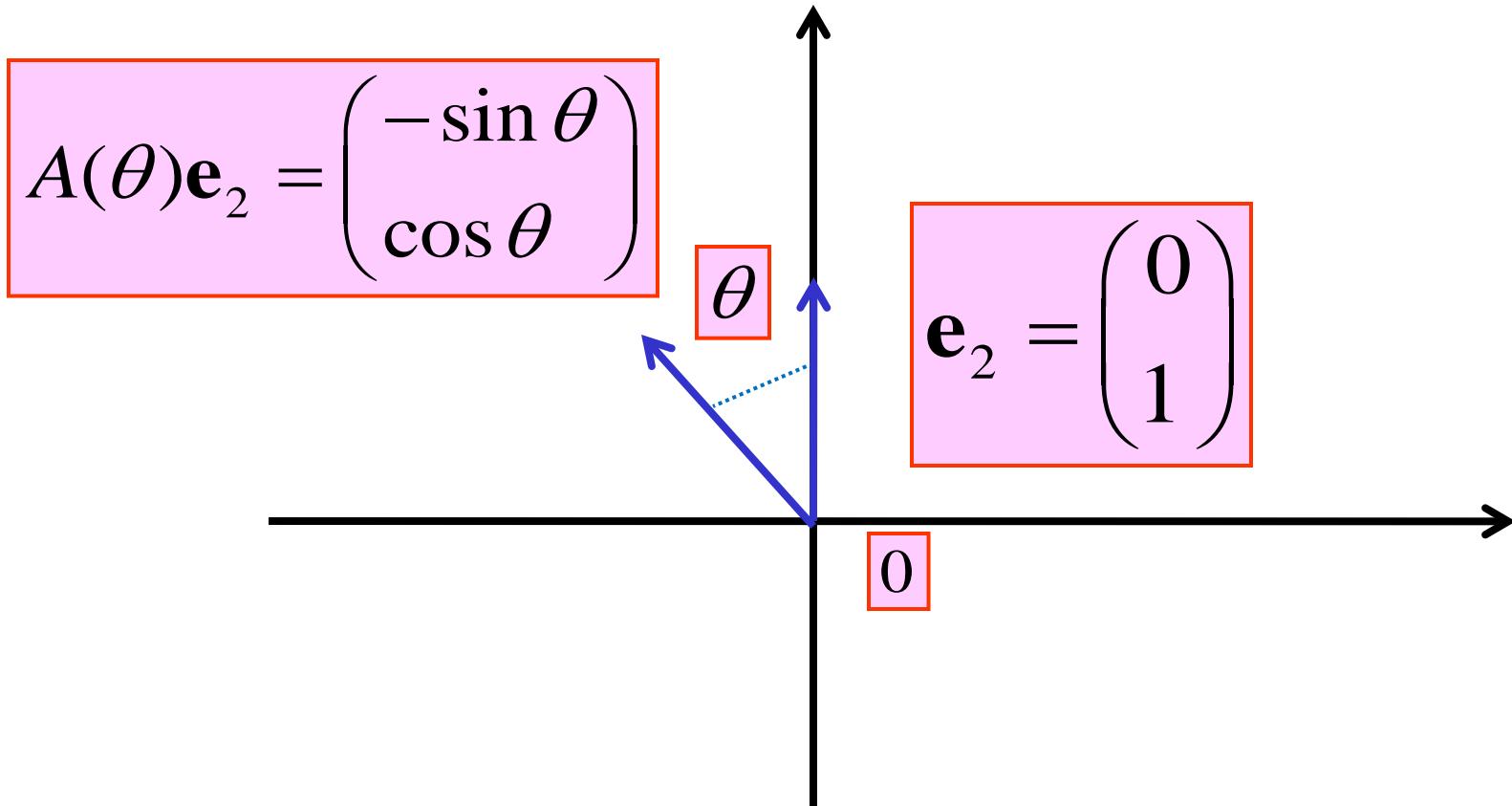
Matrix of Rotation (1)

$$A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Rotation of θ

Matrix of Rotation (2)

$$A(-\theta) = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = A(\theta)^{-1}$$

Rotation of $-\theta$

Composition of Rotations (1)

$$A(\alpha)\mathbf{e}_1 = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

Composition of Rotations (2)

$$A(\beta)\big(A(\alpha)\mathbf{e}_1\big)$$

$$= \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha \\ \sin\beta\cos\alpha + \cos\beta\sin\alpha \end{pmatrix}$$

Composition of Rotations (3)

$$A(\alpha)\mathbf{e}_2 = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}$$

Composition of Rotations (4)

$$A(\beta)\big(A(\alpha)\mathbf{e}_2\big)$$

$$= \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} -\cos\beta\sin\alpha - \sin\beta\cos\alpha \\ -\sin\beta\sin\alpha + \cos\beta\cos\alpha \end{pmatrix}$$

Composition of Rotations (5)

$$A(\beta)A(\alpha)$$

$$= \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha & -\cos\beta\sin\alpha - \sin\beta\cos\alpha \\ \sin\beta\cos\alpha + \cos\beta\sin\alpha & -\sin\beta\sin\alpha + \cos\beta\cos\alpha \end{pmatrix}$$

Composition of Rotations (6)

$$A(\beta)\big(A(\alpha)\mathbf{e}_1\big) = A(\beta)A(\alpha)\mathbf{e}_1$$

$$A(\beta)\big(A(\alpha)\mathbf{e}_2\big) = A(\beta)A(\alpha)\mathbf{e}_2$$

Composition of Rotations (7)

$$A(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$A(\beta) = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

$$\Rightarrow$$

$$A(\alpha)A(\beta) = A(\beta)A(\alpha) = A(\alpha+\beta)$$

Composition of Rotations (8)

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

$$= A(\alpha + \beta)$$

$$= \begin{pmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{pmatrix}$$

Addition Theorem (1)

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Addition Theorem (2)

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

Addition Theorem (3)

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Addition Theorem (4)

$$\sin A \,\sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$

$$\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$$

$$\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$$

Addition Theorem (5)

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$$

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$$

$$\sin 2A = 2 \sin A \cos A$$

Addition Theorem (6)

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$1 + \tan^2 A = \frac{1}{\cos^2 A}$$

Inverse Matrices

Algorithm for Inverse Matrices

Left Elementary Transformations

(1) Interchange two rows
(2) Multiply a row by a non-zero constant
(3) Add a row by a multiplied another row

Gauss' Method (1)

$$(A, E) \quad \underset{\text{Left Elemntary Transformations}}{\Longrightarrow} \quad (E, B)$$

Gauss' Method (2)

$$C(A, E) = (CA, C) = (E, B)$$

$$\Longrightarrow$$

$$\begin{cases} CA = E \\ C = B \end{cases}$$

$$\Longrightarrow$$

$$BA = E$$

$$\Longrightarrow$$

$$B = A^{-1}$$

Example of Left Elementary Transformations

(1) Interchange two rows

$$\begin{pmatrix} O & O & 1 \\ O & 1 & O \\ 1 & O & O \end{pmatrix}$$

Interchange two rows

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$= \begin{pmatrix} g & h & i \\ d & e & f \\ a & b & c \end{pmatrix}$$

(2) Multiply a row by a non-zero constant

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \lambda \neq 0$$

Multiply a row by a non-zero constant

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$= \begin{pmatrix} \lambda a & \lambda b & \lambda c \\ d & e & f \\ g & h & i \end{pmatrix}$$

(3) Add a row by a multiplied another row

$$\begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Add a row by a multiplied another row

$$\begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$= \begin{pmatrix} a+\lambda d & b+\lambda e & c+\lambda f \\ d & e & f \\ g & h & i \end{pmatrix}$$

Examples

Example 1

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ -5 & 1 & 2 \end{pmatrix}$$

$$(A, E)$$

$$= \begin{pmatrix} 3 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -5 & 1 & 2 & 0 & 0 & 1 \end{pmatrix}$$

Matrix after Left Elementary Transformations

$$(E, A^{-1})$$

$$= \begin{pmatrix} 1 & 0 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 5 & -3 & 3 \end{pmatrix}$$

Inverse Matrix

$$A^{-1} = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 0 \\ 5 & -3 & 3 \end{pmatrix}$$

Example 2

$$A = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 \end{pmatrix}$$

$$(A, \textcolor{red}{E})$$
$$= \begin{pmatrix} 2 & 0 & 1 & 0 & \textcolor{red}{1} & \textcolor{red}{0} & \textcolor{red}{0} & \textcolor{red}{0} \\ 0 & -1 & 1 & -2 & \textcolor{red}{0} & \textcolor{red}{1} & \textcolor{red}{0} & \textcolor{red}{0} \\ 1 & 0 & 1 & 0 & \textcolor{red}{0} & \textcolor{red}{0} & \textcolor{red}{1} & \textcolor{red}{0} \\ 0 & 1 & -1 & 3 & \textcolor{red}{0} & \textcolor{red}{0} & \textcolor{red}{0} & \textcolor{red}{1} \end{pmatrix}$$

Matrix after Left Elementary Transformations

$$(E, A^{-1})$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & -3 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Inverse Matrix

$$A^{-1} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & -3 & 2 & -2 \\ -1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Computational Approach

Numerical Computing with BASIC

Example

$$A = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 \end{pmatrix}$$

1 行を 2 で割る

1	0	.5	0
0	−1	1	−2
1	0	1	0
0	1	−1	3

2 行から 1 行の 0 倍を引く
3 行から 1 行の 1 倍を引く
4 行から 1 行の 0 倍を引く

1	0	.5	0
0	−1	1	−2
0	0	.5	0
0	1	−1	3

2 行を−1 で割る

1	0	.5	0
0	1	−1	2
0	0	.5	0
0	1	−1	3

1 行から 2 行の 0 倍を引く
3 行から 2 行の 0 倍を引く
4 行から 2 行の 1 倍を引く

1	0	.5	0
0	1	−1	2
0	0	.5	0
0	0	0	1

3 行を .5 で割る

1	0	.5	0
0	1	−1	2
0	0	1	0
0	0	0	1

1 行から 3 行の .5 倍を引く
2 行から 3 行の−1 倍を引く
4 行から 3 行の 0 倍を引く

1	0	0	0
0	1	0	2
0	0	1	0
0	0	0	1

4 行を 1 で割る

1	0	0	0
0	1	0	2
0	0	1	0
0	0	0	1

1 行から 4 行の 0 倍を引く
2 行から 4 行の 2 倍を引く
3 行から 4 行の 0 倍を引く

B

1	0	−1	0
−1	−3	2	−2
−1	0	2	0
0	1	0	1

Inverse Matrix

$$B = \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & -3 & 2 & -2 \\ -1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

System of Linear Equations and Ranks

System of Linear Equations

$$ax + by = \alpha$$

$$cx + dy = \beta$$

Coefficient Matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \end{pmatrix}$$

Idea of Rank (1)

$$\begin{cases} ax + by = \alpha \\ cx + dy = \beta \end{cases}$$

$$\Longleftrightarrow$$

$$x \begin{pmatrix} a \\ c \end{pmatrix} + y \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Idea of Rank (2)

$$
\begin{cases}
ax + by = \alpha \\
cx + dy = \beta
\end{cases}
$$

$$\Longleftrightarrow$$

$$\text{rank } A = \text{rank } \tilde{A}$$

Rank of Matrices

Definition of Rank

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nm} \end{pmatrix}$$

$\Longrightarrow$

Left Elemntary Transformations

Matrix after Left Elementary Transformations (Echelon Form)

$$
\begin{pmatrix}
1 & O & \cdot & O & c_{1r+1} & \cdot\cdot & c_{1n} \\
O & 1 & \cdot & \cdot & c_{2r+1} & \cdot\cdot & c_{2n} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\
O & O & \cdot & 1 & c_{rr+1} & \cdot\cdot & c_{rn} \\
O & O & \cdot & \cdot & O & \cdot\cdot & O \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\
O & O & \cdot & \cdot & O & \cdot\cdot & O
\end{pmatrix}
$$

rank A = **Number of** 1

Geometrical Meaning of Rank

Rank of Matrices

Matrix Representation

Original Form

Placement of Lines and Planes

Example 1

$$A = \begin{pmatrix} 1 & 2 & -1 & -1 \\ 2 & 4 & -1 & -1 \\ 1 & 3 & 1 & 2 \end{pmatrix}$$

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

rank $A = 3$

Example 2

$$A = \begin{pmatrix} 0 & 3 & -2 & 3 & -4 \\ 1 & 1 & 3 & 2 & 2 \\ 1 & 2 & 2 & 3 & 1 \\ 1 & 3 & 2 & 4 & -1 \end{pmatrix}$$

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 7 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

rank $A = 3$

Computational Approach

Numerical Computing with BASIC

Example 1

$$A = \begin{pmatrix} 0 & 3 & -2 & 3 \\ 1 & 1 & 3 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

2 行と 1 行を入れ替える
 1 1 3 2
 0 3 -2 3
 1 2 2 3
 1 3 2 4

2 行を 1 倍し，1 行の 0 倍を引く
3 行を 1 倍し，1 行の 1 倍を引く
4 行を 1 倍し，1 行の 1 倍を引く
 1 1 3 2
 0 3 -2 3
 0 1 -1 1
 0 2 -1 2

3 行を 3 倍し，2 行の 1 倍を引く
4 行を 3 倍し，2 行の 2 倍を引く
 1 1 3 2
 0 3 -2 3
 0 0 -1 0
 0 0 1 0

4 行を-1 倍し，3 行の 1 倍を引く
 1 1 3 2
 0 3 -2 3
 0 0 -1 0
 0 0 0 0

Rank A = 3

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

rank $A = 3$

Example 2

$$B = \begin{pmatrix} 0 & 3 & -2 & 3 & -4 \\ 1 & 1 & 3 & 2 & 2 \\ 1 & 2 & 2 & 3 & 1 \\ 1 & 3 & 2 & 4 & -1 \end{pmatrix}$$

2 行と 1 行を入れ替える
1 1 3 2 2
0 3 -2 3 -4
1 2 2 3 1
1 3 2 4 -1

2 行を 1 倍し，1 行の 0 倍を引く
3 行を 1 倍し，1 行の 1 倍を引く
4 行を 1 倍し，1 行の 1 倍を引く
1 1 3 2 2
0 3 -2 3 -4
0 1 -1 1 -1
0 2 -1 2 -3

3 行を 3 倍し，2 行の 1 倍を引く
4 行を 3 倍し，2 行の 2 倍を引く
1 1 3 2 2
0 3 -2 3 -4
0 0 -1 0 1
0 0 1 0 -1

4 行を-1 倍し，3 行の 1 倍を引く
1 1 3 2 2
0 3 -2 3 -4
0 0 -1 0 1
0 0 0 0 0

Rank B = 3

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 7 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{rank } B = 3$$

Example 3

$$C = \begin{pmatrix} 1 & -2 & -3 & 4 \\ 2 & 3 & 1 & 1 \\ 3 & -4 & -7 & 10 \end{pmatrix}$$

2 行を 1 倍し，1 行の 2 倍を引く
3 行を 1 倍し，1 行の 3 倍を引く
1 -2 -3 4
0 7 7 -7
0 2 2 -2

3 行を 7 倍し，2 行の 2 倍を引く
1 -2 -3 4
0 7 7 -7
0 0 0 0

Rank C = 2

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{rank } C = 2$$

Example 4

$$D = \begin{pmatrix} 0 & 3 & -2 & 3 & -4 \\ 1 & 1 & 3 & 2 & 2 \\ 1 & 2 & 2 & 3 & 1 \\ 1 & 3 & 2 & 4 & -1 \end{pmatrix}$$

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 7 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

rank $D = 3$

System of Linear Equations and Geometry

Idea of Linear Algebra

System of Linear Equations

Matrix Representation

Original Form

Placement of Lines

Classification of Intersections

$\text{rank } A = \text{rank } \tilde{A} = 2$	**One-Point**
$\text{rank } A = 1 < \text{rank } \tilde{A} = 2$	**Parallel Two Lines**
$\text{rank } A = \text{rank } \tilde{A} = 1 < 2$	**Superposed Two Lines**

$$\text{rank } A \le \text{rank } \tilde{A} \le \text{rank } A + 1$$

Equation of a Line (1)

$$ax + by = c$$

Equation of a Line (2)

$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = c$$

inner product

One-Point Intersection

$$2x + 3y = 3$$

$$3x - 8y = 17$$

One-Point Intersection
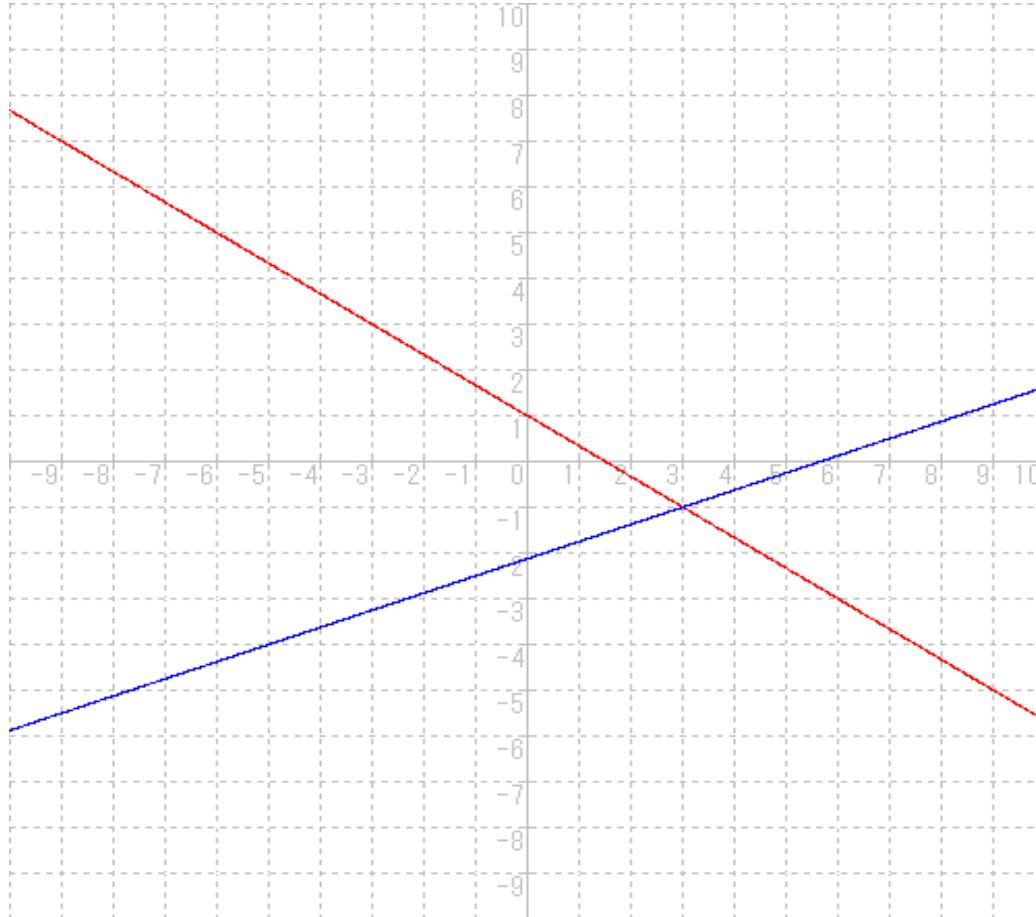
$$\text{rank } A = \text{rank } \tilde{A} = 2$$

