Singular Integrals and Feller Semigroups

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Part I
Pseudo-Differential Operators and Feller Semigroups

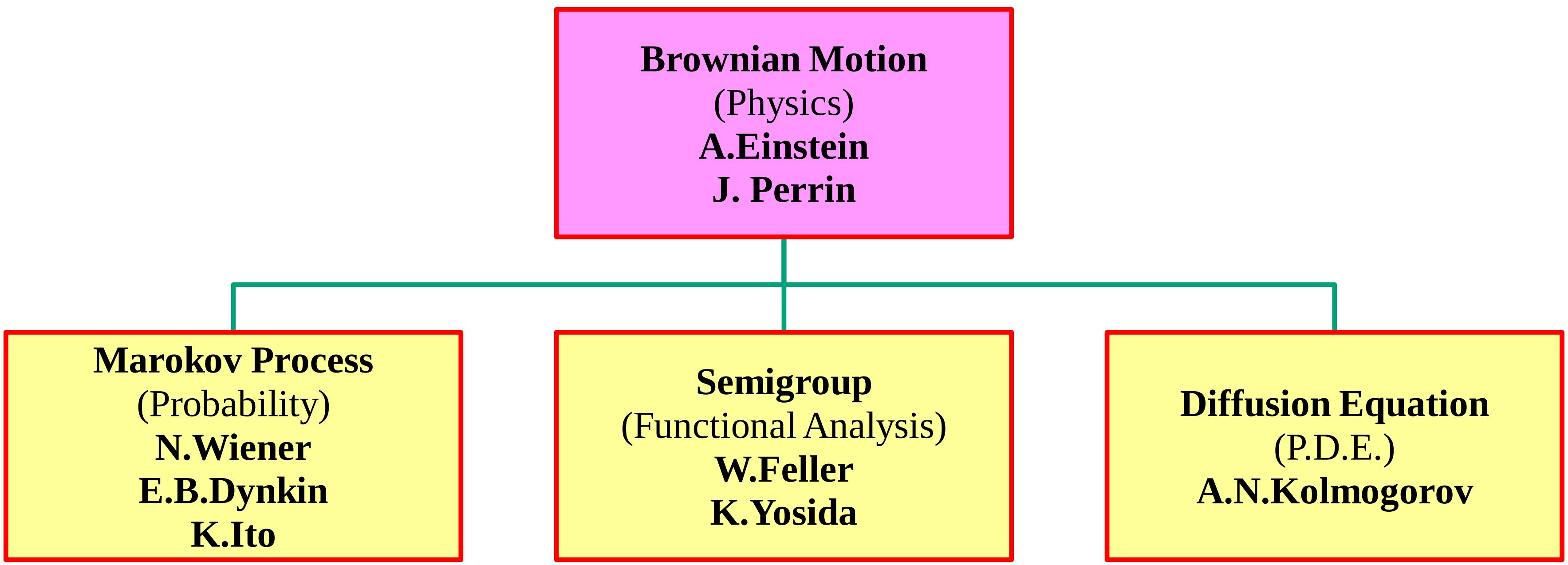
Abstract

- This talk is devoted to the functional analytic approach to the problem of construction of Markov processes for second-order elliptic integro-differential operators with smooth coefficients.
- By using the theory of pseudo-differential operators, we construct a Feller semigroup corresponding to such a diffusion phenomenon that a Markovian particle moves both by jumps and continuously in the state space.

Abstract

- **This talk is devoted to the semigroup approach to the problem of construction of Markov processes in probability theory.**

Mathematical Study of Brownian Motion



Brief History
(one-dimensional case)

- **1931: A.N. Kolmogorov (analytic approach)**
- **1952: W. Feller (semigroup approach)**
- **1965: E.B. Dynkin (probabilistic approach)**
- **1965: K. Ito and H.P. McKean, Jr. (probabilistic approach)**

References

- **Kolmogorov: Math. Ann. 104 (1931), 415-458.**
- **Feller: Ann. Math. 55 (1952), 468-519.**
- **Dynkin: Springer-Verlag, 1965.**
- **Ito and McKean, Jr. : Springer-Verlag, 1965.**
- **Ikeda and Watanabe: Kodansha, 1981.**

Brief History (1)
(multi-dimensional case)

- **1959: A.D. Wentzell (Ventcel')**
- **1964: W.v. Waldenfels**
- **1965: K. Sato and T. Ueno (semigroup approach, abstract setting)**
- **1968: J.M.Bony, P.Courrege and P.Priouret (semigroup approach, non-degenerate case)**

Brief History (2)
(multi-dimensional case)

- **1982: K. Taira (semigroup approach, degenerate case, pseudo-differential operators)**
- **1986: C. Cancelier (semigroup approach, degenerate case, elliptic regularizations)**
- **1988: S. Takanobu and S. Watanabe (stochastic approach, degenerate case)**

References (1)

- Wentzell: Theory Prob. and its Appl. 4 (1959), 164-177.
- Sato and Ueno: J. Math. Kyoto Univ. 14 (1965), 529-605.
- Bony, Courrege and Priouret : Ann. Inst. Fourier 19 (1969), 277-304.
- Taira: Academic Press, 1988.
- Cancelier: Comm. P. D. E. 11 (1986), 1677-1726.
- Takanobu and Watanabe: J. Math. Kyoto Univ. 28 (1988), 71-80.

Strategy

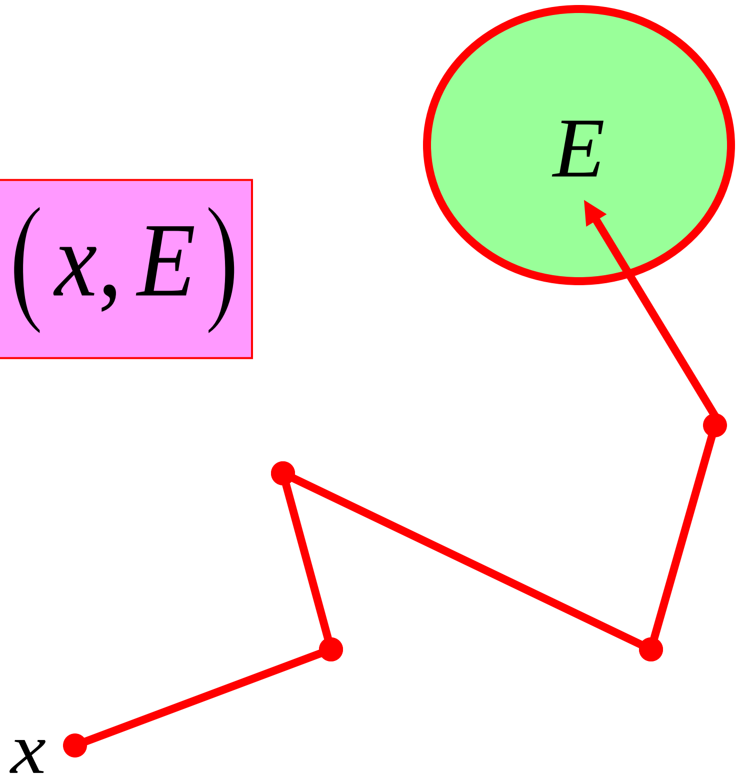
- **Existence and uniqueness theorems for Waldenfels operators with Wentzell boundary conditions (Partial Differential Equations)**
- **Generation theorems for Feller semigroups (Functional Analysis)**
- **Existence theorems for Markov processes (Probability)**

Bird's Eye View

Probability Theory	Functional Analysis	Partial Differential Equations
Markov Process	Feller Semigroup	Infinitesimal Generator
Markov Property	Semigroup Property	•Waldenfels Operator •Wentzell Condition

Transition Functions

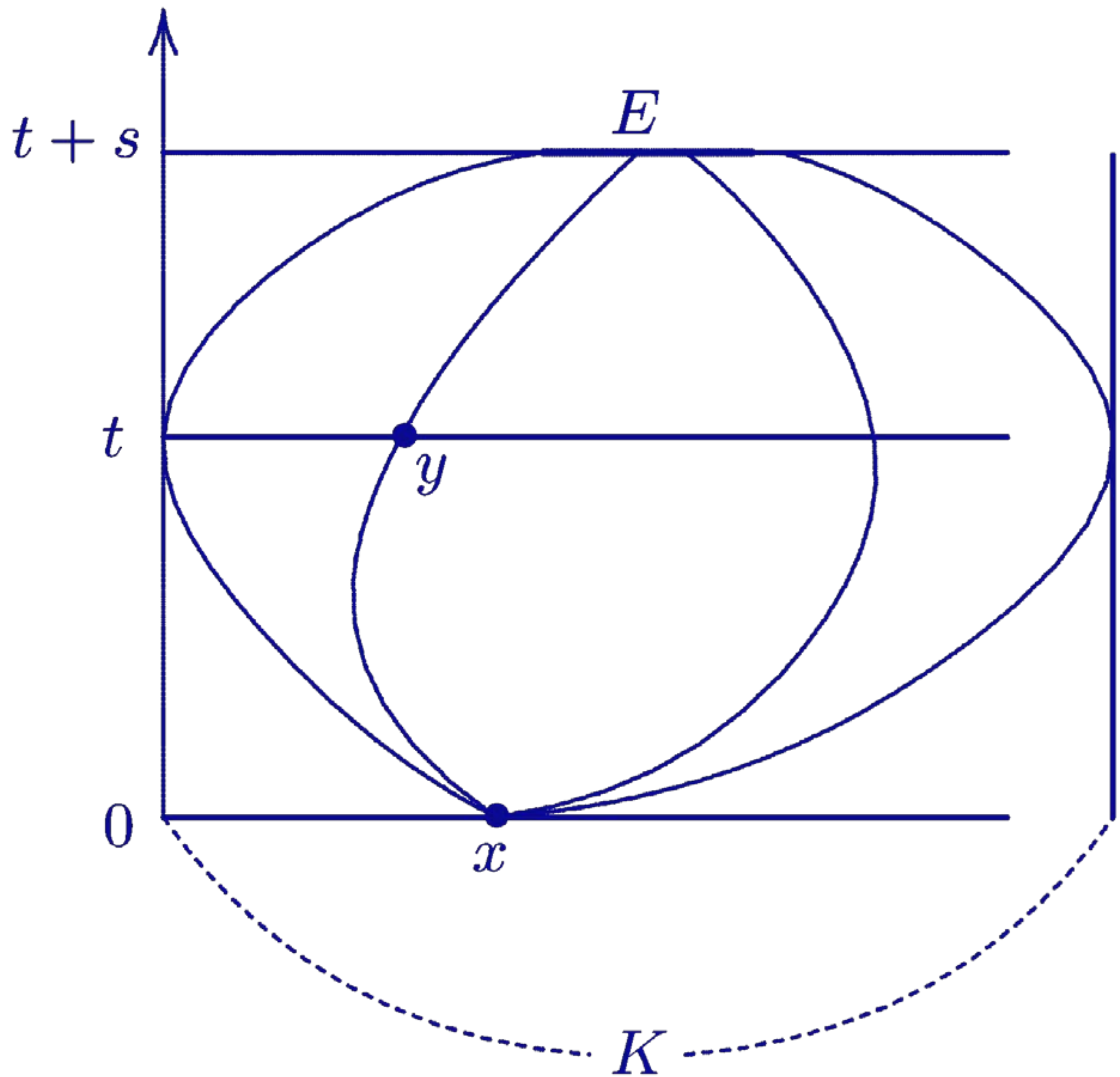
$$p_t\left(x, E\right)$$



Chapman-Kolmogorov Equation

$$p_{t+s}(x,E) = \int_K p_t(x,dy)\,p_s(y,E), \ \ \forall t,s \geq 0$$

A transition from x **to** E **in time** $t+s$ **is composed of a transition from** x **to some** y **in time** t, **followed by a transition from** y **to** E **in time** s.



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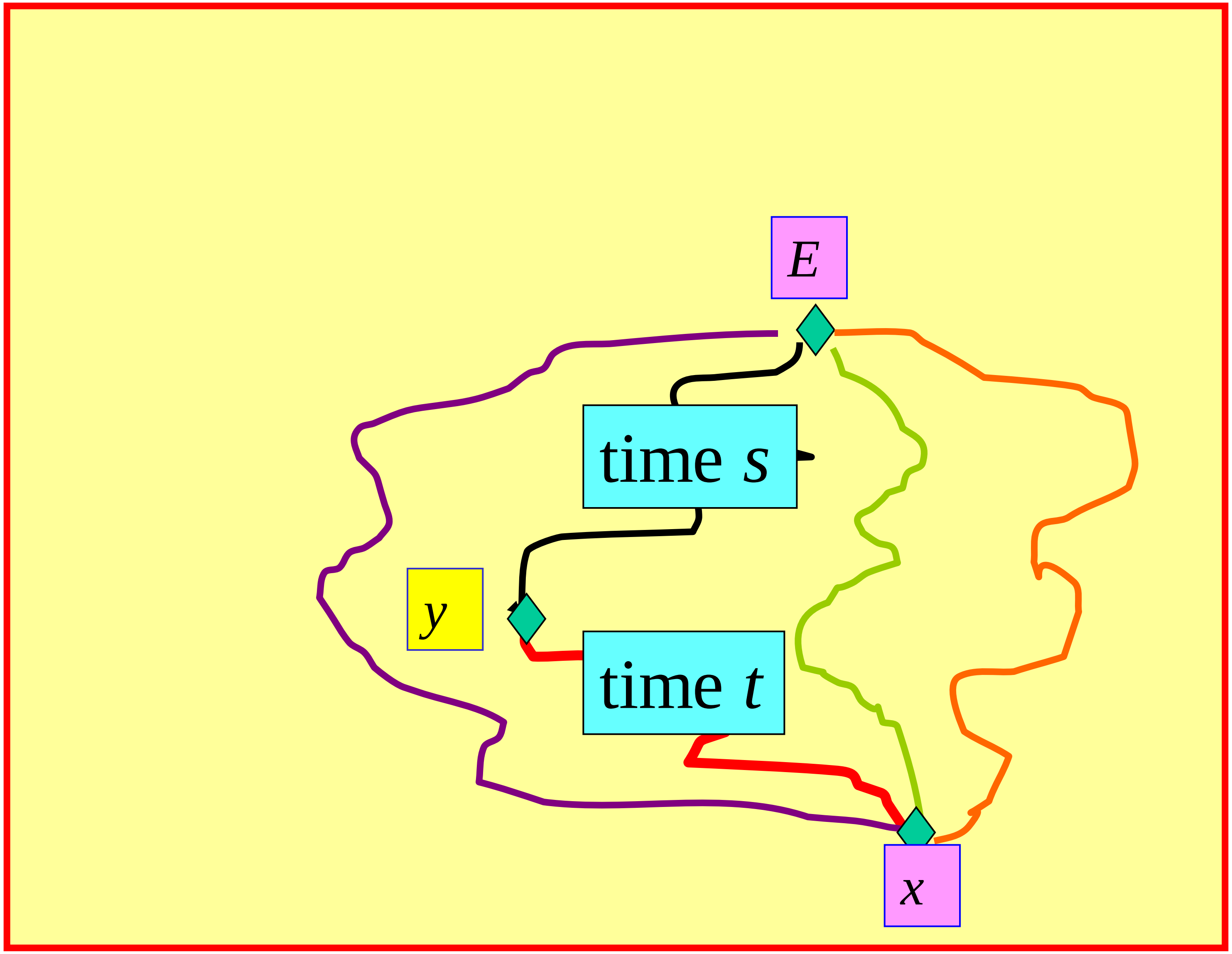
E
time s
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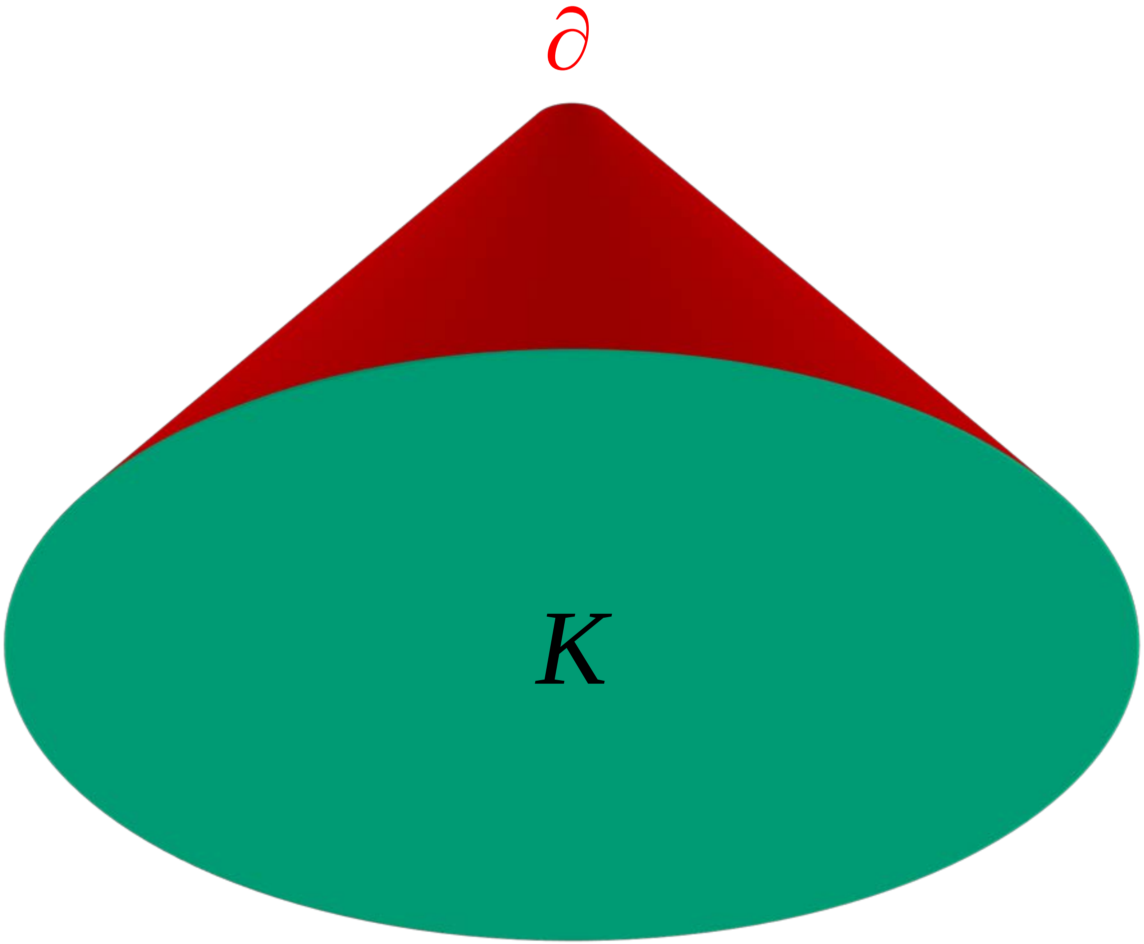
State Space
(general case)

$K =$ **locally compact**, **separable metric space**,

$\partial =$ **point at infinity**,

$K_{\partial} = K \cup \{\partial\}$, **one-point compactification**

One-Point Compactification



Function Space (1) (general case)

$K =$ locally compact, separable metric space,

$C_0(K) =$ space of real-valued, continuous functions

on K vanishing at the point at infinity ∂

with the supremum norm

$$\|u\|_\infty = \sup_{x \in K} |u(x)|$$

Function Space (2) (general case)

$$C_0(K) \cong \{u \in C(K_\partial) : u(\partial) = 0\}$$

with the maximum norm

$$\|u\| = \max_{x \in K_\partial} |u(x)|$$

$$K_\partial = K \cup \{\partial\}$$

Feller Semigroups
(general case)

A family of bounded linear operators $\{T_t\}_{t\geq 0}$ is called a **Feller semigroup** if it satisfies the following three conditions:

(1) $T_{t+s} = T_t \bullet T_s, \ \forall t, s \geq 0.$

(2) $\lim_{s\downarrow 0} \|T_{t+s} f - T_t f\| = 0, \ \forall f \in C_0(K).$

(3) $\forall f \in C_0(K), 0 \leq f \leq 1 \text{ on } K \Rightarrow 0 \leq T_t f \leq 1 \text{ on } K.$

Function Space
(compact case)

$C(K) =$ space of real-valued, continuous functions on the **compact** metric space K

with the maximum norm

$$\|u\| = \max_{x \in K} |u(x)|$$

Feller Semigroups
(compact case)

A family of bounded linear operators $\{T_t\}_{t\geq 0}$ is called a **Feller semigroup** if it satisfies the following three conditions:

(1) $T_{t+s} = T_t \bullet T_s, \ \ \forall t, s \geq 0.$

(2) $\lim_{s\downarrow 0} \left\| T_{t+s} f - T_t f \right\| = 0, \ \forall f \in C(K).$

(3) $\forall f \in C(K), 0 \leq f \leq 1 \text{ on } K \Rightarrow 0 \leq T_t f \leq 1 \text{ on } K.$

Riesz-Markov Representation Theorem

$$T_t f(x) = \int_K p_t(x, dy) f(y), \ \forall f \in C(K)$$

$$\Leftrightarrow$$

$$0 \le \exists! \, p_t(x, \bullet) \le 1, \ \forall t \ge 0, \forall x \in K$$

Semigroup Property

$$T_{t+s} = T_t \bullet T_s, \quad \forall t, s \geq 0$$

$$\Leftrightarrow$$

$$p_{t+s}(x, E) = \int_K p_t(x, dy)\, p_s(y, E), \quad \forall t, s \geq 0$$

(**Chapman - Kolmogorov Equation**)

Markov Transition Functions

(a) $p_t(x,\bullet)$ is a **measure on** $\mathfrak{B}(K)$ and

$0 \le p_t(x,K) \le 1,\ \forall x \in K, \forall t \ge 0$

(b) $p_0(x,\{x\}) = 1, \forall x \in K$

(c) **Chapman - Kolmogorov equation**

$$p_{t+s}(x,E) = \int_K p_t(x,dy)\, p_s(y,E),\ \ \forall t,s \ge 0$$

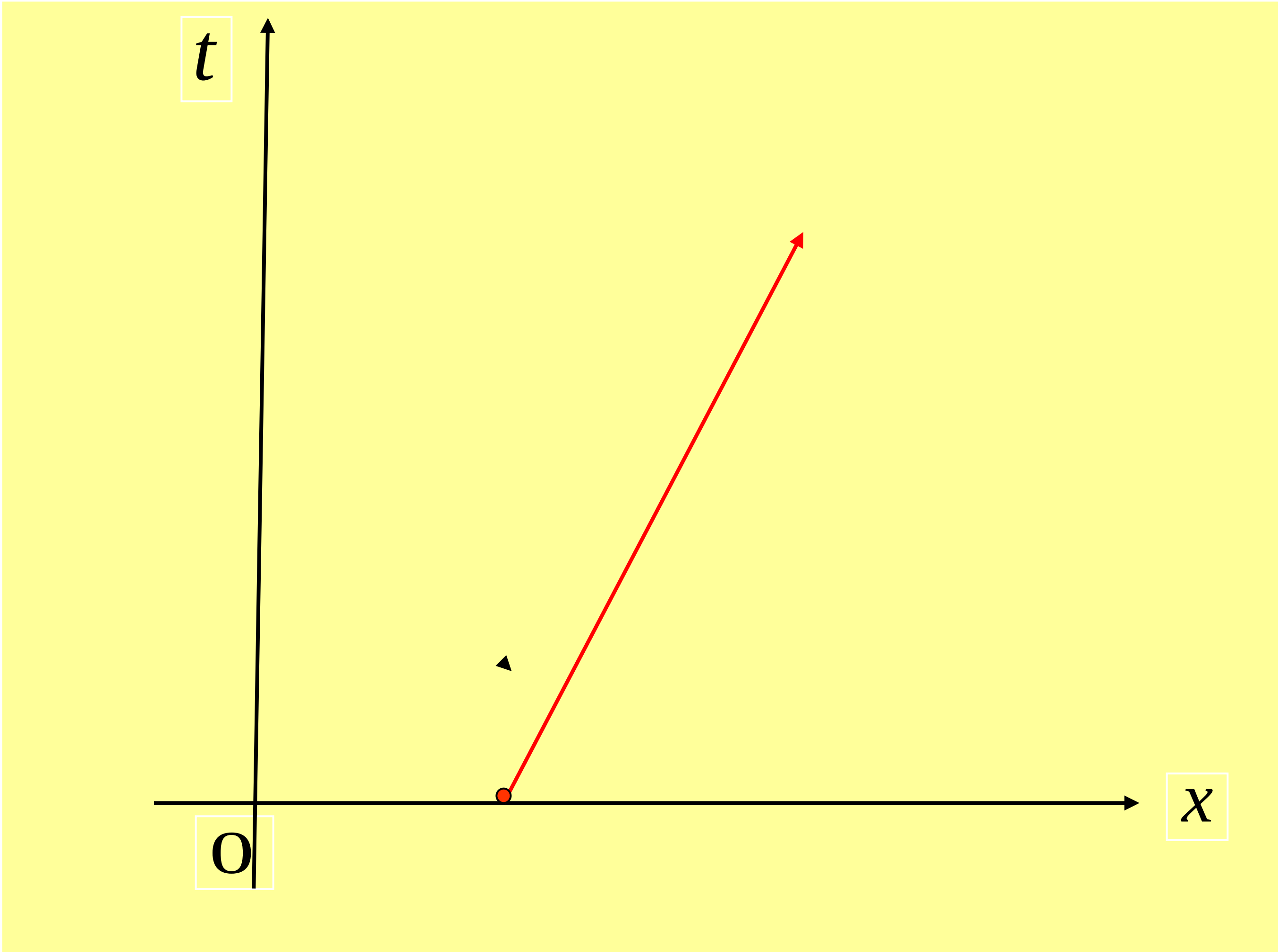
Examples (1)

Example 1 (uniform motion)

$K = \mathbf{R}$

$$p_t(x, E) = \chi_E(x + vt), \ \forall x \in \mathbf{R}, \forall E \in \mathfrak{B}(K)$$

This process, starting at x, **moves deterministically with constant velocity** v.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Examples (2)

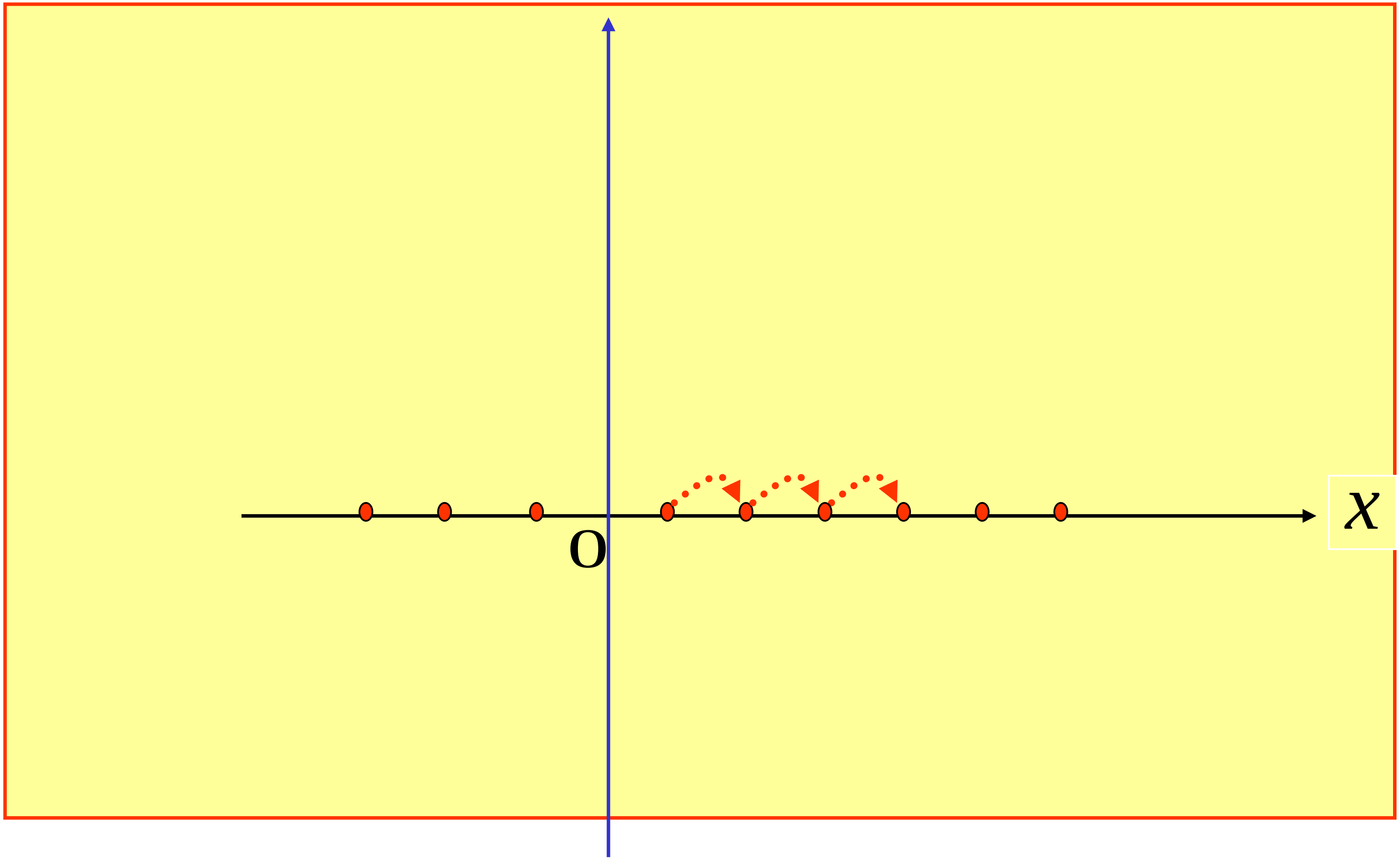
Example 2 (Poisson process)

$K = \mathbf{R}$

$$p_t(x, E) = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \chi_E(x+n), \;\; \forall x \in \mathbf{R}, \; \forall E \in \mathfrak{B}(K)$$

This process, starting at x, advances one unit by jumps.

