Calculus I

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Introduction to Calculus

Contents

Contents

1. **The purpose of this lectures is to provide students with basic knowledge and skills of calculus.**
2. **To understand natural sciences, students should learn some basic mathematics. Therefore, students will study also various topics from Physics, Biology, Chemistry and Technology.**

Why do you study Mathematics ?

The Role of Mathematics

in

Natural Sciences

Mechanism of Mathematical Analysis

Natural Phenomenon

Mathematical Analysis

Mathematical Modeling

Differential Equations ⇒ Solution

Weather Forecast

Mechanism of Weather Forecast

Weather

Weather Forecast

Mathematical Modeling

Navier - Stokes Equations $\underset{\text{Numerical Analysis}}{\Rightarrow}$ Approximation Solution

Navier-Stokes Equations
in
Fluid Dynamics

$$\rho \frac{D\mathbf{V}}{Dt}$$

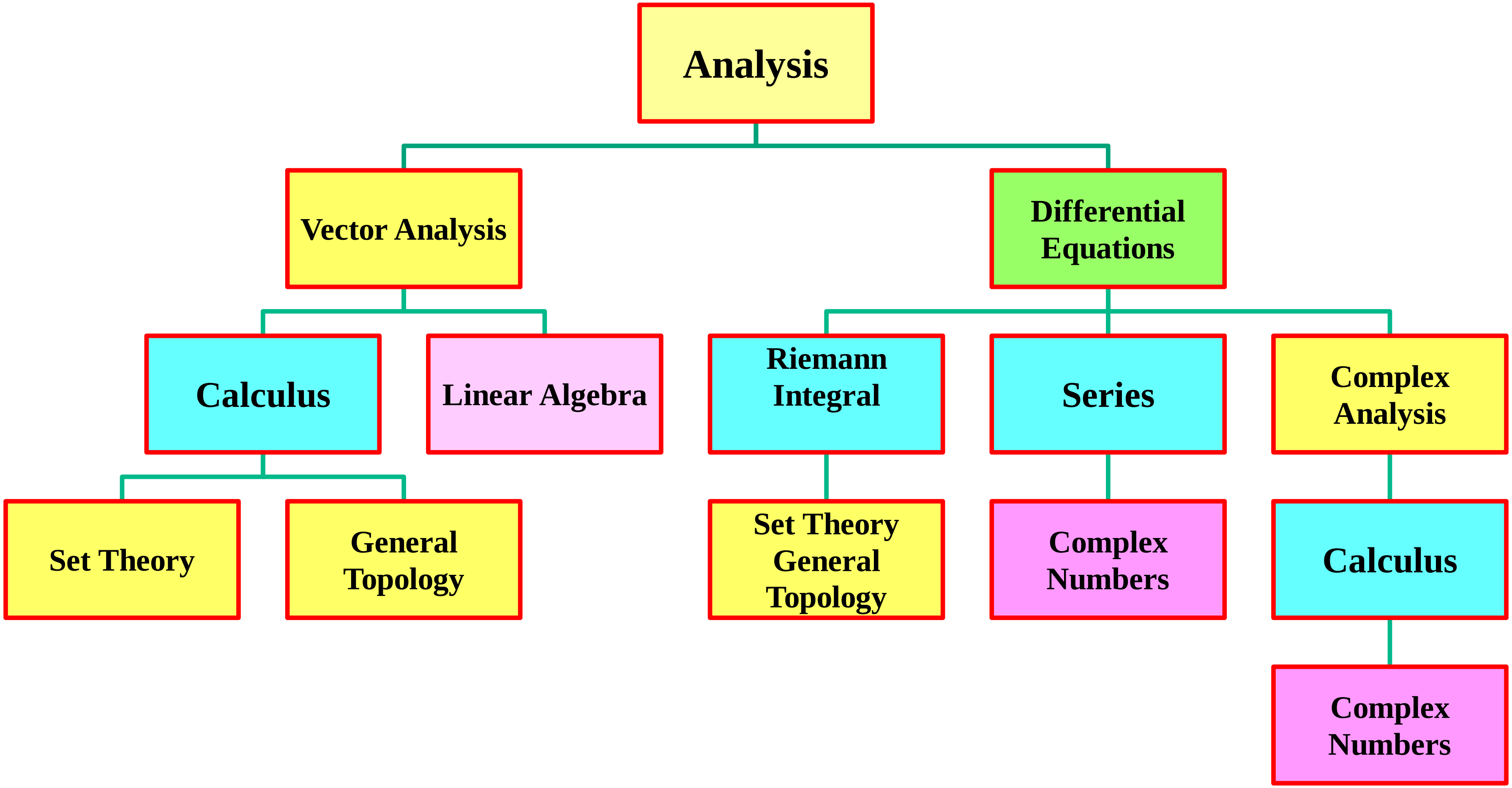
$$= -\nabla p + \rho\mathbf{B} + \mu\Delta\mathbf{V} + \frac{1}{3}\mu\nabla \cdot \mathrm{div}\,\mathbf{V}$$

Inertia Force

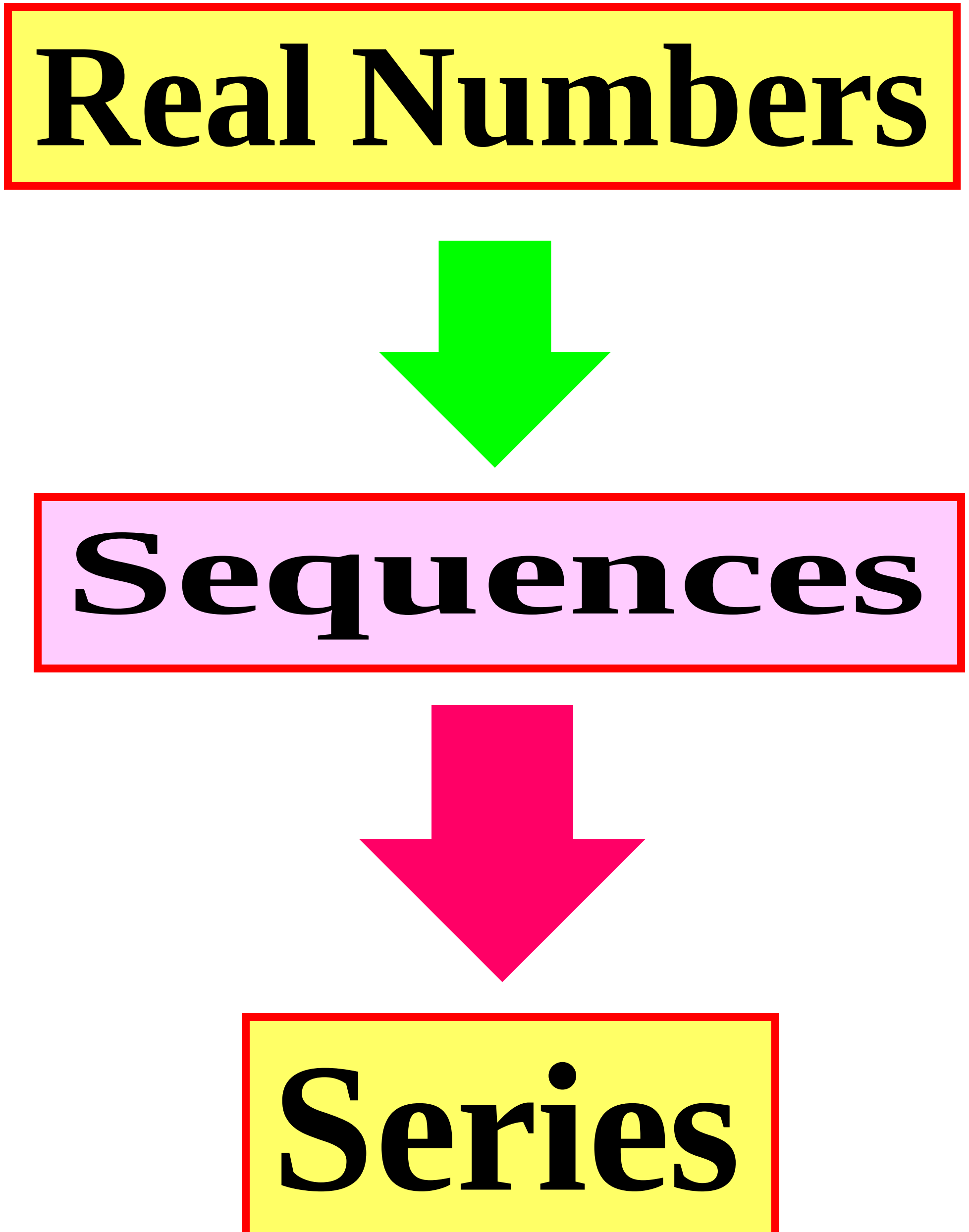
= **Pressure + Force + Viscosity + Stress**