Coefficient Matrix

$$A = \begin{pmatrix} 2 & 3 \\ 3 & -8 \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} 2 & 3 & 3 \\ 3 & -8 & 17 \end{pmatrix}$$

Unique Solution

$$\widetilde{A} = \begin{pmatrix} 2 & 3 & 3 \\ 3 & -8 & 17 \end{pmatrix}$$

$$\Longrightarrow$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\text{rank } A = \text{rank } \tilde{A} = 2$$

Parallel Two Lines

$$x + 2y = 2$$
$$x + 2y = 3$$

Parallel Two Lines
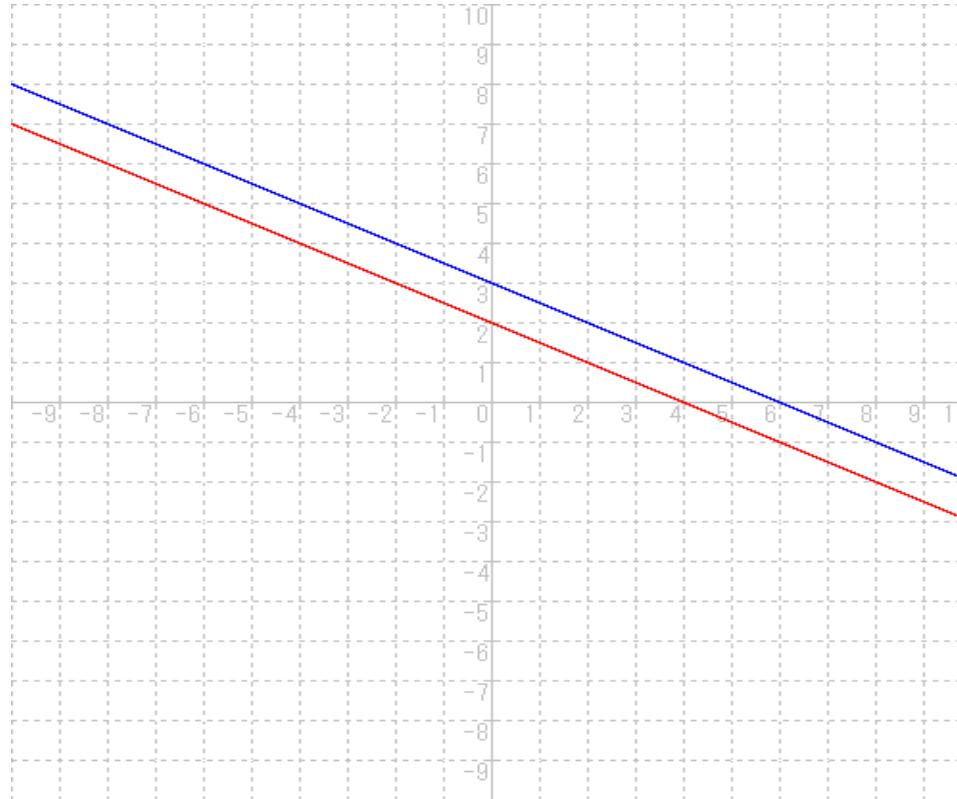
$$\operatorname{rank} A = 1 < \operatorname{rank} \tilde{A} = 2$$

Coefficient Matrix

$$A = \begin{pmatrix} 12 \\ 12 \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

No Solution

$$\widetilde{A} = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\Longrightarrow$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \textbf{(Impossible)}$$

$$\text{rank } A = 1 < \text{rank } \tilde{A} = 2$$

Superposed Two Lines

$$6x - 2y = -8$$
$$3x - y = -4$$

Superposed Two Lines
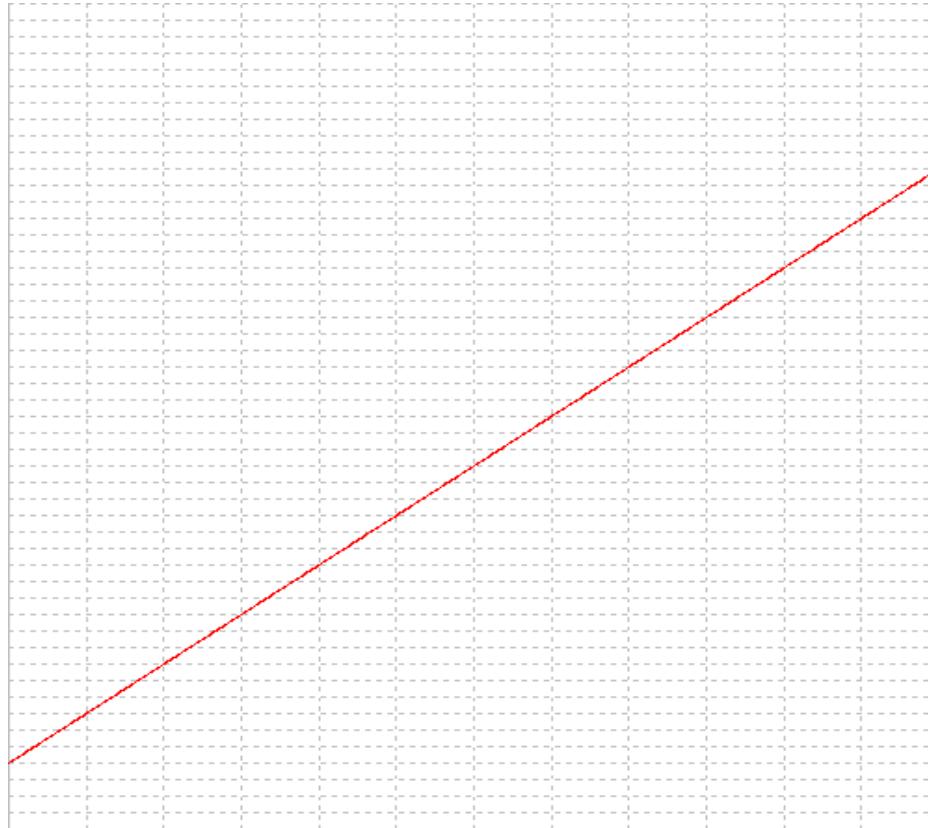
$$\text{rank } A = \text{rank } \tilde{A} = 1 < 2$$

Coefficient Matrix

$$A = \begin{pmatrix} 6 & -2 \\ 3 & -1 \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} 6 & -2 & -8 \\ 3 & -1 & -4 \end{pmatrix}$$

Many Solutions

$$\widetilde{A} = \begin{pmatrix} 6 & -2 & -8 \\ 3 & -1 & -4 \end{pmatrix}$$

$$\Longrightarrow$$

$$\begin{pmatrix} 1 & -1/3 & -4/3 \\ 0 & 0 & 0 \end{pmatrix}$$

(Indefinite)

$$\text{rank } A = \text{rank } \tilde{A} = 1 < 2$$

Geometrical Meaning of Rank

System of Linear Equations

$$a_1 x + b_1 y + c_1 z = \alpha$$

$$a_2 x + b_2 y + c_2 z = \beta$$

$$a_3 x + b_3 y + c_3 z = \gamma$$

Equation of a Plane (1)

$$ax + by + cz = d$$

Equation of a Plane (2)

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = d$$

inner product

Plane

Geometrical Meaning of Rank

Rank of Matrices

Matrix Representation

Original Form

Placement of Planes

Classification of Intersections

rank $A = $ rank $\tilde{A} = 3$	**One-Point**
rank $A = $ rank $\tilde{A} = 2 < 3$	**One Line**
rank $A = 2 < $ rank $\tilde{A} = 3$	**Parallel Two Lines** **Parallel Three Lines**
rank $A = $ rank $\tilde{A} = 1 < 3$	**Superposed Three Planes**
rank $A = 1 < $ rank $\tilde{A} = 2$	**Parallel Two Planes** **Parallel Three Planes**

$$\text{rank } A \leq \text{rank } \tilde{A} \leq \text{rank } A + 1$$

Classification of Intersections

rank A = rank $\tilde{A}$ = 3	**One-Point**
rank A = rank $\tilde{A}$ = 2 < 3	**One Line**
rank A = 2 < rank $\tilde{A}$ = 3	**Parallel Two Lines** **Parallel Three Lines**
rank A = rank $\tilde{A}$ = 1 < 3	**Superposed Three Planes**
rank A = 1 < rank $\tilde{A}$ = 2	**Parallel Two Planes** **Parallel Three Planes**

$$\text{rank } A \leq \text{rank } \tilde{A} \leq \text{rank } A + 1$$

One-Point Intersection

$$2x + 3y - z = -3$$
$$-x + 2y + 2z = 1$$
$$x + y - z = -2$$

$$\text{rank } A = \text{rank } \tilde{A} = 3$$

One-Point Intersection
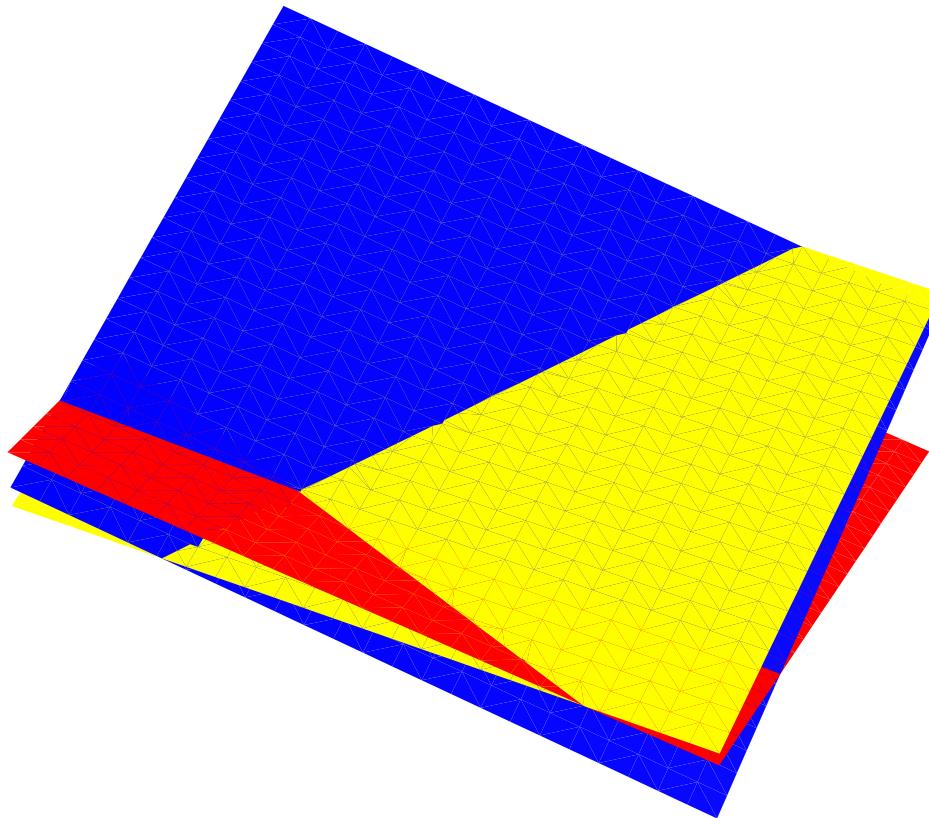
$$x = 1, \, y = -1, \, z = 2$$

One-Line Intersection

$$x - 2y - 3z = 4$$
$$2x + 3y + z = 1$$
$$3x - 4y - 7z = 10$$

$$\text{rank } A = \text{rank } \tilde{A} = 2 < 3$$

One-Line Intersection
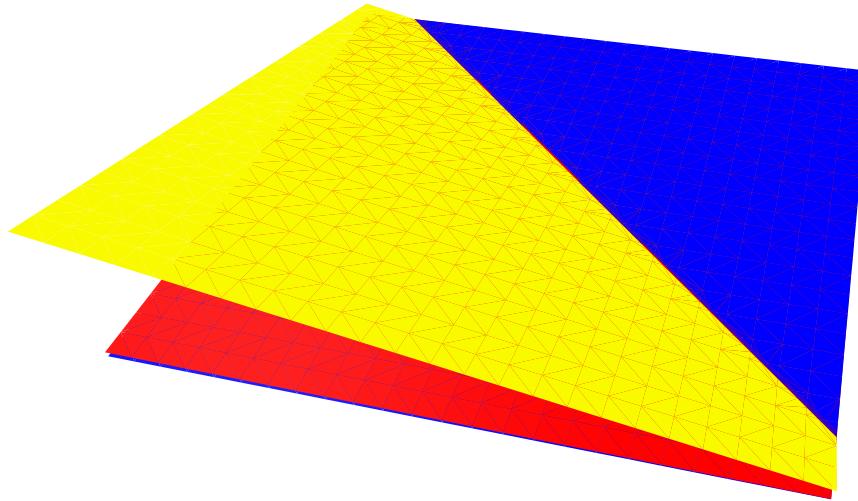
$$x = 2 + t, \, y = -1 - t, \, z = t$$

Parallel Two-Lines Intersection

$$x - 2y - 3z = 4$$
$$2x + 3y + z = 4$$
$$3x - 4y - 7z = 10$$

$$\text{rank } A = 2 < \text{rank } \tilde{A} = 3$$
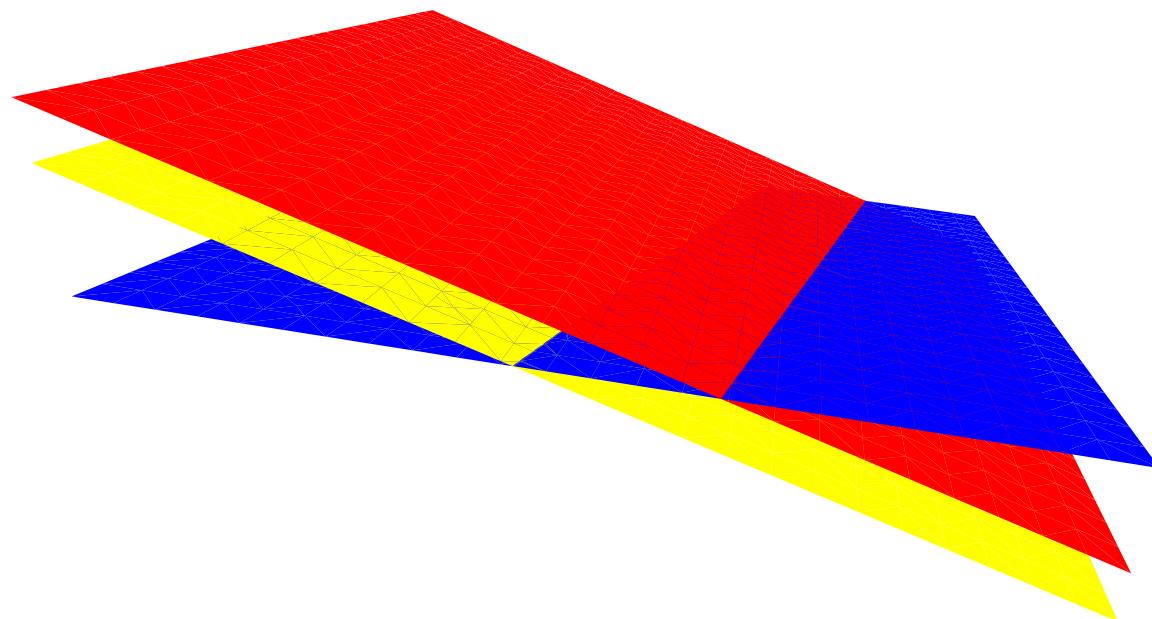
Parallel Two-Lines Intersection

Parallel Two Planes

$$x - y + 3z = 1$$
$$3x - 3y + 9z = 3$$
$$x - y + 3z = 0$$

$$\text{rank } A = 1 < \text{rank } \tilde{A} = 2$$

Parallel Two Planes
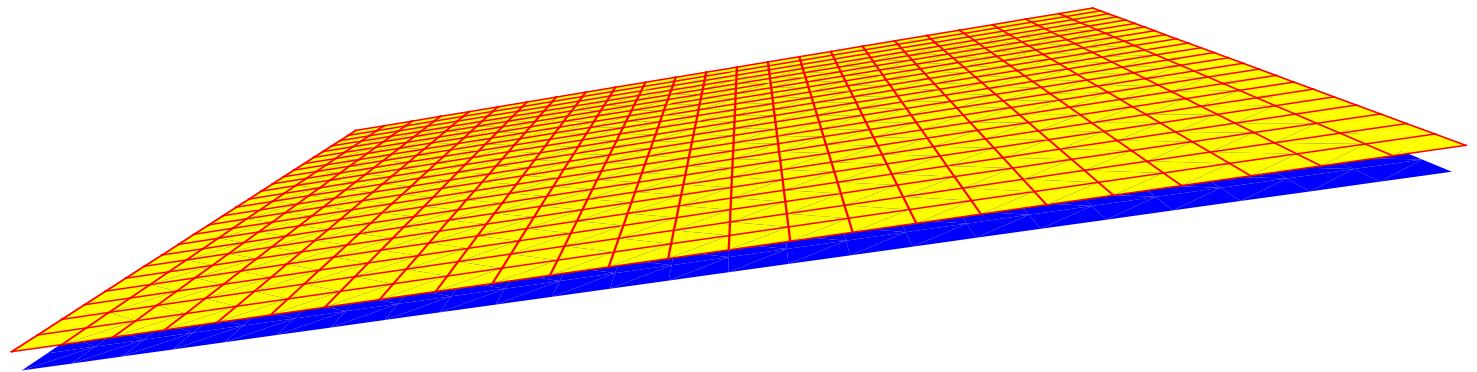
Parallel Three Planes