Poisson Process



Examples (3)

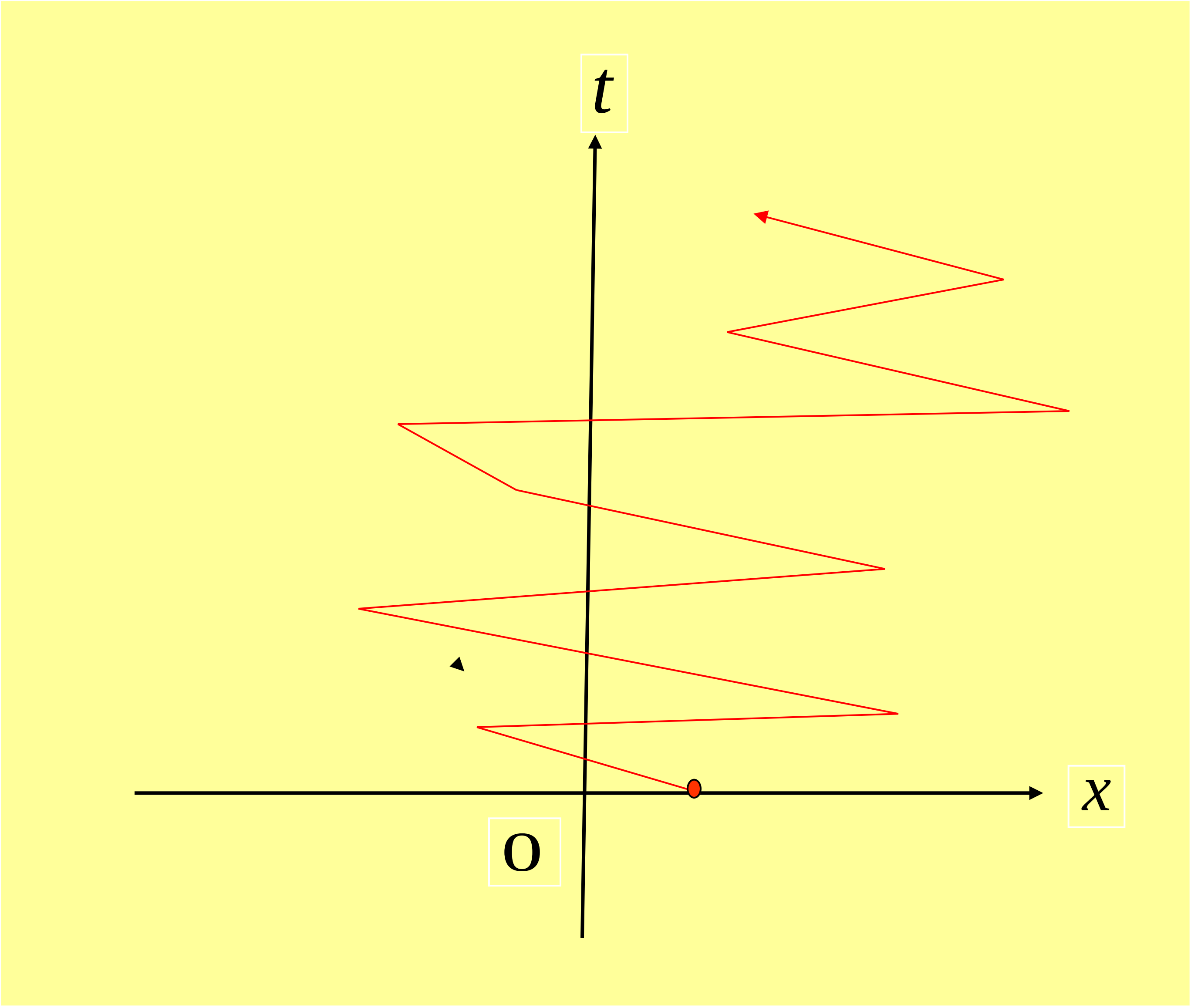
Example 3 (Brownian motion)

$K = \mathbf{R}$

$$p_t(x, E) = \frac{1}{\sqrt{2\pi t}} \int_E \exp\left[-\frac{(y-x)^2}{2t} \right] dy,$$

$\forall x \in \mathbf{R}, \ \forall E \in \mathfrak{B}(K).$

This is a mathematical model of one-dimensional Brwonian motion.



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Examples (4)

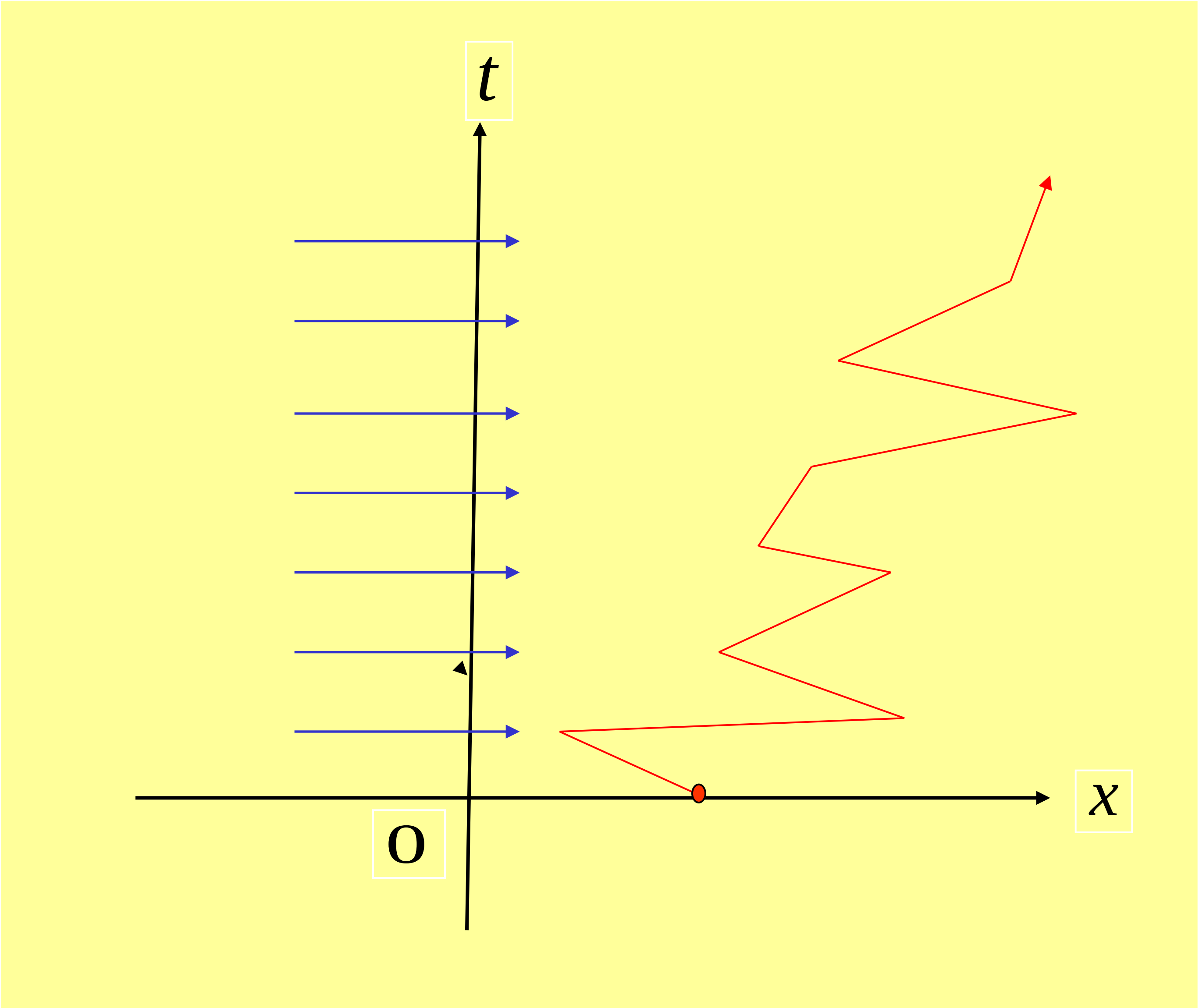
Example 4 (Brownian motion with constant drift)

$K = \mathbf{R}$

$$p_t(x, E) = \frac{1}{\sqrt{2\pi t}} \int_E \exp\left[-\frac{(y - mt - x)^2}{2t}\right] dy,$$

$\forall x \in \mathbf{R}, \forall E \in \mathfrak{B}(K).$

This is Brwonian motion with constant drift.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Examples (5)

Example 5 (Brownian motion with reflecting barrier)

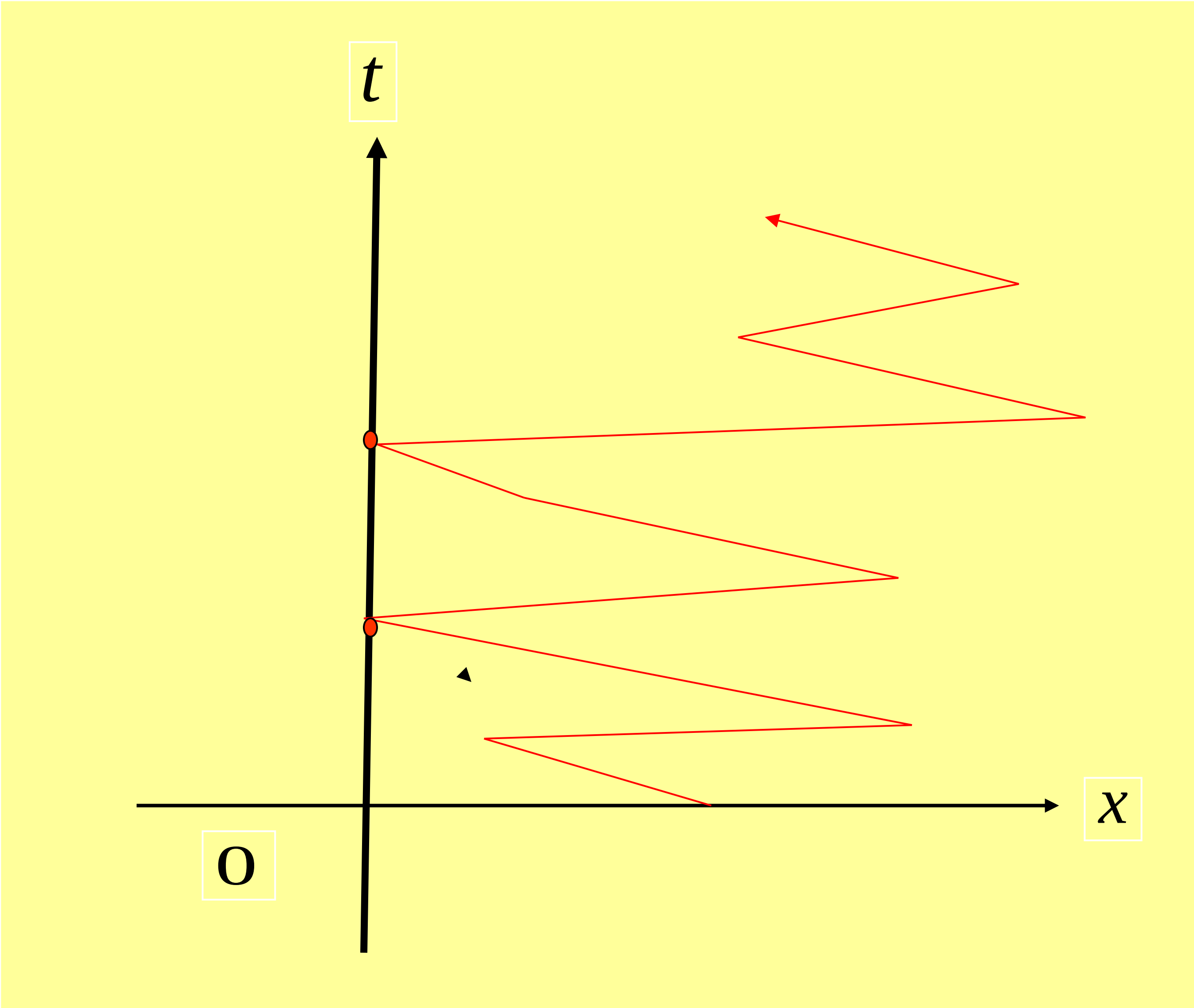
$K = [0, \infty)$

$p_t(x, E)$

$$= \frac{1}{\sqrt{2\pi t}} \left(\int_E \exp\left[-\frac{(y-x)^2}{2t} \right] dy + \int_E \exp\left[-\frac{(y+x)^2}{2t} \right] dy \right),$$

$\forall x \in \mathbf{R}, \ \forall E \in \mathfrak{B}(K).$

This is Brwonian motion with reflecting barrier at $x = 0$.



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Hille-Yosida Theorem

The operator

$\mathfrak{A} : C(K) \to C(K)$

generates a Feller semigroup if it satisfies the following three conditions :

(a) $D(\mathfrak{A})$ **is dense in** $C(K)$.

(b) $\exists! u \in D(\mathfrak{A})$ s.t. $(\alpha - \mathfrak{A})u = f, \ \forall f \in C(K)$.

(c) $\forall f \in C(K), \ f \geq 0 \text{ in } K \Rightarrow (\alpha - \mathfrak{A})^{-1} f \geq 0 \text{ in } K$.

(d) $\left\|(\alpha - \mathfrak{A})^{-1}\right\| \leq \dfrac{1}{\alpha}, \ \forall \alpha > 0$.

Hille-Yosida-Ray Theorem

The operator

$\mathfrak{A} : C(K) \to C(K)$

generates a Feller semigroup if it satisfies the following three conditions :

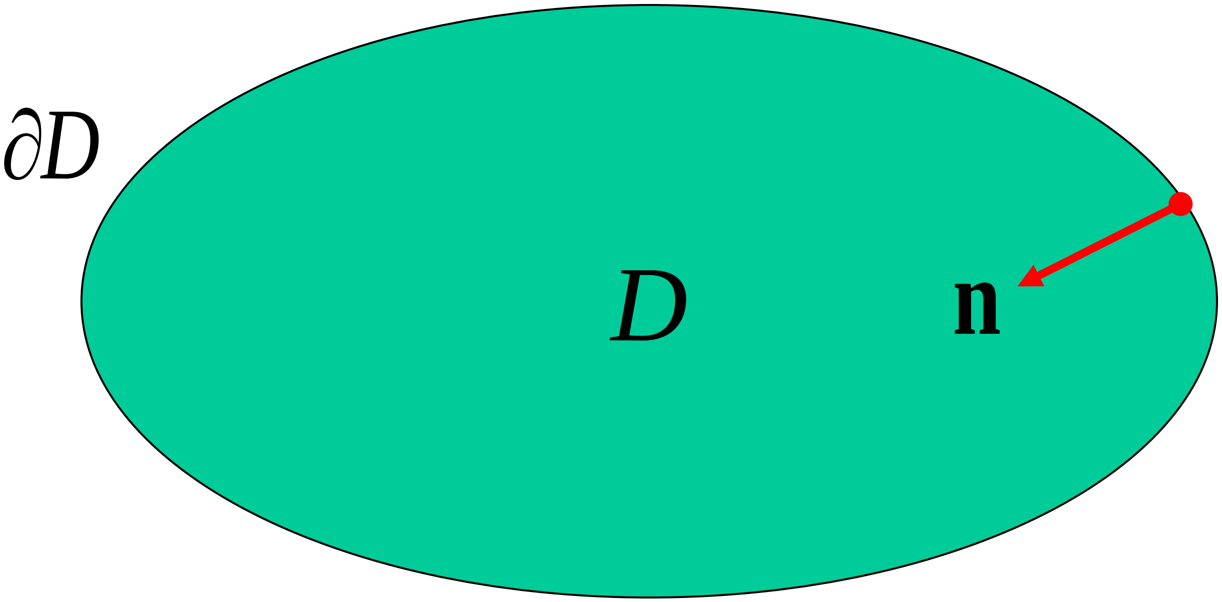
(a) $D(\mathfrak{A})$ **is dense in** $C(K)$

(b) $\exists u \in D(\mathfrak{A})$ s.t. $(\alpha - \mathfrak{A})u = f, \ \forall f \in C(K)$

(c) **If** $u \in D(\mathfrak{A})$ **attains its positive maximum. at a point** $x_0 \in K$, **then** $\mathfrak{A}u(x_0) \le 0.$

Bounded Domain

$\mathbf{R}^N, \quad N \geq 2$



Function Space

$C(\bar{D}) =$ space of real-valued, continuous functions

on the closure $\bar{D} = D \cup \partial D$

with the maximum norm

$$\|u\| = \max_{x \in \bar{D}} |u(x)|$$

Feller Semigroups

A family of bounded linear operators $\{T_t\}_{t\geq 0}$ is called a **Feller semigroup** if it satisfies the following three conditions:

(1) $T_{t+s} = T_t \bullet T_s, \ \forall t, s \geq 0.$

(2) $\lim_{s\downarrow 0} \| T_{t+s} f - T_t f \| = 0, \ \forall f \in C(\overline{D}).$

(3) $\forall f \in C(\overline{D}), 0 \leq f \leq 1 \text{ on } \overline{D} \Rightarrow 0 \leq T_t f \leq 1 \text{ on } \overline{D}.$

Hille-Yosida-Ray Theorem

The operator

$\mathfrak{A} : C(\bar{D}) \to C(\bar{D})$

generates a **Feller semigroup** if it satisfies the following three conditions:

(a) $D(\mathfrak{A})$ is dense in $C(\bar{D})$.

(b) $\exists u \in D(\mathfrak{A})$ s.t. $(\alpha - \mathfrak{A})u = f, \ \forall f \in C(\bar{D})$.

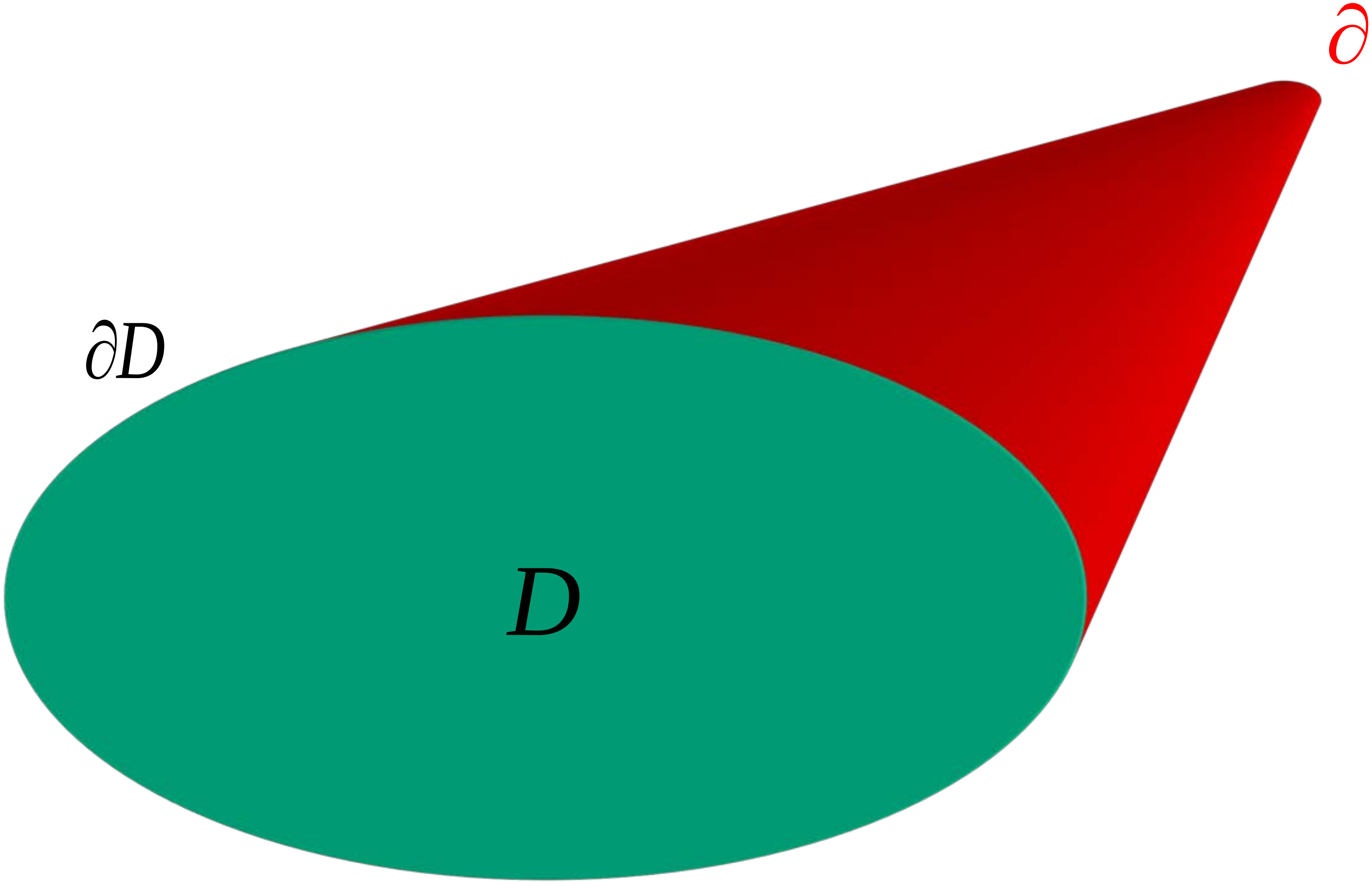
(c) If $u \in D(\mathfrak{A})$ attains its positive maximum at a point $x_0 \in \bar{D}$, then $\mathfrak{A}u(x_0) \leq 0$.

State Space
(Dirichlet case)

$\partial := \partial D$ **one-point compactification**

$$x \sim y \overset{def}{\Longleftrightarrow} \begin{array}{l} (a)\ \ x = y, \\ (b)\ \ x, y \in \partial D \end{array}$$

One-Point Compactification



Function Space
(Dirichlet case)

$$C_0(\bar{D}) = \{u \in C(\bar{D}) : u = 0 \text{ on } \partial D\}$$

with the maximum norm

$$\|u\| = \max_{x \in \bar{D}} |u(x)|$$

Feller Semigroups (Dirichlet case)

A family of bounded linear operators $\{T_t\}_{t\geq 0}$

is called a **Feller semigroup** if it satisfies

the following three conditions:

(1) $T_{t+s} = T_t \bullet T_s, \ \forall t, s \geq 0.$

(2) $\lim_{s\downarrow 0} \left\| T_{t+s} f - T_t f \right\|_\infty = 0, \ \forall f \in C_0(\overline{D}).$

(3) $\forall f \in C_0(\overline{D}), 0 \leq f \leq 1 \text{ on } \overline{D} \Rightarrow 0 \leq T_t f \leq 1 \text{ on } \overline{D}.$

Hille-Yosida-Ray Theorem (Dirichlet case)

The operator

$\mathfrak{A} : C_0(\bar{D}) \to C_0(\bar{D})$

generates a Feller semigroup if it satisfies the following three conditions :

(a) $D(\mathfrak{A})$ **is dense in** $C_0(\bar{D})$.

