ESTIMATES OF ANTIPODAL SETS IN ORIENTED REAL GRASSMANN MANIFOLDS

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Dedicated to the memory of Professor Shoshichi Kobayashi

Abstract. We estimate the cardinalities of antipodal sets in oriented real Grassmann manifolds of low ranks. The author reduced the classification of antipodal sets in oriented real Grassmann manifolds to a certain combinatorial problem in a previous paper. So we can reduce estimates of the antipodal sets to those of certain combinatorial objects. The sequences of antipodal sets we obtained in previous papers show that the estimates we obtained in this paper are the best.

1. Introduction

An antipodal set in a Riemannian symmetric space was introduced by Chen-Nagano [1]. A subset S of a Riemannian symmetric space is an *antipodal set*, if $s_x(y) = y$ holds for any x and y in S, where s_x is the geodesic symmetry at x. We denote by $\tilde{G}_k(\mathbb{R}^n)$ the oriented real Grassmann manifold consisting of oriented subspaces of dimension k in $\mathbb{R}^n$, which is a compact Riemannian symmetric space. The main theorem of this paper is the following:

Theorem 1.1. *If $n \geq 87$, then antipodal sets of maximal cardinality in $\tilde{G}_5(\mathbb{R}^n)$ are unique up to isometries of $\tilde{G}_5(\mathbb{R}^n)$.*

The author [2] defined an antipodal subset of

$$P_k(n) = \{\alpha \mid \alpha \subset \{1, \dots, n\}, |\alpha| = k\}.$$

Two elements α and β in $P_k(n)$ are *antipodal*, if the cardinality $|\beta - \alpha|$ is even, where $\beta - \alpha = \{i \in \beta \mid i \notin \alpha\}$. A subset A of $P_k(n)$ is *antipodal*, if any α and β in A are antipodal. The author reduced the classification of antipodal sets in $\tilde{G}_k(\mathbb{R}^n)$ to that of antipodal subsets in $P_k(n)$ in [2] and showed the classification of antipodal subsets of $P_k(n)$ for $k \leq 4$. This is the reason why we consider $\tilde{G}_5(\mathbb{R}^n)$. Theorem 1.1 is equivalent to the following:

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Theorem 1.2. *If $n \geq 87$, then antipodal sets of maximal cardinality in $P_5(n)$ are unique up to permutations of $\{1, \dots, n\}$.*

More detailed statement of this theorem is described in Theorem 3.1, which we prove by the use of sequences of antipodal subsets investigated in [2] and [3].

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2. Antipodal subsets

We denote by $\mathrm{Sym}(n)$ the symmetric group on $\{1, \dots, n\}$. Two subsets X and Y in $P_k(n)$ are *congruent*, if X is transformed to Y by an element of $\mathrm{Sym}(n)$. If X in $P_k(n)$ is antipodal, then a subset congruent with X is also antipodal.

In order to describe antipodal subsets we prepare some notation. For a set I we denote by $P_k(I)$ the set consisting of all subsets of cardinality k in I. We simply write $P_k(n)$ instead of $P_k(\{1, \dots, n\})$. When $I = I_1 \cup \cdots \cup I_m$ is a disjoint union, we put

$$A_1 \times \cdots \times A_m = \{\alpha_1 \cup \cdots \cup \alpha_m \mid \alpha_i \in A_i\}$$

for subsets A_i of $P_{k_i}(I_i)$. We get

$$A_1 \times \cdots \times A_m \subset P_{k_1 + \cdots + k_m}(I).$$

If each A_i is an antipodal subset of $P_{k_i}(I_i)$, then $A_1 \times \cdots \times A_m$ is an antipodal subset of $P_{k_1 + \cdots + k_m}(I)$.

We define some sequences of antipodal subsets according to [2] and [3]. We put

$$\begin{aligned} A(2, 2l) &= \{\{1, 2\}, \dots, \{2l - 1, 2l\}\}, \\ A(2k, 2l) &= \{\alpha_1 \cup \cdots \cup \alpha_k \in P_{2k}(2l) \mid \\ &\qquad \alpha_1, \dots, \alpha_k \text{ are distinct elements of } A(2, 2l)\}, \end{aligned}$$

which is an antipodal subset of $P_{2k}(2l)$ and

$$A(2k + 1, 2l + 1) = A(2k, 2l) \times \{\{2l + 1\}\},$$

which is an antipodal subset of $P_{2k+1}(2l + 1)$. By the definition

$$A(2k + 1, 2l + 1) = \{\alpha \cup \{2l + 1\} \mid \alpha \in A(2k, 2l)\}.$$

Their cardinalities are

$$|A(2k, 2l)| = |A(2k + 1, 2l + 1)| = \binom{l}{k}.$$

We define

$$a(k,n) = \max\{|A| \mid A \text{ is antipodal in } P_k(n).\}$$

and estimate it in the next section for $k = 5$.