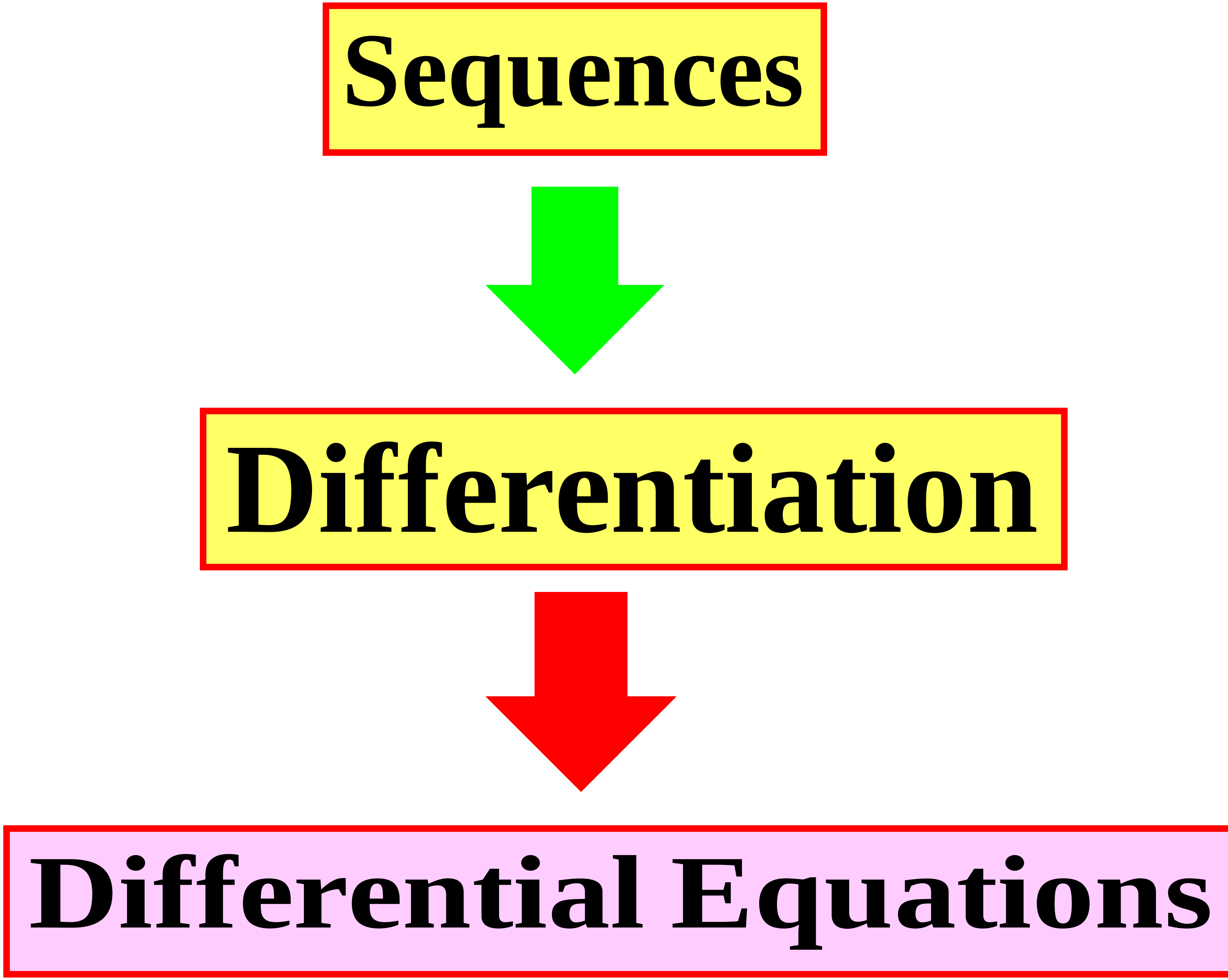
Bird's-Eye View

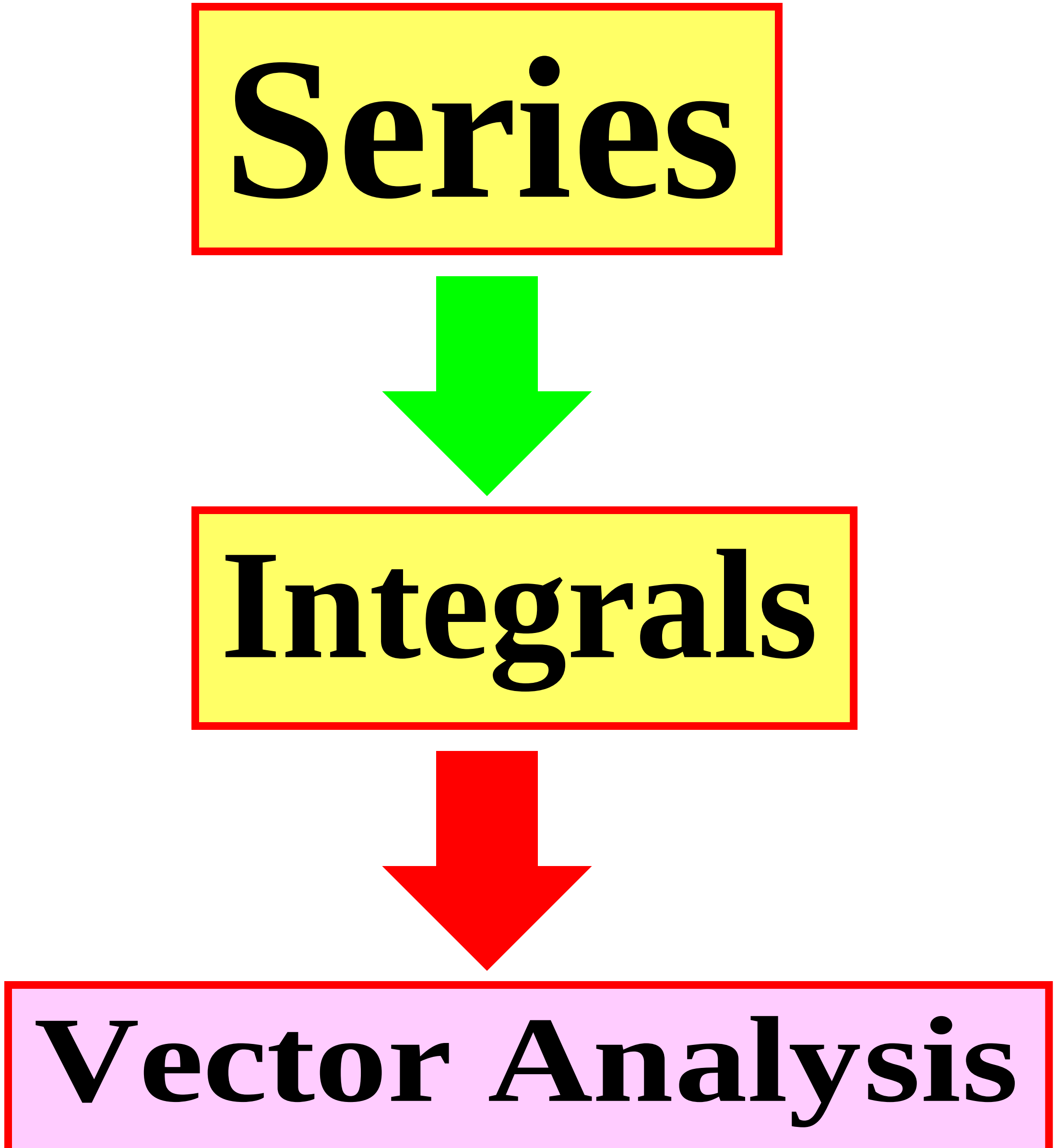
Bird's- Eye View



Bird's-Eye View
of
Calculus


Real Numbers
Sequences
Series


Sequences
Differentiation
Differential Equations


Series
Integrals
Vector Analysis

List of Mathematicians

List (1)

- **Archimedes**（Ｂ．Ｃ．２８７－Ｂ．Ｃ．２１２）**Greece**
- **Newton**（１６４２－１７２７）**England**
- **Leibniz**（１６４６－１７１６）**Germany**
- **Machin**（１６８５－１７５１）**England**
- **Fourier**（１７３６－１８１３）**France**
- **Lagrange**（１７３６－１８１３）**Italy, France**
- **Gauss**（１７７７－１８５５）**Germany**
- **Cauchy**（１７８９－１８５７）**France**
- **Abel**（１８０２－１８２９）**Norway**