(b) $\exists u \in D(\mathfrak{A})$ s.t. $(\alpha - \mathfrak{A})u = f, \ \forall f \in C_0(\bar{D})$.

(c) **If** $u \in D(\mathfrak{A})$ **attains its positive maximum at a point** $x_0 \in D$, **then** $\mathfrak{A}u(x_0) \le 0$.

Transition Functions and Semigroups

$$T_t f(x) = e^{t\mathfrak{A}} f(x)$$

$$= \int_{\overline{D}} p_t(x, dy) f(y), \ \forall f \in C_0(\overline{D})$$

Semigroups and Green Operators

$$T_t = e^{t\mathfrak{A}}$$

Laplace Transform

$$G_\alpha := \int_0^\infty e^{-\alpha t} T_t \, dt = \int_0^\infty e^{-\alpha t} e^{t\mathfrak{A}} dt = \left(\alpha - \mathfrak{A}\right)^{-1}$$

Examples (1)

Example 1 (uniform motion)

$K = \mathbf{R}$

$$D(\mathfrak{A}) = \left\{ f \in C_0(K) \cap C^1(K) : f' \in C_0(K) \right\}$$

$$\mathfrak{A} f = v f', \quad \forall f \in D(\mathfrak{A})$$

$C_0(K) =$ **space of real-valued, continuous functions on R vanishing at** $\pm\infty$.

Examples (2)

Example 2 (Poisson process)

$$K = \mathbf{R}$$

$$D(\mathfrak{A}) = C_0(K)$$

$$\mathfrak{A}f = \lambda(f(x+1) - f(x)), \quad \forall f \in D(\mathfrak{A})$$

Examples (3)

Example 3 (Brownian motion)

$K = \mathbf{R}$

$$D(\mathfrak{A}) = \{ f \in C_0(K) \cap C^2(K) : f' \in C_0(K), \quad f'' \in C_0(K) \}$$

$$\mathfrak{A} f = \frac{1}{2} f'', \quad \forall f \in D(\mathfrak{A})$$

Examples (4)

Example 4 (Brownian motion with constant drift)

$$K = \mathbf{R}$$

$$D(\mathfrak{A}) = \{ f \in C_0(K) \cap C^2(K) : f' \in C_0(K), \quad f'' \in C_0(K) \}$$

$$\mathfrak{A}f = \frac{1}{2} f'' + m f', \quad \forall f \in D(\mathfrak{A})$$

Examples (5)

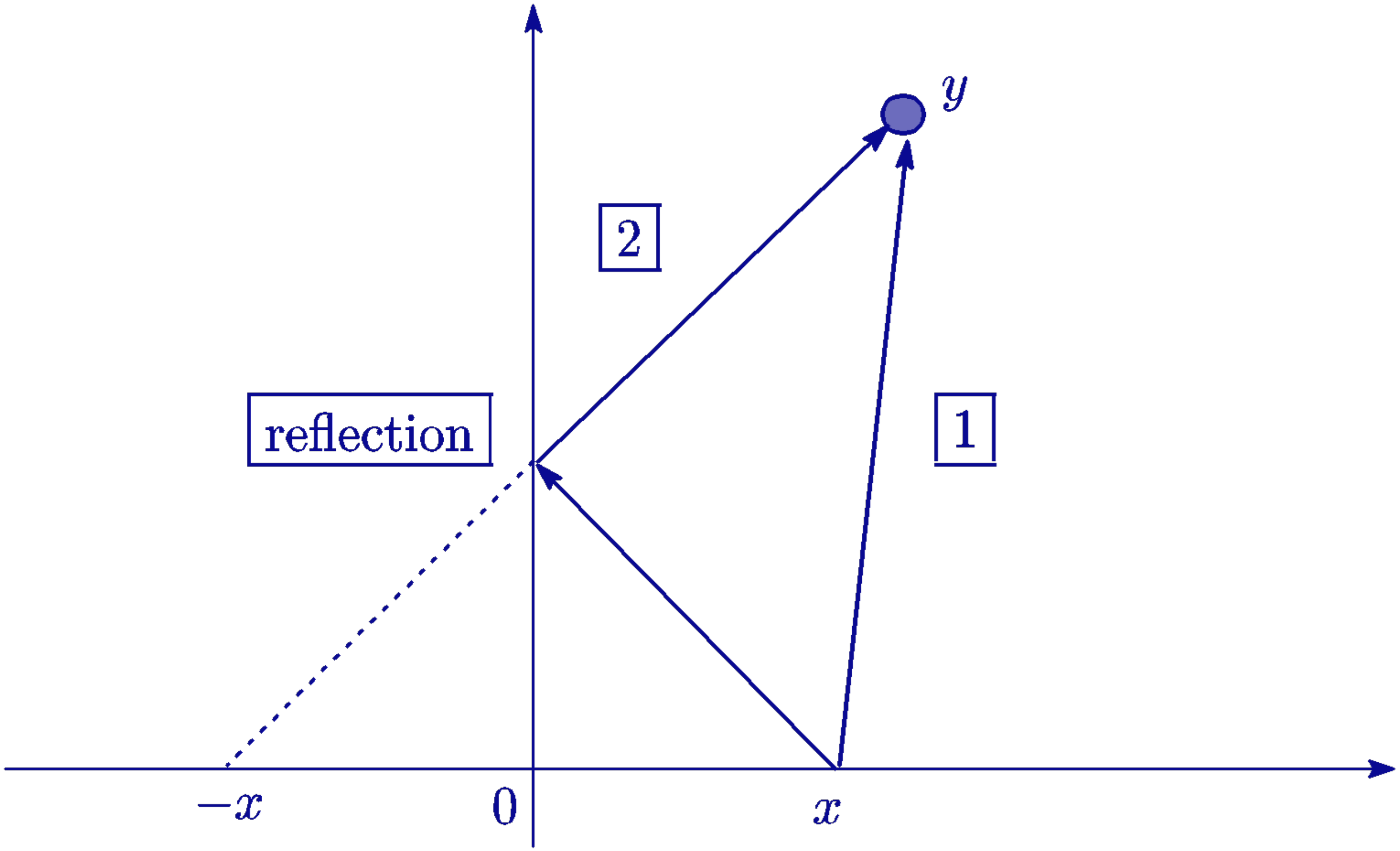
Example 5 (reflecting barrier Brownian motion)

$$K = [0, \infty)$$

$$D(\mathfrak{A}) = \{ f \in C_0(K) \cap C^2(K) : f' \in C_0(K), \quad f'' \in C_0(K), \ f'(0) = 0 \}$$

$$\mathfrak{A} f = \frac{1}{2} f'', \quad \forall f \in D(\mathfrak{A})$$

$C_0(K) =$ **space of real-valued, continuous functions on** $[0, \infty)$ **vanishing at** ∞.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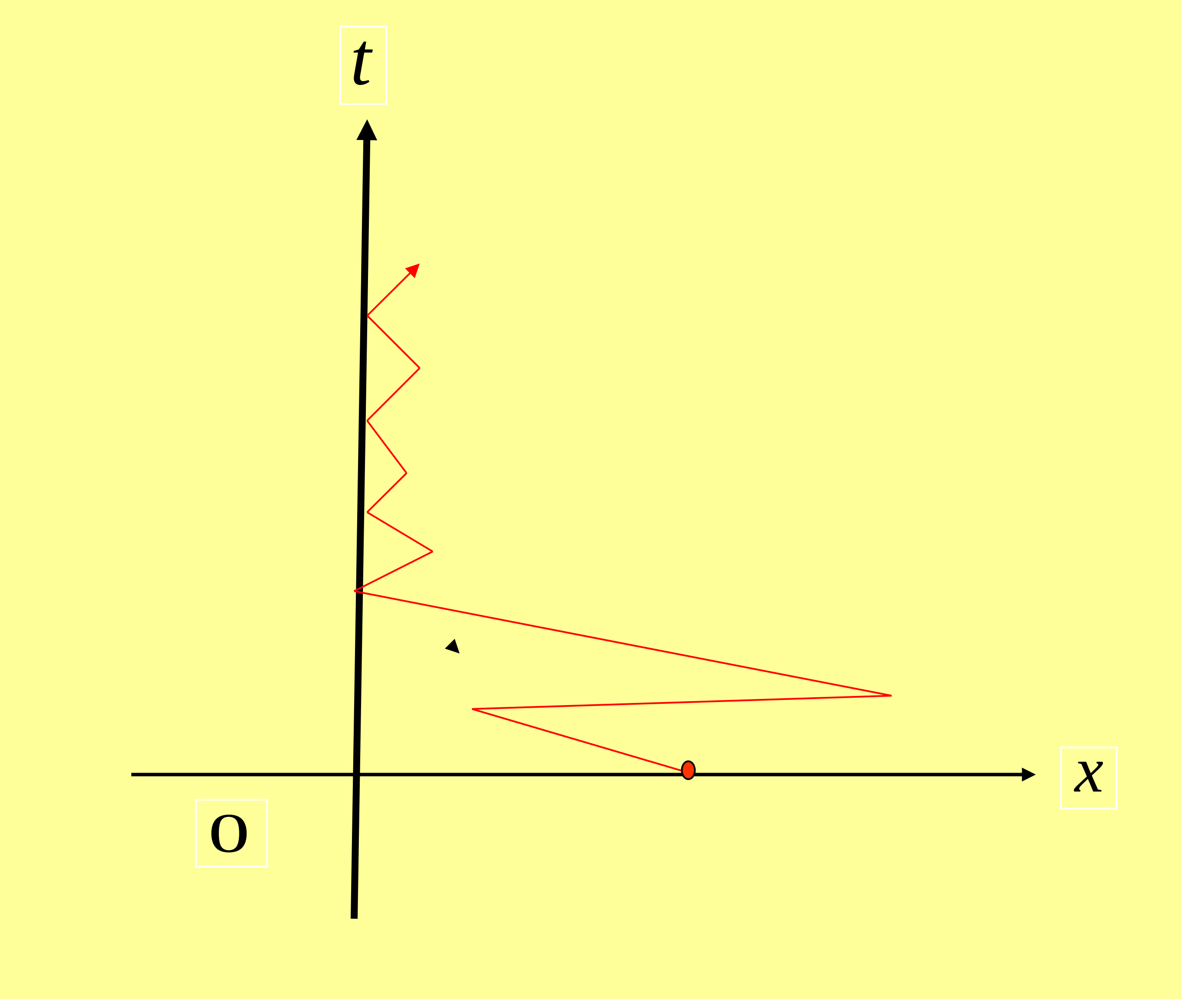
Examples (6)

Example 6 (sticking barrier Brownian motion)

$$K = [0, \infty)$$

$$D(\mathfrak{A}) = \{ f \in C_0(K) \cap C^2(K) : f' \in C_0(K), \\ f'' \in C_0(K),\ f''(0) = 0 \}$$

$$\mathfrak{A} f = \frac{1}{2} f'', \quad \forall f \in D(\mathfrak{A})$$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Examples (7)

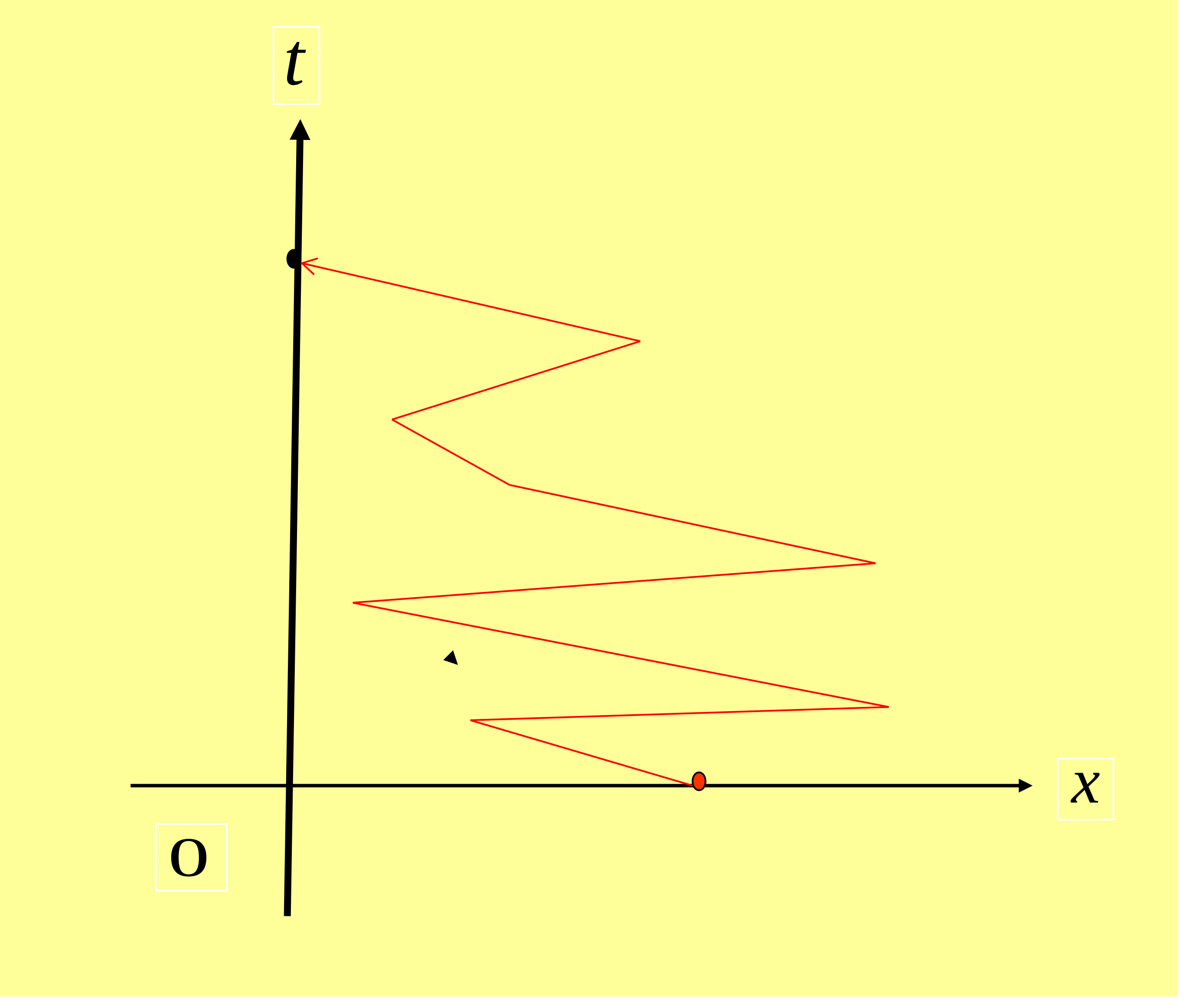
Example 7 (absorbing barrier Brownian motion)

$K = [0, \infty)$ **where** 0 **and** ∞ **are identified.**

$$D(\mathfrak{A}) = \{ f \in C_0(K) \cap C^2(K) : f' \in C_0(K), \quad f'' \in C_0(K), \ f(0) = 0 \}$$

$$\mathfrak{A} f = \frac{1}{2} f'', \quad \forall f \in D(\mathfrak{A})$$

$C_0(K) =$ **space of real-valued, continuous functions on** $[0, \infty)$ **vanishing at** ∞.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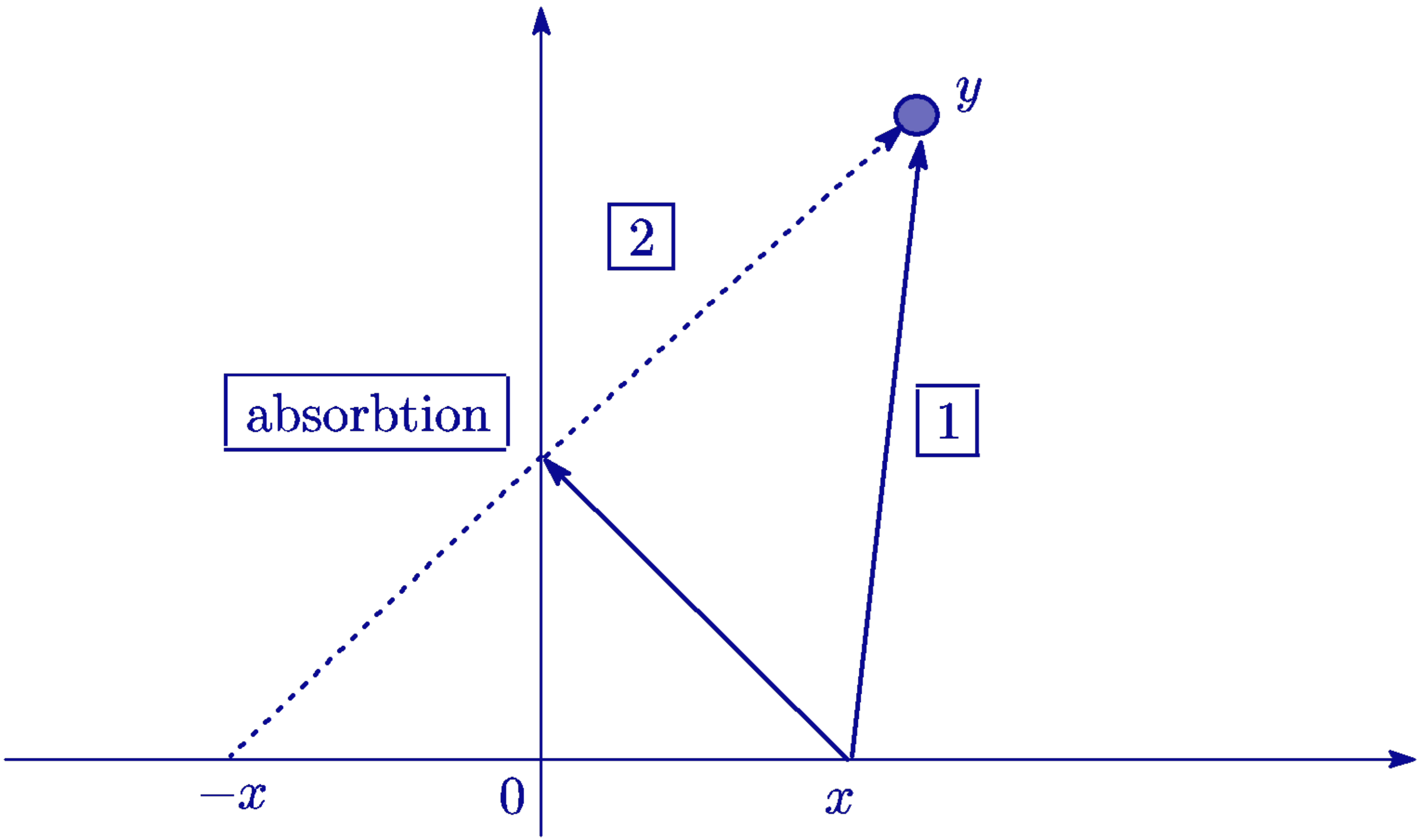
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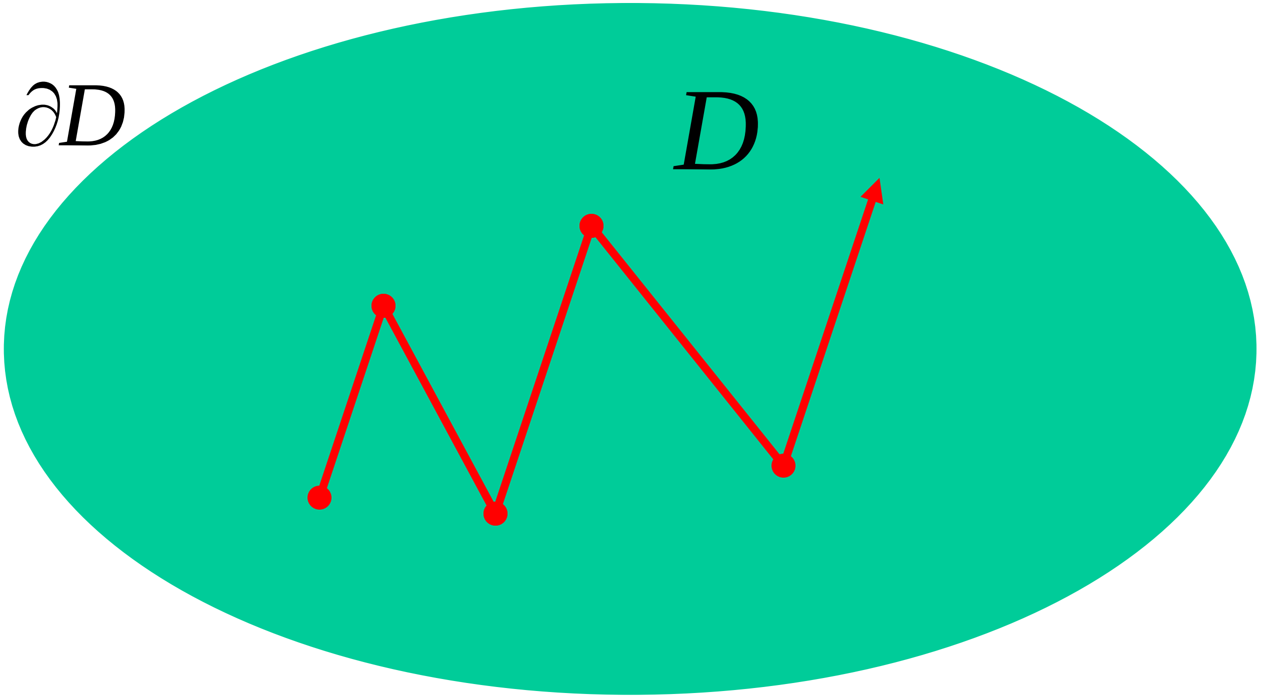
Diffusion Operators
(differential operators)

$$Au = \sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

Diffusion Phenomenon
(continuous motion)



Drift Term

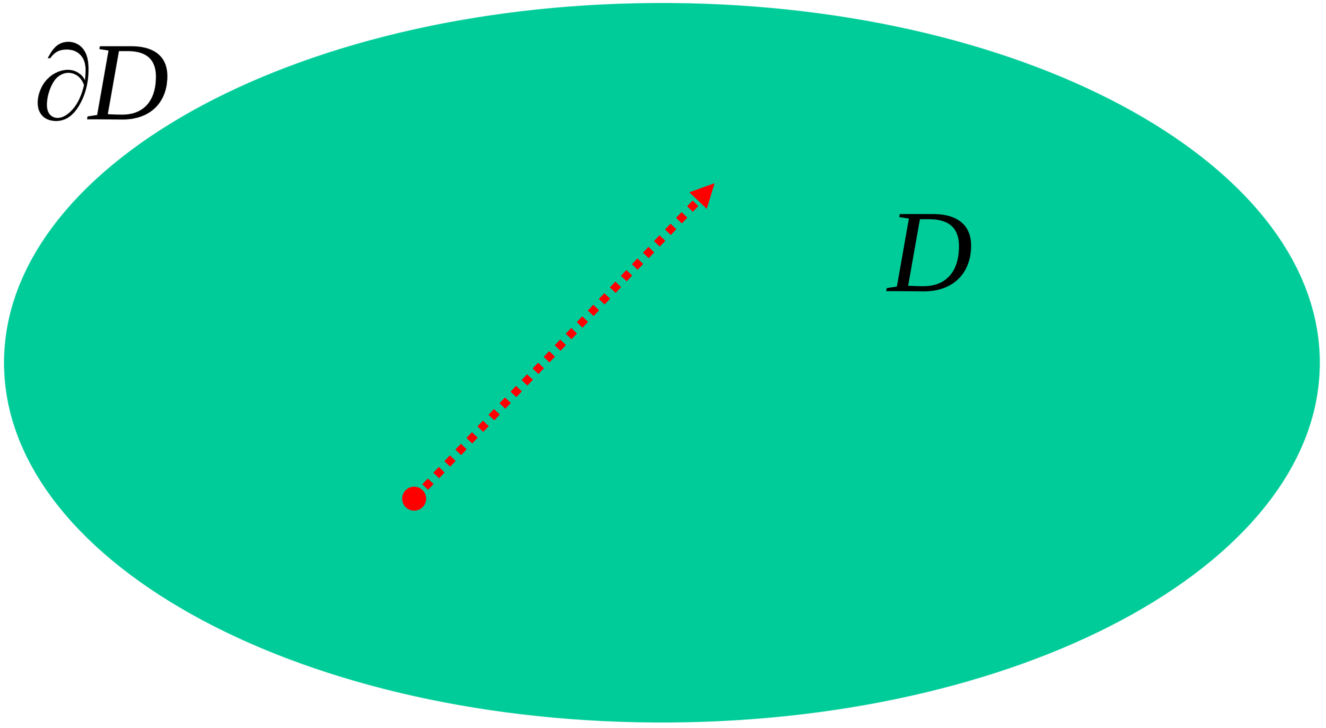
(Subprincipal Symbol)

$$\sum_{i=1}^{N}\left(b^{i}(x) - \frac{1}{2}\sum_{j=1}^{N}\frac{\partial a^{ij}}{\partial x_{j}}(x) \right)\frac{\partial}{\partial x_{i}}$$

Lévy Operators of first order
(integro-differential operators)