Lemma 2.1.

$$a(k,n+1) \geq a(k,n), \quad a(k+1,n+1) \geq a(k,n),$$
$$a(2k,n) \geq \binom{\lfloor \frac{n}{2} \rfloor}{k}, \quad a(2k+1,n) \geq \binom{\lfloor \frac{n-1}{2} \rfloor}{k}.$$

Proof. If A is an antipodal subset of $P_k(n)$, then A is also an antipodal subset of $P_k(n+1)$. Thus we have $a(k,n+1) \geq a(k,n)$. $A \times \{\{n+1\}\}$ is also an antipodal subset of $P_{k+1}(n+1)$. Thus we have $a(k+1,n+1) \geq a(k,n)$.

$A\left(2k, 2\left\lfloor \frac{n}{2} \right\rfloor\right)$ is an antipodal subset of $P_{2k}(n)$, hence $a(2k,n) \geq \binom{\lfloor \frac{n}{2} \rfloor}{k}$. $A\left(2k+1, 2\left\lfloor \frac{n-1}{2} \right\rfloor + 1\right)$ is an antipodal subset of $P_{2k+1}(n)$, hence $a(2k+1,n) \geq \binom{\lfloor \frac{n-1}{2} \rfloor}{k}$. □

We can get the values of $a(k,n)$ for $k \leq 4$ from the classifications of maximal antipodal subsets of $P_k(n)$ obtained in [2] as follows. We have $a(1,n) = 1$ and any nonempty antipodal subset of $P_1(n)$ is congruent with $\{\{1\}\}$ by Proposition 4.1 and Corollary 4.1 in [2]. We have $a(2,n) = \lfloor \frac{n}{2} \rfloor$ and any antipodal subset of $P_2(n)$ which attains $a(2,n)$ is congruent with $A(2, 2\lfloor \frac{n}{2} \rfloor)$ by Proposition 4.2 and Corollary 4.2 in [2]. We have

n	4	5	6	$7,\ldots,16$	more than 16
$a(3,n)$	1	2	4	7	$\lfloor \frac{n-1}{2} \rfloor$

and any antipodal subset of $P_3(n)$ which attains $a(3,n)$ for $n > 16$ is congruent with $A(3, 2\lfloor \frac{n-1}{2} \rfloor + 1)$ by Theorem 5.1 and Corollary 5.1 in [2]. We have

n	5	6	7	$8,\ldots,11$	more than 11
$a(4,n)$	1	3	7	14	$\binom{\lfloor \frac{n}{2} \rfloor}{2}$

and any antipodal subset of $P_4(n)$ which attains $a(4,n)$ for $n > 11$ is congruent with $A(4, 2\lfloor \frac{n}{2} \rfloor)$ by Theorem 6.1 and Corollary 6.2 in [2]. These phenomena stimulate us to formulate Theorem 3.1 in the next section.

3. Estimates of the cardinalities of antipodal subsets

In this section we show the following main theorem.

Theorem 3.1. *If $n \geq 87$, then*

$$a(5, n) = \left| A\left(5, 2\left\lfloor \frac{n-1}{2} \right\rfloor + 1\right) \right| = \binom{\lfloor \frac{n-1}{2} \rfloor}{2}.$$

If an antipodal subset A of $P_5(n)$ for $n \geq 87$ attains $a(5, n)$, then A is congruent with $A\left(5, 2\left\lfloor \frac{n-1}{2} \right\rfloor + 1\right)$.

Proof. We estimate $|A|$ for an antipodal subset A of $P_5(n)$. We can suppose A contains $\alpha_0 = \{1, 2, 3, 4, 5\}$ without loss of generality. For an element α in $P_k(n)$ and a subset B in $P_k(n)$ we write

$$A_\alpha(B) = \{\beta \in B \mid \alpha, \beta \text{ are antipodal}\} - \{\alpha\}.$$

We have

$$A_{\alpha_0}(P_5(n)) = P_3(\{1, 2, 3, 4, 5\}) \times P_2(\{6, \ldots, n\}) \\ \cup P_1(\{1, 2, 3, 4, 5\}) \times P_4(\{6, \ldots, n\}),$$

which is a disjoint union. We put

$$M_j = P_{5-j}(\{1, 2, 3, 4, 5\}) \times P_j(\{6, \ldots, n\}) \quad (j = 2, 4).$$

So $A_{\alpha_0}(P_5(n)) = M_2 \cup M_4$. Since $A \subset \{\alpha_0\} \cup A_{\alpha_0}(P_5(n))$, we get

$$A = \{\alpha_0\} \cup (A \cap M_2) \cup (A \cap M_4),$$

which is also a disjoint union. We estimate the cardinalities of $A \cap M_2$ and $A \cap M_4$ in the following propositions.