$$x + 2y + 3z = 10$$
$$x + 2y + 3z = 20$$
$$x + 2y + 3z = 30$$

$$\text{rank } A = 1 < \text{rank } \tilde{A} = 2$$

Parallel Three Planes
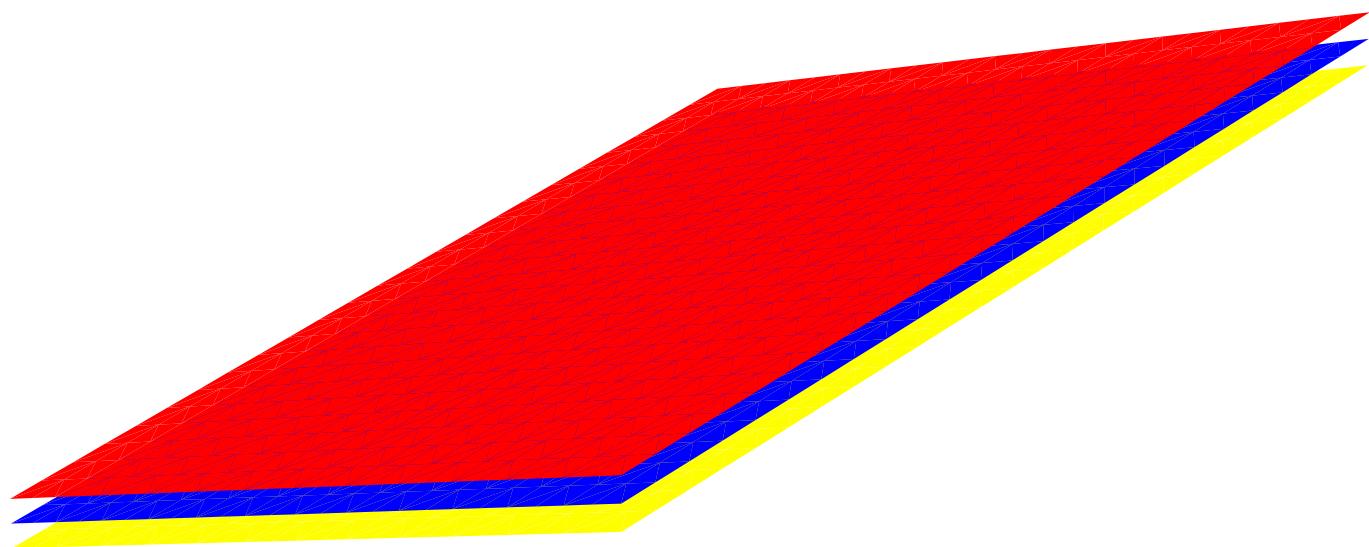
Superposed Three Planes

$$x + 2y + 3z = 20$$
$$2x + 4y + 6z = 40$$
$$3x + 6y + 9z = 60$$

$$\text{rank } A = \text{rank } \tilde{A} = 1 < 3$$

Superposed Three Planes
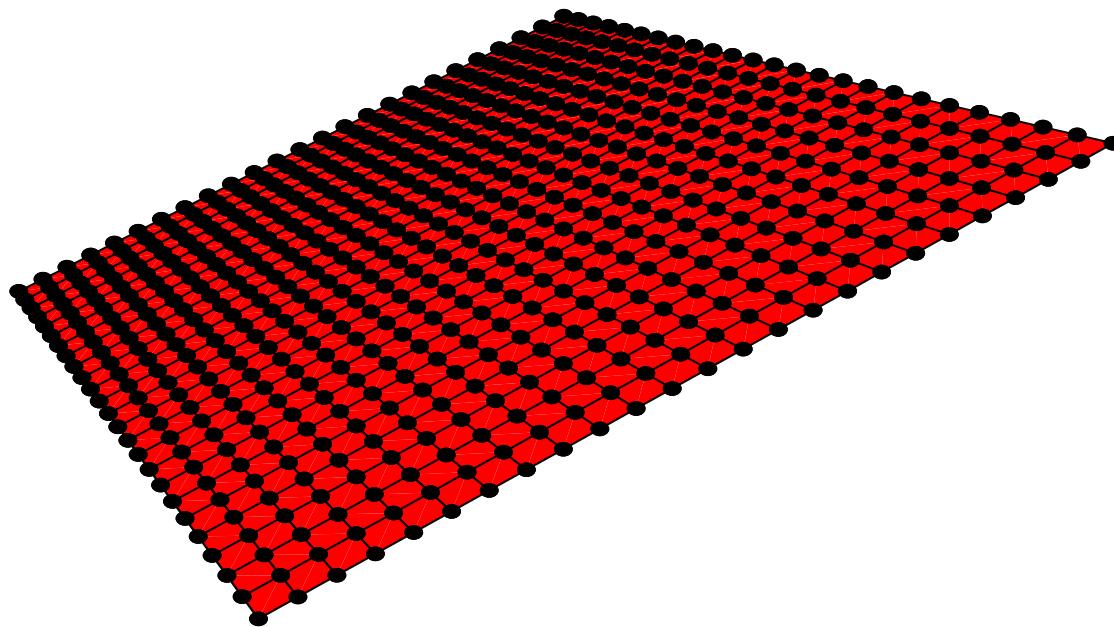
System of Linear Equations

General Form

$$\sum_{j=1}^{n} a_{ij} x_j = b_i$$

Direct Solution

$$3x_2 - 2x_3 + 3x_4 = -4$$
$$x_1 + x_2 + 3x_3 + 2x_4 = 2$$
$$x_1 + 2x_2 + 2x_3 + 3x_4 = 1$$
$$x_1 + 3x_2 + 2x_3 + 4x_4 = -1$$

Transformation of Equations

$$x_1 + x_4 = 7$$
$$x_2 + x_4 = -2$$
$$x_3 = -1$$
$$x_4 = \alpha \quad \text{(Indefinite)}$$

Gaussian Sweeping Out

Original Form

$$\sum_{j=1}^{n} a_{ij} x_j = b_i$$

Idea of Gauss

$$A\mathbf{x} = \mathbf{b}$$

$$J\mathbf{x} = \mathbf{c}$$

Matrix Representation

Original Form

$$\tilde{A} = (A, \mathbf{b}) \qquad \Longrightarrow \qquad \tilde{J} = (J, \mathbf{c})$$

Left Elemntary Transformations

Matrix Representation (1)

$$\begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \end{pmatrix}$$

Matrix Representation (2)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{pmatrix}$$

Matrix Representation (3)

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \end{pmatrix}$$

Matrix Representation (4)

$$\sum_{j=1}^{n} a_{ij} x_j = b_i$$

$$\Downarrow$$

$$A\mathbf{x} = \mathbf{b}$$

Coefficient Matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} & b_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} & b_n \end{pmatrix}$$

Left Elementary Transformations

(1) Interchange two rows
(2) Multiply a row by a non-zero constant
(3) Add a row by a multiplied another row

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & \cdot & 0 & c_{1r+1} & \cdots & c_{1n} & d_1 \\ 0 & 1 & \cdot & \cdot & c_{2r+1} & \cdots & c_{2n} & d_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdot & 1 & c_{rr+1} & \cdots & c_{rn} & d_r \\ 0 & 0 & \cdot & \cdot & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Equation after Left Elementary Transformations

$$\begin{cases} x_1 + c_{1r+1}x_{r+1} + \cdots + c_{1n}x_n = d_1 \\ x_2 + c_{2r+1}x_{r+1} + \cdots + c_{2n}x_n = d_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_r + c_{rr+1}x_{r+1} + \cdots + c_{rn}x_n = d_r \\ 0x_1 + \cdots + 0x_r + 0x_{r+1} + \cdots + 0x_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0x_1 + \cdots + 0x_r + 0x_{r+1} + \cdots + 0x_n = 0 \end{cases}$$

Examples

Example (n=3)

$$x_1 + 2x_2 - x_3 = -1$$
$$2x_1 + 4x_2 - x_3 = -1$$
$$x_1 + 3x_2 + x_3 = 2$$

Coefficient Matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -1 \\ 1 & 3 & 1 \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} 1 & 2 & -1 & -1 \\ 2 & 4 & -1 & -1 \\ 1 & 3 & 1 & 2 \end{pmatrix}$$

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\text{rank } A = \text{rank } \tilde{A} = 3$$

Equation after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

Unique Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

Example (n=4)

$$3x_2 - 2x_3 + 3x_4 = -4$$
$$x_1 + x_2 + 3x_3 + 2x_4 = 2$$
$$x_1 + 2x_2 + 2x_3 + 3x_4 = 1$$
$$x_1 + 3x_2 + 2x_3 + 4x_4 = -1$$

General Solution

$$x_1 + x_4 = 7$$

$$x_2 + x_4 = -2$$

$$x_3 = -1$$

$$x_4 = \alpha \quad \text{(Indefinite)}$$

Coefficient Matrix

$$A = \begin{pmatrix} 0 & 3 & -2 & 3 \\ 1 & 1 & 3 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} 0 & 3 & -2 & 3 & -4 \\ 1 & 1 & 3 & 2 & 2 \\ 1 & 2 & 2 & 3 & 1 \\ 1 & 3 & 2 & 4 & -1 \end{pmatrix}$$

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 7 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{rank } A = \text{rank } \tilde{A} = 3 < 4$$

Equation after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \\ -1 \\ 0 \end{pmatrix}$$

$$x_1 + x_4 = 7$$

$$x_2 + x_4 = -2$$

$$x_3 = -1$$

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 0$$

General Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \\ -1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Recurrence Formula for Sequences

Matrix Form

$$x_n = 4x_{n-1} + 10y_{n-1}, \quad x_0 = 3$$

$$y_n = -3x_{n-1} - 7y_{n-1}, \quad y_0 = 1$$

$$\Longrightarrow$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 4 & 10 \\ -3 & -7 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}$$

System of Differential Equations

Linear Case

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\[2em] \dfrac{dy}{dt} = cx + dy \end{cases}$$

Matrix Form

$$U(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Rightarrow$$

$$\frac{dU}{dt} = A\,U(t)$$

Example

$$\frac{dx}{dt} = 4x + 10y, \quad x(0) = 3$$

$$\frac{dy}{dt} = -3x - 7y, \quad y(0) = 1$$

Matrix Form

$$U(t) := \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A := \begin{pmatrix} 4 & 10 \\ -3 & -7 \end{pmatrix}$$

$$\Rightarrow$$

$$\frac{dU}{dt} = A\,U(t)$$

Diagonalization

$$A = \begin{pmatrix} 4 & 10 \\ -3 & -7 \end{pmatrix}, P = \begin{pmatrix} -2 & -5 \\ 1 & 3 \end{pmatrix}$$

$$\Rightarrow$$

$$P^{-1}AP = \Lambda = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$

Reduction (1)

$$V(t) = \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}$$

$$= P^{-1}U(t)$$

$$= \begin{pmatrix} -3 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Reduction (2)

$$\frac{dV}{dt} = P^{-1}\frac{dU}{dt} = P^{-1}AU(t)$$

$$= \left(P^{-1}AP\right)V(t) = \Lambda V(t)$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} V(t)$$

Reduction (3)

$$\begin{cases} \dfrac{dz}{dt} = -z \\[2mm] \dfrac{dw}{dt} = -2w \end{cases}$$

$$\begin{pmatrix} z(0) \\ w(0) \end{pmatrix} = P^{-1} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} -3 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -14 \\ 5 \end{pmatrix}$$

Reduction (4)

$$\begin{cases} \dfrac{dz}{dt} = -z, & z(0) = -14 \\[2em] \dfrac{dw}{dt} = -2w, & w(0) = 5 \end{cases}$$

$$\Rightarrow$$

$$\begin{pmatrix} z(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} -14e^{-t} \\ 5e^{-2t} \end{pmatrix}$$

Solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = P \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -14e^{-t} \\ 5e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} 28e^{-t} - 25e^{-2t} \\ -14e^{-t} + 15e^{-2t} \end{pmatrix}$$

Stable Node
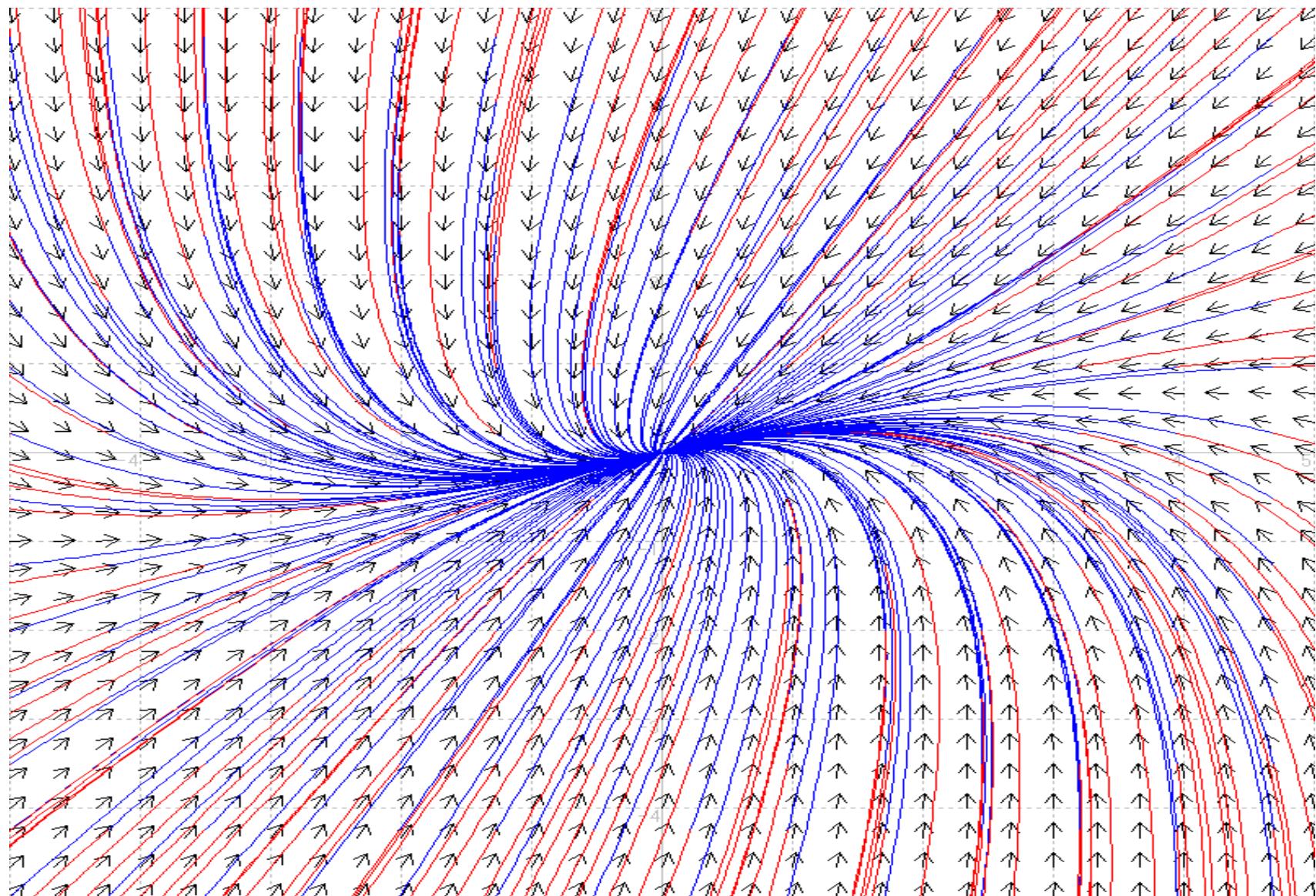
Canonical Forms
of
Quadratic Forms