List (2)

- **Taylor**（1685－1731）**England**
- **Bolzano**（1781－1848）**Italy**
- **Hermite**（1822－1901）**France**
- **Maclaurin**（1698－1746）**Scotland**
- **Borel**（1871－1956）**France**
- **Dirichlet**（1805－1859）**Germany**
- **Weierstrass**（1815－1897）**Germany**
- **Dedekind**（1831－1916）**Germany**

List (3)

- Rolle（1652－1719）France
- Laplace（1749－1827）France
- Riemann（1826－1866）Germany
- Hilbert（1862－1943）Germany
- Hadamard（1865－1963）France
- Lebesgue（1875－1941）France
- Euler（1707－1783）Switzerland
- Poincare（1854－1912）France

List (4)

- **Bernouille**（1667－1748）**Switzerland**
- **Bessel**（1784－1846）**Germany**
- **Cantor**（1845－1918) **Denmark/ Germany**
- **D’Alembert**（1717－1783）**France**
- **Darboux**（1842－1917） **France**
- **De Morgan**（1806－1871） **France**
- **Fubini**（1879－1943）**Italy**
- **de L’Hospital**（1661－1704） **France**

List (5)

- **Stokes**（1819－1903）**England**
- **Stirling**（1962－1770）
- **Simpson**（1710－1761）**England**
- **Schwarz**（1843－1921）**Germany**
- **Peano**（1858－1932）**Italy**
- **Napier**（1550－1617）**Scotland**
- **Jordan**（1838－1922）**France**
- **Landau**（1887－1938）

Mathematical Thoughts

Mathematical Thoughts

(Ⅰ) Mathematical Reasoning
(Ⅱ) Mathematical Ideas
(Ⅲ) Mathematical Image

Numerical Analysis

Role of Numerical Analysis

Mathematics	Analysis	Numerical Analysis
Physics	Theoretical Physics	Physical Experiments

Mathematics

versus

Physics

Bird's-Eye View

Theme	Mathematics	Physics
Differential Equations	Ordinary Differential Equations	Newton's Equation of Motion
Infinite Series	Fourier Series	Eigenfunction Expansions （Principle of Superposition）
Vector Analysis	Calculus on Surfaces	Continuum Mechanics

Elasticity

Importance of Elasticity

A human body is an elastic material

Thoughts and Methods
in
Analysis

Four Thoughts in Analysis

(Ⅰ) Discrete Case and Continuous Case

(Ⅱ) Principle of Superposition

(Ⅲ) Completeness

(Ⅳ) Numerical Analysis

Discrete Case
versus
Continuous Case

Vectors and Functions

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad \left(\textbf{Finite - Dimensional Case}\right)$$

Discrete Case

Continuous Case

$$\int_a^b K(t, s)x(s)ds = y(t)$$

(Infinite - Diemnsional Case)

Principle of Superposition

Principle of Superposition

Theme	Mathematics	Kinetics
Infinite Series	Fourier Series	Eigenfunction Expansions

Principle of Superposition

$$Pu = f, \quad u = \sum_i u_i$$

Decomposition into Fundamental Elements

$$f = \sum_i f_i$$

Find a solution

$$Pu_i = f_i$$

Superposition of Solutions

Jean Baptiste Joseph Fourier

Fourier

- Jean Baptiste Joseph Fourier (1768-1830)

French Mathematician and Physicist

La theorie analytique de la chaleur (1822)

Fourier's These

Every function of period 2π can be approximated in terms of trigonometric functions.

Fourier Series Expansion (1)

$$f(x) = \sum_{j=0}^{\infty} f_j(x)$$

$$= \frac{a_0}{2} + a_1 \cos x + b_1 \sin x$$

$$+ a_2 \cos 2x + b_2 \sin 2x + \cdots$$

$$+ a_j \cos jx + b_j \sin jx + \cdots$$

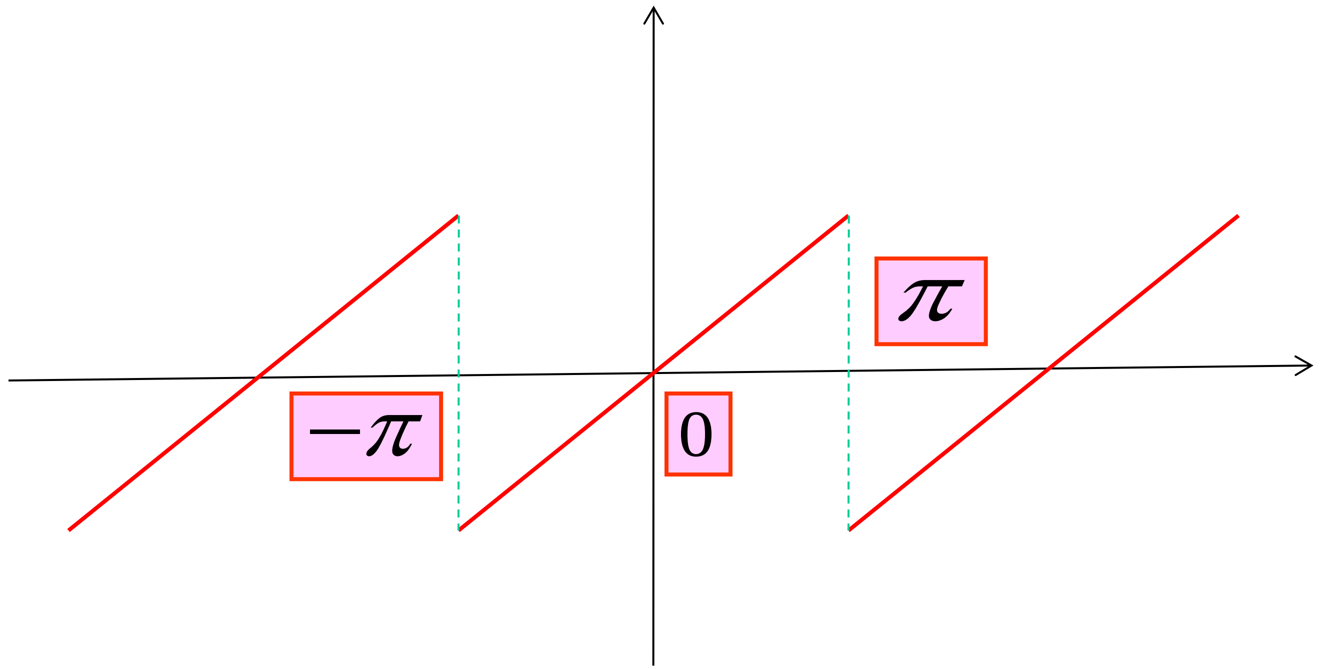
Fourier Series Expansion (2)

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos jx\, dx$$

$$b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin jx\, dx$$

Example

$$f(x) = x, \quad -\pi < x < \pi$$



Fourier Coefficients

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos jx\,dx = 0$$

$$b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin jx\,dx = \frac{2}{j}(-1)^{j+1}$$