Proposition 3.2. *For $A = A \cap M_2$, the following holds:*

(1) *If $A \cap M_2$ is contained in a product of antipodal subsets in $P_3(\{1, 2, 3, 4, 5\})$ and $P_2(\{6, \ldots, n\})$, then*

$$|A \cap M_2| \leq 2\left\lfloor \frac{n-1}{2} \right\rfloor - 4.$$

The equality holds if and only if $A \cap M_2$ is a product of maximal antipodal subsets in $P_3(\{1, 2, 3, 4, 5\})$ and $P_2(\{6, \ldots, n\})$.

(2) *If $A \cap M_2$ is not contained in a product of antipodal subsets in $P_3(\{1, 2, 3, 4, 5\})$ and $P_2(\{6, \ldots, n\})$, then*

$$|A \cap M_2| \leq \left\lfloor \frac{n}{2} \right\rfloor + 11.$$

Proposition 3.3. *For $A = A \cap M_4$, the following holds:*

(1) *If $A \cap M_4$ is contained in a product of antipodal subsets in $P_1(\{1,2,3,4,5\})$ and $P_4(\{6,\dots,n\})$, then*

$$|A \cap M_4| \leq a(4, n-5).$$

In particular, if $n \geq 17$,

$$|A \cap M_4| \leq \binom{\lfloor \frac{n-5}{2} \rfloor}{2}.$$

The equality holds if and only if $A \cap M_4$ is a product of maximal antipodal subsets in $P_1(\{1,2,3,4,5\})$ and $P_4(\{6,\dots,n\})$.

(2) *If $n \geq 29$ and $A \cap M_4$ is not contained in a product of antipodal subsets in $P_1(\{1,2,3,4,5\})$ and $P_4(\{6,\dots,n\})$, then*

$$|A \cap M_4| \leq 11 \left\lfloor \frac{n}{2} \right\rfloor + 9 \left\lfloor \frac{n-1}{2} \right\rfloor - 68.$$

Proof of Proposition 3.2. (1) By the assumption of this case we can estimate the cardinality as follows:

$$|A \cap M_2| \leq 2 \left\lfloor \frac{n-5}{2} \right\rfloor = 2 \left\lfloor \frac{n-1}{2} \right\rfloor - 4,$$

because $a(3,5) = 2$ and $a(2, n-5) = \lfloor \frac{n-5}{2} \rfloor$. Moreover the equality holds if and only if $A \cap M_2$ is a product of maximal antipodal subsets in $P_3(\{1,2,3,4,5\})$ and $P_2(\{6,\dots,n\})$.

(2) The assumption of this case means that there exist two elements $\alpha = \alpha_1 \cup \alpha_2, \beta = \beta_1 \cup \beta_2$ in $A \cap M_2$ such that $\alpha_1, \beta_1 \in P_3(\{1,2,3,4,5\})$ are not antipodal or that $\alpha_2, \beta_2 \in P_2(\{6,\dots,n\})$ are not antipodal. Since

$$(\alpha_1 \cup \alpha_2) - (\beta_1 \cup \beta_2) = (\alpha_1 - \beta_1) \cup (\alpha_2 - \beta_2),$$

α_1, β_1 are not antipodal in $P_3(\{1,2,3,4,5\})$ and α_2, β_2 are not antipodal in $P_2(\{6,\dots,n\})$. This condition is equivalent with $|\alpha_1 \cap \beta_1| = 2, |\alpha_2 \cap \beta_2| = 1$. We can suppose that $\alpha = \{1,2,3,6,7\}, \beta = \{1,2,4,6,8\}$ without loss of generality. Let B be a maximal antipodal subset of M_2 containing $A \cap M_2$. We estimate $|B|$. Since $\alpha, \beta \in A \cap M_2$, we have

$$A \cap M_2 - \{\alpha, \beta\} \subset B - \{\alpha, \beta\} \subset A_\beta(A_\alpha(M_2)).$$

In order to estimate $|B|$ we describe $A_\beta(A_\alpha(M_2))$. We have

$$\begin{aligned} A_\alpha(M_2) = {} & P_3(\{1,2,3\}) \times P_2(\{8,...,n\}) \\ & \cup P_2(\{1,2,3\}) \times P_1(\{4,5\}) \times P_1(\{6,7\}) \times P_1(\{8,...,n\}) \\ & \cup P_1(\{1,2,3\}) \times P_2(\{4,5\}) \times P_2(\{6,7\}) \\ & \cup P_1(\{1,2,3\}) \times P_2(\{4,5\}) \times P_2(\{8,...,n\}) \end{aligned}$$

and

$$\begin{aligned}
&A_\beta(A_\alpha(M_2))\\
&=\{\{1,2,5,7,8\}\}\\
&\cup\{\{1,2,3,8\},\{1,2,4,7\},\{1,2,5,6\}\}\times P_1(\{9,...,n\})\\
&\cup P_1(\{1,2\})\times\{\{3,4,7,8\},\{3,5,6,8\},\{4,5,6,7\}\}\\
&\cup P_1(\{1,2\})\times\{\{3,4,6\},\{3,5,7\},\{4,5,8\}\}\times P_1(\{9,...,n\})\\
&\cup\{\{3,4,5\}\}\times P_2(\{9,...,n\}).
\end{aligned}$$