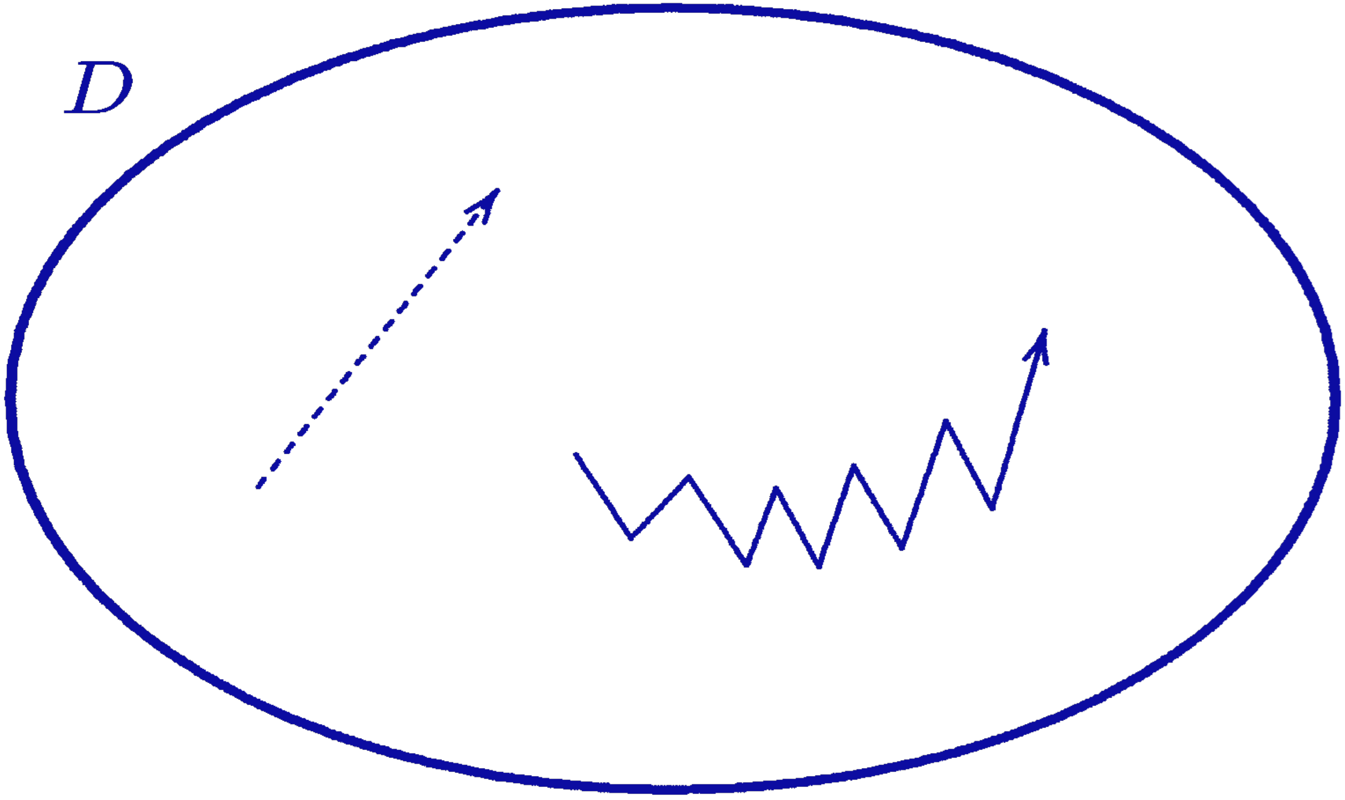
$$Su = \int_D s(x, dy) \left[u(y) - u(x) - \sum_{j=1}^{N} (y_j - x_j) \frac{\partial u}{\partial x_j}(x) \right]$$

Jump Phenomenon
(discontinuous motion)



Waldenfels Operators
(integro-differential operators)

$$
\begin{aligned}
Wu &:= Au + Su \\
&= \sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x) \frac{\partial u}{\partial x_i} + c(x)u \\
&\quad + \int_D s(x,dy) \left[u(y) - u(x) - \sum_{j=1}^{N} (y_j - x_j) \frac{\partial u}{\partial x_j}(x) \right]
\end{aligned}
$$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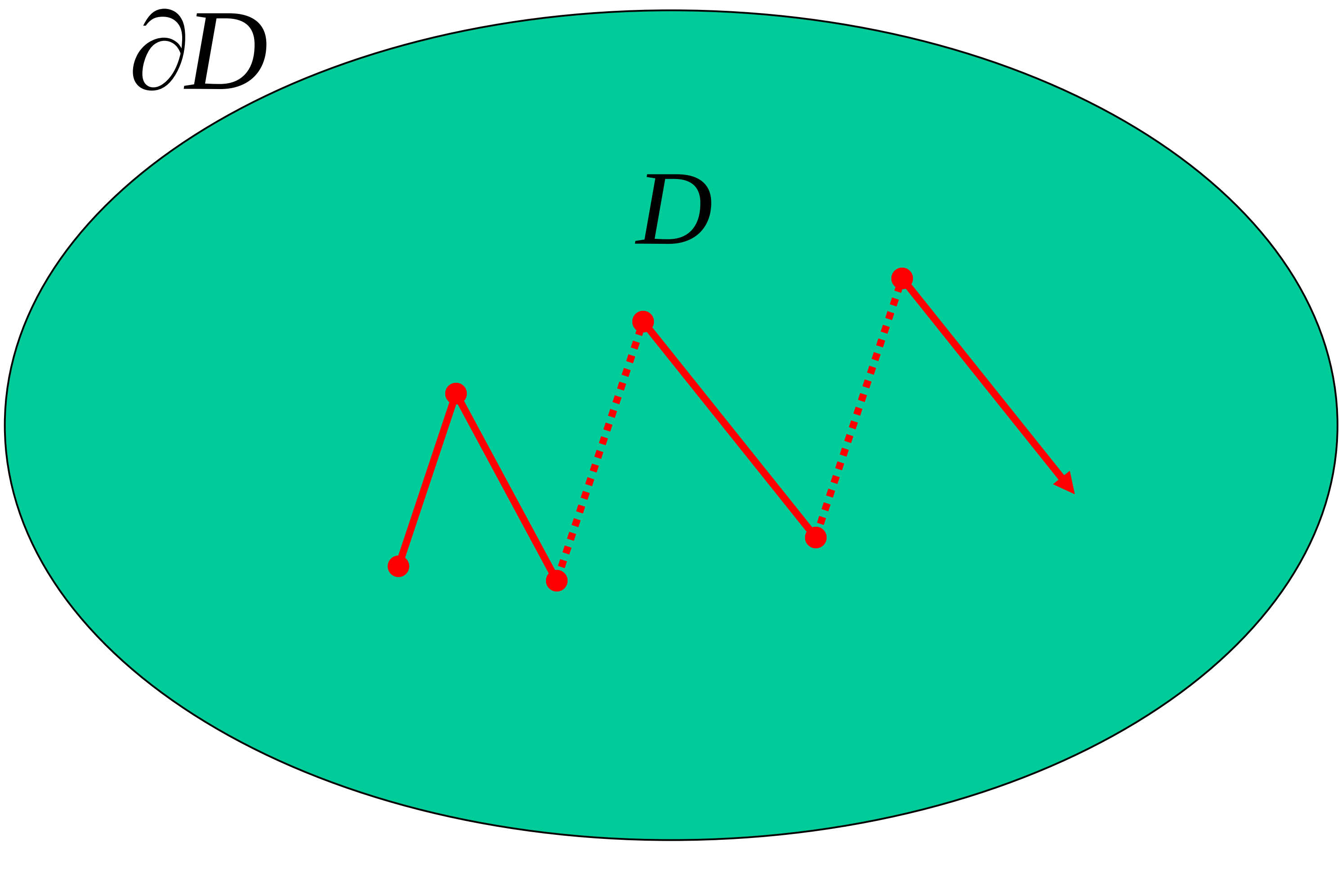

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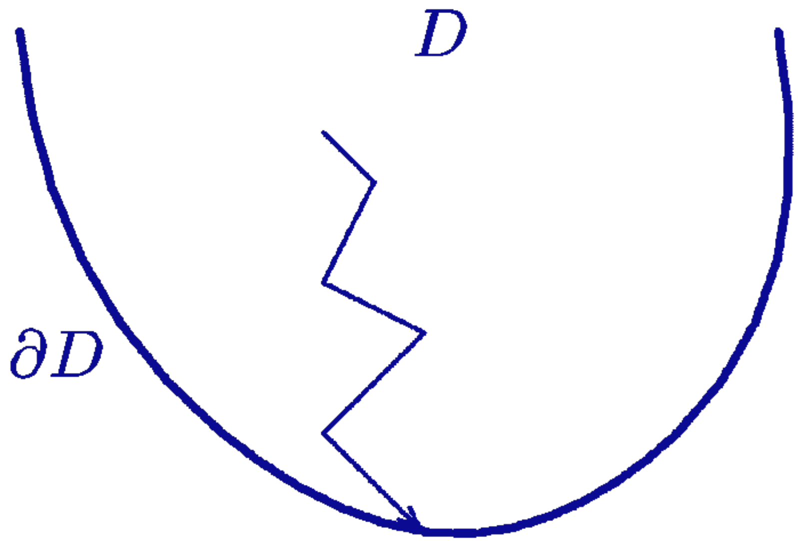
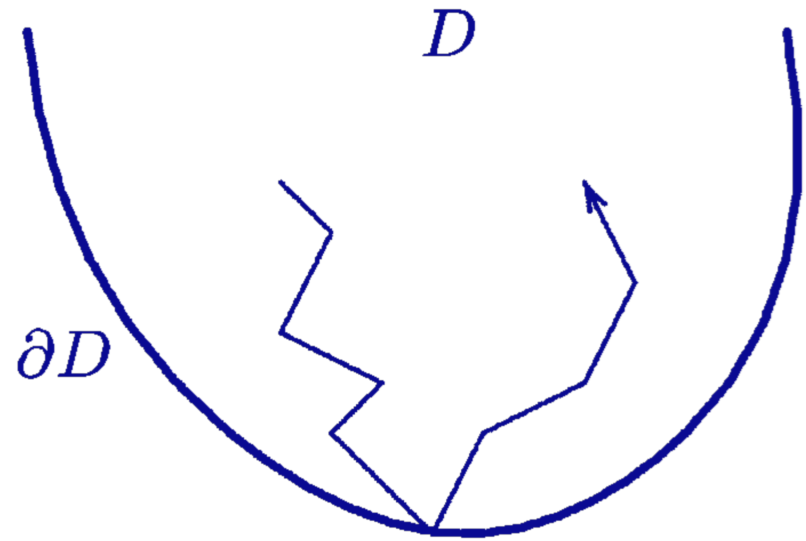
Wentzell boundary conditions
(general form)

$$
\begin{aligned}
Lu &= \sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N-1} \beta^i(x') \frac{\partial u}{\partial x_i} + \gamma(x')u \\
&+ \mu(x') \frac{\partial u}{\partial \mathbf{n}} - \delta(x') W u \\
&+ \int_{\partial D} r(x', dy') \left[u(y') - u(x') - \sum_{j=1}^{N-1} (y_j - x_j) \frac{\partial u}{\partial x_j}(x') \right] \\
&+ \int_{D} t(x', dy) \left[u(y) - u(x') - \sum_{j=1}^{N-1} (y_j - x_j) \frac{\partial u}{\partial x_j}(x') \right]
\end{aligned}
$$

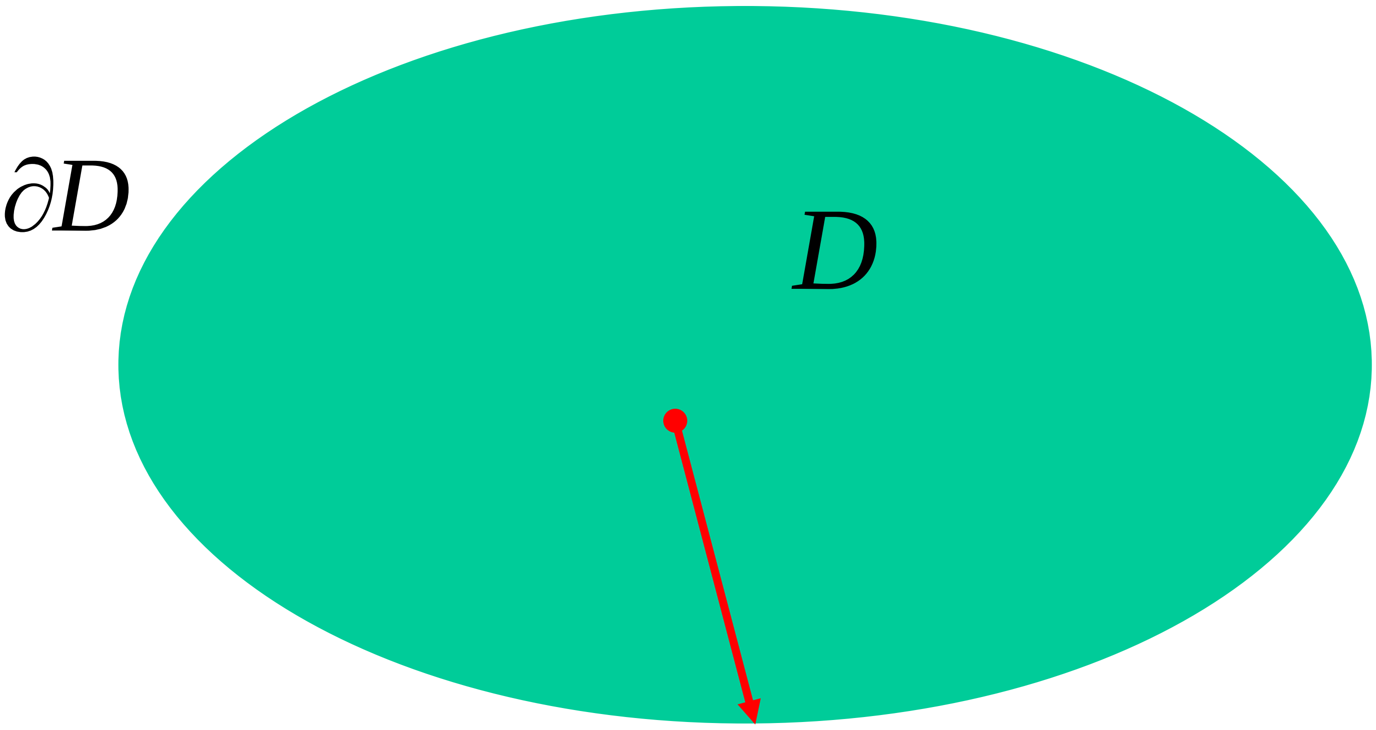


absorption

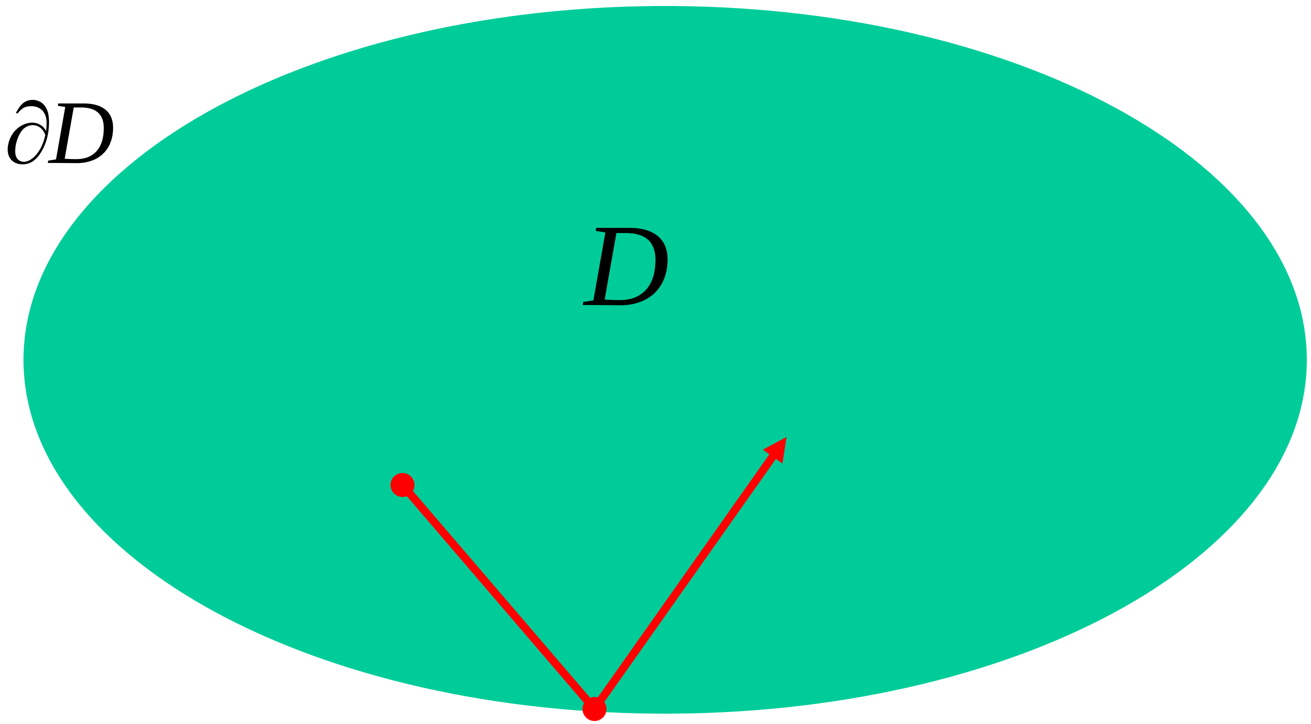


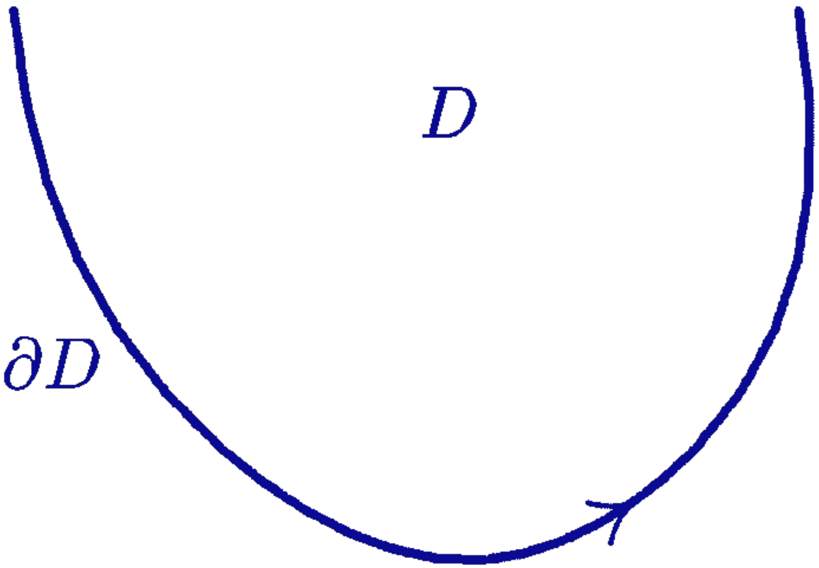
reflection

Absorption Phenomenon
(Dirichlet condition)

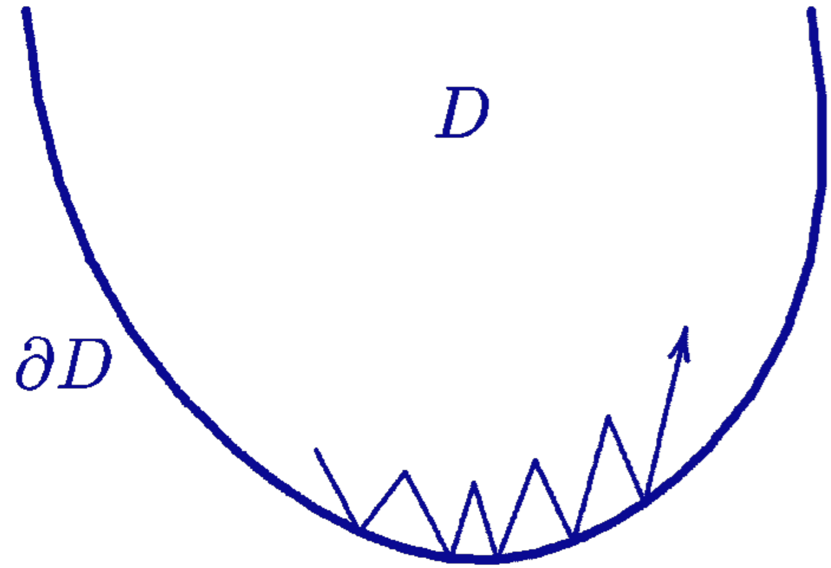


Reflection Phenomenon
(Neumann condition)