$$(j \neq 0)$$

Example of a Fourier Series

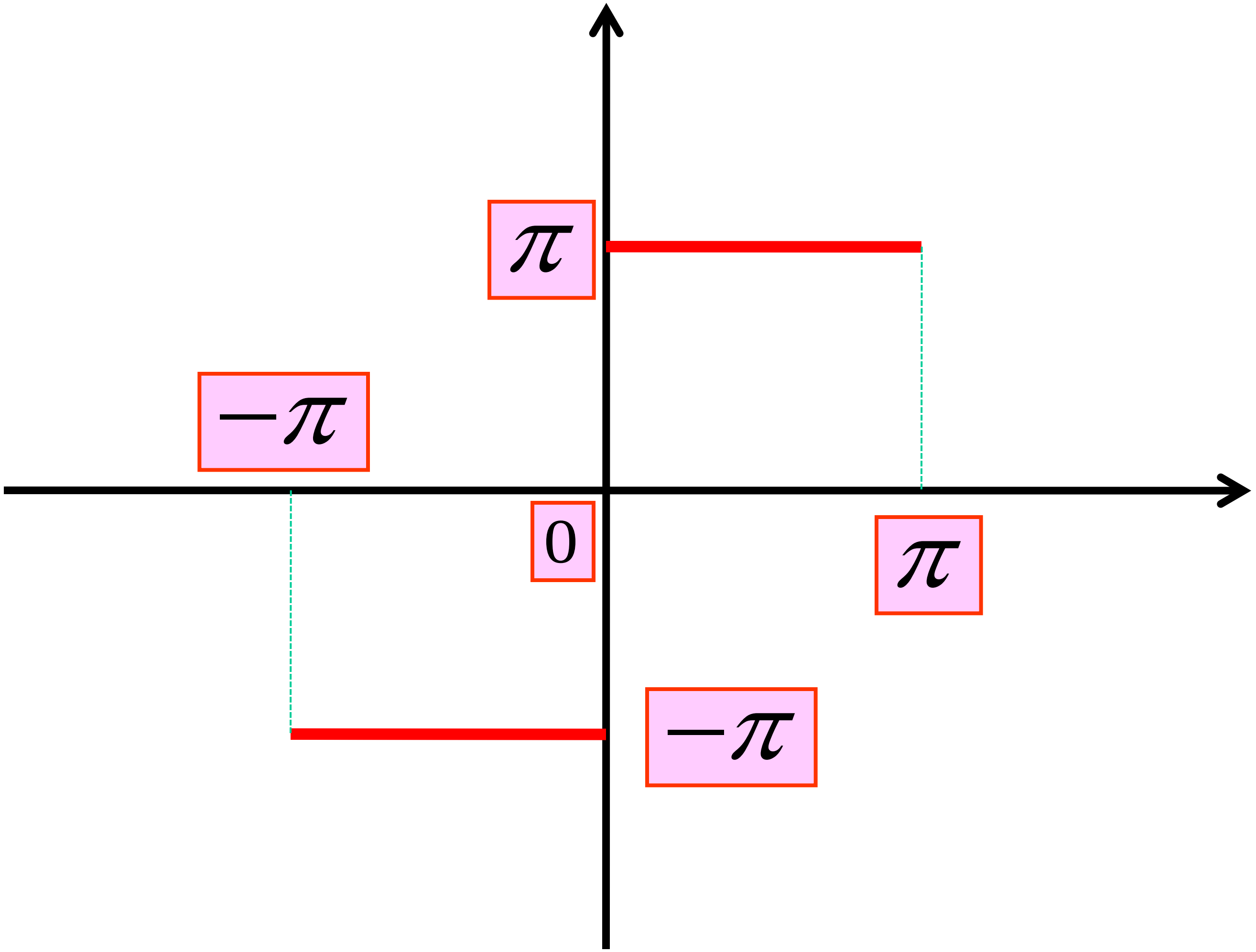
$$x = 2\sin x - 1\sin 2x + \cdots$$

$$+ \frac{2}{j}(-1)^{j+1}\sin jx + \cdots$$

$$(-\pi < x < \pi)$$

Fourier Series of Step Functions

Example of Step Functions



Example of Fourier Series

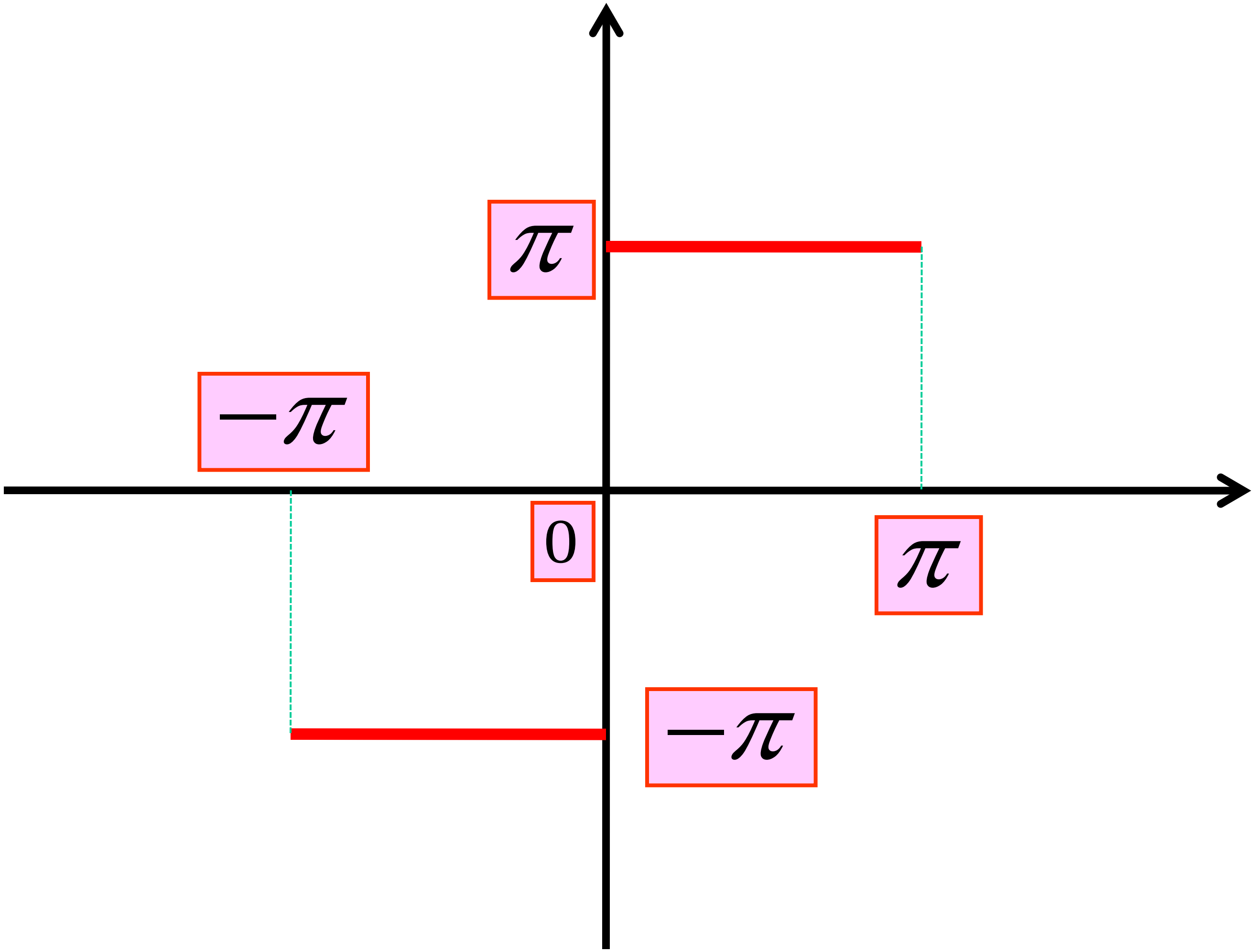
$$\sum_{j=0}^{\infty} \frac{1}{2j-1} \sin(2j-1)x$$

$$= \begin{cases} \dfrac{\pi}{4} & 0 < x < \pi \\ 0 & x = 0, \pi \\ -\dfrac{\pi}{4} & -\pi < x < 0 \end{cases}$$