The element $\{1,2,5,7,8\}$ is antipodal with all elements in $A_\beta(A_\alpha(M_2))$, hence $\{1,2,5,7,8\}$ is contained in B because of the maximal property of B. We put $\gamma=\{1,2,5,7,8\}$. We get

$$B-\{\alpha,\beta,\gamma\}\subset A_\gamma(A_\beta(A_\alpha(M_2)))$$

and

$$\begin{aligned}
&A_\gamma(A_\beta(A_\alpha(M_2)))\\
&=\{\{1,2,3,8\},\{1,2,4,7\},\{1,2,5,6\}\}\times P_1(\{9,...,n\})\\
&\cup P_1(\{1,2\})\times\{\{3,4,7,8\},\{3,5,6,8\},\{4,5,6,7\}\}\\
&\cup P_1(\{1,2\})\times\{\{3,4,6\},\{3,5,7\},\{4,5,8\}\}\times P_1(\{9,...,n\})\\
&\cup\{\{3,4,5\}\}\times P_2(\{9,...,n\}).
\end{aligned}$$

For simplicity, we set $B'=B\cap A_\gamma(A_\beta(A_\alpha(M_2)))$ and

$$\begin{aligned}
C_1&=\{\{1,2,3,8\},\{1,2,4,7\},\{1,2,5,6\}\}\times P_1(\{9,...,n\}),\\
C_2&=P_1(\{1,2\})\times\{\{3,4,7,8\},\{3,5,6,8\},\{4,5,6,7\}\},\\
C_3&=P_1(\{1,2\})\times\{\{3,4,6\},\{3,5,7\},\{4,5,8\}\}\times P_1(\{9,...,n\})\\
C_4&=\{\{3,4,5\}\}\times P_2(\{9,...,n\}).
\end{aligned}$$

Then $B=B'\cup\{\alpha,\beta,\gamma\}$ and $B'\subset A_\gamma(A_\beta(A_\alpha(M_2)))=C_1\cup C_2\cup C_3\cup C_4$. We have a disjoint union

$$B'=(B'\cap C_1)\cup(B'\cap C_2)\cup(B'\cap C_3)\cup(B'\cap C_4).$$

In order to estimate $|B|$ we estimate each $|B'\cap C_i|$. For $i=1,\dots,4$ each $B'\cap C_i$ is an antipodal subset of C_i.

Any maximal antipodal subset of C_1 is congruent with

$$\{\{1,2,3,8,9\},\{1,2,4,7,9\},\{1,2,5,6,9\}\},$$

thus we have $|B'\cap C_1|\le 3$. Any maximal antipodal subset of C_2 is congruent with

$$\{\{1,3,4,7,8\},\{1,3,5,6,8\},\{1,4,5,6,7\}\},$$

thus we have $|B' \cap C_2| \leq 3$. Any maximal antipodal subset of C_3 is congruent with

$$\{\{1,3,4,6,9\},\{1,3,5,7,9\},\{1,4,5,8,9\},$$
$$\{2,3,4,6,10\},\{2,3,5,7,10\},\{2,4,5,8,10\}\},$$

thus we have $|B' \cap C_3| \leq 6$. Any maximal antipodal subset of C_4 is congruent with

$$\{\{3,4,5\}\} \times \left\{\{9,10\},\dots,\left\{2\left\lfloor\frac{n}{2}\right\rfloor - 1, 2\left\lfloor\frac{n}{2}\right\rfloor\right\}\right\},$$

thus we have $|B' \cap C_4| \leq \left\lfloor\frac{n-8}{2}\right\rfloor$. Therefore we have

$$|A \cap M_2| \leq |B| \leq |B'| + 3 \leq 3 + 3 + 6 + \left\lfloor\frac{n-8}{2}\right\rfloor + 3 = \left\lfloor\frac{n}{2}\right\rfloor + 11.$$

□

Proof of Proposition 3.3. (1) By the assumption of this case we can estimate its cardinality as follows:

$$|A \cap M_4| \leq a(4, n-5).$$

If $n \geq 17$, then

$$|A \cap M_4| \leq \binom{\left\lfloor\frac{n-5}{2}\right\rfloor}{2}.$$

Moreover the equality holds if and only if $A \cap M_4$ is a product of maximal antipodal subsets in $P_1(\{1,2,3,4,5\})$ and $P_4(\{6,\dots,n\})$.