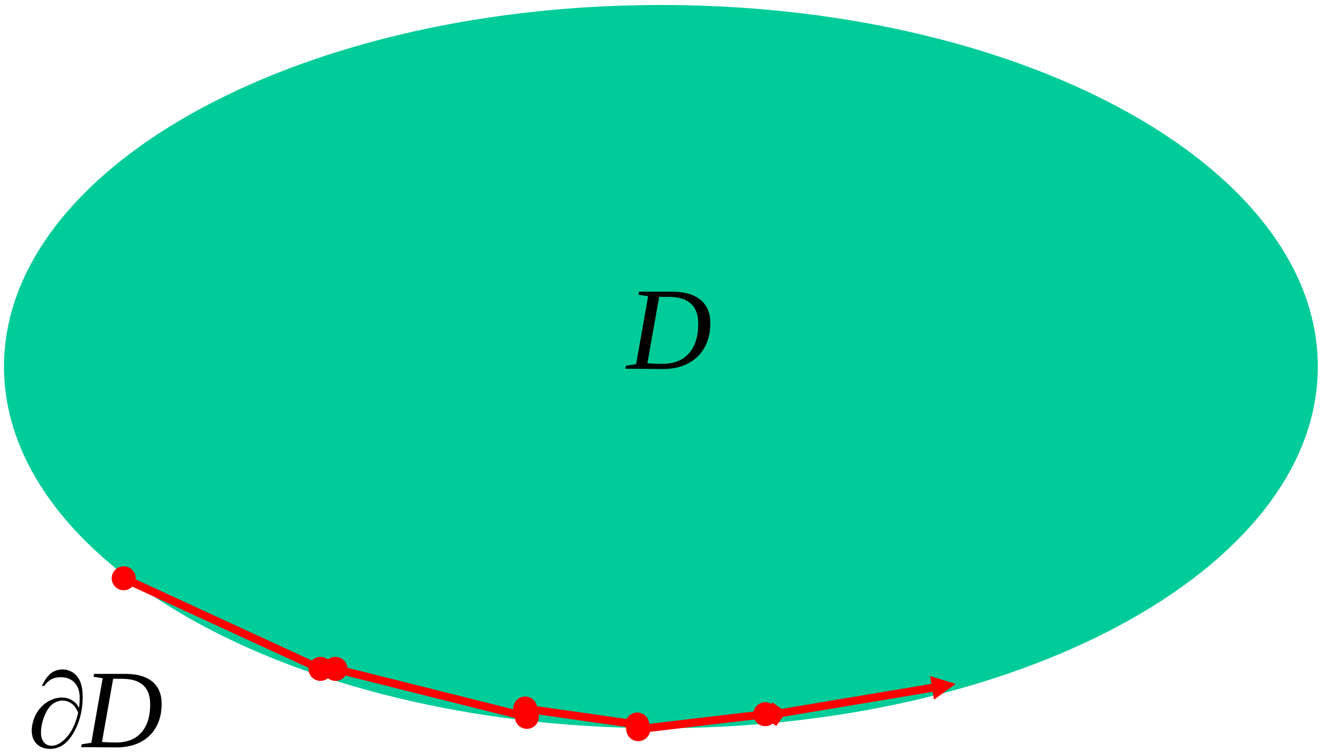


diffusion along the boundary

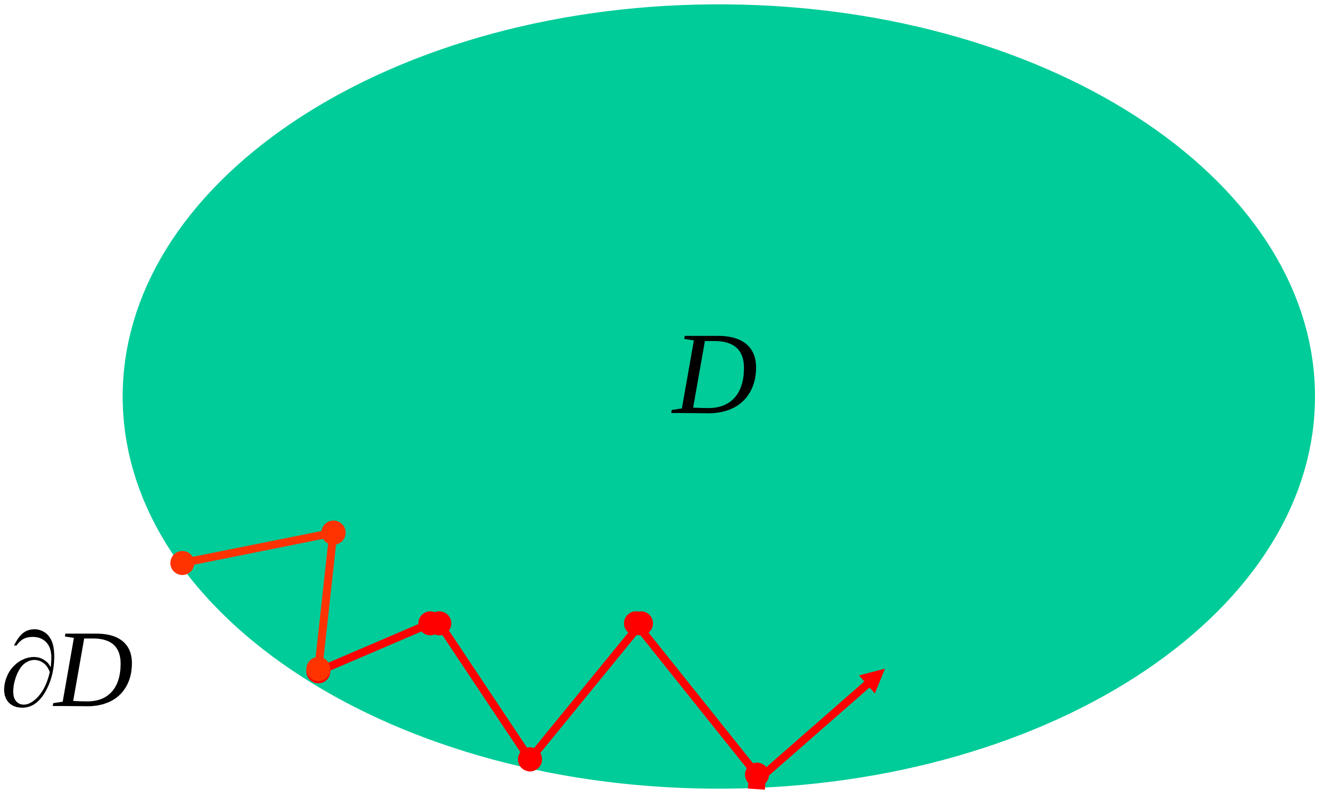


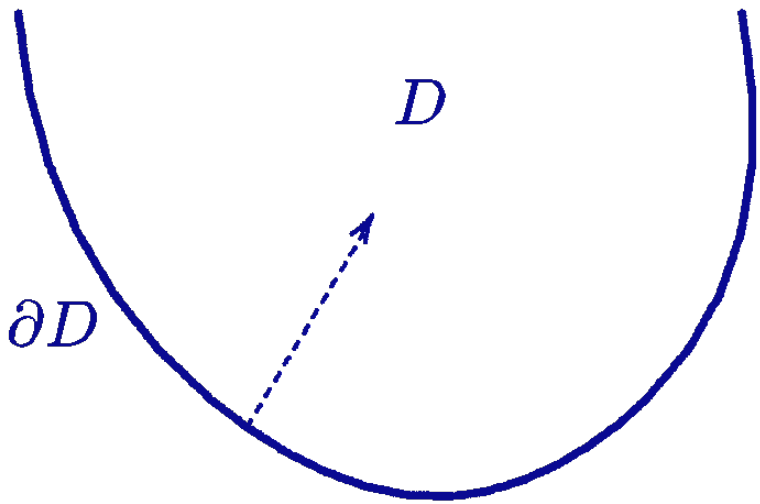
viscosity

Diffusion on the Boundary

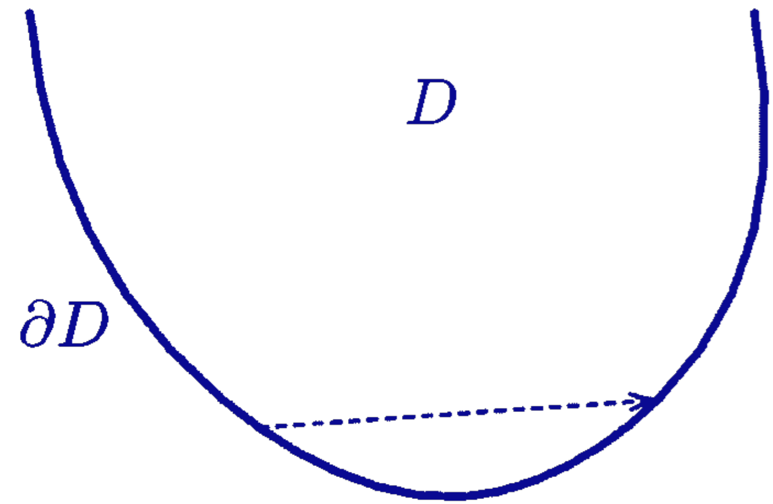


Viscosity Phenomenon



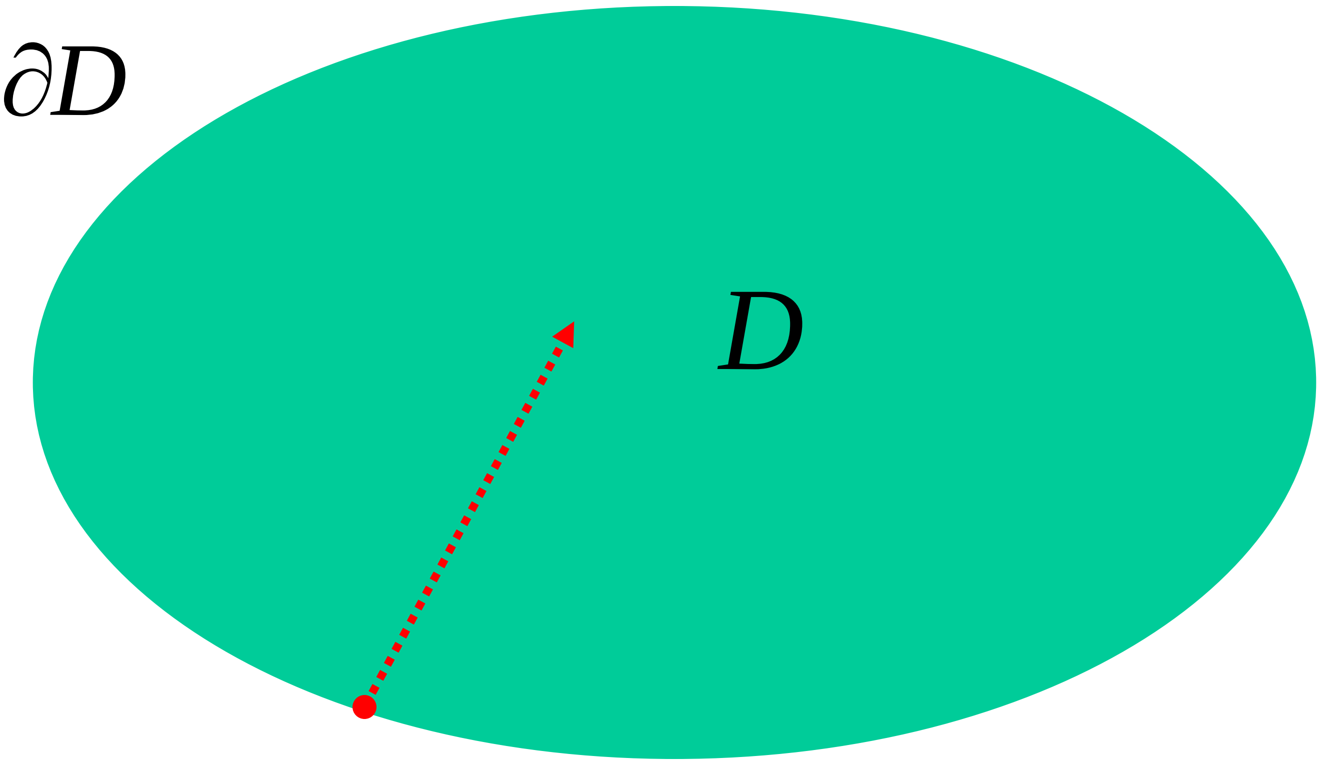


jump into the interior

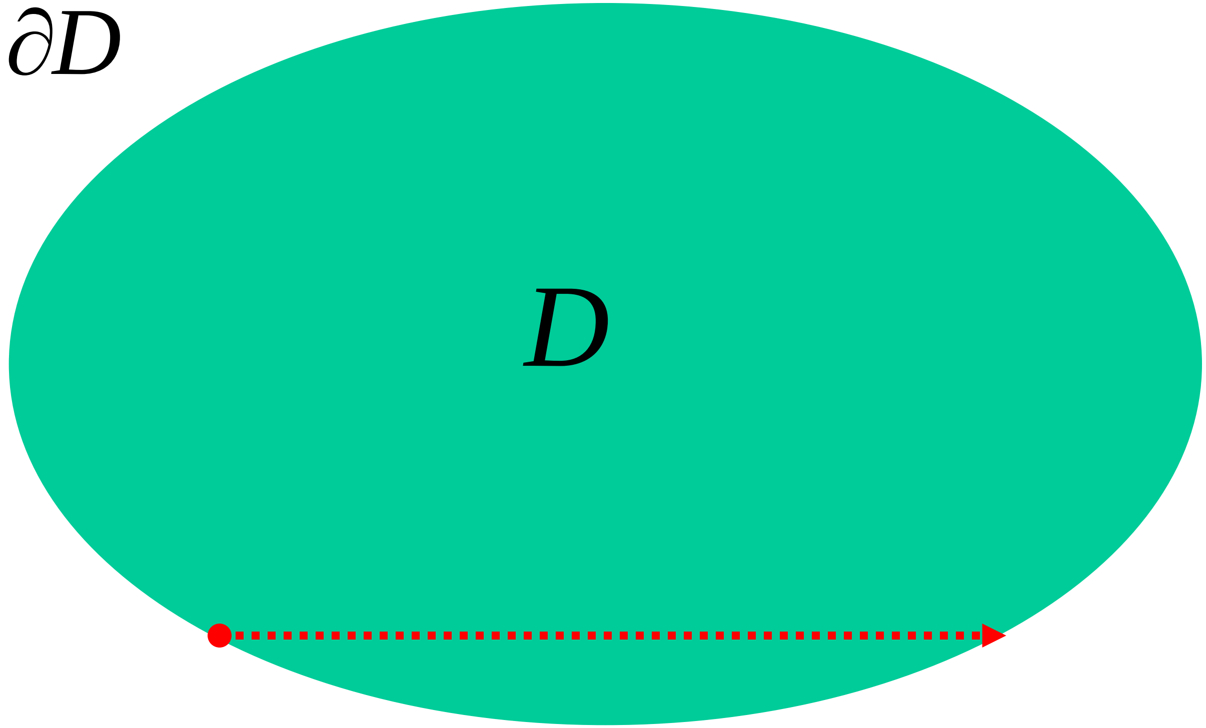


jump on the boundary

Jump Phenomenon (1)



Jump Phenomenon (2)



Purpose of Talk

This talk is devoted to the functional analytic approach to the problem of construction of Feller semigroups with Wentzell boundary conditions. More precisely we consider the following problem:

Problem

Given analytic data (W,L), can we construct a Feller semigroup whose infinitesimal generator is characterized by (W,L) ?

Bird's Eye View

Probability Theory	Functional Analysis	Partial Differential Equations
Markov Process	Feller Semigroup	Infinitesimal Generator
Markov Property	Semigroup Property	•Waldenfels Operator •Wentzell Condition

Waldenfels Operators

$$Wu := Au + Su$$

$$= \sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

$$+ \int_D s(x,y)\left[u(y) - u(x) - \sum_{j=1}^{N} (y_j - x_j)\frac{\partial u}{\partial x_j}(x) \right] dy$$

Diffusion Operators

$$Au = \sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

Here:

(1) $a^{ij}(x) \in C^\infty(\mathbf{R}^N),\ a^{ij}(x) = a^{ji}(x)$

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq \exists\lambda\left|\xi\right|^2,\ \forall x \in \mathbf{R}^N, \forall \xi \in \mathbf{R}^N$$

(2) $b^i(x) \in C^\infty(\mathbf{R}^N)$

(3) $c(x) \in C^\infty(\mathbf{R}^N),\ c(x) \leq 0,\ \forall x \in D$

Lévy Operators of first order

$$Su = \int_D s(x,y)\left[u(y) - u(x) - \sum_{j=1}^{N}(y_j - x_j)\frac{\partial u}{\partial x_j}(x)\right]dy$$

Here:

(1) $s(x,y)$, **distribution kernel** of

$$S \in L_{cl}^{2-\kappa}\left(\mathbf{R}^N\right), \kappa > 0$$

(2) $\boxed{s(x,y) \geq 0, \ \forall x \neq y}$

Wentzell Boundary Conditions (1)

$$Lu = \sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N-1} \beta^i(x') \frac{\partial u}{\partial x_i} + \gamma(x')u$$

$$+\mu(x') \frac{\partial u}{\partial \mathbf{n}} - \delta(x')Wu$$

$$+\int_{\partial D} r(x', y') \left[u(y') - u(x') - \sum_{j=1}^{N-1} (y_j - x_j) \frac{\partial u}{\partial x_j}(x') \right] dy'$$

$$+\int_{D} t(x', y) \left[u(y) - u(x') - \sum_{j=1}^{N-1} (y_j - x_j) \frac{\partial u}{\partial x_j}(x') \right] dy$$

Wentzell Boundary Conditions (2)

$$(1)\ \alpha^{ij}(x) \in C^{\infty}(\partial D),\ \alpha^{ij}(x') = \alpha^{ji}(x')$$

$$\sum_{i,j=1}^{N-1} \alpha^{ij}(x')\eta_i\eta_j \geq 0,\ \forall x' \in \partial D, \forall \eta' \in T^*_{x'}(\partial D)$$

$$(2)\ \gamma(x') \in C^{\infty}(\partial D),\ \gamma(x') \leq 0,\ \forall x' \in \partial D$$

$$(3)\ \mu(x') \in C^{\infty}(\partial D),\ \mu(x') \geq 0,\ \forall x' \in \partial D$$

$$(4)\ \delta(x') \in C^{\infty}(\partial D),\ \delta(x') \geq 0,\ \forall x' \in \partial D$$

Wentzell Boundary Conditions (3)

(1) $r(x', y')$, **distribution kernel of**

$$R \in L_{cl}^{2-\kappa_1}(\partial D),\ \kappa_1 > 0$$

(2) $\boxed{r(x', y') \geq 0,\ \ \forall x' \neq y'}$

(3) $t(x, y)$, **distribution kernel of**

$$T \in L_{cl}^{2-\kappa_2}\left(\mathbf{R}^N\right),\ \kappa_2 > 0$$

(4) $\boxed{t(x, y) \geq 0,\ \ \forall x \neq y}$

Transversal Condition (1)

$$\int_D t(x', y)dy = +\infty \text{ if } \mu(x') = \delta(x') = 0$$

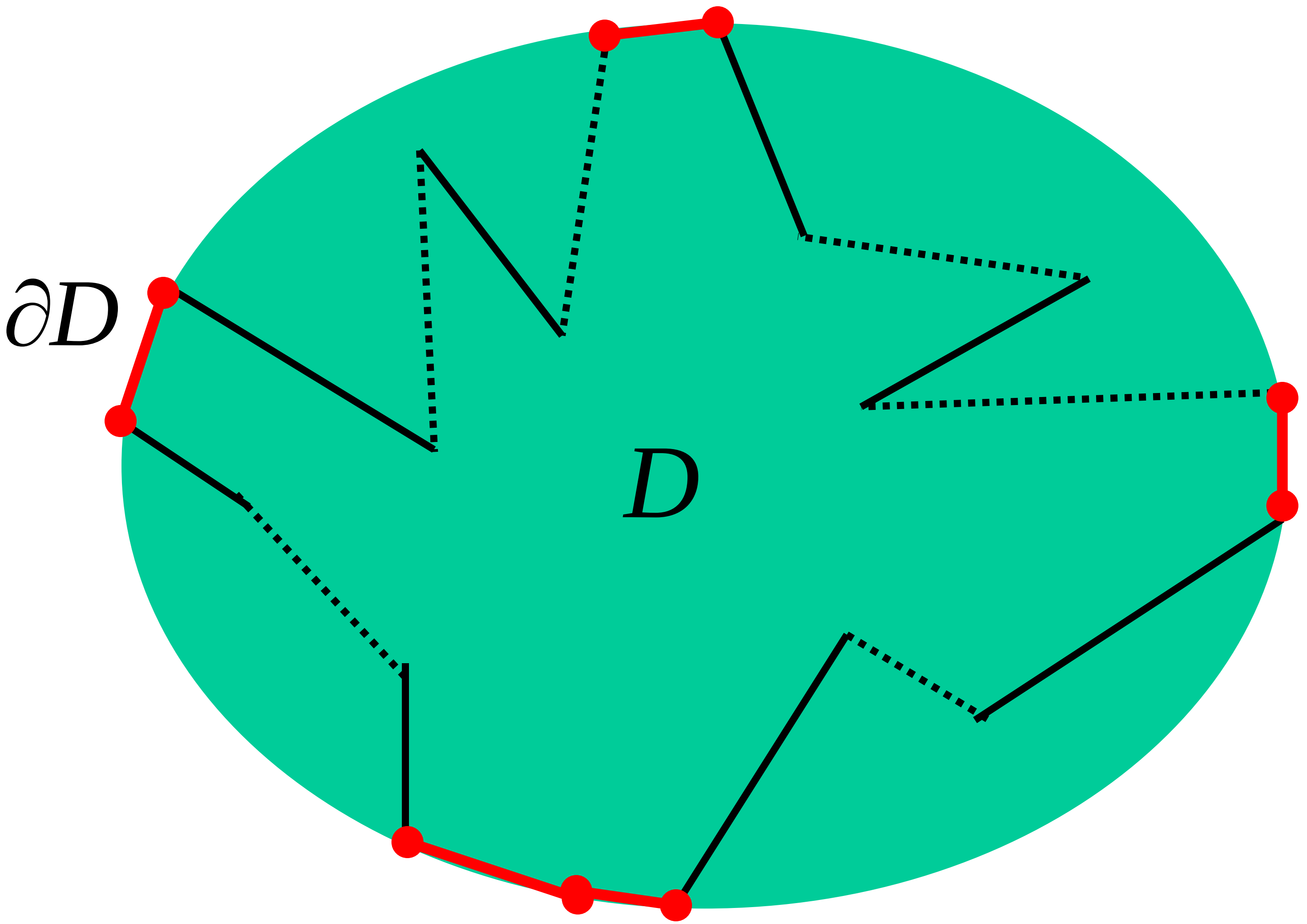
Transversal Condition (2)

Intuitively, the transversality condition implies that a Markovian particle **jumps away instantaneously** from the points $x' \in \partial D$ where neither reflection nor viscosity phenomenon occurs (which is similar to the reflection phenomenon).

Transversal Condition (3)

Probabilistically, this means that every Markov process on the boundary ∂D is the **trace** on ∂D of trajectories of some Markov process on the closure

$$\overline{D} = D \cup \partial D.$$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Transversal Condition (2)
-Reduction to the Boundary-

Probability Theory	Partial Differential Equations
Markov processes on the boundary	Fredholm integral equations
Markov processes on the domain	Boundary value problems

Main Theorem (general case)

We define a linear operator

$\mathfrak{W} : C(\overline{D}) \to C(\overline{D})$

as follows:

(a) $D(\mathfrak{W}) = \left\{ u \in C(\overline{D}) : Wu \in C(\overline{D}),\, Lu = 0 \right\}$

(b) $\mathfrak{W}u = Wu = (A + S)u, \ \forall u \in D(\mathfrak{W})$

If L is **transversal,** then $\mathfrak{W}$ generates a **Feller semigroup.**

Hille-Yosida-Ray Theorem (general case)

The operator

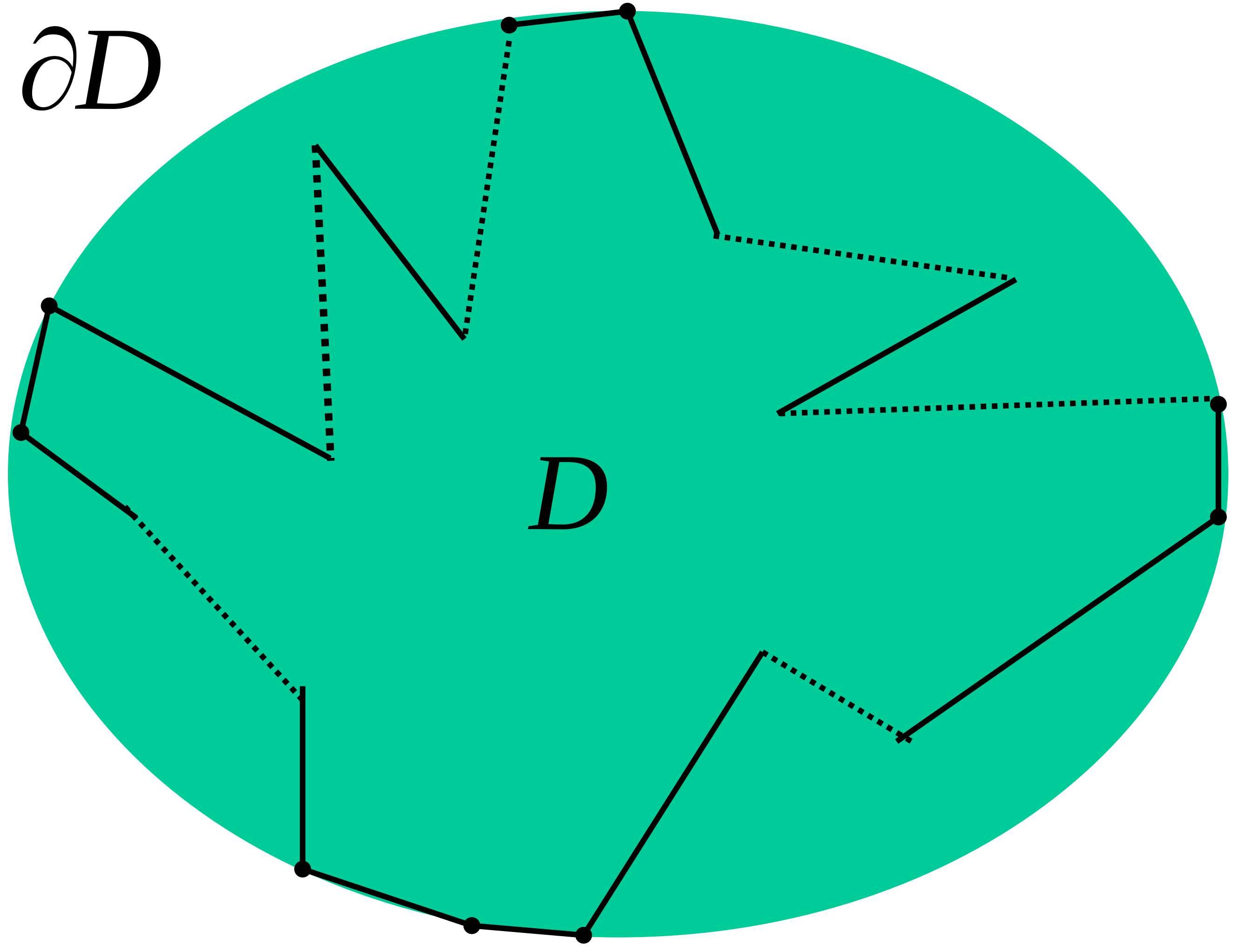
$\mathfrak{W} : C(\overline{D}) \to C(\overline{D})$

generates a **Feller semigroup** if it satisfies the following three conditions:

(a) $D(\mathfrak{W})$ is dense in $C(\overline{D})$

(b) $\exists u \in D(\mathfrak{W})$ s.t. $(\alpha - \mathfrak{W})u = f, \ \forall f \in C(\overline{D})$

(c) If $u \in D(\mathfrak{W})$ attains its positive maximum at a point $x_0 \in \overline{D}$, then $\mathfrak{W}u(x_0) \leq 0$.



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Main Theorem (Dirichlet case)

We define a linear operator

$\mathfrak{W} : C_0(\overline{D}) \to C_0(\overline{D})$

as follows:

(a) $D(\mathfrak{W}) = \left\{ u \in C_0(\overline{D}) : Wu \in C_0(\overline{D}) \right\}$

(b) $\mathfrak{W} u = Wu = (A+S)u, \ \forall u \in D(\mathfrak{W})$

Then $\mathfrak{W}$ generates a **Feller semigroup.**

Hille-Yosida-Ray Theorem (Dirichlet case)

The operator

$\mathfrak{W} \ : C_0(\bar{D}) \to C_0(\bar{D})$

generates a **Feller semigroup** if it satisfies the following three conditions:

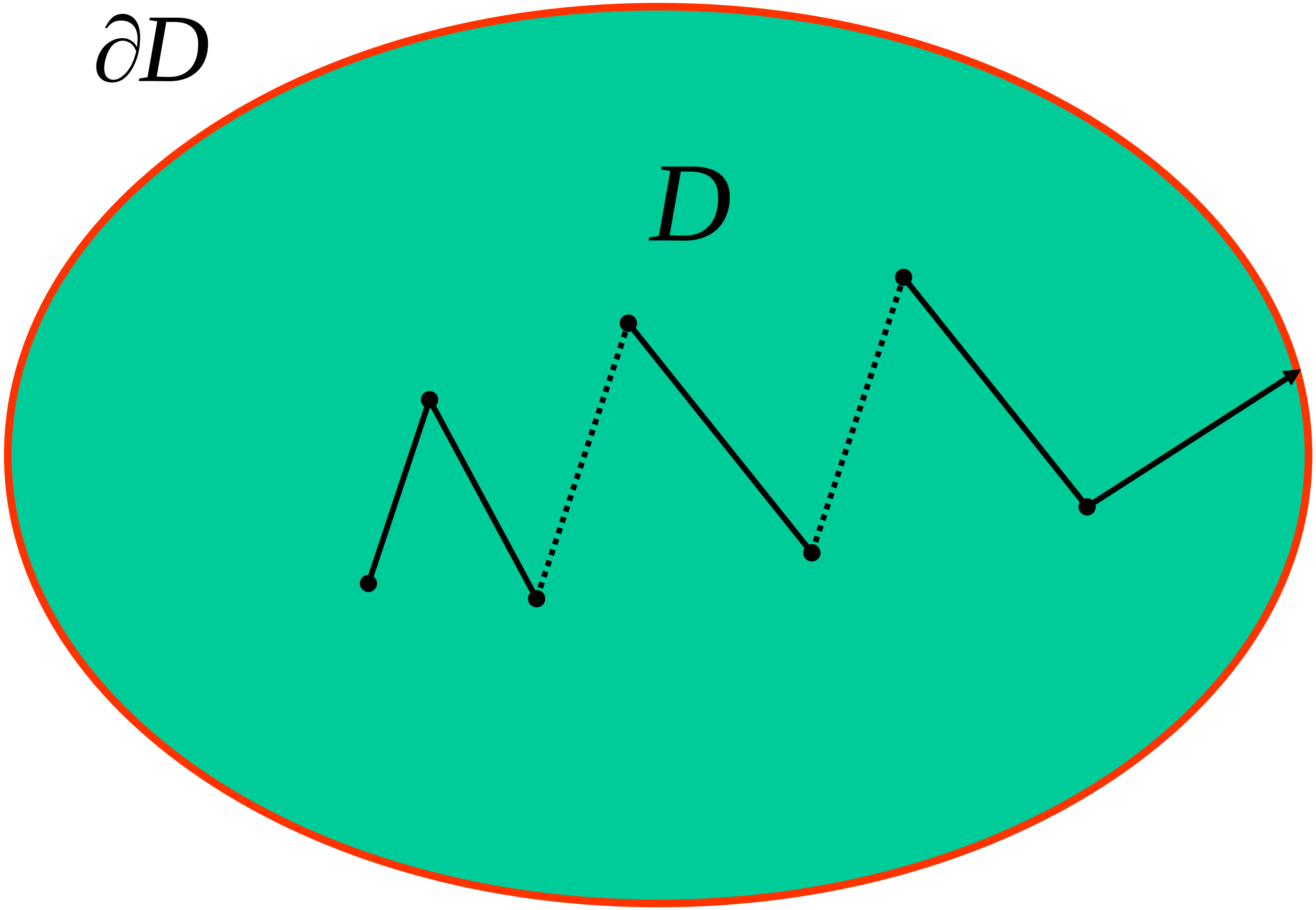
(a) $D(\mathfrak{W})$ is dense in $C_0(\bar{D})$.

(b) $\exists u \in D(\mathfrak{W})$ s.t. $(\alpha - \mathfrak{W})u = f, \ \forall f \in C_0(\bar{D})$.

(c) If $u \in D(\mathfrak{W})$ attains its **positive maximum** at a point $x_0 \in D$, then $\mathfrak{W}u(x_0) \leq 0$.