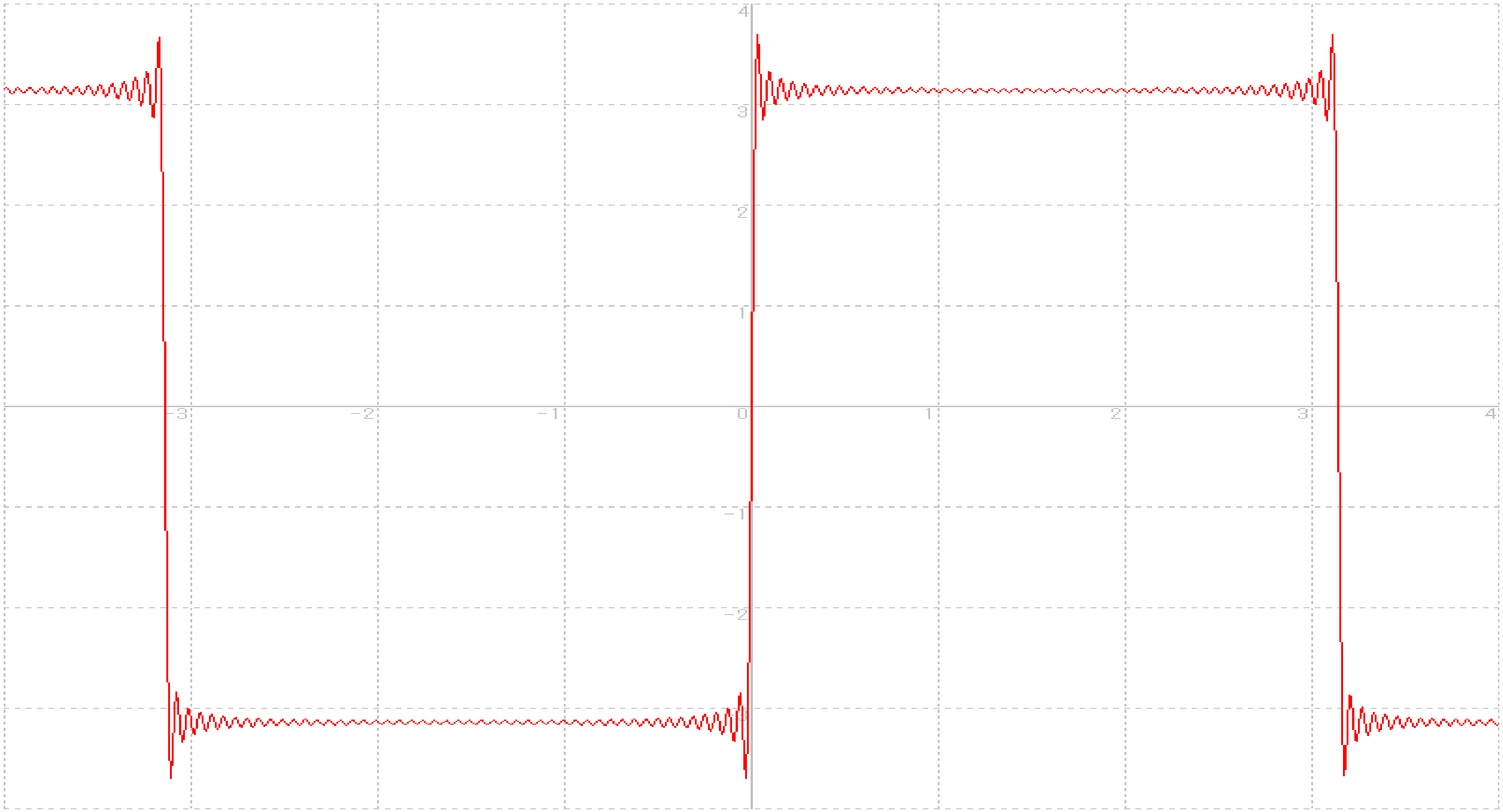
Gibbs Phenomenon

Numerical Computing with BASIC

Example of Step Functions



Example of Gibbs Phenomenon



Weierstrass’ Continuous Function

Weierstrass's Function

$$f(x) = \sum_{k=0}^{\infty} a^k \cos\left(b^k x\right)$$

$$0 < a < 1, \quad ab \geq 1$$

Numerical Computing with BASIC

Example

$$f(x) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \cos\left(3^k x\right)$$

$$a = \frac{1}{2}, b = 3 \Rightarrow ab = \frac{3}{2} > 1$$

$$s_0(x) = \cos x$$

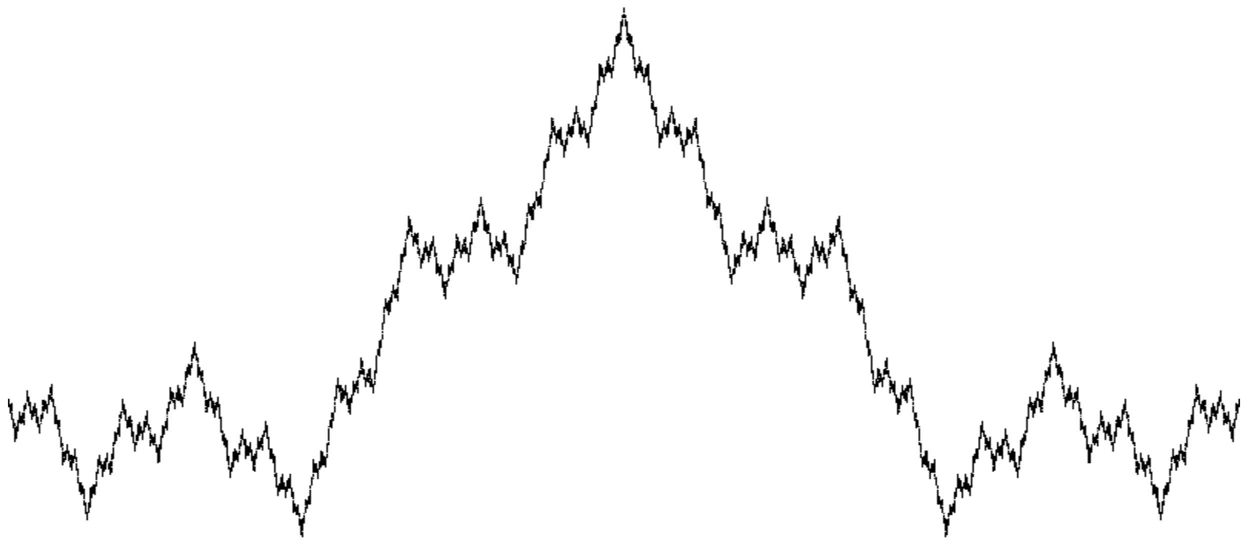
$$s_1(x) = \cos x + \frac{1}{2}\cos 3x$$

$$s_2(x) = \cos x + \frac{1}{2}\cos 3x + \frac{1}{4}\cos 9x$$

$$s_3(x) = \cos x + \frac{1}{2}\cos 3x + \frac{1}{4}\cos 9x + \frac{1}{8}\cos 27x$$

$$s_4(x) = \cos x + \frac{1}{2}\cos 3x + \frac{1}{4}\cos 9x + \frac{1}{8}\cos 27x + \frac{1}{16}\cos 81x$$

Weierstrass Function



Heat Conduction (Fourier's Work)

Formulation of a Problem

Steel bar of length π

Zero temperature on its ends

Initial temperature $f(x)$

Initial-Boundary Value Problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \; t > 0$$

$u(0,t) = u(\pi,t) = 0, \quad t > 0$ （**Boundary Condition**）

$u(x,0) = f(x), \quad 0 < x < \pi$ （**Initial Condition**）

Fourier's Method
（Separation of Variables）

Representation of a Solution (Heat Kernel)

$$u(x,t) = \int_0^{\pi} p(t,x,y) f(y) dy$$

$$p(t,x,y) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} \sin nx \sin ny$$

(Heat Kernel)

Application to Series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

Trace of a Matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{pmatrix}$$

$$\Rightarrow$$

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$$ (**Sum of Eigenvalues**)

Trace Formula (1)

$$\int_0^{\pi} p(t,x,x)\,dx$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-n^2 t}\left(\int_0^{\pi} \sin^2 nx\,dx\right)$$

$$= \sum_{n=1}^{\infty} e^{-n^2 t}$$

Stationary Boundary Value Problem

$$v''(x) = g(x), \quad 0 < x < \pi$$

$$v(0) = v(\pi) = 0$$ （**Boundary Condition**）

Representation of a Solution (Green's Function)

$$u(x,t) = \int_0^{\pi} G(x,y)g(y)dy$$

$G(x,y)$ **Green Function**

Green's Function
(Series Version)

$$G(x,y) = -\int_0^\infty p(t,x,y)\,dt$$

$$= -\frac{2}{\pi}\sum_{n=1}^{\infty}\left(\int_0^\infty e^{-n^2 t}\,dt\right)\sin nx \sin ny$$

$$= -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin nx \sin ny$$

Trace Formula (2)

$$\int_0^{\pi} G(x,x)dx = -\int_0^{\infty}\int_0^{\pi} p(t,x,x)dxdt$$

$$= -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\int_0^{\pi}\sin^2 nx\,dx\right)$$

$$= -\sum_{n=1}^{\infty}\frac{1}{n^2}$$ （**Sum of Eigenvalues**）

Green's Function (Integral Kernel Version)

$$G(x,y)=\begin{cases}\left(\dfrac{y}{\pi}-1\right)x & 0\le x\le y\le\pi\\[2ex]\left(\dfrac{x}{\pi}-1\right)y & 0\le y\le x\le\pi\end{cases}$$

Trace Formula (3)

$$\int_0^{\pi} G(x,x)\,dx$$

$$= \int_0^{\pi} \left(\frac{x^2}{\pi} - x \right) dx = -\frac{\pi^2}{6}$$

Trace Formula (4)

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = -\int_0^{\pi} G(x,x)\,dx = \frac{\pi^2}{6}$$

Mathematical System of Numbers

Set	Algebra	Analysis
Complex Numbers	$+ - \times \div$	Complete
Real Numbers	$+ - \times \div$	Complete
Rational Numbers	$+ - \times \div$	
Integers	$+ - \times$	
Natural Numbers	$+ \times$	