(2) The assumption of this case means that there exist two elements $\alpha = \alpha_1 \cup \alpha_2, \beta = \beta_1 \cup \beta_2$ in $A \cap M_4$ such that $\alpha_1, \beta_1 \in P_1(\{1,2,3,4,5\})$ are not antipodal and that $\alpha_2, \beta_2 \in P_4(\{6,\dots,n\})$ are not antipodal. This condition is equivalent with $|\alpha_1 \cap \beta_1| = 0, |\alpha_2 \cap \beta_2| = 1, 3$. We divide the argument to two cases of $|\alpha_2 \cap \beta_2| = 1$ and $|\alpha_2 \cap \beta_2| = 3$.

(i) In the case of $|\alpha_2 \cap \beta_2| = 3$, we can suppose $\alpha = \{1,6,7,8,9\}, \beta = \{2,6,7,8,10\}$ without loss of generality. Let B be a maximal antipodal subset of M_4 containing $A \cap M_4$. We estimate $|B|$. Since $\alpha, \beta \in A \cap M_4$, we have

$$A \cap M_4 - \{\alpha,\beta\} \subset B - \{\alpha,\beta\} \subset A_\beta(A_\alpha(M_4)).$$

In order to estimate $|B|$ we describe $A_\beta(A_\alpha(M_4))$. We have

$$\begin{aligned} A_\alpha(M_4) = &\{\{1\}\} \times P_2(\{6,7,8,9\}) \times P_2(\{10,\dots,n\}) \\ &\cup \{\{1\}\} \times P_4(\{10,\dots,n\}) \\ &\cup P_1(\{2,3,4,5\}) \times P_3(\{6,7,8,9\}) \times P_1(\{10,\dots,n\}) \\ &\cup P_1(\{2,3,4,5\}) \times P_1(\{6,7,8,9\}) \times P_3(\{10,\dots,n\}) \end{aligned}$$

and

$$\begin{aligned}A_\beta(A_\alpha(M_4)) = &\{\{1\}\} \times P_2(\{6,7,8\}) \times \{\{10\}\} \times P_1(\{11, ..., n\})\\ &\cup \{\{1\}\} \times P_1(\{6,7,8\}) \times \{\{9\}\} \times P_2(\{11, ..., n\})\\ &\cup \{\{1\}\} \times \{\{10\}\} \times P_3(\{11, \dots, n\})\\ &\cup \{\{2\}\} \times P_2(\{6,7,8\}) \times \{\{9\}\} \times P_1(\{11, ..., n\})\\ &\cup \{\{2\}\} \times P_1(\{6,7,8\}) \times \{\{10\}\} \times P_2(\{11, ..., n\})\\ &\cup \{\{2\}\} \times \{\{9\}\} \times P_3(\{11, ..., n\})\\ &\cup P_1(\{3,4,5\}) \times \{\{6,7,8\}\} \times P_1(\{11, \dots, n\})\\ &\cup P_1(\{3,4,5\}) \times P_2(\{6,7,8\}) \times \{\{9\}\} \times \{\{10\}\}\\ &\cup P_1(\{3,4,5\}) \times P_1(\{6,7,8\}) \times P_3(\{11, \dots, n\})\\ &\cup P_1(\{3,4,5\}) \times \{\{9\}\} \times \{\{10\}\} \times P_2(\{11, ..., n\}).\end{aligned}$$

Any maximal antipodal subset of $P_2(\{6,7,8\}) \times P_1(\{11, \dots, n\})$ is congruent with $\{\{6,7,11\}, \{6,8,12\}, \{7,8,13\}\}$ and the maximum cardinality of antipodal subsets of $P_2(\{6,7,8\}) \times P_1(\{11, \dots, n\})$ is equal to 3. According to the classification of maximal antipodal subsets of $P_3(n)$ obtained in Theorem 5.1 of [2], if $n \geq 14$, any maximal antipodal subset of $P_1(\{6,7,8\}) \times P_2(\{11, \dots, n\}) \subset P_3(\{6,7,8,11, \dots, n\})$ is congruent with

$$\{\{6\}\} \times \left\{\{11,12\}, \{13,14\}, \dots, \left\{2\left\lfloor\frac{n}{2}\right\rfloor - 1, 2\left\lfloor\frac{n}{2}\right\rfloor\right\}\right\}$$

or

$$\{\{6,11,12\}, \{6,13,14\}, \{7,11,13\}, \{7,12,14\}, \{8,11,14\}, \{8,12,13\}\}$$

and the maximum cardinality of antipodal subsets of $P_1(\{6,7,8\}) \times P_2(\{11, \dots, n\})$ is equal to $\max\left\{6, \lfloor\frac{n-10}{2}\rfloor\right\}$. Any maximal antipodal subset of $P_1(\{3,4,5\}) \times P_1(\{11, \dots, n\})$ is congruent with $\{\{3,11\}, \{4,12\}, \{5,13\}\}$ and the maximum cardinality of antipodal subsets of $P_1(\{3,4,5\}) \times P_1(\{11, \dots, n\})$ is equal to 3. The maximum cardinality of antipodal subset of

$$P_1(\{3,4,5\}) \times P_1(\{6,7,8\}) \times P_3(\{11, \dots, n\})$$

is less than or equal to $9a(3, n-10)$, because

$$|P_1(\{3,4,5\}) \times P_1(\{6,7,8\})| = 9.$$

We obtain the following estimate.