Conclusion

Rephrased, Main Theorem states that there exists a Feller semigroup corresponding to such a diffusion phenomenon that a Markovian particle moves both by jumps and continuously in the state space until it dies at the time when it reaches the boundary.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Waldenfels Operators

$$Wu := Au + Su$$

$$= \sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x) \frac{\partial u}{\partial x_i} + c(x)u$$

$$+ \int_D s(x,dy) \left[u(y) - u(x) - \sum_{j=1}^{N} (y_j - x_j) \frac{\partial u}{\partial x_j}(x) \right]$$

Wentzell boundary conditions

$$Lu = \sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N-1} \beta^i(x') \frac{\partial u}{\partial x_i} + \gamma(x')u$$

$$+\mu(x') \frac{\partial u}{\partial \mathbf{n}} - \delta(x') Wu$$

$$+\int_{\partial D} r(x', dy') \left[u(y') - u(x') - \sum_{j=1}^{N-1} (y_j - x_j) \frac{\partial u}{\partial x_j}(x') \right]$$

$$+\int_{D} t(x', dy) \left[u(y) - u(x') - \sum_{j=1}^{N-1} (y_j - x_j) \frac{\partial u}{\partial x_j}(x') \right]$$

Sketch of Proof (1)

We reduce the problem of construction of Feller semigroups to the problem of unique solvability of the boundary value problem

$$\begin{cases} (\alpha - W)u = f \quad \text{in } D, \\ Lu = 0 \quad \text{on } \partial D \end{cases}$$

Sketch of Proof (2)

We consider the Dirichlet problem

$$(\alpha - W)v = f \quad \text{in } D,$$
$$v = 0 \quad \text{on } \partial D$$

Let

$$v := G_\alpha^0 f \quad (\textbf{Green operator})$$

Sketch of Proof (3)

Let

$$\boxed{w := u - v = u - G_\alpha^0 f}$$

Sketch of Proof (4)

Then :

$$(\alpha - W)u = f \ \text{ in } D,$$
$$Lu = 0 \ \text{ on } \partial D$$

$$\Longleftrightarrow$$

$$(\alpha - W)w = 0 \ \text{ in } D,$$
$$Lw = -Lv = -LG_{\alpha}^{0} f \ \text{ on } \partial D$$

Sketch of Proof (5)

Every solution *w* of the equation

$$(\alpha - W)w = 0 \ \text{ in } D$$

can be expressed by means of

a single layer potential as follows

$$w = H_\alpha \psi \ \ (\textbf{Harmonic operator})$$

Sketch of Proof (6)

Then :

$$(\alpha - W)u = f \quad \text{in } D,$$
$$Lu = 0 \quad \text{on } \partial D$$

$$\Longleftrightarrow$$

$$LH_{\alpha}\psi = Lw = -LG_{\alpha}^{0} f \quad \text{on } \partial D$$

(**Fredholm integral equation**)

Fredholm Boundary Operator (1)

$$
\begin{aligned}
LH_\alpha\varphi = {} & \sum_{i,j=1}^{N-1}\alpha^{ij}(x')\frac{\partial^2\varphi}{\partial x_i\partial x_j}+\sum_{i=1}^{N-1}\beta^i(x')\frac{\partial\varphi}{\partial x_i}+\gamma(x')\varphi \\
& -\alpha\delta(x')\varphi+\mu(x')\frac{\partial}{\partial\mathbf{n}}\left(H_\alpha\varphi\right) \\
& +\int_{\partial D} r(x',y')\left[\varphi(y')-\varphi(x')-\sum_{j=1}^{N-1}(y_j-x_j)\frac{\partial\varphi}{\partial x_j}(x')\right]dy' \\
& +\int_D t(x',y)\left[H_\alpha\varphi(y)-\varphi(x')-\sum_{j=1}^{N-1}(y_j-x_j)\frac{\partial\varphi}{\partial x_j}(x')\right]dy
\end{aligned}
$$

Dirichlet-Neumann Operator

$$\frac{\partial}{\partial \mathbf{n}}\left(H_\alpha \varphi\right)(x')$$

$$= \int_{\partial D} \pi_\alpha(x', y') \left[\varphi(y') - \sigma(x', y') \left(\varphi(x') + \sum_{j=1}^{N-1} (y_j - x_j) \frac{\partial \varphi}{\partial x_j}(x') \right) \right] dy'$$

Here :

(1) $\sigma(x', y') \in C^\infty(\partial D \times \partial D)$ **such that**

$0 \le \sigma(x', y') \le 1$ on $\partial D \times \partial D$.

$\sigma(x', y') = 1$ near $x' = y'$.

(2) $\pi_\alpha(x', y') \ge 0, \ \forall x' \ne y'$.

Fredholm Boundary Operator (2)

$$LH_\alpha \varphi$$

$$= \sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \sum_{i=1}^{N-1} \beta^i(x') \frac{\partial \varphi}{\partial x_i} + \left(\gamma(x') - \alpha\delta(x') \right) \varphi$$

$$+ \int_{\partial D} \tilde{\pi}_\varepsilon(x', y') \left[\varphi(y') - \tilde{\sigma}(x', y') \left(\varphi(x') + \sum_{j=1}^{N-1} (y_j - x_j) \frac{\partial \varphi}{\partial x_j}(x') \right) \right] dy'.$$

Fredholm Boundary Operator (3)

Here :

(1) $\tilde{\sigma}(x', y') \in C^{\infty}(\partial D \times \partial D)$ **such that**

$$0 \le \tilde{\sigma}(x', y') \le 1 \text{ on } \partial D \times \partial D.$$

$$\tilde{\sigma}(x', y') = 1 \text{ near } x' = y'.$$

(2) $\tilde{\pi}_{\alpha}(x', y') \ge 0, \quad \forall x' \ne y'.$

(3) $\gamma(x') - \alpha\delta(x') + \int_{\partial D} \tilde{\pi}_{\varepsilon}(x', y')\left[1 - \tilde{\sigma}(x', y')\right] dy' \le 0, \quad \forall x' \in \partial D.$

Unique Solvability Theorem

For $\forall k \geq 1,\ \exists \lambda = \lambda(k) > 0$ such that

$$LH_\alpha - \lambda : C^{k+\theta}(\partial D) \to C^{k+\theta}(\partial D)$$

is **surjective**.

Fredholm Boundary Operator (4)

The closed extension

$$\overline{LH_\alpha} : C(\partial D) \to C(\partial D)$$

generates a **Feller semigroup** on $C(\partial D)$.

(**Hille - Yosida - Ray Theorem**)

Fredholm Boundary Operator (5)

If L is **transversal**, then

$$\overline{LH}_{\alpha} : C(\partial D) \to C(\partial D)$$

is **bijective**.

Sketch of Proof (i)

$$
\begin{aligned}
& LH_\alpha 1(x') \\
& \le \mu(x') \frac{\partial}{\partial \mathbf{n}} \left(H_\alpha 1 \right)(x') - \alpha \delta(x') \\
& \quad + \int_D t(x', y) \left[H_\alpha 1(y) - 1 \right] dy \\
& < 0, \quad \forall x' \in \partial D.
\end{aligned}
$$

$$\int_D t(x', y) dy = +\infty \ \ \textbf{if} \ \ \mu(x') = \delta(x') = 0$$

Sketch of Proof (ii)

$$\ell_\alpha = -\sup_{\partial D} LH_\alpha 1 > 0.$$

$\Longrightarrow$

$$\overline{LH_\alpha} + \ell_\alpha : C(\partial D) \to C(\partial D)$$

generates a **Feller semigroup** on $C(\partial D)$.

(**Perturbation Theorem**)

Hille-Yosida Theorem

The operator

$\mathfrak{A} : C(K) \to C(K)$

generates a Feller semigroup if it satisfies the following three conditions :

(a) $D(\mathfrak{A})$ is dense in $C(K)$.

(b) $\exists! u \in D(\mathfrak{A})$ s.t. $(\alpha - \mathfrak{A})u = f, \ \forall f \in C(K)$.

(c) $\forall f \in C(K), \ f \geq 0$ in $K \Rightarrow (\alpha - \mathfrak{A})^{-1} f \geq 0$ in K.

(d) $\left\| (\alpha - \mathfrak{A})^{-1} \right\| \leq \frac{1}{\alpha}, \ \forall \alpha > 0.$

Sketch of Proof (iii)

$$\ell_\alpha = -\sup_{\partial D} LH_\alpha 1 > 0.$$

$$\Longrightarrow$$

$$\exists -\overline{LH}_\alpha^{-1} = \left(\ell_\alpha - \left(\overline{LH}_\alpha + \ell_\alpha\right)\right)^{-1}.$$

$$\left\| -\overline{LH}_\alpha^{-1} \right\| = \left\| \left(\ell_\alpha - \left(\overline{LH}_\alpha + \ell_\alpha\right)\right)^{-1} \right\| \leq \frac{1}{\ell_\alpha}.$$

Sketch of Proof (7)

Then :

$$\begin{cases} (\alpha - W)u = f \quad \text{in } D, \\ Lu = 0 \quad \text{on } \partial D \end{cases}$$

$$\Longleftrightarrow$$

$$u = G_\alpha f$$
$$:= G_\alpha^0 f - H_\alpha \left(\overline{LH_\alpha}^{-1} (LG_\alpha^0 f) \right)$$

Reduction to the Boundary

Probability Theory	Partial Differential Equations
Markov processes on the boundary	Fredholm integral equations
Markov processes on the domain	Boundary value problems

Green Operators

$$u = G_\alpha f$$
$$:= G_\alpha^0 f - H_\alpha \left(\overline{LH_\alpha}^{-1} (LG_\alpha^0 f) \right)$$

$$G_\alpha f = (\alpha - \mathfrak{W})^{-1} f$$

Integral Representation of Green Operators

$$u(x) = G_\alpha f(x) = \int_{\overline{D}} G_\alpha(x, y) f(y) dy$$

$$u = G_\alpha f = (\alpha - \mathfrak{W})^{-1} f$$

Transition Probability and Green kernels

$$p_t(x, dy) = p_t(x, y)dy$$

Laplace Transform

$$G_\alpha(x, y) = \int_0^\infty e^{-\alpha t} p_t(x, y)dt$$

Transition Probability and Green Operators

$$p_t(x, dy) = p_t(x, y)dy$$

Laplace Transform

$$G_\alpha f := \int_0^\infty e^{-\alpha t} T_t f \, dt = \int_0^\infty e^{-\alpha t} e^{t\mathfrak{A}} f \, dt$$

$$= \int_{\overline{D}} \left(\int_0^\infty e^{-\alpha t} p_t(x, y) \, dt \right) f(y) dy$$

Sketch of Proof (8)

The Green operators

$$G_\alpha : C(\overline{D}) \to C(\overline{D}), \quad \forall \alpha > 0$$

are **nonnegative**.

$$G_\alpha f = (\alpha - \mathfrak{W})^{-1} f$$

$$\forall f \in C(\overline{D}),\ f \geq 0 \text{ on } \overline{D} \Rightarrow G_\alpha f \geq 0 \text{ on } \overline{D}.$$

Weak Maximum Principle
(Aleksandrov-Bakel'man)

Assume that:

$$u \in C(\bar{D}) \cap W^{2,N}_{\text{loc}}(D),$$

$$(W-\alpha)u(x) \geq 0 \quad \text{for almost all } x \in D.$$

Then:

$$\boxed{\sup_D u \leq \sup_{\partial D} u^+}$$

Strong Maximum Principle

Assume that:

$$u \in C(\bar{D}) \cap W^{2,N}_{\mathrm{loc}}(D),$$

$$(W-\alpha)u\left(x\right) \geq 0 \quad \text{for almost all } x \in D,$$

$$m = \sup_D u \geq 0.$$

Then:

$$\exists x_0 \in D \text{ s.t. } u(x_0) = m \Longrightarrow u(x) \equiv m, \quad \forall x \in D.$$

Hopf Boundary Point Lemma

Assume that:

(1) $u \in C^1(\bar{D}) \cap W^{2,N}_{\text{loc}}(D)$,

$(W - \alpha)u(x) \geq 0$ for almost all $x \in D$.

(2) $\exists x_0' \in \partial D$ such that

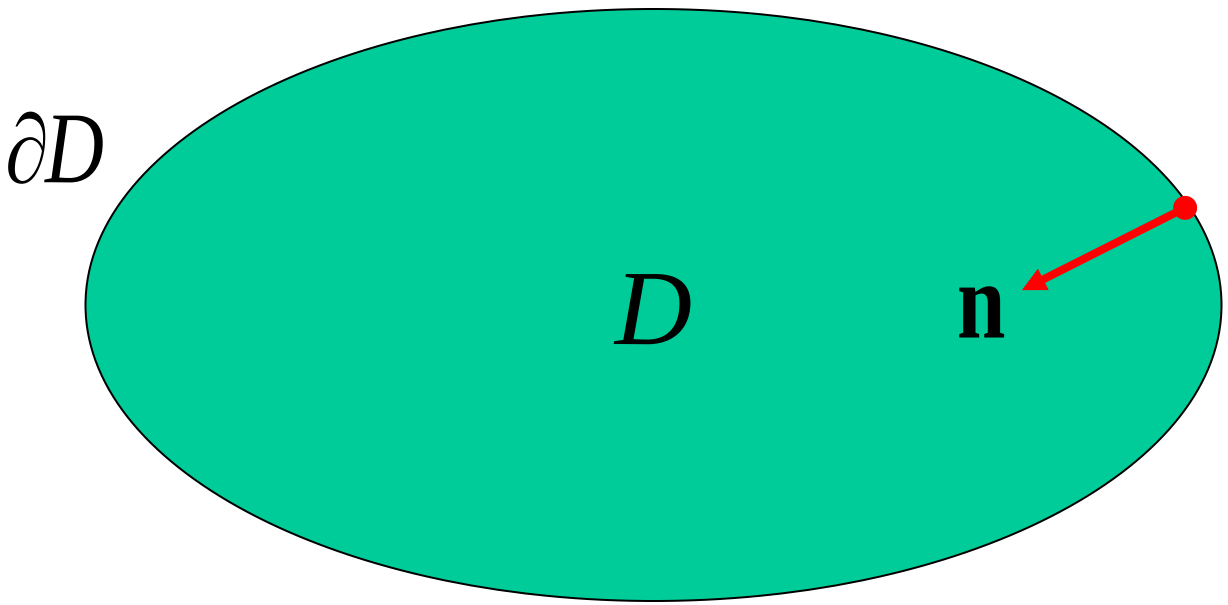
$$\begin{cases} u(x_0') = \sup\limits_{D} u = m \geq 0, \\ \quad u(y) < m, \ \forall y \in D. \end{cases}$$

Then:

$$\frac{\partial u}{\partial \mathbf{n}}(x_0') < 0.$$

Bounded Domain

$\mathbf{R}^N, \quad N \geq 2$



Sketch of Proof (9)

The Green operators

$$G_\alpha : C(\overline{D}) \to C(\overline{D}), \quad \forall \alpha > 0$$

are **contractive**.

$$G_\alpha f = (\alpha - \mathfrak{W})^{-1} f$$

$$\|G_\alpha\| \leq \frac{1}{\alpha}, \quad \forall \alpha > 0.$$

Sketch of Proof (10)

The domain $D(\mathfrak{W})$ is **dense** in $C(\overline{D})$:

$$\lim_{\alpha \to +\infty} \left\| \alpha G_\alpha u - u \right\| = 0, \quad \forall u \in C(\overline{D})$$

Sketch of Proof (11)

If $\boldsymbol{L}$ is **transversal**, then

$$\lim_{\alpha\to+\infty}\left\|\overline{LH}_{\alpha}^{\,-1}\right\|=0$$

$$\int_D t(x',y)dy=+\infty \ \textbf{ if } \ \mu(x')=\delta(x')=0$$

Main Theorem (general case)

We define a linear operator

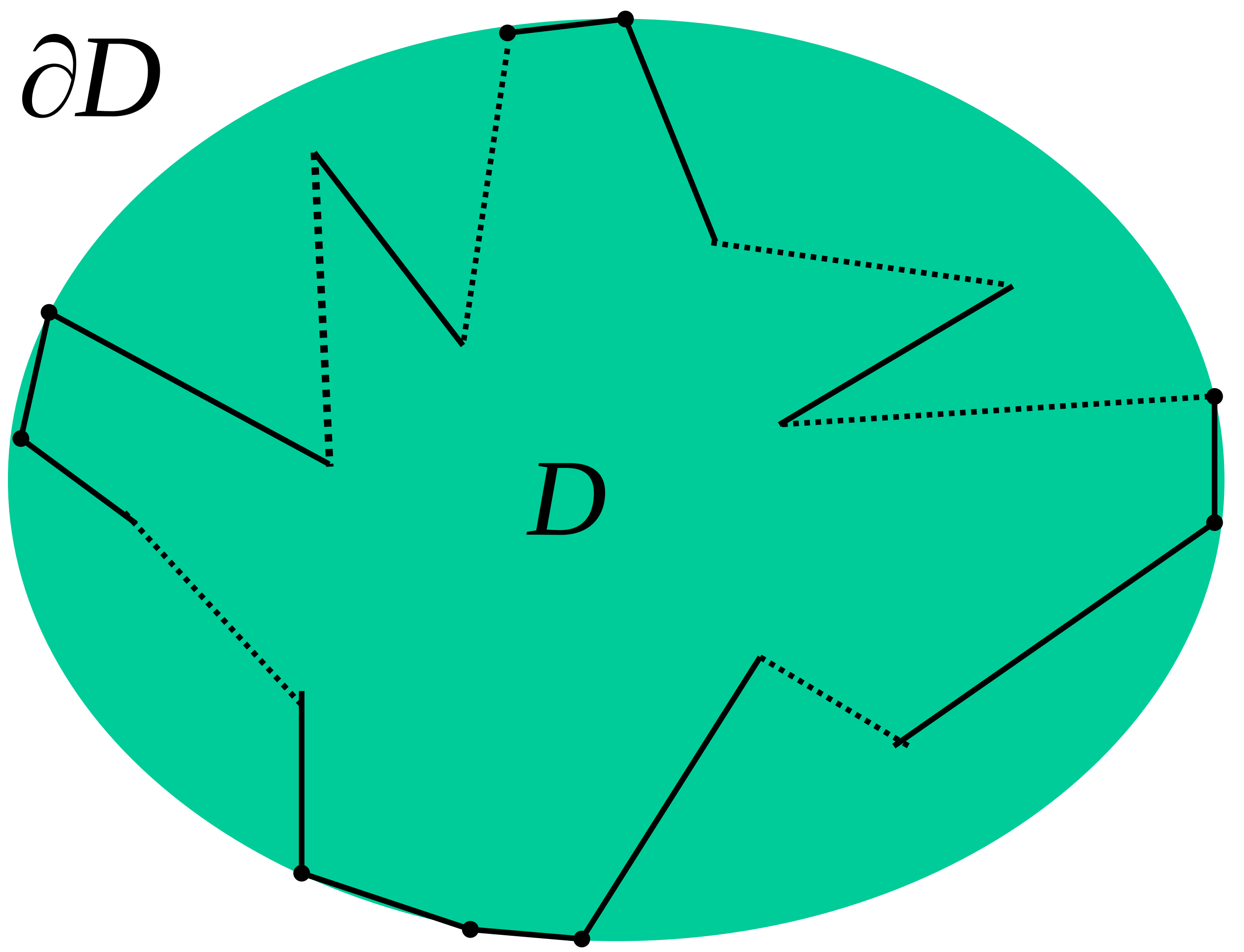
$$\mathfrak{W}: C(\overline{D}) \to C(\overline{D})$$

as follows:

(a) $D(\mathfrak{W}) = \left\{ u \in C(\overline{D}) : Wu \in C(\overline{D}),\, Lu = 0 \right\}$

(b) $\mathfrak{W}u = Wu = (A + S)u, \ \forall u \in D(\mathfrak{W})$

If L is **transversal**, then $\mathfrak{W}$ generates a **Feller semigroup.**



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Main Theorem (Dirichlet case)

We define a linear operator

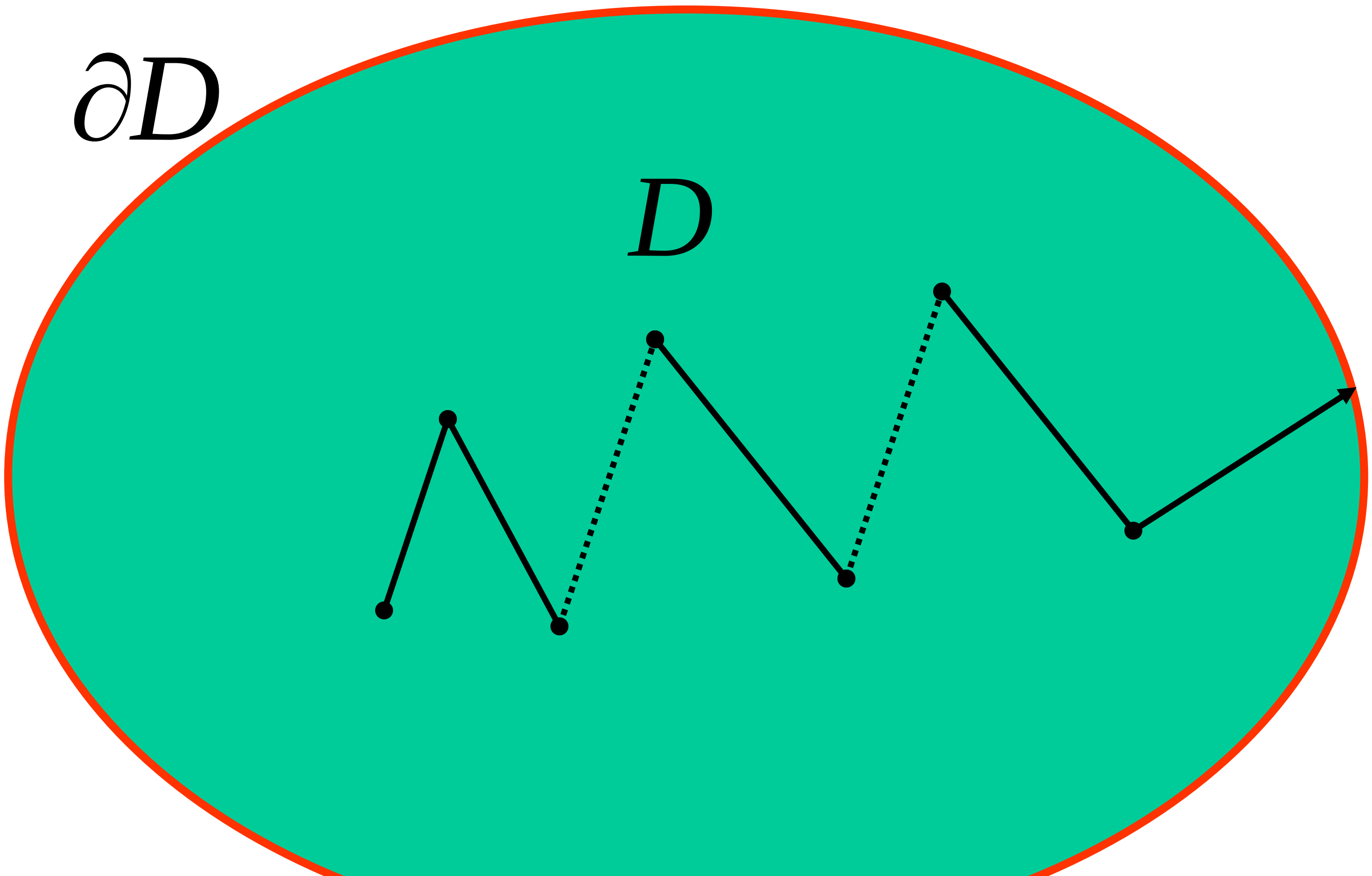
$$\mathfrak{W} : C_0(\overline{D}) \to C_0(\overline{D})$$

as follows:

(a) $D(\mathfrak{W}) = \left\{ u \in C_0(\overline{D}) : Wu \in C_0(\overline{D}) \right\}$

(b) $\mathfrak{W}u = Wu = (A + S)u, \ \forall u \in D(\mathfrak{W})$

Then $\mathfrak{W}$ generates a **Feller semigroup.**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Open Problems

(1) Generalization of Boundary Conditions

Non-Transversal Case

(2) Generalization of Elliptic Operators

(a) Degenerate Case

(b) Discontinuous Case

References

K. Taira: Semigroups, boundary value problems and Markov processes, Springer-Verlag, Springer Monographs in Mathematics, 2014

Singular Integrals and Feller Semigroups

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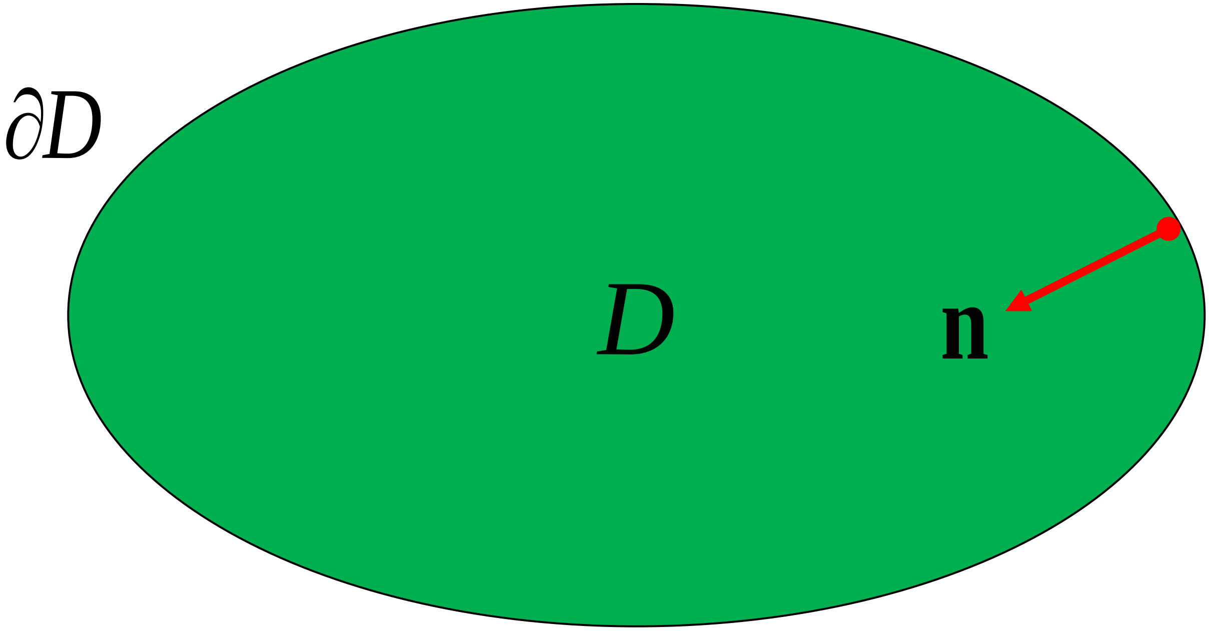
Part II
Singular Integral Operators and Feller Semigroups

Abstract

- This talk is devoted to the functional analytic approach to the problem of construction of Markov processes for second-order elliptic integro-differential operators with discontinuous coefficients.
- By using the theory of singular integral operators, we construct a Feller semigroup corresponding to such a diffusion phenomenon that a Markovian particle moves both by jumps and continuously in the state space.