Quadratic Form of Two Variables

$$z = f(x, y)$$
$$= ax^2 + 2bxy + cy^2$$

Matrix Form

$$z = f(x, y)$$

$$= ax^2 + 2bxy + cy^2$$

$$\Longrightarrow$$

$$ax^2 + 2bxy + cy^2$$

$$= \left\langle \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle$$

Example 1

$$z = f(x, y)$$

$$= 3x^2 - 2xy + 3y^2$$

$$\Longrightarrow$$

$$3x^2 - 2xy + 3y^2$$

$$= \left\langle \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

Eigenvalues : 2, 4

$$z = x^2 + 2y^2 \quad \text{（ellipse）}$$

Example 2

$$z = f(x, y)$$

$$= x^2 - 6xy + y^2$$

$$\implies$$
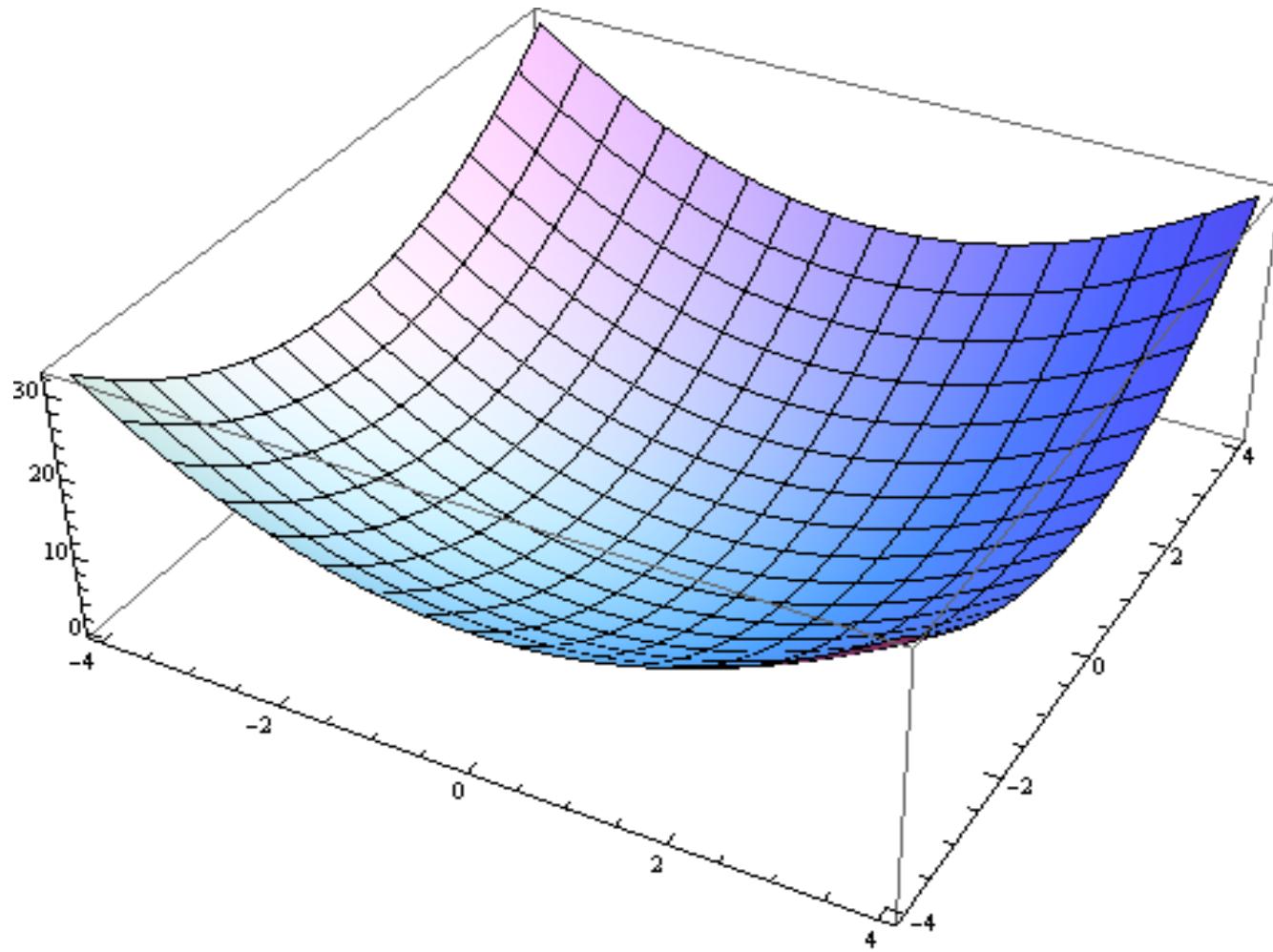
$$x^2 - 6xy + y^2$$

$$= \left\langle \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}$$

Eigenvalues : - 2, 4

$$z = -2x^2 + 4y^2 \quad \text{(hyperbola)}$$
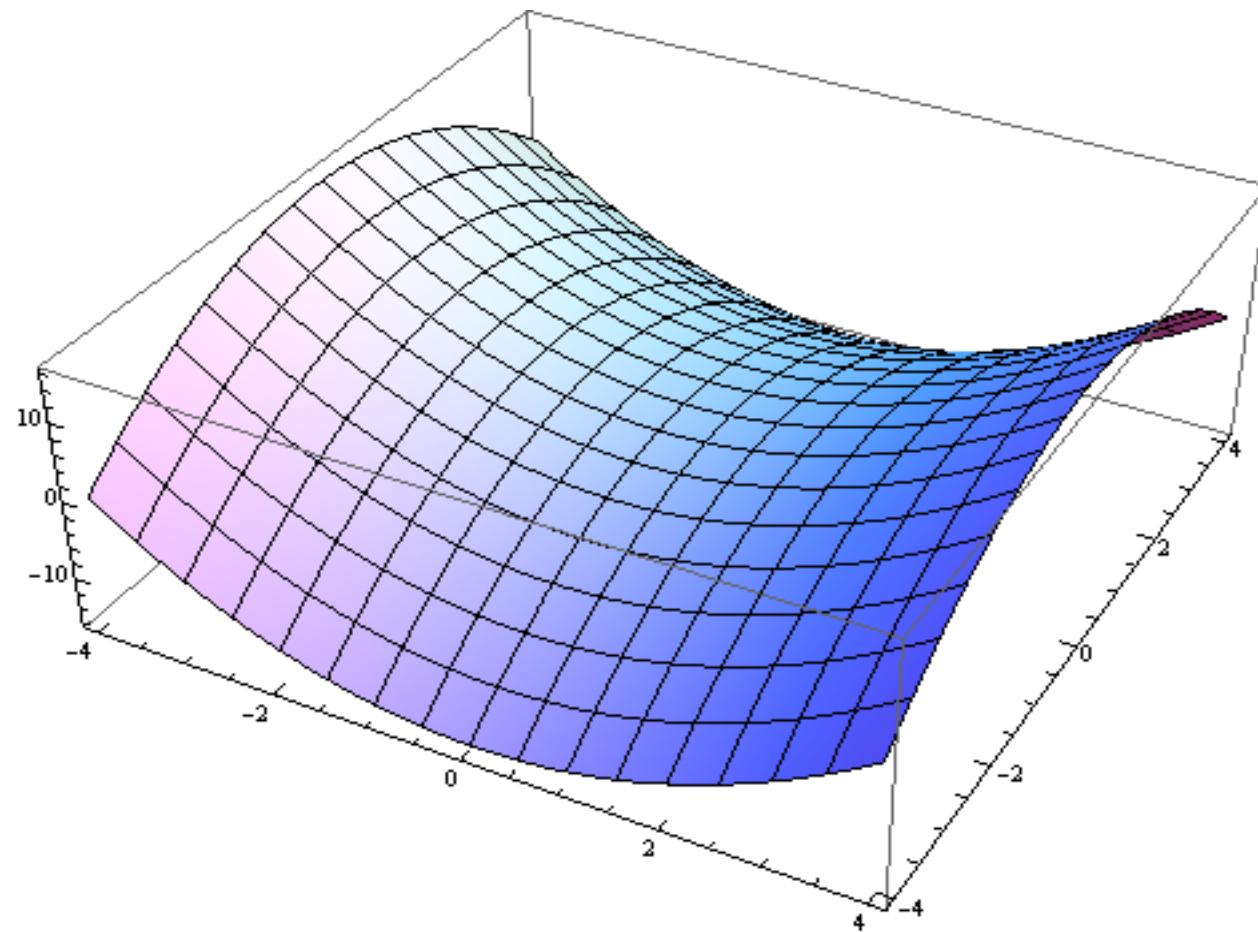
Example 3

$$z = f(x, y)$$

$$= 4x^2 - 4xy + y^2$$

$$\Longrightarrow$$

$$4x^2 - 4xy + y^2$$

$$= \left\langle \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$$

Eigenvalues : 0, 5

$$z = 5y^2 + \sqrt{5}\,x \quad \text{(parabola)}$$
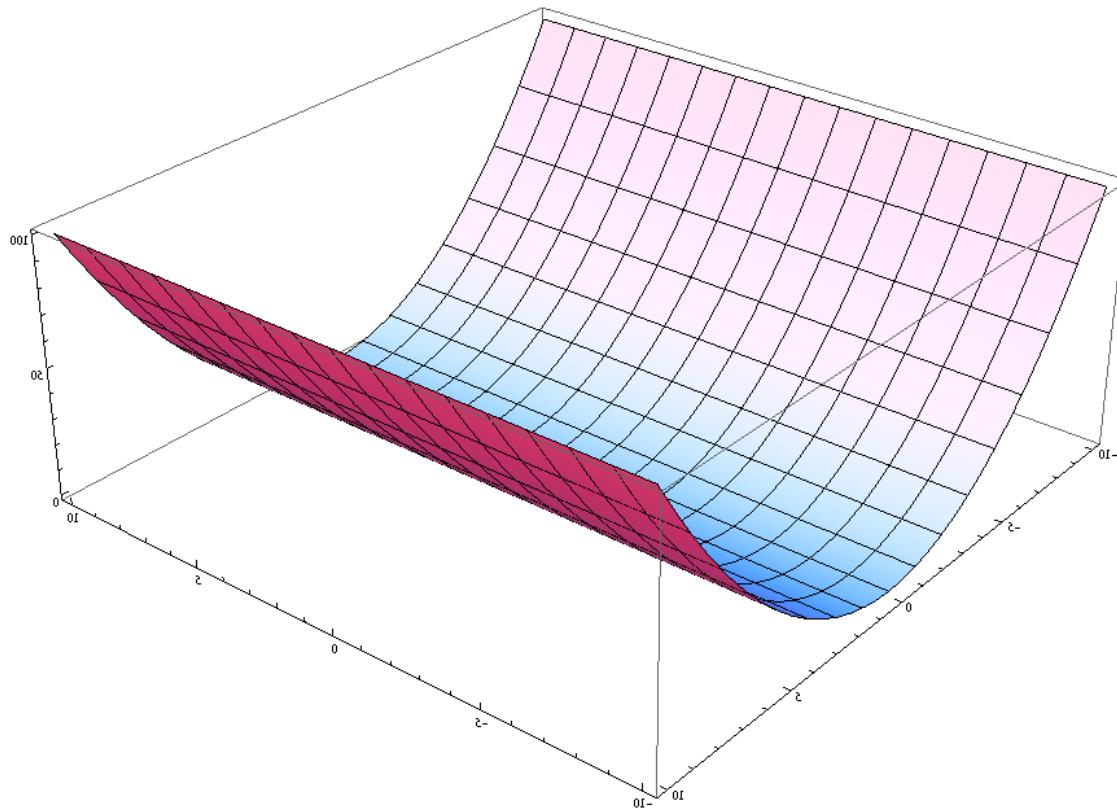
Linea Algebra
and
Differential Equations

2-dimensional Autonomous System

Linear Case

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\[2em] \dfrac{dy}{dt} = cx + dy \end{cases}$$

Matrix Form

$$U(t) := \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Longrightarrow$$

$$\frac{d}{dt} U(t) = A \, U(t)$$

Stability
of
Solutions

Computational Approach

Numerical Computing with BASIC

Example 1 (Unstable Node)

$$\begin{cases} \dfrac{dx}{dt} = 2x \\[2mm] \dfrac{dy}{dt} = y \end{cases}$$

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Eigenvalues : 2, 1

Unstable Node
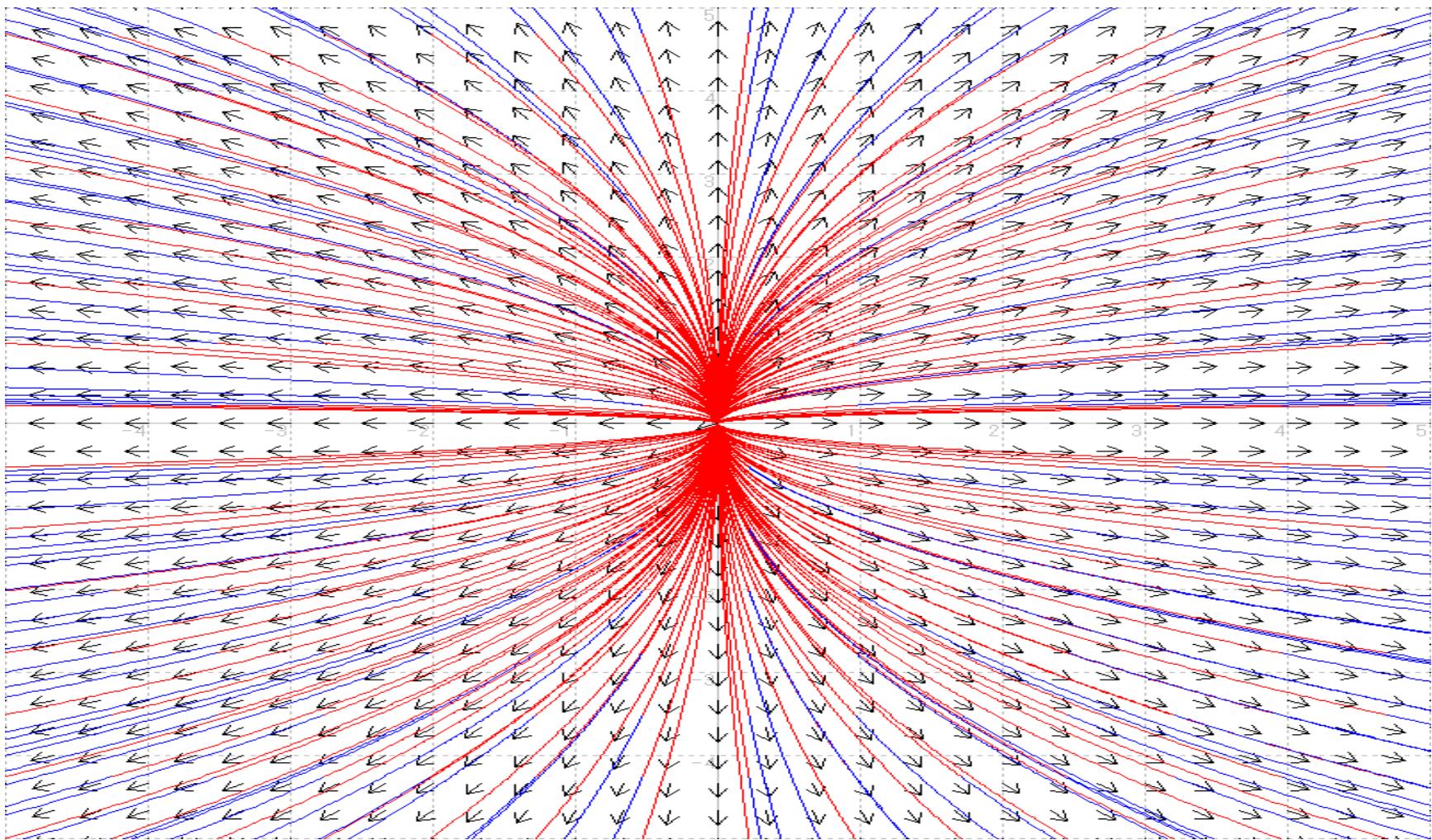
Example 2 (Saddle Point)

$$\begin{cases} \dfrac{dx}{dt} = x \\[2em] \dfrac{dy}{dt} = -y \end{cases}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Eigenvalues : 1, -1

Saddle Point
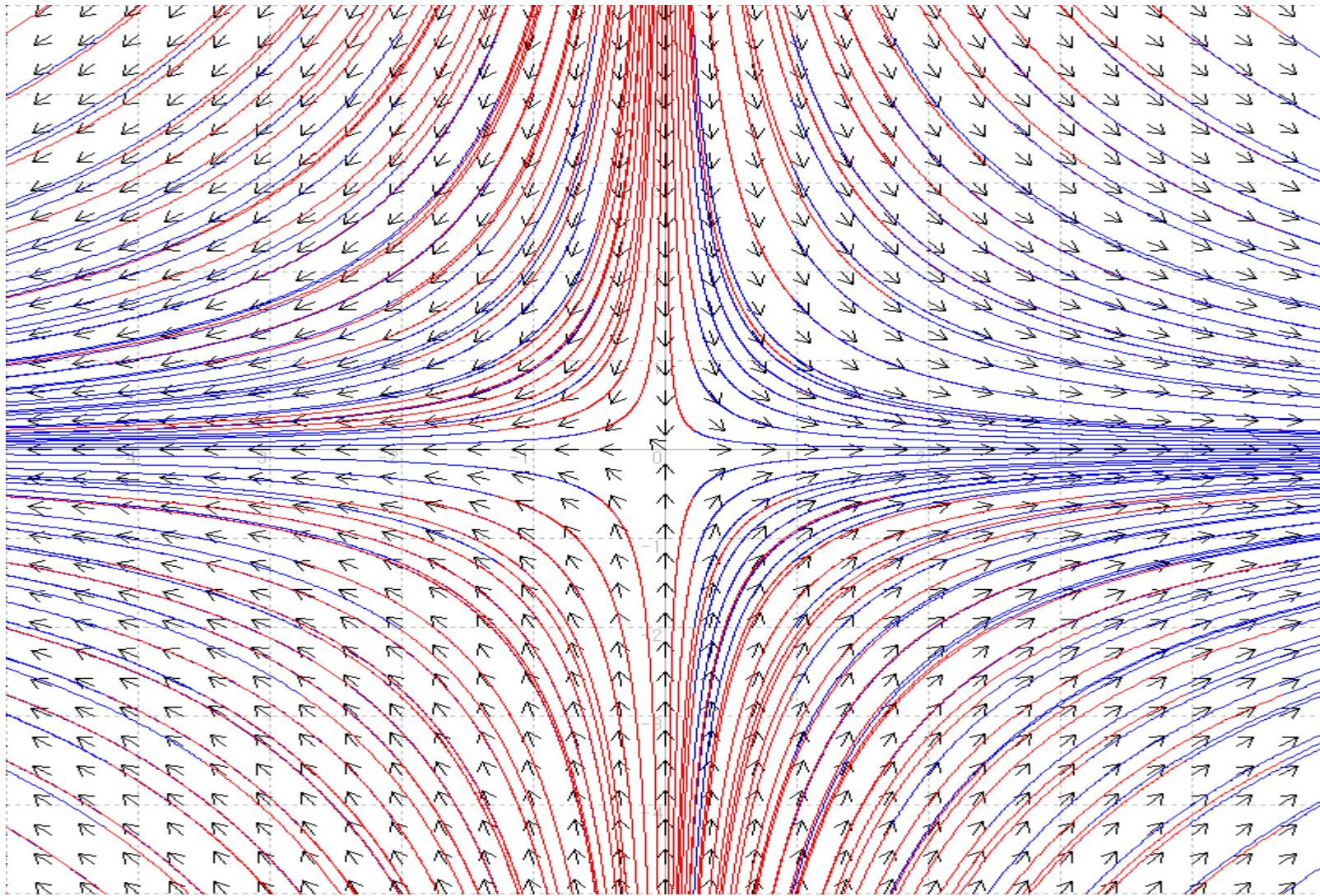
Example 3 (Unstable Node)

$$\begin{cases} \dfrac{dx}{dt} = x \\[3mm] \dfrac{dy}{dt} = 3x + 2y \end{cases}$$

$$A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$$

**Eigenvalues : 1, 2**

Unstable Node
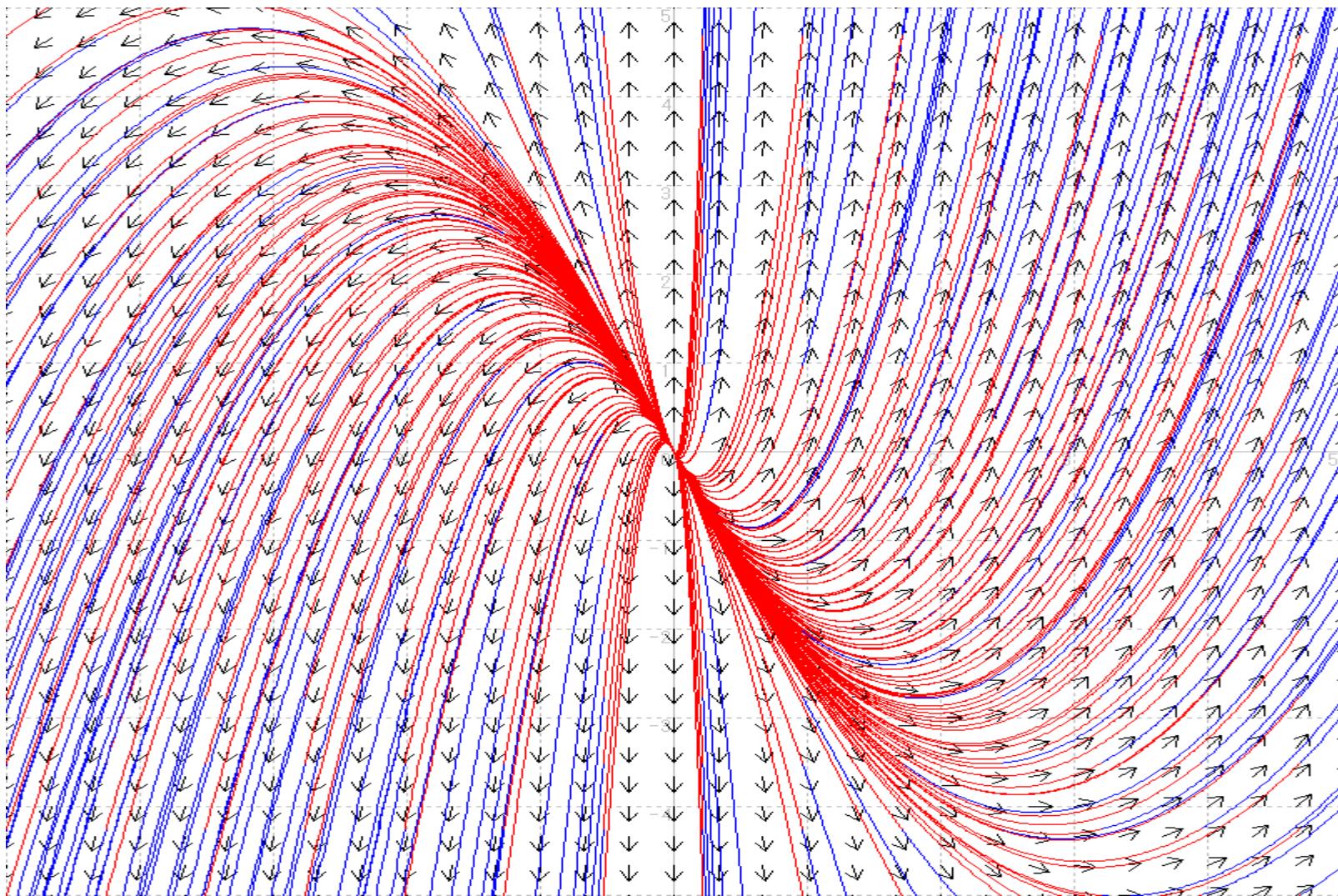
Example 4 (Stable Node)

$$\begin{cases} \dfrac{dx}{dt} = -2x - 1.5\,y \\[4mm] \dfrac{dy}{dt} = x - 5.5\,y \end{cases}$$

$$A = \begin{pmatrix} -2 & -1.5 \\ 1 & -5.5 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} -2 & -1.5 \\ 1 & -5.5 \end{pmatrix}$$