$$\begin{aligned}|B| &\leq 2+3+\max\left\{6, \left\lfloor \frac{n-10}{2} \right\rfloor\right\} + a(3,n-10)+3 \\ &\quad + \max\left\{6, \left\lfloor \frac{n-10}{2} \right\rfloor\right\} + a(3,n-10)+3+3+9a(3,n-10) \\ &\quad + \max\left\{6, \left\lfloor \frac{n-10}{2} \right\rfloor\right\} \\ &= 3\max\left\{6, \left\lfloor \frac{n-10}{2} \right\rfloor\right\} + 11a(3,n-10)+14.\end{aligned}$$

Hence, if $n \geq 27$, we have

$$|B| \leq 3\left\lfloor \frac{n-10}{2} \right\rfloor + 11\left\lfloor \frac{n-11}{2} \right\rfloor + 14 = 3\left\lfloor \frac{n}{2} \right\rfloor + 11\left\lfloor \frac{n-1}{2} \right\rfloor - 56.$$

(ii) In the case of $|\alpha_2 \cap \beta_2| = 1$, we can suppose $\alpha = \{1,6,7,8,9\}, \beta = \{2,6,10,11,12\}$ without loss of generality. α is the same in the case (i). There exists a maximal antipodal subset B of M_4 containing $A \cap M_4$. We estimate $|B|$. Since $\alpha, \beta \in A \cap M_4$, we have

$$A \cap M_4 - \{\alpha, \beta\} \subset B - \{\alpha, \beta\} \subset A_\beta(A_\alpha(M_4)).$$

In order to estimate $|B|$ we describe $A_\beta(A_\alpha(M_4))$. We have

$$\begin{aligned}A_\alpha(M_4) &= \{\{1\}\} \times P_2(\{6,7,8,9\}) \times P_2(\{10,...,n\}) \\ &\quad \cup \{\{1\}\} \times P_4(\{10,...,n\}) \\ &\quad \cup P_1(\{2,3,4,5\}) \times P_3(\{6,7,8,9\}) \times P_1(\{10,...,n\}) \\ &\quad \cup P_1(\{2,3,4,5\}) \times P_1(\{6,7,8,9\}) \times P_3(\{10,...,n\})\end{aligned}$$

and

$$\begin{aligned}&A_\beta(A_\alpha(M_4)) \\ &= \{\{1,6\}\} \times P_1(\{7,8,9\}) \times P_2(\{10,11,12\}) \\ &\cup \{\{1,6\}\} \times P_1(\{7,8,9\}) \times P_2(\{13,\dots,n\}) \\ &\cup \{\{1\}\} \times P_2(\{7,8,9\}) \times P_1(\{10,11,12\}) \times P_1(\{13,\dots,n\}) \\ &\cup \{\{1\}\} \times \{\{10,11,12\}\} \times P_1(\{13,\dots,n\}) \\ &\cup \{\{1\}\} \times P_1(\{10,11,12\}) \times P_3(\{13,\dots,n\}) \\ &\cup \{\{2\}\} \times \{\{6\}\} \times P_2(\{7,8,9\}) \times P_1(\{10,11,12\}) \\ &\cup \{\{2\}\} \times \{\{7,8,9\}\} \times P_1(\{13,\dots,n\}) \\ &\cup P_1(\{3,4,5\}) \times \{\{6\}\} \times P_2(\{7,8,9\}) \times P_1(\{13,\dots,n\})\end{aligned}$$

$$\cup P_1(\{3,4,5\}) \times \{\{7,8,9\}\} \times P_1(\{10,11,12\})$$
$$\cup \{\{2\}\} \times \{\{6\}\} \times P_1(\{10,11,12\}) \times P_2(\{13,\dots,n\})$$
$$\cup \{\{2\}\} \times P_1(\{7,8,9\}) \times P_2(\{10,11,12\}) \times P_1(\{13,\dots,n\})$$
$$\cup \{\{2\}\} \times P_1(\{7,8,9\}) \times P_3(\{13,\dots,n\})$$
$$\cup P_1(\{3,4,5\}) \times \{\{6\}\} \times P_2(\{10,11,12\}) \times P_1(\{13,\dots,n\})$$
$$\cup P_1(\{3,4,5\}) \times \{\{6\}\} \times P_3(\{13,\dots,n\})$$
$$\cup P_1(\{3,4,5\}) \times P_1(\{7,8,9\}) \times \{\{10,11,12\}\}$$
$$\cup P_1(\{3,4,5\}) \times P_1(\{7,8,9\}) \times P_1(\{10,11,12\}) \times P_2(\{13,\dots,n\}).$$