Completeness

Convergence of Sequences

Definition of Convergence

$\{a_n\}$ **sequence of real numbers**

$\{a_n\}$ **converges to** a

$$\overset{\text{def}}{\Leftrightarrow}$$

$\forall \varepsilon > 0,\ \exists N = N(\varepsilon) \in \mathbf{N}$ **such that**

$$\forall n \geq N \Rightarrow |a_n - a| < \varepsilon$$

Cauchy's Test

Cauchy's Test

$\{a_n\}$ **converges**

$\Leftrightarrow$

$$\lim_{n,m\to\infty} |a_n - a_m| = 0$$

Complex Numbers

Carl Friedrich Gauss

Gauss

◆ **Carl Friedrich Gauss (1777-1855)**

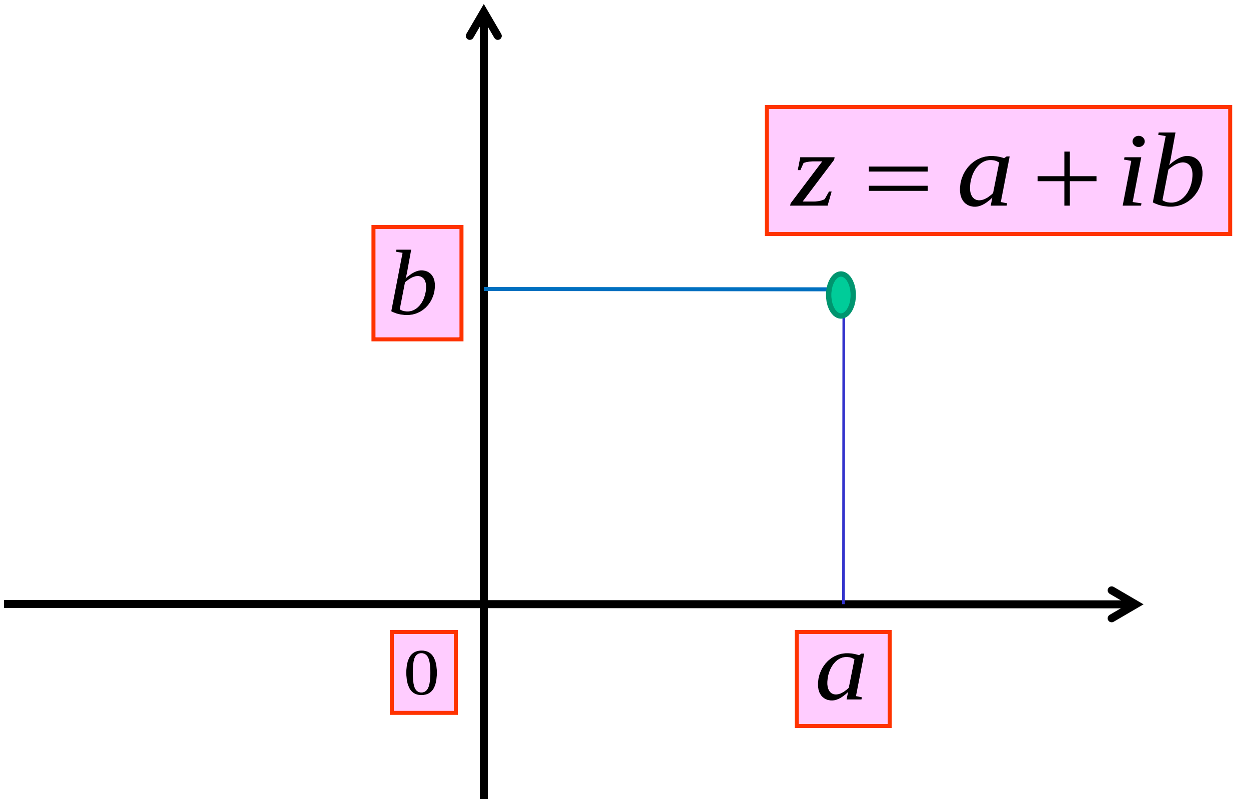
German Mathematician and Physicist

Complex Number

$$a + ib = c + id$$
$$\Leftrightarrow$$
$$a = c, b = d$$

$$i = \sqrt{-1}$$

Complex Plane

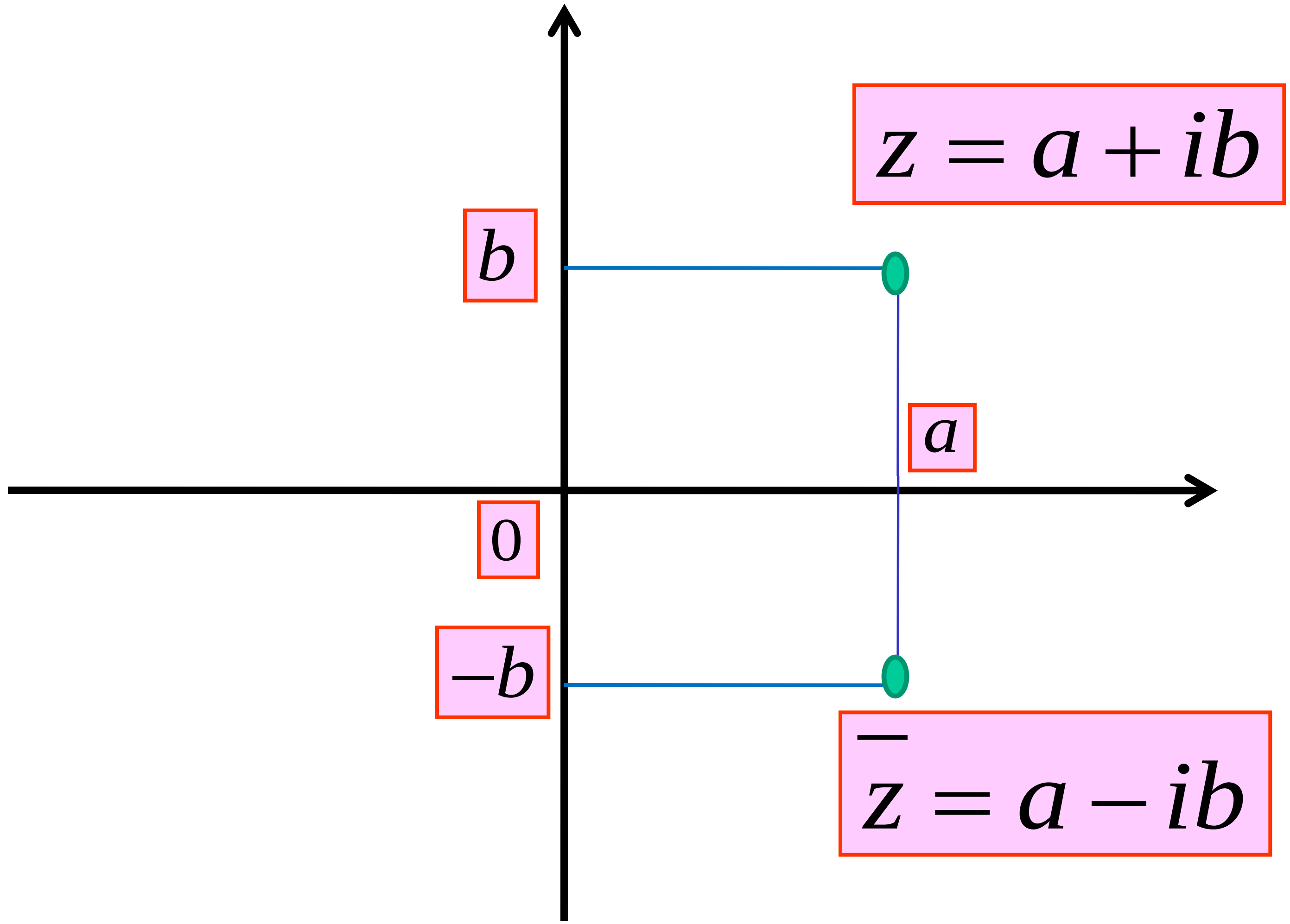


Conjugate of a Complex Number

$$z = a + ib$$

$$\Rightarrow$$

$$\bar{z} = a + i(-b) = a - ib$$



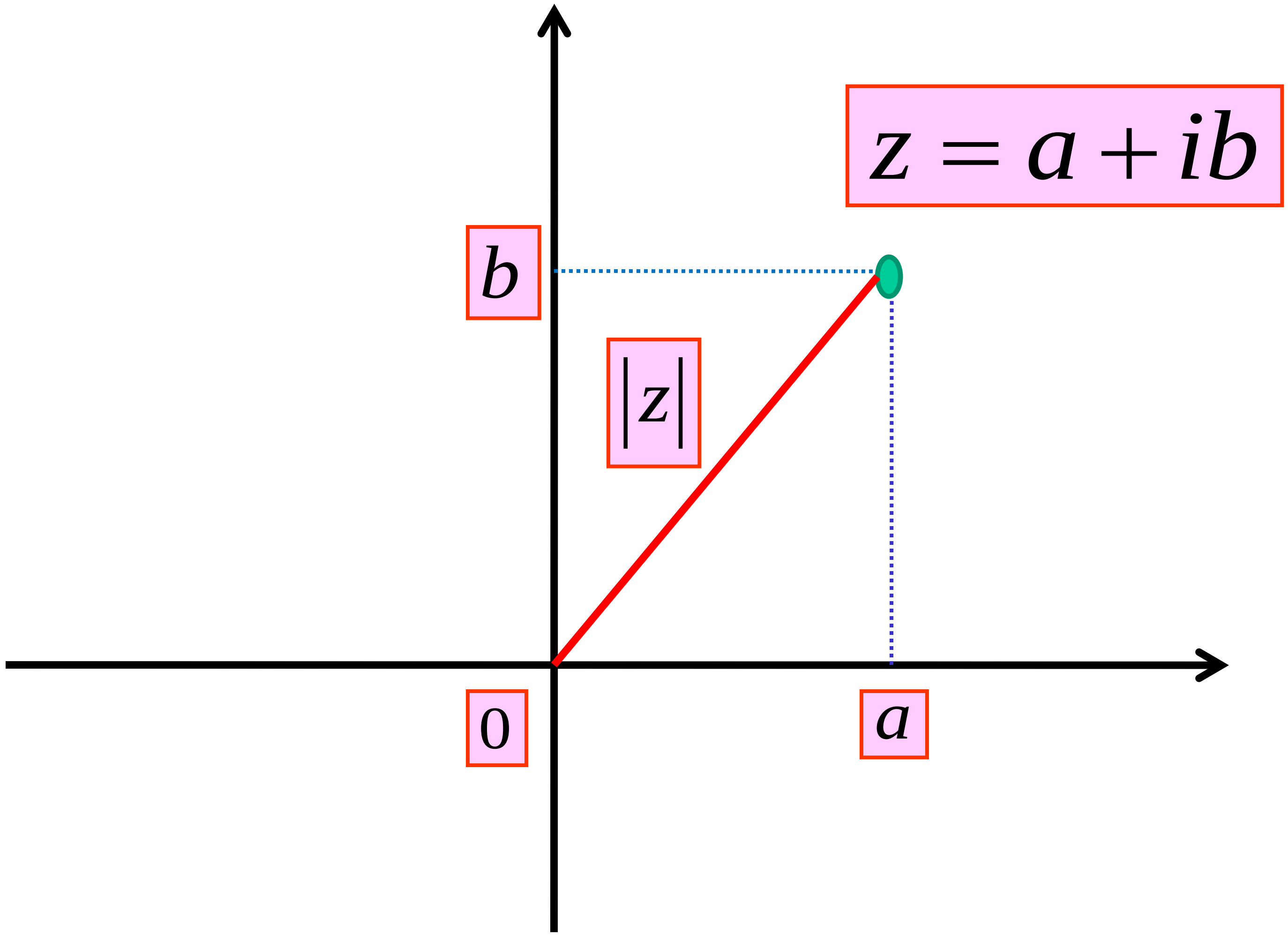
$z = a + ib$
b
a
0
$-b$
$\bar{z} = a - ib$