Eigenvalues : $-2.5, \ -5$

Stable Node
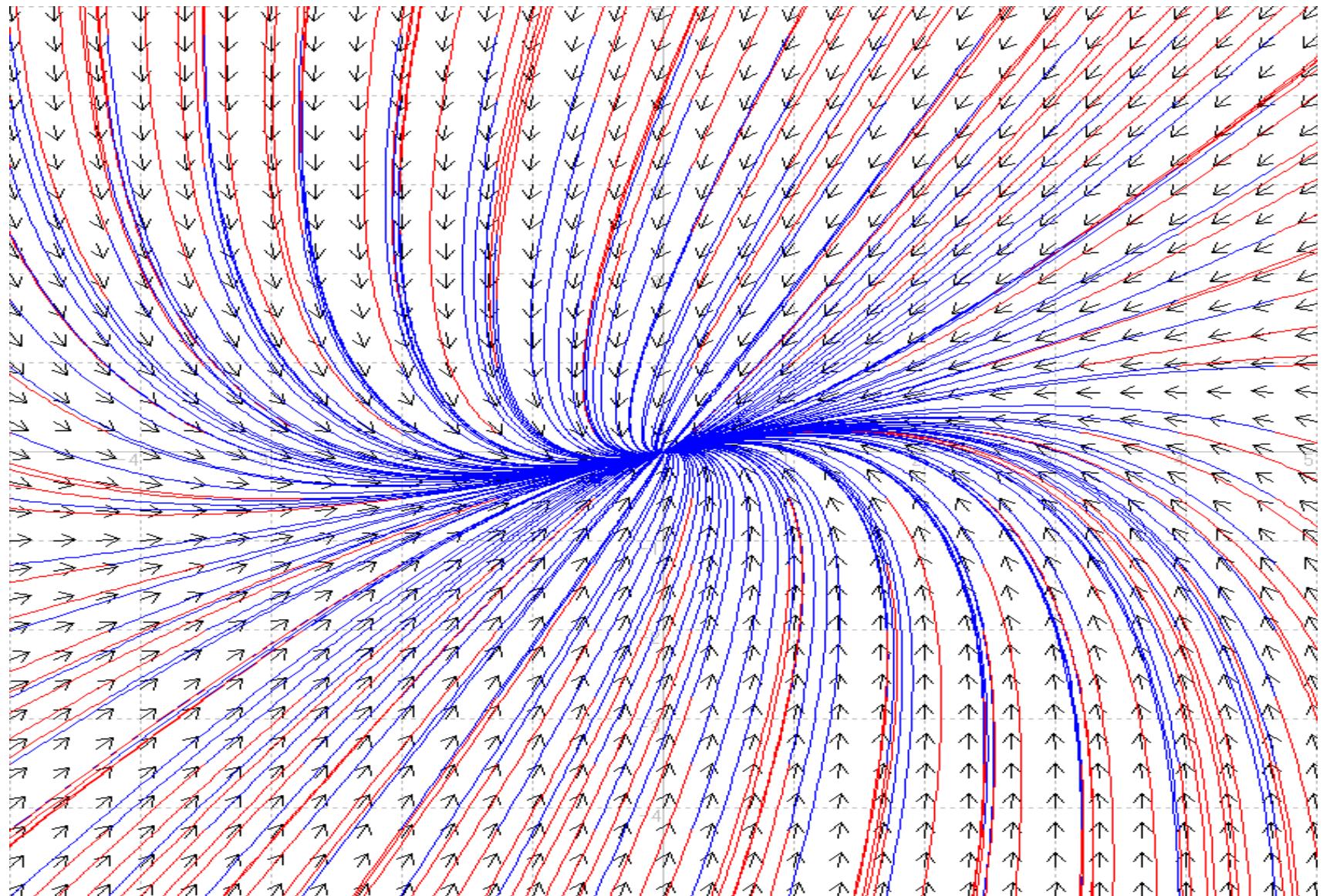
Example 5 (Saddle Point)

$$\begin{cases} \dfrac{dx}{dt} = -2x + 2y \\[2mm] \dfrac{dy}{dt} = -2x + 3y \end{cases}$$

$$A = \begin{pmatrix} -2 & 2 \\ -2 & 3 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} -2 & 2 \\ -2 & 3 \end{pmatrix}$$

Eigenvalues : $2, \quad -1$

Saddle Point
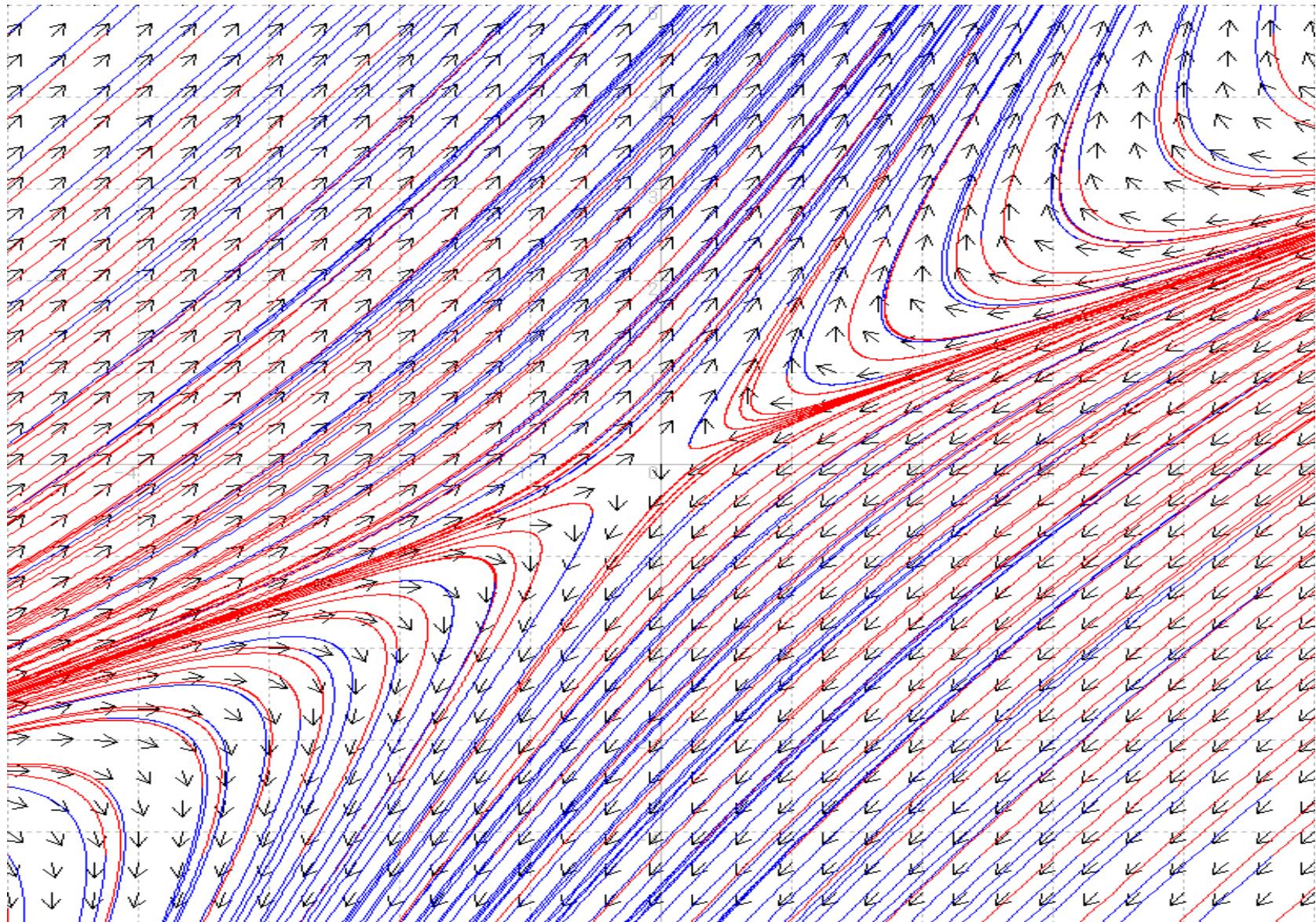
Example 6 (Unstable Node)

$$\begin{cases} \dfrac{dx}{dt} = 2x + y \\[2em] \dfrac{dy}{dt} = 2y \end{cases}$$

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Eigenvalues: 2, 2

Unstable Node
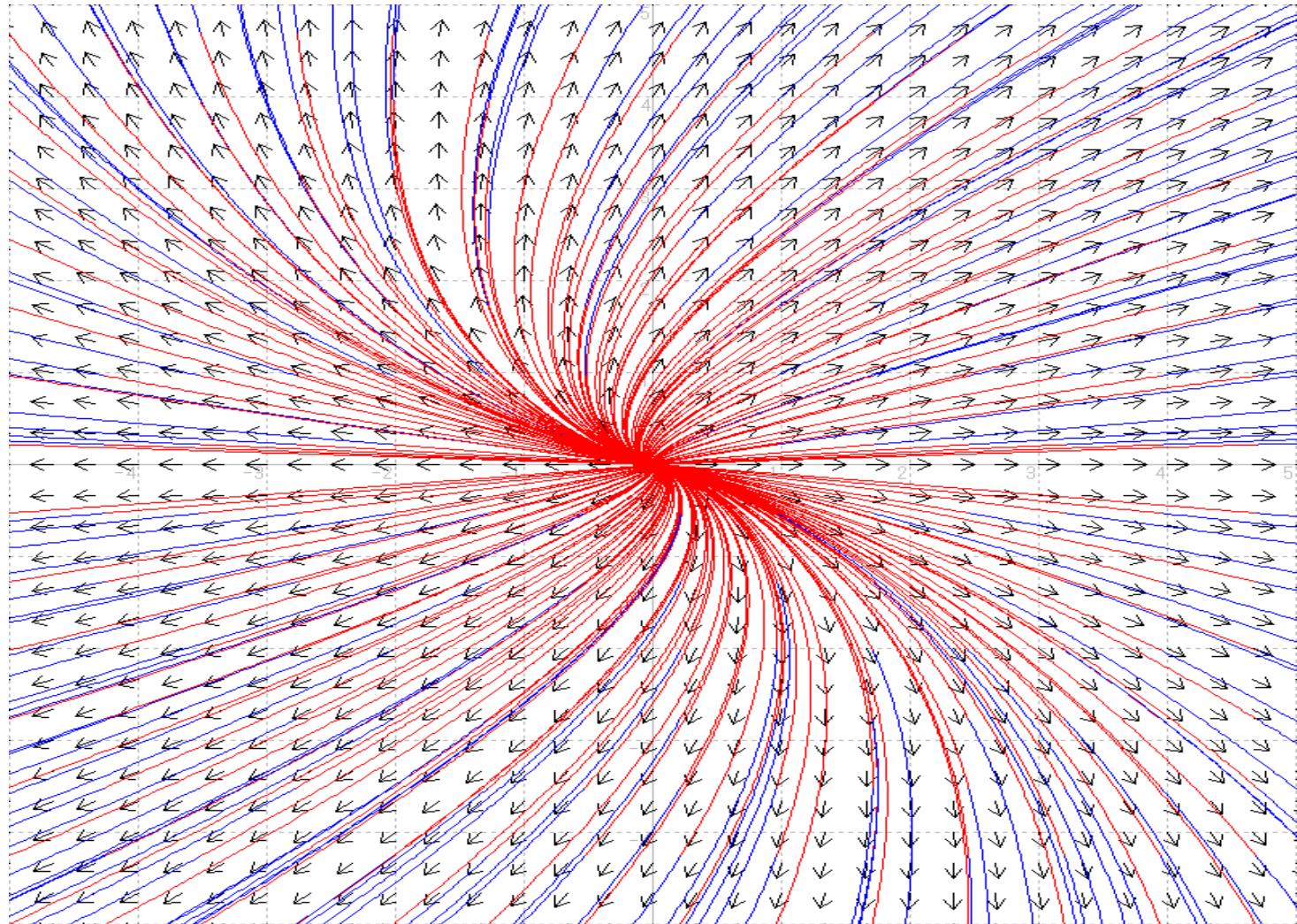
Example 7 (Center)

$$\begin{cases} \dfrac{dx}{dt} = x + 2\,y \\[2mm] \dfrac{dy}{dt} = -x - y \end{cases}$$

$$A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

Eigenvalues: $\sqrt{-1}, \ -\sqrt{-1}$

Center
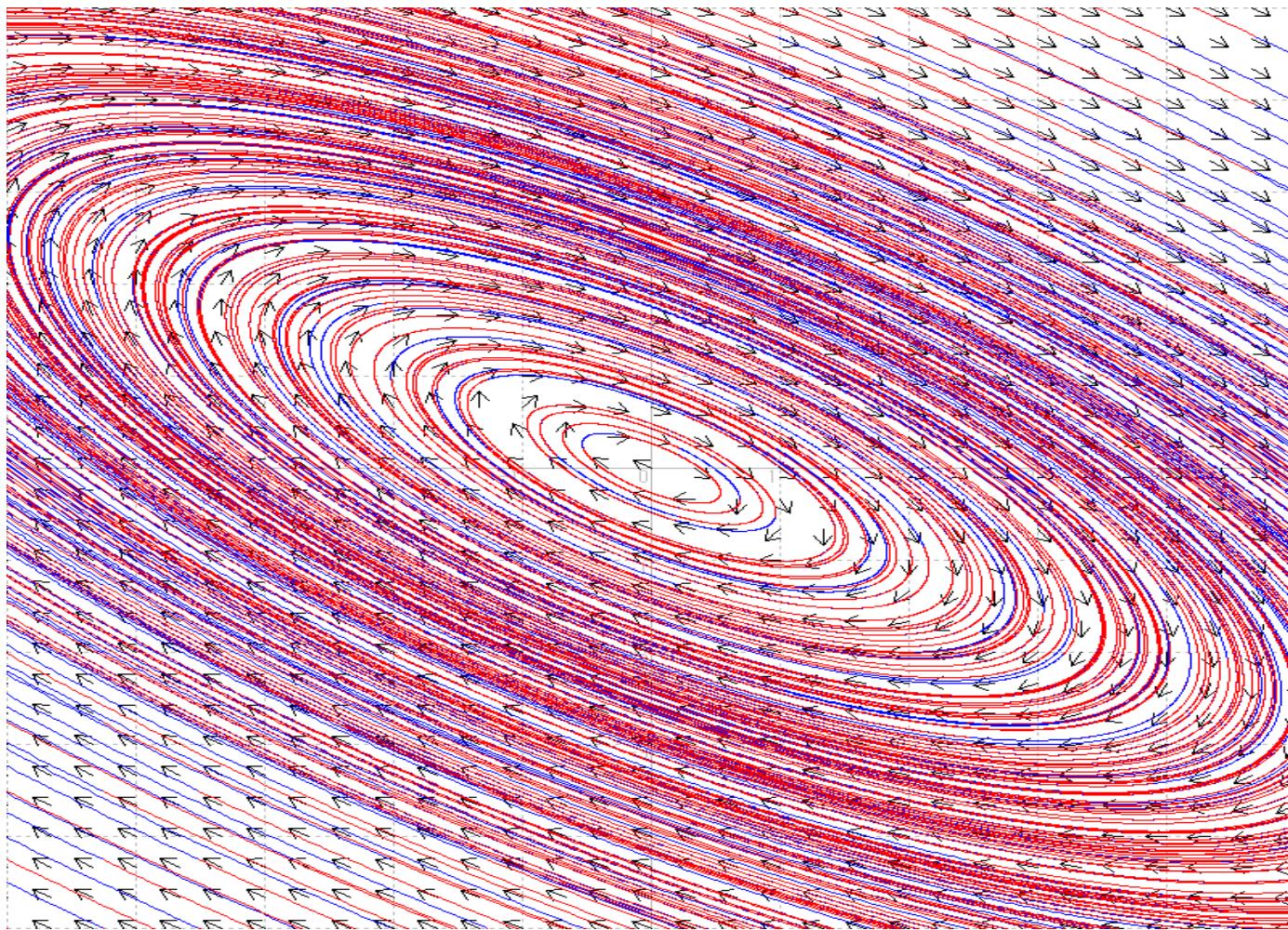
Example 8 (Unstable Focus)

$$\begin{cases} \dfrac{dx}{dt} = x + y \\[2mm] \dfrac{dy}{dt} = -2x + y \end{cases}$$