The maximum cardinality of antipodal subsets of

$$P_2(\{7,8,9\}) \times P_1(\{10,11,12\}) \times P_1(\{13,\dots,n\})$$

is less than or equal to

$$|P_2(\{7,8,9\})| \times 3 = 9.$$

The maximum cardinality of antipodal subsets of

$$P_1(\{3,4,5\}) \times P_1(\{7,8,9\}) \times P_1(\{10,11,12\}) \times P_2(\{13,\dots,n\})$$

is less than or equal to

$$|P_1(\{3,4,5\}) \times P_1(\{7,8,9\})| \max\left\{6, \left\lfloor \frac{n-12}{2} \right\rfloor\right\} = 9 \max\left\{6, \left\lfloor \frac{n-12}{2} \right\rfloor\right\}.$$

We obtain the following estimate for $n \geq 29$.

$$\begin{aligned}
|B| &\leq 2+3+\max\left\{6, \left\lfloor \frac{n-12}{2} \right\rfloor\right\} + 9 + 1 + 3a(3, n-12) + 3 + 1 + 9 + 3 \\
&\quad + \max\left\{6, \left\lfloor \frac{n-12}{2} \right\rfloor\right\} + 9 + 3a(3, n-12) + 9 + 3a(3, n-12) + 3 \\
&\quad + 9\max\left\{6, \left\lfloor \frac{n-12}{2} \right\rfloor\right\} \\
&= 11\max\left\{6, \left\lfloor \frac{n-12}{2} \right\rfloor\right\} + 9a(3, n-12) + 52 \\
&= 11\left\lfloor \frac{n-12}{2} \right\rfloor + 9\left\lfloor \frac{n-13}{2} \right\rfloor + 52 \\
&= 11\left\lfloor \frac{n}{2} \right\rfloor + 9\left\lfloor \frac{n-1}{2} \right\rfloor - 68.
\end{aligned}$$

We can see

$$3\left\lfloor \frac{n}{2} \right\rfloor + 11\left\lfloor \frac{n-1}{2} \right\rfloor - 56 \leq 11\left\lfloor \frac{n}{2} \right\rfloor + 9\left\lfloor \frac{n-1}{2} \right\rfloor - 68,$$

in the case $n \geq 4$. Therefore in the case $n \geq 29$ we have

$$|A \cap M_4| \leq 11 \left\lfloor \frac{n}{2} \right\rfloor + 9 \left\lfloor \frac{n-1}{2} \right\rfloor - 68.$$

□

Now using Propositions 3.2 and 3.3, we prove Theorem 3.1. By the disjoint union

$$A = \{\{1,2,3,4,5\}\} \cup (A \cap M_2) \cup (A \cap M_4)$$

we estimate $|A|$. We divide the argument to four cases whether $A \cap M_2$ is contained in the product of two antipodal subsets in $P_3(\{1,2,3,4,5\})$ and $P_2(\{6,\dots,n\})$ or not, and whether $A \cap M_4$ is contained in the product of two antipodal subsets in $P_1(\{1,2,3,4,5\})$ and $P_4(\{6,\dots,n\})$ or not. We suppose that $n \geq 29$ to use the result on $A \cap M_4$ obtained above.

If $A \cap M_2$ is not contained in the product and if $A \cap M_4$ is not contained in the product, then we have

$$\begin{aligned} |A| &= 1 + |A \cap M_2| + |A \cap M_4| \\ &\leq 1 + \left\lfloor \frac{n}{2} \right\rfloor + 11 + 11 \left\lfloor \frac{n}{2} \right\rfloor + 9 \left\lfloor \frac{n-1}{2} \right\rfloor - 68 \\ &= 12 \left\lfloor \frac{n}{2} \right\rfloor + 9 \left\lfloor \frac{n-1}{2} \right\rfloor - 56. \end{aligned}$$

The last term is less than $\left|A\left(5, 2\left\lfloor \frac{n-1}{2} \right\rfloor + 1\right)\right|$ if $n \geq 83$. Hence A can not attain $a(5,n)$.

If $A \cap M_2$ is not contained in the product and if $A \cap M_4$ is contained in the product, then we have

$$\begin{aligned} |A| &= 1 + |A \cap M_2| + |A \cap M_4| \\ &\leq 1 + \left\lfloor \frac{n}{2} \right\rfloor + 11 + a(4, n-5). \end{aligned}$$

The last term is less than $\left|A\left(5, 2\left\lfloor \frac{n-1}{2} \right\rfloor + 1\right)\right|$ if $n \geq 35$. Hence A can not attain $a(5,n)$.