Bounded Domain

$\mathbf{R}^N, \quad N \geq 3$

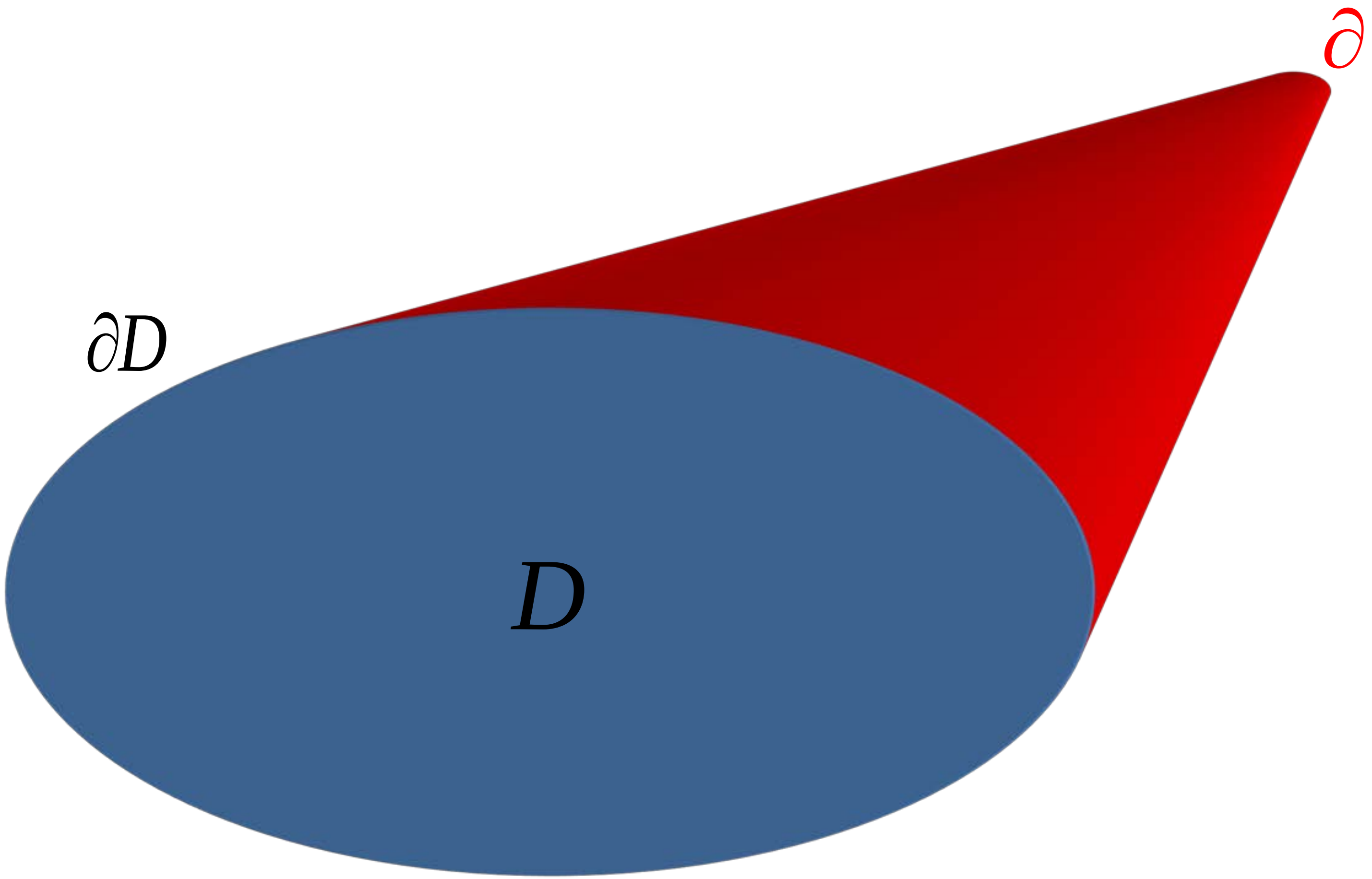


State Space
(Dirichlet case)

$\partial := \partial D$ **one-point compactification**

$$x \sim y \overset{def}{\Leftrightarrow} (a)\ x = y, \quad (b)\ x, y \in \partial D$$

One-Point Compactification



Function Space
(Dirichlet case)

$$C_0(\bar{D}) = \{u \in C(\bar{D}) : u = 0 \ \text{ on } \partial D\}$$

with the maximum norm

$$\|u\| = \max_{x \in \bar{D}} |u(x)|$$

Feller Semigroups (Dirichlet case)

A family of bounded linear operators $\{T_t\}_{t \geq 0}$ is called a **Feller semigroup** if it satisfies the following three conditions:

(1) $T_{t+s} = T_t \bullet T_s, \ \forall t, s \geq 0.$

(2) $\lim_{s \downarrow 0} \| T_{t+s} f - T_t f \| = 0, \ \forall f \in C_0(\bar{D}).$

(3) $\forall f \in C_0(\bar{D}), 0 \leq f \leq 1 \text{ on } \bar{D} \Rightarrow 0 \leq T_t f \leq 1 \text{ on } \bar{D}.$

Hille-Yosida-Ray Theorem (Dirichlet case)

The operator

$\mathfrak{W} : C_0(\bar{D}) \to C_0(\bar{D})$

generates a **Feller semigroup** if it satisfies the following three conditions:

(a) $D(\mathfrak{W})$ is dense in $C_0(\bar{D})$.

(b) $\exists u \in D(\mathfrak{W})$ s.t. $(\alpha - \mathfrak{W})u = f, \ \forall f \in C_0(\bar{D})$.

(c) If $u \in D(\mathfrak{W})$ attains its **positive maximum** at a point $x_0 \in D$, then $\mathfrak{W}u(x_0) \leq 0$.

Waldenfels Operators
(Integro-differential Operators)

$$Wu = Au + Su$$

$$= \sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x) \frac{\partial u}{\partial x_i} + c(x)u$$

$$+ \int_D s(x,dy) \left[u(y) - u(x) - \sum_{j=1}^{N} (y_j - x_j) \frac{\partial u}{\partial x_j}(x) \right]$$

Diffusion Operators
(Differential Operators)

$$Au = \sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x) \frac{\partial u}{\partial x_i} + c(x)u$$

Here:

(1) $\boxed{a^{ij}(x) \in \mathbf{VMO} \cap L^\infty(\mathbf{R}^N)}$,

$a^{ij}(x) = a^{ji}(x)$ for a.a. $x \in D$ and

$$\exists \lambda^{-1} |\xi|^2 \leq \sum_{i,j=1}^{N} a^{ij}(x) \xi_i \xi_j \leq \lambda |\xi|^2 .$$

(2) $b^i(x) \in L^\infty(\mathbf{R}^N)$.

(3) $c(x) \in L^\infty(\mathbf{R}^N)$ and $c(x) \leq 0$ for a.a. $x \in D$.

Lévy Operators
(Integro-differential operators of first order)

$$Su = \int_D s(x,dy)\left[u(y) - u(x) - \sum_{j=1}^{N} (y_j - x_j) \frac{\partial u}{\partial x_j}(x) \right]$$

Here:

(1) $\forall \varepsilon > 0: \quad \int_{D \cap \{|y-x| \le \varepsilon\}} s(x,dy) |y-x|^2 \le \exists \omega(\varepsilon),$

(2) $\boxed{\lim_{\varepsilon \downarrow 0} \omega(\varepsilon) = 0.}$

Main Theorem (Dirichlet case)

Let $p > N$. We define a linear operator

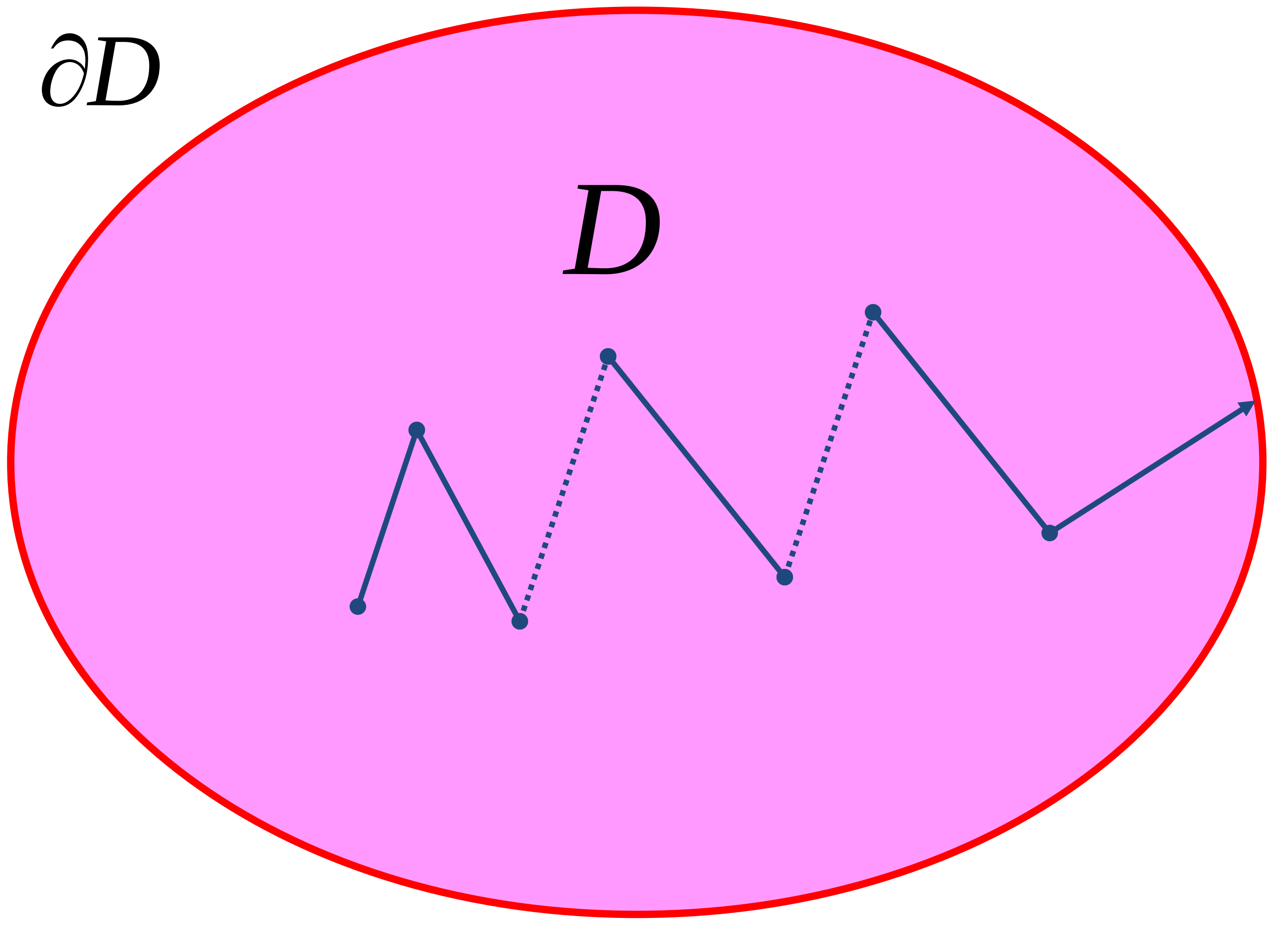
$$\mathfrak{W} : C_0(\overline{D}) \to C_0(\overline{D})$$

as follows:

(a) $D(\mathfrak{W}) = \left\{ u \in C_0(\overline{D}) \cap W^{2,p}(D) : Wu \in C_0(\overline{D}) \right\}$

(b) $\mathfrak{W}u = Wu = (A+S)u, \ \forall u \in D(\mathfrak{W})$

Then $\mathfrak{W}$ generates a **Feller semigroup.**



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- **Chiarenza, Frasca and Longo**: Trans. Amer. Math. Soc. 336 (1993), 841-853.
- **Gilbarg and Trudinger**: Springer-Verlag, 1998 edition.
- **John and Nirenberg**: Comm. Pure and Appl. Math. 14 (1961), 175-188.
- **Sarason**: Trans. Amer. Math. Soc. 207 (1975), 391-405

BMO Functions

A function

$$f \in L^1_{\text{loc}}\left(\mathbf{R}^n\right)$$

is said to be of **bounded mean oscillation** (BMO) if it satisfies the condition

$$\|f\|_* = \sup_B \frac{1}{|B|}\int_B |f(x) - f_B|\,dx < \infty.$$

Here:

f_B is the **average** of f over the ball B.

VMO Functions

A function

$$f \in L^1_{\text{loc}}\left(\mathbf{R}^n\right)$$

is said to have **vanishing mean oscillation** (VMO)

if it satisfies the condition

$$\lim_{r \downarrow 0} \eta\left(r\right) = 0.$$

Here:

$$\eta\left(r\right) = \sup_{\rho \le r} \frac{1}{|B|} \int_B \left| f(x) - f_B \right| dx.$$

Examples (1)

$$(1)\ L^{\infty}\left(\mathbf{R}^n\right) \subset \mathrm{BMO}$$

$$(2)\ \mathrm{BMO} \cap \mathrm{UC} \subset \mathrm{VMO}$$

$$(3)\ W^{1,n}\left(\mathbf{R}^n\right) \subset \mathrm{VMO}$$

Examples (2)

$$(1)\ \log|x| \in \mathrm{BMO},\ \ \log|x| \notin \mathrm{VMO}$$

$$(2)\ \log\big|\log|x|\big| \in \mathrm{VMO}$$

Hille-Yosida-Ray Theorem (Dirichlet case)

The operator

$\mathfrak{W} : C_0(\bar{D}) \to C_0(\bar{D})$

generates a **Feller semigroup** if it satisfies the following three conditions:

(a) $D(\mathfrak{W})$ is dense in $C_0(\bar{D})$.

(b) $\exists u \in D(\mathfrak{W})$ s.t. $(\alpha - \mathfrak{W})u = f, \ \forall f \in C_0(\bar{D})$.

(c) If $u \in D(\mathfrak{W})$ attains its **positive maximum** at a point $x_0 \in D$, then $\mathfrak{W}u(x_0) \leq 0$.

Transition Functions and Semigroups

$$T_t f(x) = e^{t\mathfrak{W}} f(x)$$

$$= \int_{\overline{D}} p_t(x, dy) f(y), \ \forall f \in C_0(\overline{D})$$

Semigroups and Green Operators

$$T_t = e^{t\mathfrak{W}}$$

Laplace Transform

$$G_\alpha^0 := \int_0^\infty e^{-\alpha t} T_t \, dt = \int_0^\infty e^{-\alpha t} e^{t\mathfrak{W}} dt = \left(\alpha - \mathfrak{W}\right)^{-1}$$

Differential Operators

$$Au = \sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x)\frac{\partial u}{\partial x_i} + c(x)u.$$

Here:

(1) $a^{ij}(x) \in$ **VMO** $\cap L^\infty(\mathbf{R}^N)$,

$a^{ij}(x) = a^{ji}(x)$ for a.a. $x \in D$ and

$$\exists \lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \le \lambda|\xi|^2.$$

(2) $b^i(x) \in L^\infty(\mathbf{R}^N)$.

(3) $c(x) \in L^\infty(\mathbf{R}^N)$ and $c(x) \le 0$ for a.a. $x \in D$.

Dirichlet Problem (VMO)

Let $N < p < \infty$. If $\alpha \geq 0$, then the Dirichlet problem

$$\begin{cases} (A - \alpha)u = f & \text{in } D, \\ u = \varphi & \text{on } \partial D \end{cases}$$

has a solution $\exists! u \in W^{2,p}(D)$ for $\forall f \in L^p(D)$, $\forall \varphi \in B^{2-1/p,p}(\partial D)$.

Uniqueness Theorem

If a function

$$u \in W^{2,p}(D), \quad N < p < \infty,$$

is a solution of the homogeneous problem

$$\begin{cases} (W - \alpha)u = 0 & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases}$$

then it follows that

$$\boxed{u = 0 \text{ in } D.}$$

Weak Maximum Principle
(Aleksandrov-Bakel'man)

Assume that:

$$u \in C(\bar{D}) \cap W^{2,N}_{\text{loc}}(D),$$

$$(A - \alpha)u(x) \geq 0 \quad \text{for almost all } x \in D.$$

Then:

$$\sup_D u \leq \sup_{\partial D} u^+.$$

Strong Maximum Principle

Assume that:

$$u \in C(\bar{D}) \cap W^{2,N}_{\text{loc}}(D),$$

$$(A - \alpha)u(x) \geq 0 \quad \text{for almost all } x \in D,$$

$$m = \sup_D u \geq 0.$$

Then:

$$\exists x_0 \in D \text{ s.t. } u(x_0) = m \Rightarrow u(x) \equiv m, \quad \forall x \in D$$

Hopf Boundary Point Lemma

Assume that:

(1) $u \in C^1(\bar{D}) \cap W^{2,N}_{\mathrm{loc}}(D)$,

$(A - \alpha)u(x) \geq 0$ for almost all $x \in D$.

(2) $\exists x_0' \in \partial D$ such that $u(x_0') = \sup_D u = m \geq 0$,

$u(y) < m = \sup_D u, \ \forall y \in D$.

Then:

$$\boxed{\frac{\partial u}{\partial \mathbf{n}}(x_0') < 0}.$$

A Priori Estimates (VMO)

$$\|u\|_{W^{2,p}(D)} \leq \exists C\left(\|Au\|_{L^p(D)} + \|u\|_{L^p(D)}\right),$$

$$\forall u \in W^{2,p}(D) \cap W_0^{1,p}(D).$$

$$Au = \sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

Calderón-Zygmund Kernels

A function

$$k(x): \mathbf{R}^n \setminus \{0\} \to \mathbf{R}$$

is called a **Calderon - Zygmund kernel**

if it satisfies the following three conditions:

(1) $k(x) \in C^\infty(\mathbf{R}^n \setminus \{0\})$.

(2) $k(x)$ is homogeneous of degree $-n$.

(3) $\displaystyle\int_{\{|x|=1\}} k(x)\, d\sigma = 0.$

Example

$$h(x) \in C^{\infty}\left(\mathbf{R}^n \setminus \{0\}\right),$$

$$h(tx) = t^{1-n}\, h(x), \ \ \forall t > 0.$$

Then:

$$\frac{\partial h}{\partial x_i}(x): \textbf{Calderon - Zygmund kernels}$$

Calderón-Zygmund Operators (1)
-global version-

Assume that a function

$$k(x,z):\mathbf{R}^n \times \left(\mathbf{R}^n \setminus \{0\}\right) \to \mathbf{R}$$

satisfies the following two conditions:

(1) $k(x,\bullet)$ is a **Calderon - Zygmund kernel**

for almost all $x \in \mathbf{R}^n$.

(2) $\max_{|\alpha|\le 2n} \left\| \partial_z^\alpha k(x,z) \right\|_{L^\infty(\mathbf{R}^n \times \Sigma)} \le \exists M < \infty.$

Calderón-Zygmund Operators (2)

Then:

$$Kf := \exists \lim_{\varepsilon \downarrow 0} \int_{|x-y|>\varepsilon} k(x, x-y) f(y) dy \quad \text{in } L^p(\mathbf{R}^n).$$

$$C[\varphi, f] := \exists \lim_{\varepsilon \downarrow 0} \int_{|x-y|>\varepsilon} k(x, x-y)[\varphi(x) - \varphi(y)] f(y) dy$$

$$\text{in } L^p(\mathbf{R}^n).$$

$$\varphi \in L^\infty\left(\mathbf{R}^n\right).$$

Calderón-Zygmund Operators (3)
- local version -

Assume that a function

$$k(x,z):\Omega\times\left(\mathbf{R}^n\setminus\{0\}\right)\to\mathbf{R}$$

satisfies the following two conditions:

(1) $k(x,\bullet)$ is a **Calderon - Zygmund kernel**

for almost all $x\in\Omega$.

(2) $\max_{|\alpha|\le 2n}\left\|\partial_z^\alpha k(x,z)\right\|_{L^\infty(\Omega\times\Sigma)}\le\exists M<\infty.$

Calderón-Zygmund Operators (4)

Then:

$$Kf := \exists \lim_{\varepsilon \downarrow 0} \int_{|x-y|>\varepsilon} k(x, x-y) f(y) dy \text{ in } L^p(\Omega).$$

$$C[\varphi, f] := \exists \lim_{\varepsilon \downarrow 0} \int_{|x-y|>\varepsilon} k(x, x-y)[\varphi(x) - \varphi(y)] f(y) dy$$

$$\text{in } L^p(\Omega).$$

$$\varphi \in L^\infty(\mathbf{R}^n).$$

Fundamental Solution (1)

$$\Gamma(x,t)$$

$$= \frac{1}{(N-2)\omega_N} \frac{1}{\sqrt{\det(a^{ij}(x))}} \left(\sum_{i,j=1}^{N} A_{ij}(x) t_i t_j \right)^{(2-N)/2}$$

Here:

$\left(A_{ij}(x) \right) =$ the **inverse matrix** of $\left(a^{ij}(x) \right)$

$$\omega_N = \frac{2\pi^{N/2}}{\Gamma(N/2)} \quad \text{(surface area)}$$

Fundamental Solution (2)

$$\Gamma_i(x,t) = \frac{\partial \Gamma}{\partial t_i}(x,t)$$

$$= -\frac{1}{\omega_N} \frac{1}{\sqrt{\det(a^{ij}(x))}} \left(\sum_{i,j=1}^{N} A_{ij}(x) t_i t_j \right)^{-N/2} \sum_{j=1}^{N} A_{ij}(x) t_j$$

The functions

$$\Gamma_{ij}(x,t) = \frac{\partial^2 \Gamma}{\partial t_i \partial t_j}(x,t)$$

are **Calderon - Zygmund kernels** in t.

Representation Formula (1)

$$u \in W_0^{2,p}(B),\ 1 < p < \infty,$$

$$Lu = \sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Here:

$$a^{ij}(x) \in \mathrm{VMO}$$

A Priori Estimates
- local version -

$$\exists \rho_0 > 0 : \ \forall u \in W_0^{2,p}(B_r), \ 0 < \forall r < \rho_0$$

$$\Longrightarrow$$

$$\left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^p(B_r)} \leq \exists C \| Lu \|_{L^p(B_r)} .$$

Representation Formula (2)

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) =$$

$$\text{v.p.}\int_B \Gamma_{ij}(x, x-y)\left[\sum_{k,h=1}^{N}\left(a^{hk}(y)-a^{hk}(x)\right)\frac{\partial^2 u}{\partial x_i \partial x_j}(y)+Lu(y)\right]dy$$

$$+Lu(x)\int_{|t|=1}\Gamma_i(x,t)t_j\,d\sigma$$

$$a^{hk}(x)\in \textbf{VMO}$$

Commutator Estimates

Assume that

$a(x) \in \mathbf{VMO} \cap \mathrm{L}^{\infty}(\mathbf{R}^n).$

Then:

$\forall \varepsilon > 0, \ \exists \rho_0 = \rho_0(\varepsilon, a) > 0$ such that

$0 < \forall r < \rho_0$

$\|C[a, f]\|_{L^p(B_r)} \leq \varepsilon \| f \|_{L^p(B_r)}, \ \forall f \in L^p(B_r).$

A Priori Estimates
- local version -

$$\exists \rho_0 > 0 : \ \forall u \in W_0^{2,p}(B_r), \ 0 < \forall r < \rho_0$$

$$\Rightarrow$$

$$\left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^p(B_r)} \leq \exists C \| Lu \|_{L^p(B_r)} .$$

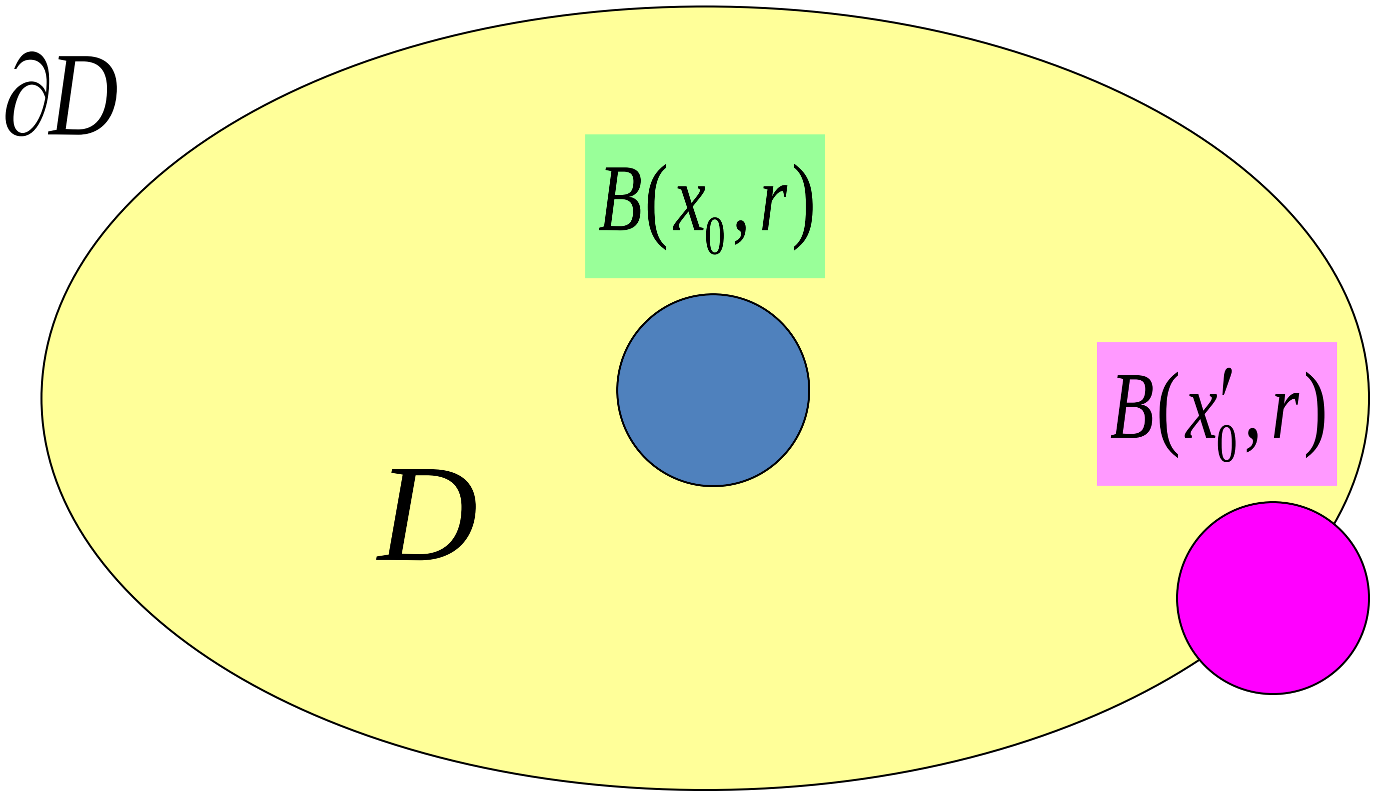
A Priori Estimates
- global version -

$$\|u\|_{W^{2,p}(D)} \leq \exists C \left(\|Au\|_{L^p(D)} + \|u\|_{L^p(D)} \right),$$

$$\forall u \in W^{2,p}(D) \cap W_0^{1,p}(D).$$

$$Au = \sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x) \frac{\partial u}{\partial x_i} + c(x)u$$

Localization Argument



Interpolation Inequality

$$\|u\|_{W^{1,p}(D)} \le \forall \varepsilon \|u\|_{W^{2,p}(D)} + \frac{\exists C}{\varepsilon} \|u\|_{L^p(D)},$$

$$\forall u \in W^{2,p}(D).$$

Uniqueness Theorem (VMO)

If a function

$$u \in W^{2,p}(D), \quad N < p < \infty,$$

is a solution of the homogeneous problem

$$\begin{cases} (A - \alpha)u = 0 \ \text{ in } D, \\ \quad u = 0 \ \text{ on } \partial D, \end{cases}$$

then it follows that

$$\boxed{u = 0 \text{ in } D.}$$

A Priori Estimates (VMO)

$$\|u\|_{W^{2,p}(D)} \leq \exists C \|(A-\alpha)u\|_{L^p(D)},$$

$$\forall u \in W^{2,p}(D) \cap W_0^{1,p}(D)$$

$$C > 0 : \text{structure constant}$$

Existence Theorem (VMO)

For any function

$$f \in L^p(D),\ N < p < \infty,$$

the Dirichlet problem

$$\begin{cases} (A-\alpha)u = f & \text{in } D, \\ u = 0 & \text{on } \partial D \end{cases}$$

has a **(unique) solution**

$$u \in W^{2,p}(D) \cap W_0^{1,p}(D).$$

Differential Operators (Uniformly Continuous Case)

$$Au = \sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x)\frac{\partial u}{\partial x_i} + c(x)u.$$

Here:

(1) $a^{ij}(x) \in C\left(\bar{D}\right)$, $a^{ij}(x) = a^{ji}(x), \forall x \in D$ and

$$\exists \lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2.$$

(2) $b^i(x) \in L^\infty(\mathbf{R}^N)$.