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

Eigenvalues : $1 + \sqrt{2}i$, $1 - \sqrt{2}i$

Unstable Node
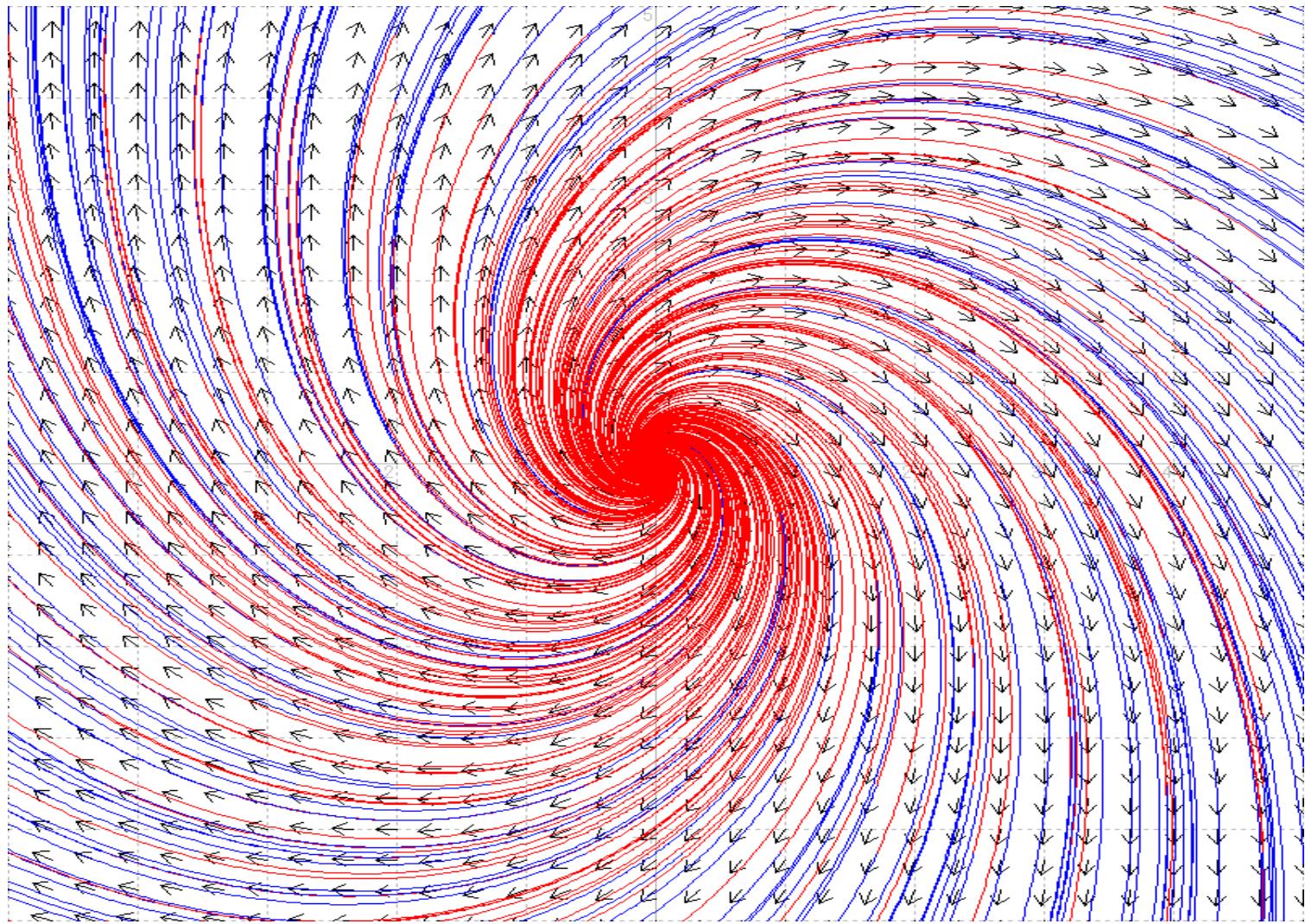
Example 9 (Degenerate Node)

$$\begin{cases} \dfrac{dx}{dt} = 2x + 2y \\[4mm] \dfrac{dy}{dt} = 3x + 3y \end{cases}$$

$$A = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$$

Eigenvalues : $0, \ 5$

Degenerate Node

Rank of Matrices Revisited

Definition of Rank

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nm} \end{pmatrix}$$

$\Longrightarrow$

Left Elemntary Transformations

Matrix after Left Elementary Transformations

$$
\begin{pmatrix}
1 & O & \cdot & O & c_{1r+1} & \cdots & c_{1n} \\
O & 1 & \cdot & \cdot & c_{2r+1} & \cdots & c_{2n} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\
O & O & \cdot & 1 & c_{rr+1} & \cdots & c_{rn} \\
O & O & \cdot & \cdot & O & \cdots & O \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\
O & O & \cdot & \cdot & O & \cdots & O
\end{pmatrix}
$$

rank A = **Number of** 1

Geometrical Meaning of Rank

Rank of Matrices

Matrix Representation

Original Form

Placement of Lines and Planes

Examples

Computational Approach

Numerical Computing with BASIC

Example 1

$$A = \begin{pmatrix} 0 & 3 & -2 & 3 \\ 1 & 1 & 3 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

2 行と 1 行を入れ替える

```
1  1  3  2
0  3 -2  3
1  2  2  3
1  3  2  4
```

2 行を 1 倍し，1 行の 0 倍を引く
3 行を 1 倍し，1 行の 1 倍を引く
4 行を 1 倍し，1 行の 1 倍を引く

```
1  1  3  2
0  3 -2  3
0  1 -1  1
0  2 -1  2
```

3 行を 3 倍し，2 行の 1 倍を引く
4 行を 3 倍し，2 行の 2 倍を引く

```
1  1  3  2
0  3 -2  3
0  0 -1  0
0  0  1  0
```

4 行を-1 倍し，3 行の 1 倍を引く

```
1  1  3  2
0  3 -2  3
0  0 -1  0
0  0  0  0
```

Rank A = 3

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

rank $A = 3$

Example 2

$$B = \begin{pmatrix} 0 & 3 & -2 & 3 & -4 \\ 1 & 1 & 3 & 2 & 2 \\ 1 & 2 & 2 & 3 & 1 \\ 1 & 3 & 2 & 4 & -1 \end{pmatrix}$$

2 行と 1 行を入れ替える

```
 1  1  3  2  2
 0  3 -2  3 -4
 1  2  2  3  1
 1  3  2  4 -1
```

2 行を 1 倍し，1 行の 0 倍を引く
3 行を 1 倍し，1 行の 1 倍を引く
4 行を 1 倍し，1 行の 1 倍を引く

```
 1  1  3  2  2
 0  3 -2  3 -4
 0  1 -1  1 -1
 0  2 -1  2 -3
```

3 行を 3 倍し，2 行の 1 倍を引く
4 行を 3 倍し，2 行の 2 倍を引く

```
 1  1  3  2  2
 0  3 -2  3 -4
 0  0 -1  0  1
 0  0  1  0 -1
```

4 行を-1 倍し，3 行の 1 倍を引く

```
 1  1  3  2  2
 0  3 -2  3 -4
 0  0 -1  0  1
 0  0  0  0  0
```

Rank B = 3

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 7 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{rank } B = 3$$

Example 3

$$C = \begin{pmatrix} 1 & -2 & -3 & 4 \\ 2 & 3 & 1 & 1 \\ 3 & -4 & -7 & 10 \end{pmatrix}$$

2 行を 1 倍し，1 行の 2 倍を引く
3 行を 1 倍し，1 行の 3 倍を引く
1 -2 -3 4
0 7 7 -7
0 2 2 -2

3 行を 7 倍し，2 行の 2 倍を引く
1 -2 -3 4
0 7 7 -7
0 0 0 0

Rank C = 2

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\operatorname{rank} C = 2$$

Example 4

$$D = \begin{pmatrix} 0 & 3 & -2 & 3 & -4 \\ 1 & 1 & 3 & 2 & 2 \\ 1 & 2 & 2 & 3 & 1 \\ 1 & 3 & 2 & 4 & -1 \end{pmatrix}$$

Matrix after Left Elementary Transformations

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 7 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

rank $D = 3$

System of Linear Equations and Ranks

General Form (n=2)

$$ax + by = \alpha$$
$$cx + dy = \beta$$

Matrix Representation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Coefficient Matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \end{pmatrix}$$

Idea of Rank (1)

$$\begin{cases} ax + by = \alpha \\ cx + dy = \beta \end{cases}$$

$$\Longleftrightarrow$$

$$x \begin{pmatrix} a \\ c \end{pmatrix} + y \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Idea of Rank (2)

$$\begin{cases} ax + by = \alpha \\ cx + dy = \beta \end{cases}$$

$$\Longleftrightarrow$$

$$\text{rank } A = \text{rank } \tilde{A}$$

Linear Algebra and Geometry

Geometrical Meaning of Rank

Rank of Matrices

Matrix Representation

Original Form

Placement of Lines

Classification of Intersections

rank A = rank $\tilde{A}$ = 2	**One-Point**
rank A = 1 < rank $\tilde{A}$ = 2	**Parallel Two Lines**
rank A = rank $\tilde{A}$ = 1 < 2	**Superposed Two Lines**

$$\text{rank } A \leq \text{rank } \tilde{A} \leq \text{rank } A + 1$$

Equation of a Line

$$ax + by = c$$

One-Point Intersection

$$2x + 3y = 3$$

$$3x - 8y = 17$$

Coefficient Matrix

$$A = \begin{pmatrix} 2 & 3 \\ 3 & -8 \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} 2 & 3 & 3 \\ 3 & -8 & 17 \end{pmatrix}$$

Unique Solution

$$\widetilde{A} = \begin{pmatrix} 2 & 3 & 3 \\ 3 & -8 & 17 \end{pmatrix}$$

$$\Longrightarrow$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\text{rank } A = \text{rank } \tilde{A} = 2$$

One-Point Intersection
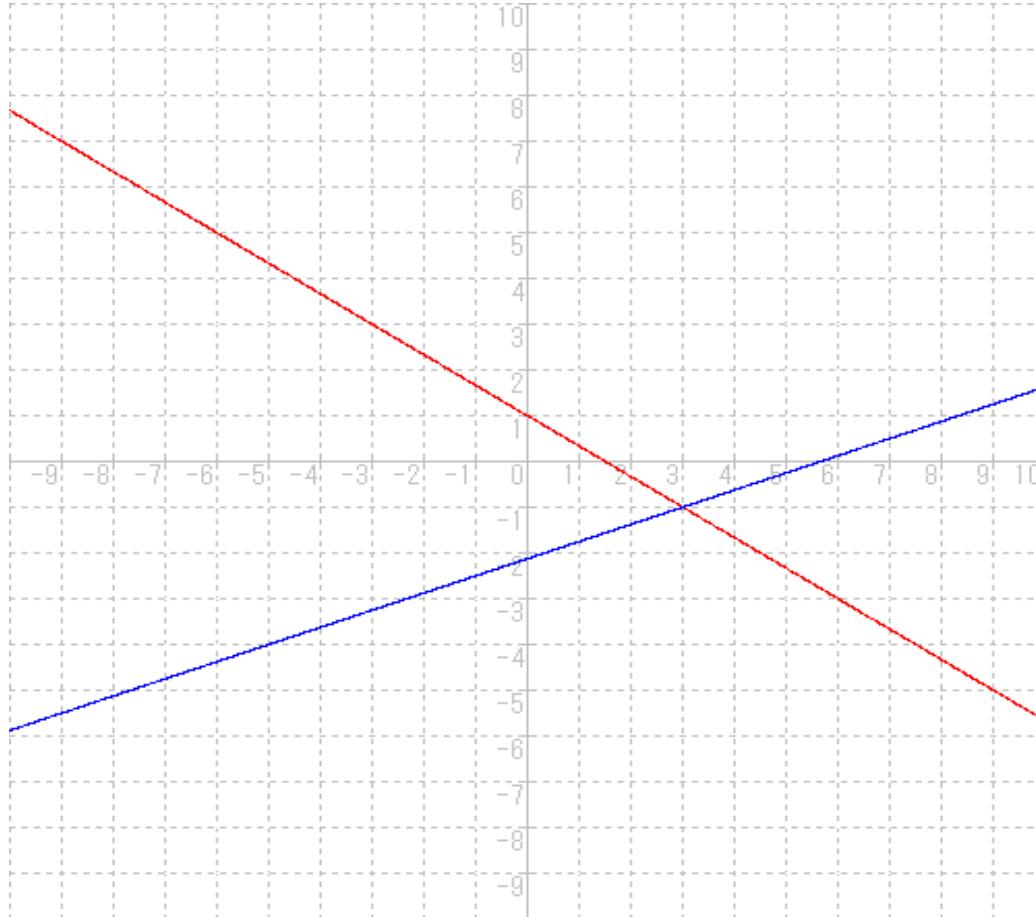
$$\text{rank } A = \text{rank } \tilde{A} = 2$$

Parallel Two Lines

$$x + 2y = 2$$
$$x + 2y = 3$$

Coefficient Matrix

$$A = \begin{pmatrix} 12 \\ 12 \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

No Solution

$$\widetilde{A} = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\Longrightarrow$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \textbf{(Impossible)}$$

$$\text{rank } A = 1 < \text{rank } \tilde{A} = 2$$

Parallel Two Lines
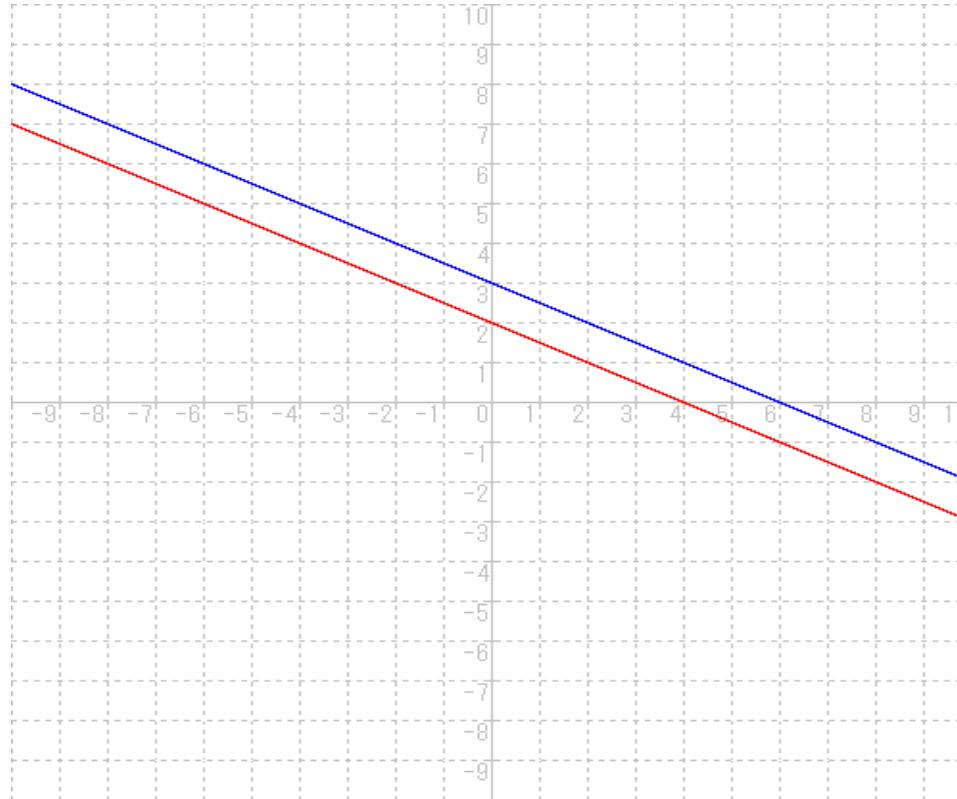
$$\text{rank } A = 1 < \text{rank } \tilde{A} = 2$$

Superposed Two Lines

$$6x - 2y = -8$$

$$3x - y = -4$$

Coefficient Matrix

$$A = \begin{pmatrix} 6 & -2 \\ 3 & -1 \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} 6 & -2 & -8 \\ 3 & -1 & -4 \end{pmatrix}$$

Many Solutions

$$\widetilde{A} = \begin{pmatrix} 6 & -2 & -8 \\ 3 & -1 & -4 \end{pmatrix}$$

$$\Longrightarrow$$

$$\begin{pmatrix} 1 & -1/3 & -4/3 \\ 0 & 0 & 0 \end{pmatrix}$$

(Indefinite)

$$\text{rank } A = \text{rank } \tilde{A} = 1 < 2$$

Superposed Two Lines
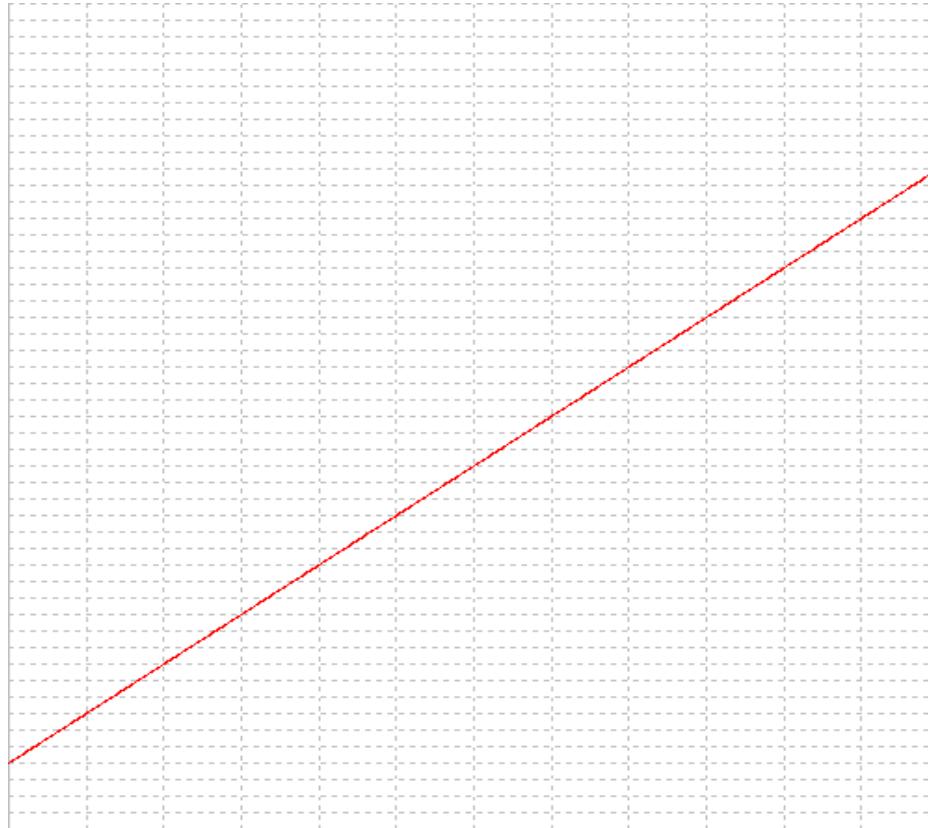
$$\text{rank } A = \text{rank } \tilde{A} = 1 < 2$$

General Form (n=3)

$$ax + by + cz = \alpha$$
$$dx + ey + fz = \beta$$
$$gx + hy + kz = \gamma$$

Matrix Representation

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

Coefficient Matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$$

Enlarged Coefficient Matrix

$$\tilde{A} = \begin{pmatrix} a & b & c & \alpha \\ d & e & f & \beta \\ g & h & k & \gamma \end{pmatrix}$$