If $A \cap M_2$ is contained in the product and if $A \cap M_4$ is not contained in the product, then for $n \geq 29$ we have

$$\begin{aligned} |A| &= 1 + |A \cap M_2| + |A \cap M_4| \\ &\leq 1 + 2\left\lfloor \frac{n-1}{2} \right\rfloor - 4 + 11 \left\lfloor \frac{n}{2} \right\rfloor + 9 \left\lfloor \frac{n-1}{2} \right\rfloor - 68 \\ &= 11 \left\lfloor \frac{n}{2} \right\rfloor + 11 \left\lfloor \frac{n-1}{2} \right\rfloor - 71. \end{aligned}$$

The last term is less than $\left|A\left(5, 2\left\lfloor\frac{n-1}{2}\right\rfloor+1\right)\right|$ if $n \geq 87$. Hence A can not attain $a(5,n)$.

If $A \cap M_2$ is contained in the product and if $A \cap M_4$ is contained in the product, then we have

$$\begin{aligned}|A| &= 1 + |A \cap M_2| + |A \cap M_4| \\ &\leq 1 + 2\left\lfloor\frac{n-1}{2}\right\rfloor - 4 + a(4, n-5) \\ &= \left|A\left(5, 2\left\lfloor\frac{n-1}{2}\right\rfloor + 1\right)\right|.\end{aligned}$$

Therefore if $n \geq 87$ then $A\left(5, 2\left\lfloor\frac{n-1}{2}\right\rfloor+1\right)$ attains $a(5,n)$ and

$$a(5,n) = \left|A\left(5, 2\left\lfloor\frac{n-1}{2}\right\rfloor + 1\right)\right| = \binom{\left\lfloor\frac{n-1}{2}\right\rfloor}{2}.$$

Finally we show that an antipodal subset A of $P_5(n)$ which attains $a(5,n)$ is congruent with $A\left(5, 2\left\lfloor\frac{n-1}{2}\right\rfloor+1\right)$ if $n \geq 87$. Since $A \cap M_2$ is the product of a maximal antipodal subset of $P_3(\{1,2,3,4,5\})$ and a maximal antipodal subset of $P_2(\{6,\dots,n\})$, by transforming A under the action of $\mathrm{Sym}(n)$ we can suppose

$$A \cap M_2 = \{\{1,2,3\},\{1,4,5\}\} \times \{\{6,7\},\{8,9\},\dots,\{2l,2l+1\}\},$$

where $l = \left\lfloor\frac{n-1}{2}\right\rfloor$. Moreover $A \cap M_4$ is the product of a maximal antipodal subset A_1 of $P_1(\{1,2,3,4,5\})$ and a maximal antipodal subset A_2 of $P_4(\{6,\dots,n\})$. There exists i in $\{1,2,3,4,5\}$ such that $A_1 = \{i\}$. We show that i is equal to 1. We suppose that i is not equal to 1. Without loss of generality we can suppose that i is equal to 2. Fix $\{a,b,c,d\} \in A_2 \subset P_4(\{6,\dots,n\})$. Then the element $\{1,4,5,6,7\} \in A \cap M_2$ and $\{2,a,b,c,d\} \in A_1 \times A_2$ are antipodal, hence one of a,b,c,d is equal to 6 or 7. The element $\{1,4,5,8,9\} \in A \cap M_2$ and $\{2,a,b,c,d\} \in A_1 \times A_2$ are antipodal, hence one of a,b,c,d is equal to 8 or 9. Iterating this we obtain

$$\{a,b,c,d\} \in P_1(\{6,7\}) \times P_1(\{8,9\}) \times P_1(\{10,11\}) \times P_1(\{12,13\}).$$

Hence $\{1,4,5,14,15\}$ and $\{2,a,b,c,d\}$ are not antipodal, which is a contradiction. Therefore i is equal to 1 and we have

$$A \cap M_4 = \{\{1\}\} \times A_2.$$

All elements of

$$A \cap M_2 = \{\{1,2,3\},\{1,4,5\}\} \times \{\{6,7\},\{8,9\},\dots,\{2l,2l+1\}\}$$

and $\{\{1\}\} \times A_2$ are antipodal, so the intersection of each element of $\{\{6,7\},\{8,9\},\ldots,\{2l,2l+1\}\}$ and each element of A_2 has an even cardinality. Hence we have

$$A_2 = \{\alpha_1 \cup \alpha_2 \in P_4(\{6,\ldots,n\}) \mid \alpha_i \in \{\{6,7\},\{8,9\},\ldots,\{2l,2l+1\}\}\}$$

and A is congruent with $A\left(5, 2\left\lfloor \frac{n-1}{2} \right\rfloor + 1\right)$. □

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