(3) $c(x) \in L^\infty(\mathbf{R}^N)$ and $c(x) \leq 0$ for a.a. $x \in D$.

Existence and Uniqueness Theorem (Uniformly Continuous Case)

Let $N < p < \infty$. If $\alpha \geq 0$, then the Dirichlet problem

$$\begin{cases} (A - \alpha)u = f \text{ in } D, \\ \quad u = \varphi \text{ on } \partial D \end{cases}$$

has a solution $\exists! u \in W^{2,p}(D)$ for $\forall f \in L^p(D)$, $\forall \varphi \in B^{2-1/p,p}(\partial D)$.

Approximation Theorem
(Mollifiers)

For $\forall a \in \mathrm{VMO}$,

$$\exists a_\varepsilon = a * \rho_\varepsilon \in C^\infty\left(\mathbf{R}^N\right) \cap \mathrm{VMO}$$

such that

$$\|a_\varepsilon - a\|_* \to 0 \ \text{ as } \varepsilon \downarrow 0$$

Approximate Solutions (1)

$$A_m = \sum_{i,j=1}^{N} a_m^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x) \frac{\partial}{\partial x_i} + c(x).$$

Here:

$$a_m^{ij}(x) = a^{ij} * \rho_{1/m}(x) \in C^{\infty} \cap \mathbf{VMO}$$

Approximate Solutions (2)

$$\forall f \in L^p(D), \exists! u_m \in W^{2,p}(D) \cap W_0^{1,p}(D)$$

such that

$$\boxed{\begin{aligned} &(A_m - \alpha) u_m = f \quad \text{in } D, \\ &u_m = 0 \quad \text{on } \partial D. \end{aligned}}$$

A Priori Estimates (VMO)

$$\|u_m\|_{W^{2,p}(D)} \leq \exists C \|(A_m - \alpha)u_m\|_{L^p(D)}$$

$$= \exists C \|f\|_{L^p(D)}.$$

Here:

$$\forall u_m \in W^{2,p}(D) \cap W_0^{1,p}(D)$$

Eberlein-Shmulyan Theorem

A Banach space *X* is **reflexive** if and only if every strongly bounded sequence contains a subsequence which converges **weakly** to an element of *X*.

$$X := W^{2,p}(D), \quad N < p < \infty$$

Rellich-Kondrachov Theorem

The injection

$$W^{2,p}(D) \to W^{1,p}(D)$$

is **compact**.

Approximate Solutions (3)

$$\exists ! u_{m'} \xrightarrow{\text{weakly}} \exists u \in W^{2,p}(D) \cap W_0^{1,p}(D)$$

and

$$\begin{cases} (A - \alpha) u = f & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases}$$

$$A = \sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x) \frac{\partial}{\partial x_i} + c(x),$$

$$a^{ij}(x) \in \mathbf{VMO}$$

Dirichlet Problem (VMO)

Let $N < p < \infty$. If $\alpha \geq 0$, then the Dirichlet problem

$$\begin{cases} (A - \alpha)u = f \text{ in } D, \\ \quad u = \varphi \text{ on } \partial D \end{cases}$$

has a solution $\exists! u \in W^{2,p}(D)$ for $\forall f \in L^p(D)$, $\forall \varphi \in B^{2-1/p,p}(\partial D)$.

Compact Perturbation (1)

$$Su = \int_D s(x, dy)\left[u(y) - u(x) - \sum_{j=1}^{N} (y_j - x_j) \frac{\partial u}{\partial x_j}(x) \right].$$

Here:

(1) $\forall \varepsilon > 0 : \quad \int_{D \cap \{|y-x| \le \varepsilon\}} s(x, dy) |y - x|^2 \le \exists \omega(\varepsilon),$

(2) $\boxed{\lim_{\varepsilon \downarrow 0} \omega(\varepsilon) = 0.}$

Compact Perturbation (2)

$$Su = \int_D s(x,dy)\left[u(y) - u(x) - \sum_{j=1}^{N}(y_j - x_j)\frac{\partial u}{\partial x_j}(x)\right]$$

$$= \int_{D\cap\{|y-x|\le\varepsilon\}} s(x,dy)\left[u(y) - u(x) - \sum_{j=1}^{N}(y_j - x_j)\frac{\partial u}{\partial x_j}(x)\right]$$

$$+ \int_{D\cap\{|y-x|>\varepsilon\}} s(x,dy)\left[u(y) - u(x) - \sum_{j=1}^{N}(y_j - x_j)\frac{\partial u}{\partial x_j}(x)\right]$$

Compact Perturbation (3)

$$S_1^{(\varepsilon)} u := \int_{D \cap \{|y-x| \le \varepsilon\}} s(x, dy) \left[u(y) - u(x) - \sum_{j=1}^{N} (y_j - x_j) \frac{\partial u}{\partial x_j}(x) \right]$$

$\Rightarrow$

$$\left\| S_1^{(\varepsilon)} u \right\|_{L^p(D)} \le \exists C_1 \omega(\varepsilon) \|u\|_{W^{2,p}(D)}, \quad \forall u \in W^{2,p}(D)$$

Fundamental Lemma

$$u \in W^{2,p}(\mathbf{R}^n), \quad \forall p > n$$

$$U(x) := \sup_{\substack{t \in \mathbf{R}^n \\ t \neq 0}} \frac{\left| u(x+t) - u(x) - \sum_{j=1}^{n} t_j \frac{\partial u}{\partial x_j}(x) \right|}{|t|^2}$$

$$\Longrightarrow$$

$$\boxed{\begin{aligned} & U \in L^p(\mathbf{R}^n), \\ & \|U\|_{L^p(\mathbf{R}^n)} \leq \exists C \|u\|_{W^{2,p}(\mathbf{R}^n)} \end{aligned}}$$

Maximal Functions

Let

$f \in L^1_{\text{loc}}\left(\mathbf{R}^n\right).$

The **maximal function** is defined as follows:

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy.$$

Here:

$B(x,r) =$ ball of radius r, centered at x.

Hardy-Littlewood Theorem

$$f \in L^p(\mathbf{R}^n),\ 1 < \forall p \leq \infty$$

$$\Longrightarrow$$

$$Mf \in L^p(\mathbf{R}^n),$$

$$\|Mf\|_{L^p(\mathbf{R}^n)} \leq \exists A_p \|f\|_{L^p(\mathbf{R}^n)}$$

Compact Perturbation (3-1)

$$\left|S_1^{(\varepsilon)}u(x)\right|$$

$$\leq \int_{D\cap\{|y-x|\leq\varepsilon\}} s(x,dy)\left|u(y)-u(x)-\sum_{j=1}^{N}(y_j-x_j)\frac{\partial u}{\partial x_j}(x)\right|$$

$$\leq \int_{D\cap\{|y-x|\leq\varepsilon\}} s(x,dy)\left|y-x\right|^2 \sup_{\substack{t\in\mathbf{R}^n \\ t\neq 0}} \frac{\left|\tilde{u}(x+t)-\tilde{u}(x)-\sum_{j=1}^{n}t_j\frac{\partial \tilde{u}}{\partial x_j}(x)\right|}{|t|^2}$$

$$\leq \omega(\varepsilon)\,\tilde{U}(x), \quad \forall x\in D.$$

Compact Perturbation (3-2)

$$\left\| S_1^{(\varepsilon)} u \right\|_{L^p(D)} \leq \omega(\varepsilon) \left\| \tilde{U} \right\|_{L^p(\mathbf{R}^N)}$$

$$\leq \exists C\, \omega(\varepsilon) \left\| \tilde{u} \right\|_{W^{2,p}(\mathbf{R}^N)}$$

$$\leq \exists C_1\, \omega(\varepsilon) \left\| u \right\|_{W^{2,p}(D)} .$$

Compact Perturbation (4)

$$S_2^{(\varepsilon)} u := \int_{D \cap \{|y-x|>\varepsilon\}} s(x,dy) \left[u(y) - u(x) - \sum_{j=1}^{N} (y_j - x_j) \frac{\partial u}{\partial x_j}(x) \right]$$

$$\Rightarrow$$

$$\left\| S_2^{(\varepsilon)} u \right\|_{L^p(D)} \leq \exists C_2(\varepsilon) \left\| u \right\|_{W^{1,p}(D)}, \quad \forall u \in W^{2,p}(D)$$

Compact Perturbation (4-1)

$$\left|S_2^{(\varepsilon)}u(x)\right| \leq \int_{D\cap\{|y-x|>\varepsilon\}} s(x,dy)\left|u(y)-u(x)\right|$$

$$+\sum_{j=1}^{N}\int_{D\cap\{|y-x|>\varepsilon\}} s(x,dy)\left|y_j - x_j\right|\left|\frac{\partial u}{\partial x_j}(x)\right|$$

$$:= A_\varepsilon u(x) + B_\varepsilon u(x)$$

Compact Perturbation (4-2)

$$B_\varepsilon u(x) = \int_{D\cap\{|y-x|>\varepsilon\}} s(x,dy)\left|y_j - x_j\right|\left|\frac{\partial u}{\partial x_j}(x)\right|$$

$$\le \frac{1}{\varepsilon}\int_{D\cap\{|y-x|>\varepsilon\}} s(x,dy)\left|y-x\right|^2\left|\frac{\partial u}{\partial x_j}(x)\right|$$

$$\le \frac{1}{\varepsilon}\sup_{\Omega}\left(\int_{D\cap\{|y-x|>\varepsilon\}} s(x,dy)\left|y-x\right|^2\right)\left|\frac{\partial u}{\partial x_j}(x)\right|$$

Compact Perturbation (4-3)

$$\left\| B_\varepsilon u \right\|_{L^p(D)} \le \frac{n}{\varepsilon} \sup_{\Omega} \left(\int_{D \cap \{|y-x| > \varepsilon\}} s(x,dy) \left| y - x \right|^2 \right) \left\| \nabla u \right\|_{L^p(D)}$$

$$\left\| B^{(\varepsilon)} u \right\|_{L^p(D)} \le C'(\varepsilon) \left\| u \right\|_{W^{1,p}(D)}, \quad \forall u \in W^{2,p}(D)$$

Morrey's imbedding Theorem

$$u \in W_0^{1,p}(\Omega), \ \ \forall p > n$$

$$\Rightarrow$$

$$u \in C^{\gamma}(\overline{\Omega}), \ \ \gamma = 1 - \frac{n}{p},$$

$$|u(x) - u(y)| \leq \exists C_{n,p} |x - y|^{\gamma} \|\nabla u\|_{L^p(\Omega)}$$

Compact Perturbation (4-4)

$$A_\varepsilon u(x) = \int_{D\cap\{|y-x|>\varepsilon\}} s(x,dy)\left|u(y)-u(x)\right|$$

$$\leq \exists C \int_{D\cap\{|y-x|>\varepsilon\}} s(x,dy)\left|y-x\right|^{\gamma} \left\|\nabla u\right\|_{L^p(\Omega)}$$

$$= C \int_{D\cap\{|y-x|>\varepsilon\}} s(x,dy)\left|y-x\right|^{2} \left(\frac{1}{\left|y-x\right|^{2-\gamma}}\right) \left\|\nabla u\right\|_{L^p(\Omega)}$$

$$\leq \frac{C}{\varepsilon^{2-\gamma}} \left(\sup_{\Omega} \int_{D\cap\{|y-x|>\varepsilon\}} s(x,dy)\left|y-x\right|^{2}\right) \left\|\nabla u\right\|_{L^p(\Omega)}$$

Compact Perturbation (4-5)

$$\left\|A_\varepsilon u\right\|_{L^p(D)} \le \frac{\exists C}{\varepsilon^{2-\gamma}} \sup_{\Omega} \left(\int_{D\cap\{|y-x|>\varepsilon\}} s(x,dy)\left|y-x\right|^2 \right) \left\|\nabla u\right\|_{L^p(D)} .$$

$$\left\|A^{(\varepsilon)}u\right\|_{L^p(D)} \le C''(\varepsilon)\left\|u\right\|_{W^{1,p}(D)}, \quad \forall u \in W^{2,p}(D)$$

Compact Perturbation (5)

The operator

$$S_2^{(\varepsilon)} = A^{(\varepsilon)} + B^{(\varepsilon)} : W^{2,p}(D) \to L^p(D)$$

is **compact**.

$$\left\| S_2^{(\varepsilon)} u \right\|_{L^p(D)} \leq \exists C_2(\varepsilon) \left\| u \right\|_{W^{1,p}(D)}, \ \forall u \in W^{2,p}(D)$$

Rellich-Kondrachov Theorem

The injection

$$W^{2,p}(D) \to W^{1,p}(D)$$

is **compact**.

Compact Perturbation (6)

The Levy operator

$$S = S_1^{(\varepsilon)} + S_2^{(\varepsilon)} : W^{2,p}(D) \to L^p(D)$$

is **compact.**

$$\left\| S_1^{(\varepsilon)} u \right\|_{L^p(D)} \le C_1 \omega(\varepsilon) \left\| u \right\|_{W^{2,p}(D)}, \ \forall u \in W^{2,p}(D)$$

$$\lim_{\varepsilon \downarrow 0} \omega(\varepsilon) = 0$$

Index Formula

The **index** of the operator

$$(W-\alpha,\gamma_0):W^{2,p}(D)\to L^p(D)\times B^{2-1/p,p}(\partial D)$$

is equal to **zero**.

$$\operatorname{ind}(W-\alpha,\gamma_0)=\operatorname{ind}(A-\alpha,\gamma_0)+\operatorname{ind}(S,0)$$
$$=\operatorname{ind}(A-\alpha,\gamma_0)=0$$

Uniqueness Theorem

If a function

$$u \in W^{2,p}(D), \quad N < p < \infty,$$

is a solution of the homogeneous problem

$$\begin{cases} (W - \alpha)u = 0 & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases}$$

then it follows that

$$\boxed{u = 0 \text{ in } D.}$$

Bony's Maximum Principle

Assume that:

$u \in W^{2,p}(D), \quad \forall p > N,$

$(W - \alpha)u(x) \geq 0$ almost all $x \in D$.

Then it follows that

$u(x)$ may take its **positive maximum**

only on the boundary ∂D.

Fundamental Lemma

Assume that:

$$u \in W^{2,p}(D), \ \forall p > N,$$

$\exists x_0 \in D$ such that

$$\begin{cases} u(x_0) = \sup\limits_{D} u = m > 0, \\ u(x) < m, \ \forall x \in D. \end{cases}$$

Then:

$$\forall V(x_0), \ \exists M \subset V(x_0), \ |M| > 0:$$

$$\exists \left(u''_{ij}(x) \right) \leq 0, \ \forall x \in M.$$

Dirichlet Problem (VMO)

Let $N < p < \infty$. If $\alpha \geq 0$, then the Dirichlet problem

$$\begin{cases} (W - \alpha)u = (A + S - \alpha)u = f \ \text{ in } D, \\ u = \varphi \ \text{ on } \partial D \end{cases}$$

has a solution $\exists ! u \in W^{2,p}(D)$ for $\forall f \in L^p(D)$, $\forall \varphi \in B^{2-1/p,p}(\partial D)$.

Main Theorem (Dirichlet case)

Let $p > N$. We define a linear operator

$\mathfrak{W} : C_0(\overline{D}) \to C_0(\overline{D})$

as follows:

(a) $D(\mathfrak{W}) = \left\{ u \in C_0(\overline{D}) \cap W^{2,p}(D) : Wu \in C_0(\overline{D}) \right\}$

(b) $\mathfrak{W}u = Wu = (A + S)u, \ \forall u \in D(\mathfrak{W})$

Then $\mathfrak{W}$ generates a **Feller semigroup.**

Integral Representation of Green Operators

$$u(x) = \boldsymbol{G}_\alpha^{\boldsymbol{0}} f(x) = \int_{\overline{D}} G_\alpha^0(x, y) f(y) dy$$

$$u = G_\alpha^0 f = \left(\alpha - \mathfrak{W}\right)^{-1} f$$

Transition Probability and Green kernels

$$p_t(x, dy) = p_t(x, y)dy$$

Laplace Transform

$$G_\alpha^0(x, y) = \int_0^\infty e^{-\alpha t} p_t(x, y)dt$$

Transition Probability and Green Operators

$$p_t(x, dy) = p_t(x, y)dy$$

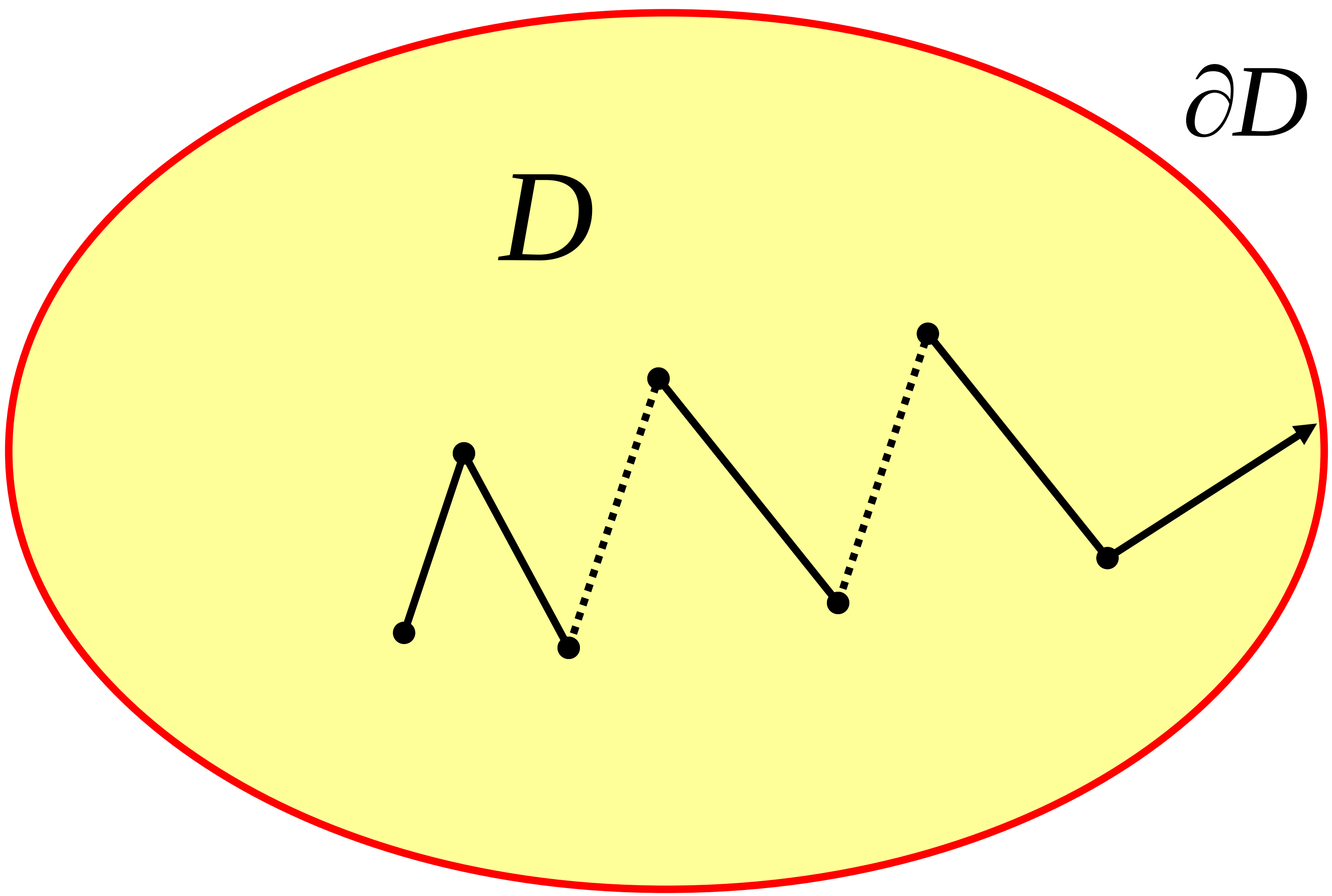
Laplace Transform

$$G_\alpha^0 f := \int_0^\infty e^{-\alpha t} T_t f \, dt = \int_0^\infty e^{-\alpha t} e^{t\mathfrak{W}} f \, dt$$

$$= \int_{\overline{D}} \left(\int_0^\infty e^{-\alpha t} p_t(x, y) \, dt \right) f(y) dy$$

Summary

We can construct a **Feller semigroup** corresponding to such a diffusion phenomenon that a Markovian particle moves both by jumps and continuously in the state space until it **dies** at the time when it reaches the boundary.



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Hille-Yosida-Ray Theorem (Dirichlet case)

The operator

$\mathfrak{W} : C_0(\overline{D}) \to C_0(\overline{D})$

generates a **Feller semigroup** if it satisfies the following three conditions:

(a) $D(\mathfrak{W})$ is dense in $C_0(\overline{D})$

(b) $\exists u \in D(\mathfrak{W})$ s.t. $(\alpha - \mathfrak{W})u = f, \ \forall f \in C_0(\overline{D})$

(c) If $u \in D(\mathfrak{W})$ attains its **positive maximum** at a point $x_0 \in D$, then $\mathfrak{W}u(x_0) \le 0$.

Sketch of Proof (1)

The Green operators

$$G_\alpha^0 : C_0(\overline{D}) \to C_0(\overline{D}), \quad \forall \alpha > 0$$

are **nonnegative**.

$$u = G_\alpha^0 f = \left(\alpha - \mathfrak{W}\right)^{-1} f$$

$$\forall f \in C_0(\overline{D}),\ f \geq 0 \text{ on } \overline{D} \Rightarrow G_\alpha^0 f \geq 0 \text{ on } \overline{D}.$$

Sketch of Proof (2)

The Green operators

$$G_\alpha^0 : C_0(\overline{D}) \to C_0(\overline{D}), \ \forall \alpha > 0$$

are **contractive**.

$$\left\| G_\alpha^0 \right\| \le \frac{1}{\alpha}, \ \ \forall \alpha > 0.$$

Sketch of Proof (3)

The domain $D(\mathfrak{W})$ is **dense** in $C_0(\overline{D})$:

$$\lim_{\alpha \to +\infty} \left\| \alpha G_\alpha^0 u - u \right\| = 0, \quad \forall u \in C_0(\overline{D})$$

Open Problem

To prove Main Theorem for **general** Wentzell boundary conditions:

$$
\begin{aligned}
&Lu(x') \\
&= \sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial^2 u}{\partial x_i \partial x_j}(x') + \sum_{i=1}^{N-1} \beta^i(x') \frac{\partial u}{\partial x_i}(x') + \gamma(x') u(x') \\
&+ \mu(x') \frac{\partial u}{\partial \mathbf{n}}(x') - \delta(x') W u(x') \\
&+ \int_{\partial D} r(x', y') \left[u(y') - u(x') - \sum_{j=1}^{N-1} (y_j - x_j) \frac{\partial u}{\partial x_j}(x') \right] dy' \\
&+ \int_{D} t(x', y) \left[u(y) - u(x') - \sum_{j=1}^{N-1} (y_j - x_j) \frac{\partial u}{\partial x_j}(x') \right] dy
\end{aligned}
$$

References (3)

- K. Taira: On the existence of Feller semigroups with discontinuous coefficients, Acta Mathematica Sinica (English Series), Vol. 22, No. 2 (2006), 595-606.
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