Summary

Idea of Linear Algebra

System of Linear Equations

Matrix Representation

Original Form

Placement of Planes

Classification of Intersections

rank A = rank $\tilde{A}$ = 2	**One-Point**
rank A = 1 < rank $\tilde{A}$ = 2	**Parallel Two Lines**
rank A = rank $\tilde{A}$ = 1 < 2	**Superposed Two Lines**

$$\text{rank } A \leq \text{rank } \tilde{A} \leq \text{rank } A + 1$$

Classification of Intersections

rank $A = $ rank $\tilde{A} = 3$	**One-Point**
rank $A = $ rank $\tilde{A} = 2 < 3$	**One Line**
rank $A = 2 < $ rank $\tilde{A} = 3$	**Parallel Two Lines** **Parallel Three Lines**
rank $A = $ rank $\tilde{A} = 1 < 3$	**Superposed Three Planes**
rank $A = 1 < $ rank $\tilde{A} = 2$	**Parallel Two Planes** **Parallel Three Planes**

$$\text{rank } A \leq \text{rank } \tilde{A} \leq \text{rank } A + 1$$

Numerical Computing
# with
MuPAD

Plane
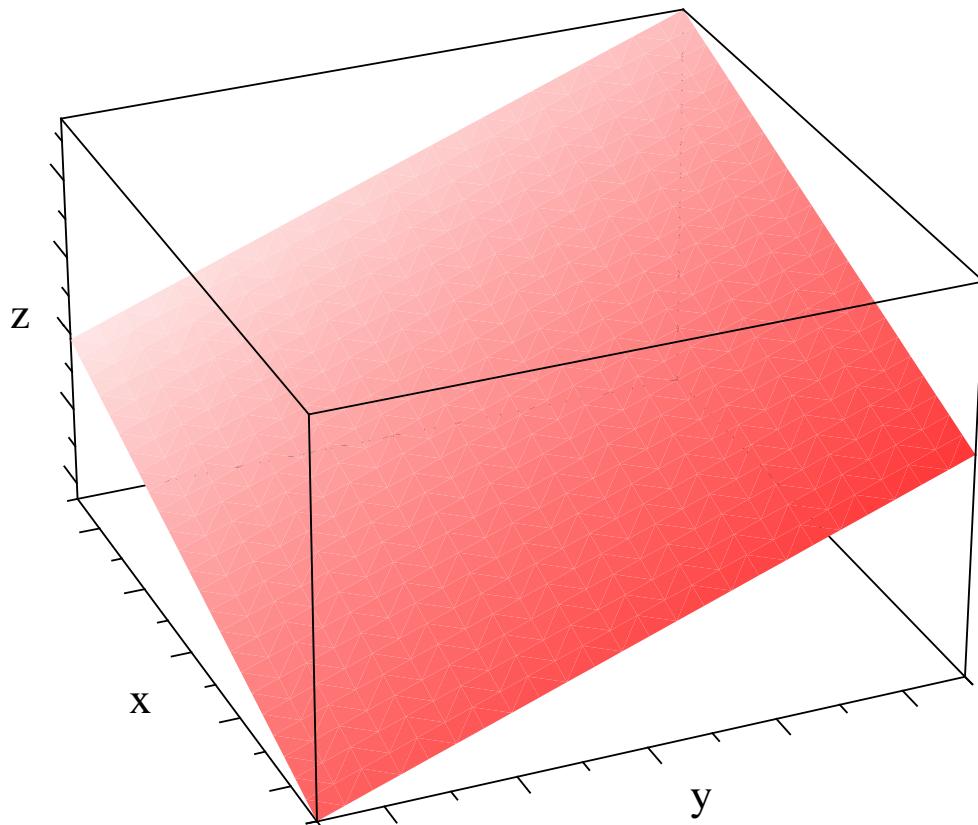
One-Point Intersection

$$x - 2y - 3z = 4$$
$$2x + 3y + 4z = 4$$
$$3x - 4y - 7z = 10$$

$$\text{rank } A = \text{rank } \tilde{A} = 3$$

One-Point Intersection
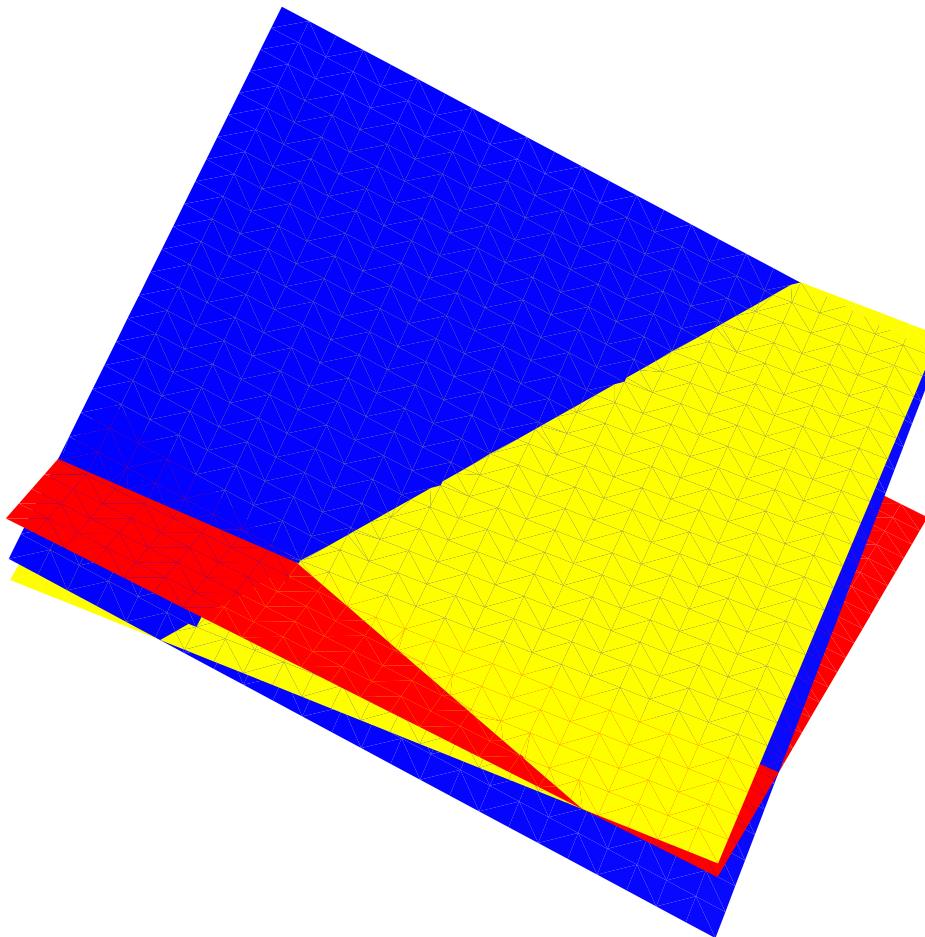
One-Line Intersection

$$x - 2y - 3z = 4$$
$$2x + 3y + z = 1$$
$$3x - 4y - 7z = 10$$

$$\text{rank } A = \text{rank } \tilde{A} = 2 < 3$$

One-Line Intersection
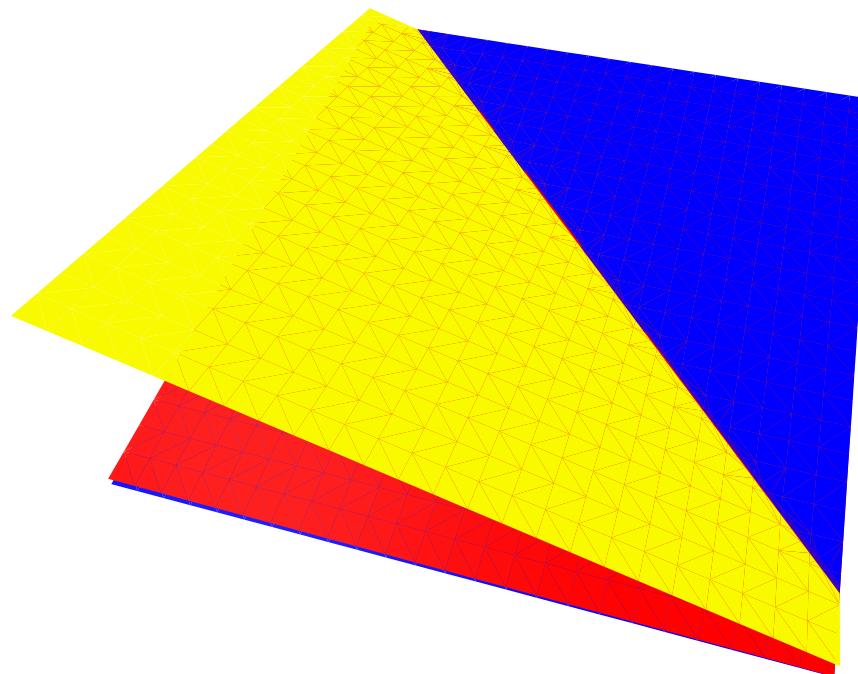
Three-Lines Intersection

$$3x + 6y + 9z = 60$$
$$2x - 4y + 6z = 40$$
$$2x + 7y - 3z = 13$$

$$\text{rank } A = 2 < \text{rank } \tilde{A} = 3$$

Three-Lines Intersection
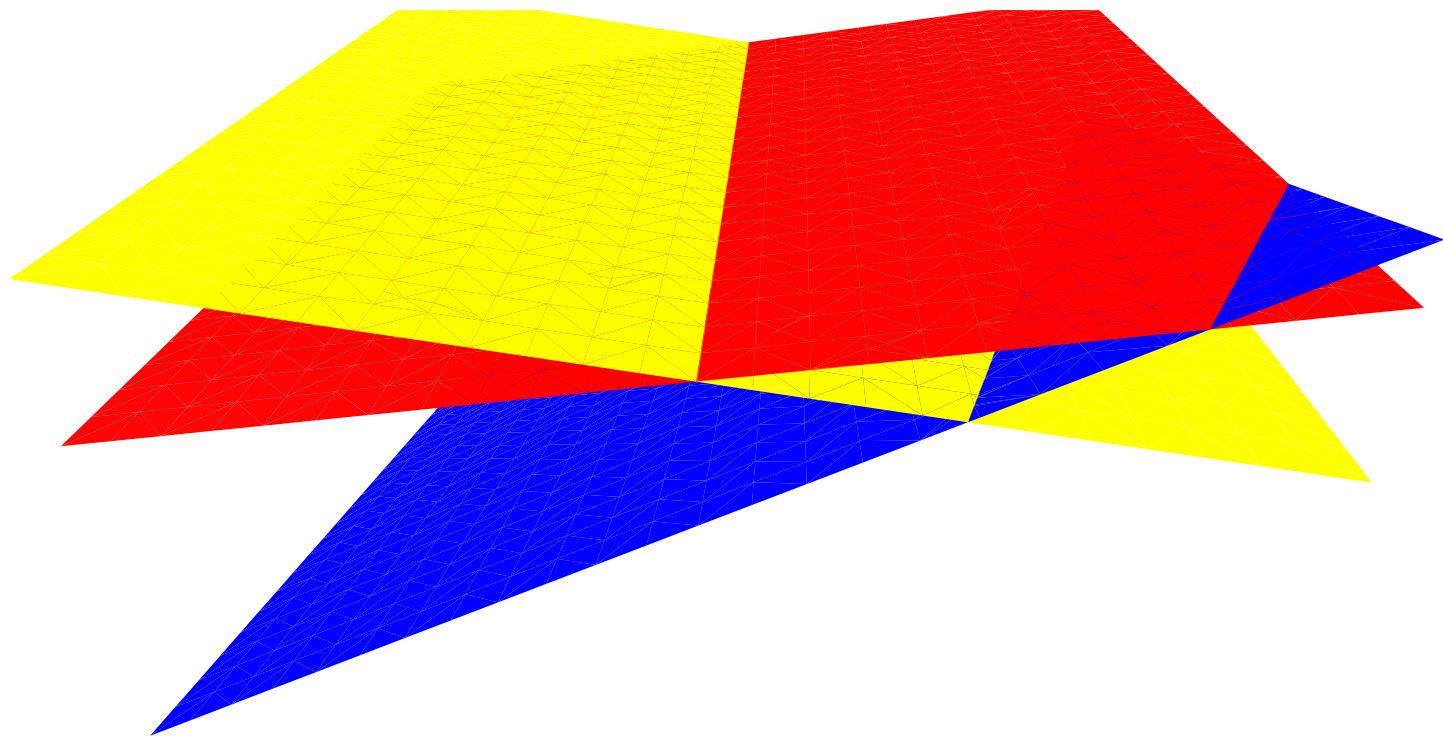
Parallel Two-Lines Intersection

$$x - 2y - 3z = 1$$
$$x - 2y - 3z = 4$$
$$3x - 4y - 7z = 10$$

$$\text{rank } A = 2 < \text{rank } \tilde{A} = 3$$

Parallel Two-Lines Intersection
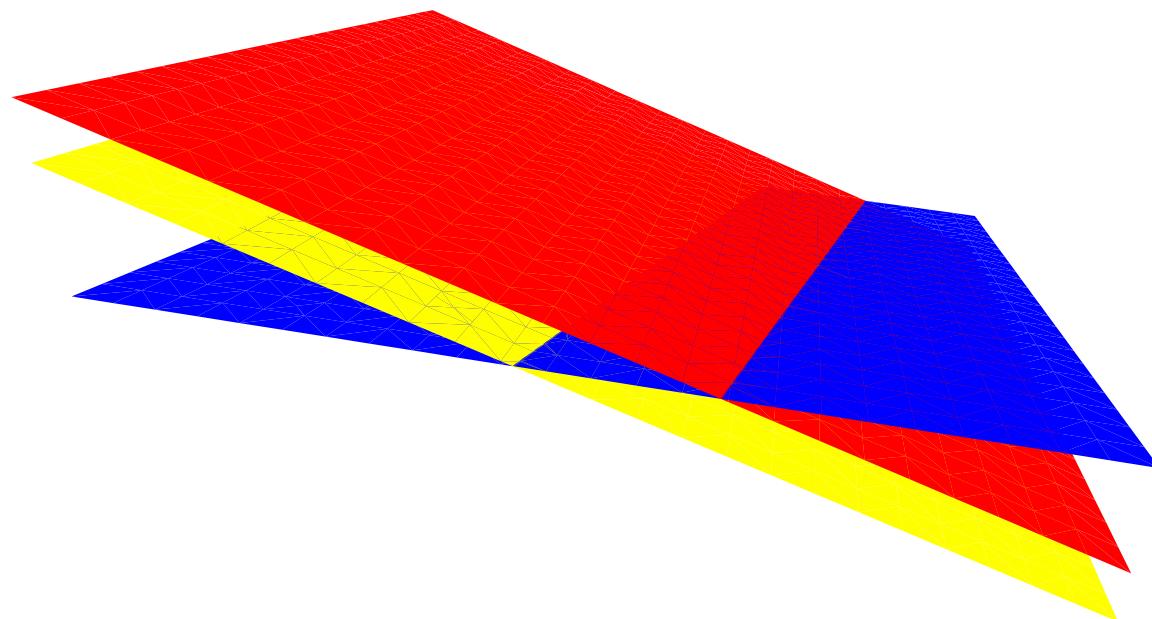
Parallel Three-Lines Intersection

$$3x + 6y + 9z = 60$$
$$2x + 7y - 3z = 13$$
$$3x + 9y \qquad = 0$$

$$\text{rank } A = 2 < \text{rank } \tilde{A} = 3$$

Parallel Three-Lines Intersection
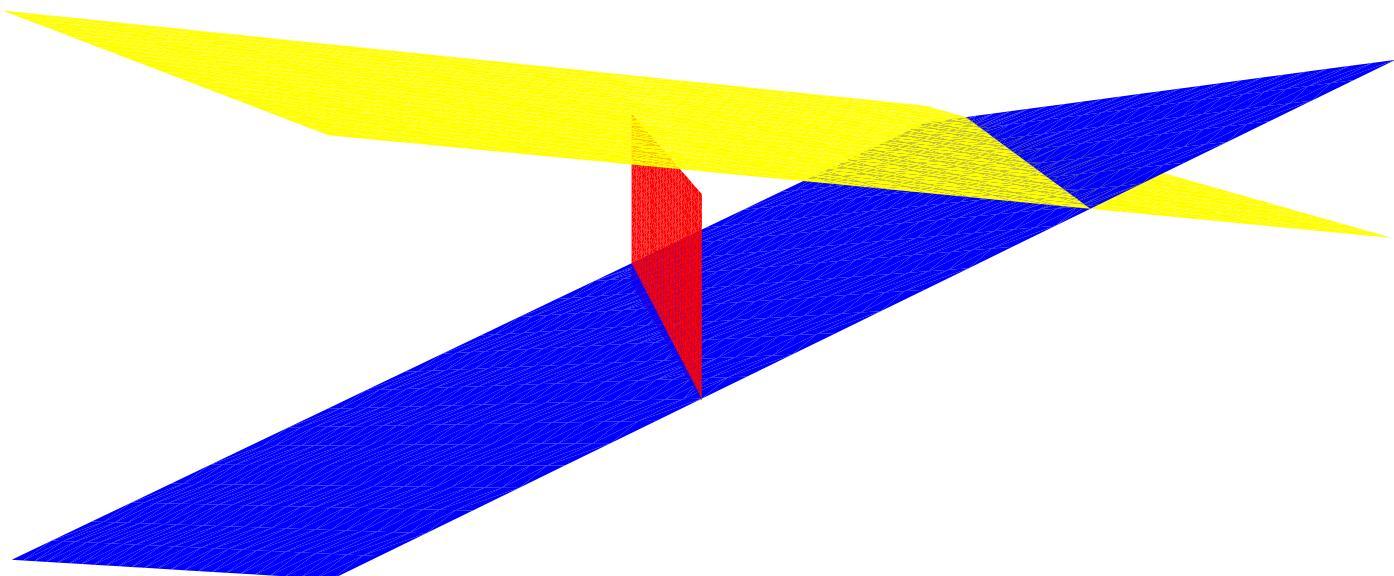
Parallel Two Planes

$$x - y + 3z = 1$$
$$3x - 3y + 9z = 3$$
$$x - y + 3z = 0$$

$$\text{rank } A = 1 < \text{rank } \tilde{A} = 2$$

# Parallel Two Planes
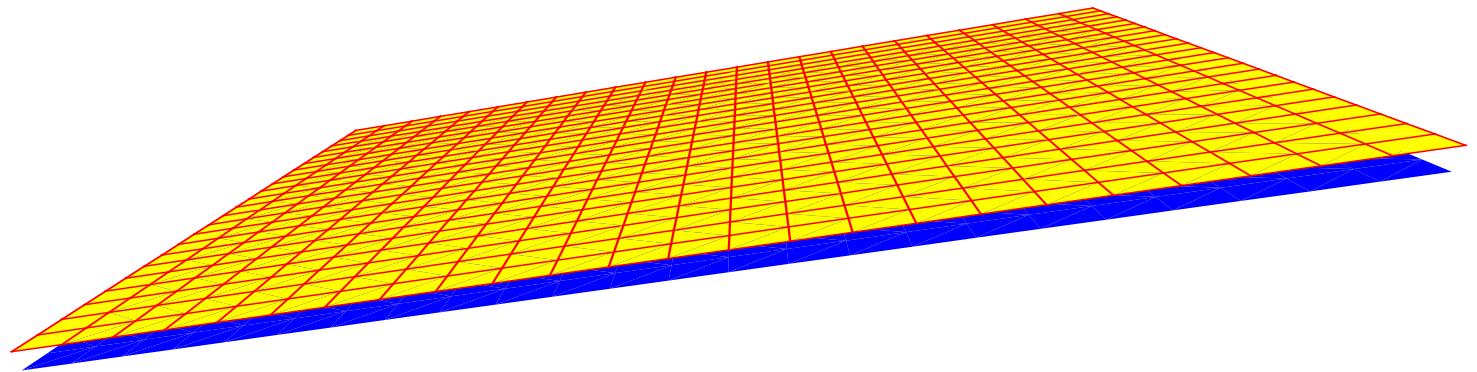
Parallel Three Planes

$$x - y + 3z = 1$$
$$x - y + 3z = 0$$
$$x - y + 3z = -1$$

$$\text{rank } A = 1 < \text{rank } \tilde{A} = 2$$

Parallel Three Planes
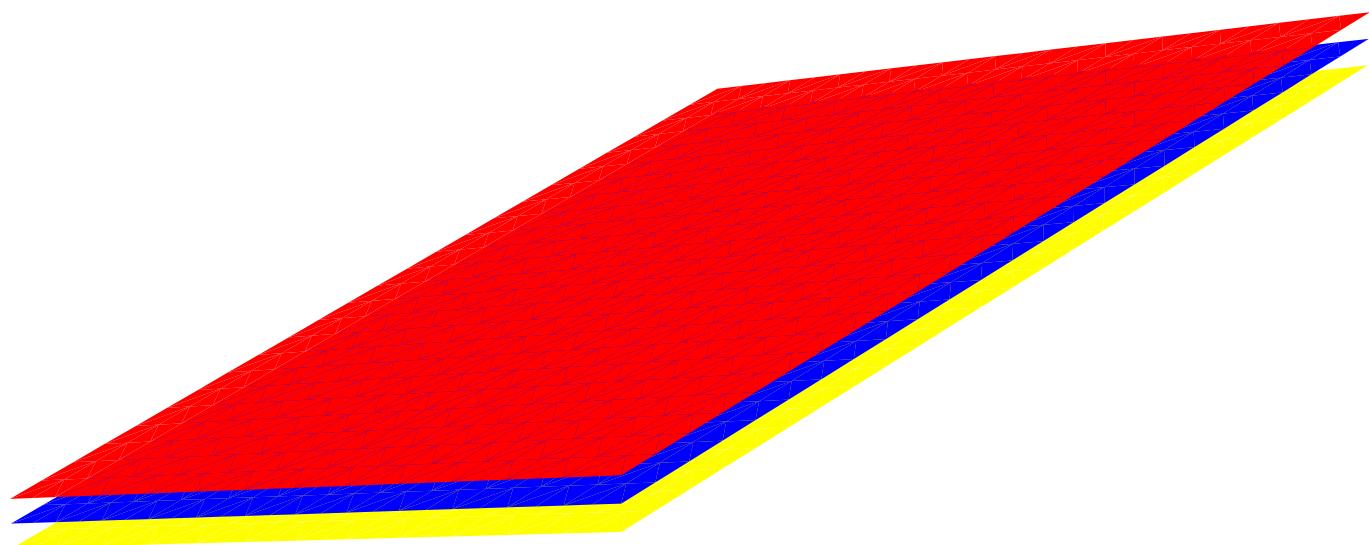
Superposed Three Planes

$$x - y + 3z = 1$$
$$3x - 3y + 9z = 3$$
$$2x - 2y + 6z = 2$$

$$\text{rank } A = \text{rank } \tilde{A} = 1 < 3$$

Superposed Three Planes
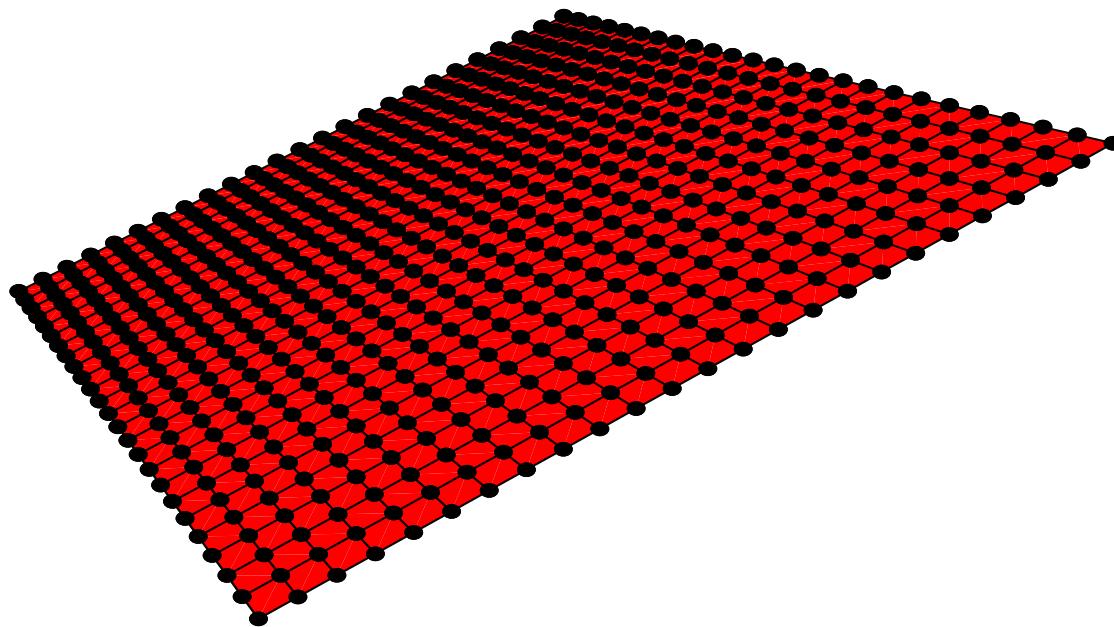
Jordan's Theory

Marie Ennemond Camille Jordan
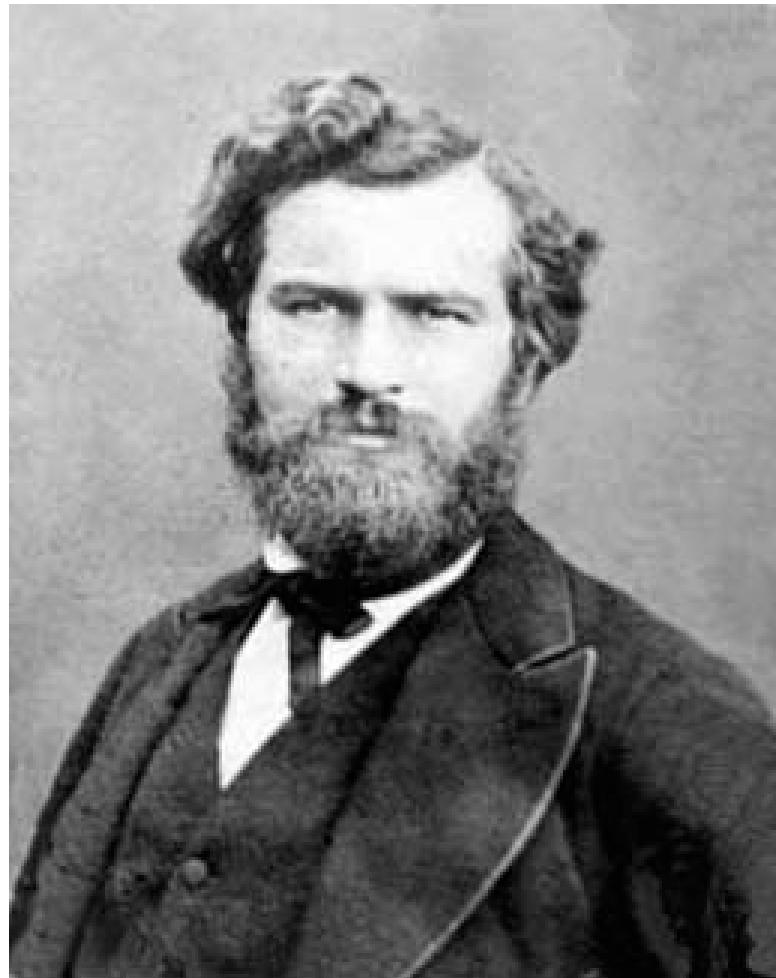
Jordan

◆ **Marie Ennemond Camille Jordan (1838-1922)**

French Mathematician

Jordan Canonical Form
of
Matrices

Jordan's Canonical Form (1)

$$A\mathbf{x} = \mathbf{b}$$

Change of Bases

Original Form

$$AP\mathbf{y} = \mathbf{b}, \, \mathbf{x} = P\mathbf{y} \implies (P^{-1}AP)\mathbf{y} = P^{-1}\mathbf{b}$$

Jordan's Canonical Form (2)

$$A\mathbf{x} = \mathbf{b}$$

Change of Bases

Original Form

$$J\mathbf{y} = \mathbf{c}, \ J = P^{-1}AP, \mathbf{c} = P^{-1}\mathbf{b}$$

Classification of
System of Linear Equations

General Form (n=3)

$$ax + by + cz = \alpha$$
$$dx + ey + fz = \beta$$
$$gx + hy + iz = \gamma$$

Matrix Representation (1)

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Matrix Representation (2)

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

Matrix Representation (3)

$$ax + by + cz = \alpha$$
$$dx + ey + fz = \beta$$
$$gx + hy + iz = \gamma$$

$$\Downarrow$$

$$A\mathbf{x} = \mathbf{b}$$

Patterns of Simultaneous Linear Equations

$$\lambda x = \alpha$$
$$\mu y = \beta$$
$$\nu z = \gamma$$

$$\lambda x + y = \alpha$$
$$\lambda y = \beta$$
$$\nu z = \gamma$$

$$\lambda x + y = \alpha$$
$$\lambda y + z = \beta$$
$$\nu z = \gamma$$

Matrix Representation

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

Jordan Canonical Form 1

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$J = P^{-1}AP = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

Pattern 1

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$3 = 1 + 1 + 1$$

Simultaneous Linear Equation 1

$$\lambda x = \alpha$$

$$\mu y = \beta$$

$$\nu z = \gamma$$

Jordan Canonical Form 2

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$J = P^{-1}AP = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

Pattern 2

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \nu \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$3 = 2 + 1$$

Simultaneous Linear Equation 2

$$\lambda x + y = \alpha$$

$$\lambda y = \beta$$

$$\nu z = \gamma$$

Jordan Canonical Form 3

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$J = P^{-1}AP = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

Pattern 3

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$3 = 3$$

Simultaneous Linear Equation 3

$$\lambda x + y = \alpha$$
$$\lambda y + z = \beta$$
$$\nu z = \gamma$$

Linea Algebra
and
Differential Equations

Linear Case

Second-Order Case

$$\begin{cases} u''(t) + 2bu'(t) + cu(t) = 0, \\ u(0) = u_0, \\ u'(0) = u_1 \end{cases}$$

General Solutions

General Solution（1）

$$D / 4 = b^2 - c > 0$$

$$u(t) = e^{-bt}\left(Ae^{t\sqrt{b^2-c}} + Be^{-t\sqrt{b^2-c}}\right)$$

A, B : **Constants**

Example

$$\begin{cases} x''(t) - x(t) = t \\ x(0) = x'(0) = 0 \end{cases}$$

Solution : $x(t) = \dfrac{1}{2}(e^t - e^{-t}) - t$

General Solution（2）

$$D / 4 = b^2 - c < 0$$

$$u(t) = e^{-bt}\left(A\cos\sqrt{c-b^2}\,t + B\sin\sqrt{c-b^2}\,t \right)$$

$A, B :$ **Constants**

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

General Solution（**3**）

$$D\ /\ 4 = b^2 - c = 0$$

$$u(t) = e^{-bt}\left(At + B\right)$$

$A, B :$ **Constants**

Exponential Matrix

Exponential Function

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$+ \frac{x^n}{n!} + \cdots$$

Main Idea

$$u''(t) + 2bu'(t) + cu(t) = 0$$

$$u''(t) + 2bu'(t) + cu(t) = 0$$

Matrix Representation

Original Form
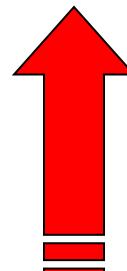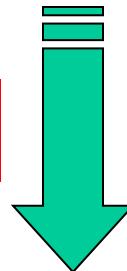
$$\frac{dU(t)}{dt} = AU(t) \implies \textbf{Calculation of } e^{tA}$$

Solution (1)

$$\begin{cases} u_1(t) = u(t), \\ u_2(t) = u'(t) \end{cases}$$

$$\begin{cases} u_1'(t) = u'(t) = u_2(t), \\ u_2'(t) = u''(t) = -2bu'(t) - cu(t) \\ \qquad = -2bu_2(t) - cu_1(t) \end{cases}$$

Solution (2)

$$
\begin{cases}
\dfrac{d}{dt}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c & -2b \end{pmatrix}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \\[2em]
\begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}
\end{cases}
$$

Solution (3)

$$U(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ -c & -2b \end{pmatrix}$$

$$\begin{cases} \dfrac{d}{dt} U(t) = A\,U(t), \\ U(0) = U_0 \end{cases}$$

Solution (4)

$$U(t) = e^{tA} U_0$$

$$e^{tA} = I + tA + \frac{(tA)^2}{2!} + \cdots + \frac{(tA)^n}{n!} + \cdots$$

(Exponential Matrix)

Example
of
Exponential Matrices

Simple Eigenvalue Case

Calculation (1)

$$A = \begin{pmatrix} 0 & 1 \\ -c & -2b \end{pmatrix}$$

$$\left| \lambda I - A \right| = \begin{vmatrix} \lambda & -1 \\ c & \lambda + 2b \end{vmatrix} = \lambda^2 + 2b\lambda + c$$

Calculation (2)

$$\textbf{Case}: D\,/\,4 = b^2 - c \neq 0$$

$$\begin{cases} \lambda_1 = -b + \sqrt{b^2 - c}\,, \\ \lambda_2 = -b - \sqrt{b^2 - c} \end{cases}$$

Calculation (3)

$$P = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -b + \sqrt{b^2 - c} & -b - \sqrt{b^2 - c} \end{pmatrix}$$

$P^{-1}AP = \Lambda$ （Diagonal）

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -b + \sqrt{b^2 - c} & 0 \\ 0 & -b - \sqrt{b^2 - c} \end{pmatrix}$$

Calculation (4)

$$P^{-1}e^{tA}P$$

$$= P^{-1}\left(I + tA + \frac{(tA)^2}{2!} + \cdots + \frac{(tA)^n}{n!} + \cdots \right) P$$

$$= P^{-1}P + t\left(P^{-1}AP \right) + \frac{t^2}{2!}\left(P^{-1}AP \right)\left(P^{-1}AP \right) + \cdots +$$

$$+ \frac{t^n}{n!}\underbrace{\left(P^{-1}AP \right)\left(P^{-1}AP \right)\cdots\left(P^{-1}AP \right)}_{n-\text{times}} + \cdots$$

$$= I + t\Lambda + \frac{(t\Lambda)^2}{2!} + \cdots + \frac{(t\Lambda)^n}{n!} + \cdots$$

$$= e^{t\Lambda}$$

Calculation (5)

$$e^{t\Lambda} = I + t\Lambda + \frac{(t\Lambda)^2}{2!} + \cdots + \frac{(t\Lambda)^n}{n!} + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} \lambda_1{}^2 & 0 \\ 0 & \lambda_2{}^2 \end{pmatrix} + \cdots$$

$$+ \frac{t^n}{n!}\begin{pmatrix} \lambda_1{}^n & 0 \\ 0 & \lambda_2{}^n \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

Calculation (6)

$$e^{tA} = Pe^{t\Lambda}P^{-1}$$

$$= \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix}$$

$$= \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} & -e^{\lambda_1 t} + e^{\lambda_2 t} \\ \lambda_1 \lambda_2 \left(e^{\lambda_1 t} - e^{\lambda_2 t} \right) & -\lambda_1 e^{\lambda_1 t} + \lambda_2 e^{\lambda_2 t} \end{pmatrix}$$

Calculation (7)

$$\textbf{Case:} \; D/4 = b^2 - c \neq 0$$

$$U(t) = e^{tA} U_0,$$

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} & -e^{\lambda_1 t} + e^{\lambda_2 t} \\ \lambda_1 \lambda_2 \left(e^{\lambda_1 t} - e^{\lambda_2 t} \right) & -\lambda_1 e^{\lambda_1 t} + \lambda_2 e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$$

Double Eigenvalue Case

Jordan's Canonical Form

$$P^{-1}AP = \Lambda \quad \text{(Jordan Form)}$$

$$\Lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Calculation (1)

$$A = \begin{pmatrix} 0 & 1 \\ -c & -2b \end{pmatrix}$$

$$\left| \lambda I - A \right| = \begin{vmatrix} \lambda & -1 \\ c & \lambda + 2b \end{vmatrix} = \lambda^2 + 2b\lambda + c$$

Calculation (2)

$$\textbf{Case}: D/4 = b^2 - c = 0$$

$$\lambda = -b \quad \textbf{(Double Root)}$$

$$P = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}$$

Calculation (3)

$$P^{-1}AP = \Lambda \quad \text{(Jordan Form)}$$

$$\Lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} -b & 1 \\ 0 & -b \end{pmatrix}$$

Calculation (4)

$$P^{-1} e^{tA} P$$

$$= P^{-1} \left(I + tA + \frac{(tA)^2}{2!} + \cdots + \frac{(tA)^n}{n!} + \cdots \right) P$$

$$= P^{-1} P + t \left(P^{-1} A P \right) + \frac{t^2}{2!} \left(P^{-1} A P \right) \left(P^{-1} A P \right) + \cdots +$$

$$+ \frac{t^n}{n!} \underbrace{\left(P^{-1} A P \right) \left(P^{-1} A P \right) \cdots \left(P^{-1} A P \right)}_{n-\text{times}} + \cdots$$

$$= I + t\Lambda + \frac{(t\Lambda)^2}{2!} + \cdots + \frac{(t\Lambda)^n}{n!} + \cdots$$

$$= e^{t\Lambda}$$

Calculation (5)

$$e^{t\Lambda} = I + t\Lambda + \frac{(t\Lambda)^2}{2!} + \cdots + \frac{(t\Lambda)^n}{n!} + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} + \frac{t^2}{2!} \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix} + \cdots$$

$$+ \frac{t^n}{n!} \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

Calculation (6)

$$e^{tA} = P e^{t\Lambda} P^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda t} - \lambda te^{\lambda t} & te^{\lambda t} \\ -\lambda^2 + e^{\lambda t} & (\lambda t + 1)e^{\lambda t} \end{pmatrix}$$

Calculation (7)

Case : $D / 4 = b^2 - c = 0$

$$U(t) = e^{tA} U_0,$$

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} e^{\lambda t} - \lambda t e^{\lambda t} & t e^{\lambda t} \\ -\lambda^2 + e^{\lambda t} & (\lambda t + 1) e^{\lambda t} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$$

END