Absolute Value of a Complex Number

$$z = a + ib$$

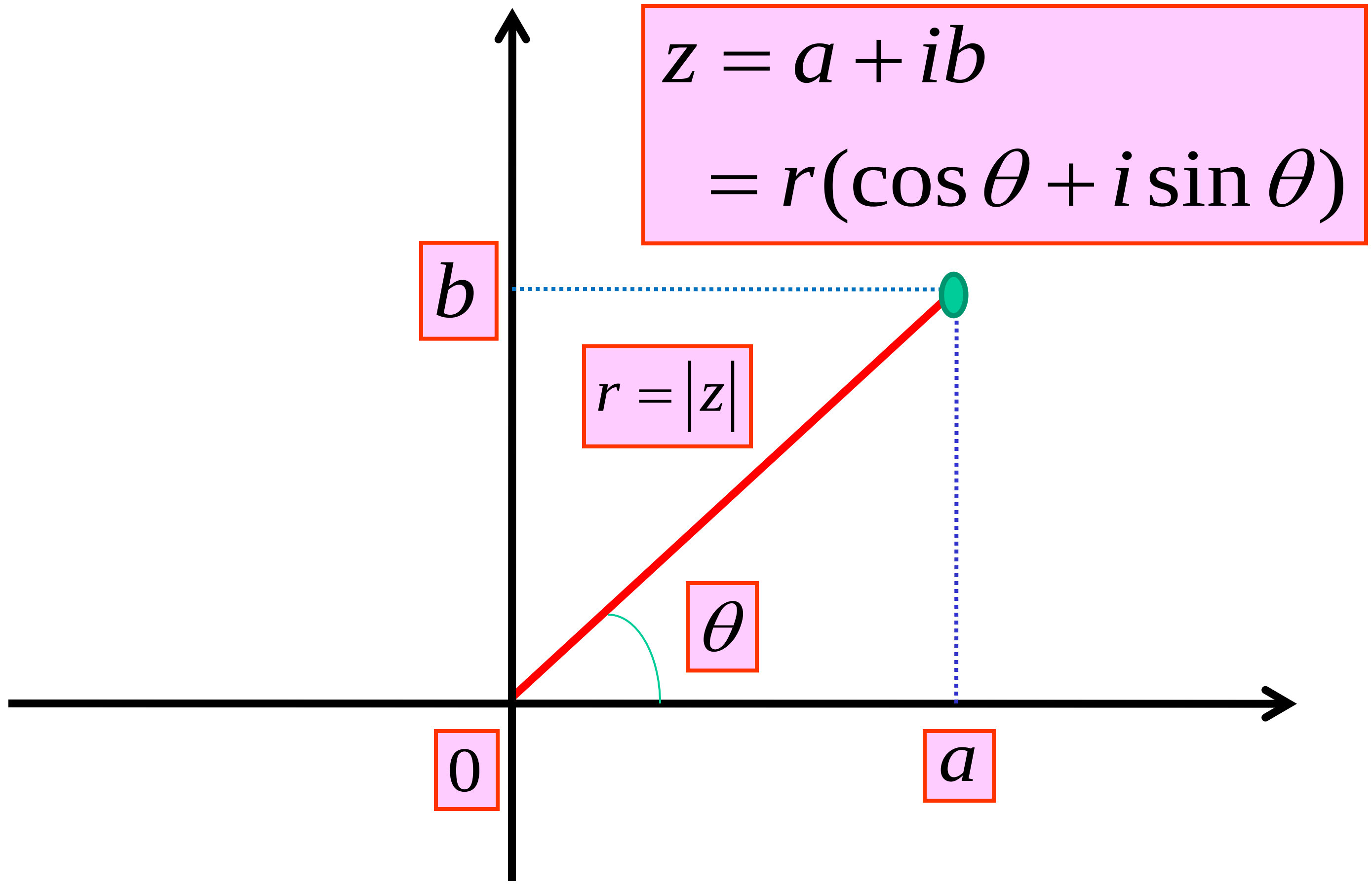
$$\Longrightarrow$$

$$|z| = |a + ib| = \sqrt{a^2 + b^2}$$



$z = a + ib$
b
$|z|$
0
a

Polar Coordinates of a Complex Number

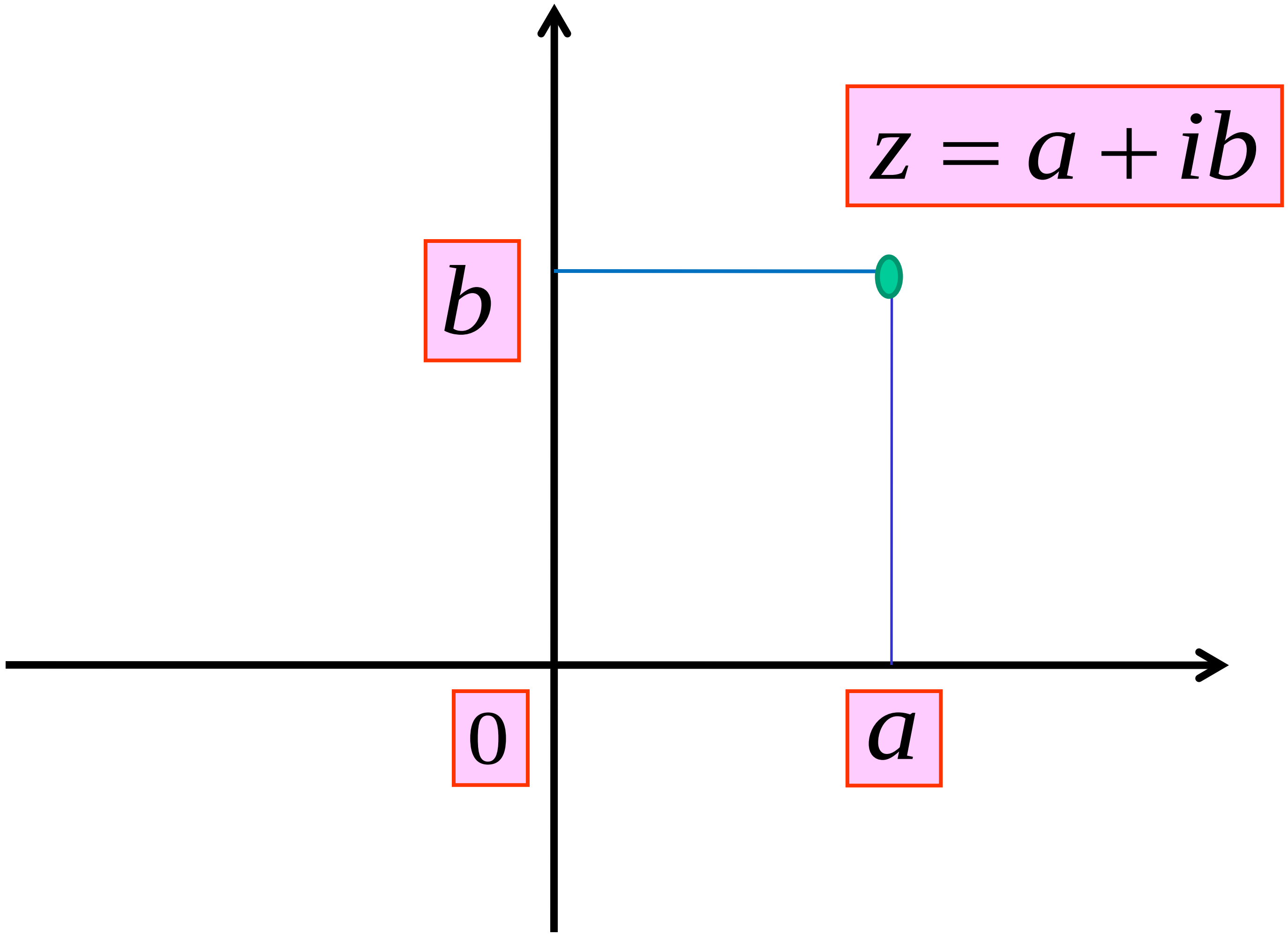


Sum of Complex Numbers

$$z = a + ib, \quad w = c + id$$

$$\Rightarrow$$

$$z + w = (a + c) + i(b + d)$$



$z = a + ib$
b
0
a

Difference of Complex Numbers

$$z = a + ib, \quad w = c + id$$

$$\Rightarrow$$

$$z - w = (a - c) + i(b - d)$$

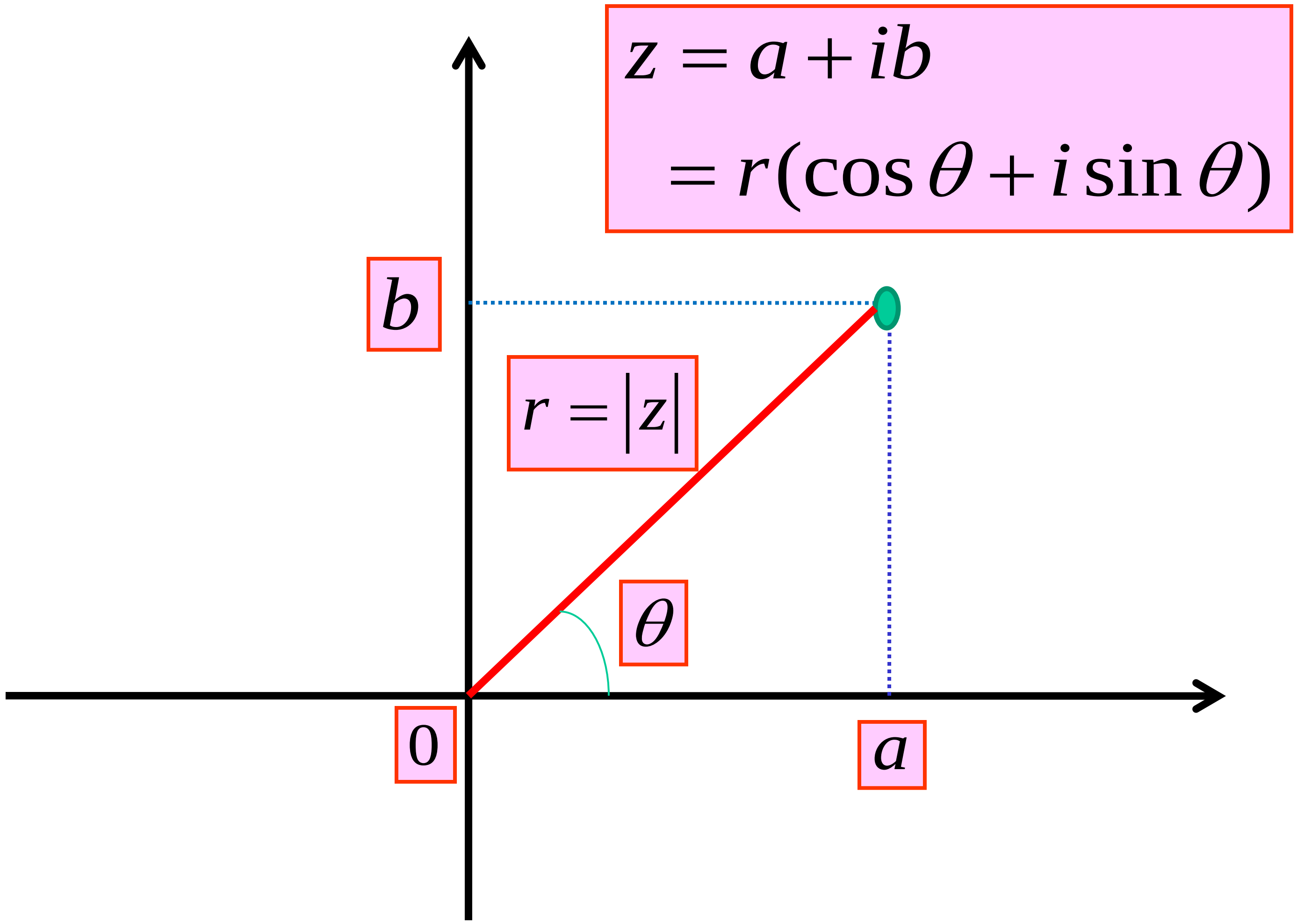
Product of Complex Numbers

$$z = a + ib, \quad w = c + id$$

$$\Rightarrow$$

$$zw = (ac - bd) + i(ad + bc)$$

$$i = \sqrt{-1} \Rightarrow i^2 = -1$$



$z = a + ib$
$= r(\cos\theta + i\sin\theta)$
b
$r = |z|$
θ
0
a

Product of Complex Numbers

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$w = s(\cos\omega + i\sin\omega) = se^{i\omega}$$

$$\Longrightarrow$$

$$zw = rs(\cos(\theta+\omega) + i\sin(\theta+\omega))$$

$$= rse^{i(\theta+\omega)}$$

De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$\forall n \in \mathbf{Z}$$

Leonhard Euler (1707-1783)

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{i\pi} = \cos\pi + i\sin\pi = -1$$

Euler + De Moivre

$$
\begin{aligned}
(e^{i\theta})^n &= (\cos\theta + i\sin\theta)^n \\
&= \cos n\theta + i\sin n\theta \\
&= e^{in\theta} \qquad (\forall n \in \mathbf{Z})
\end{aligned}
$$

Algebraic Equation

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0$$

$$a_i \in \mathbf{C}$$

Fundamental Theorem of Algebra (Gauss)

Every algebraic equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0,\ a_0 \neq 0$$

has *n* roots in C counted with multiplicity.

Example (1)

$$ax + b = 0,\ a \neq 0$$

$$\Longrightarrow$$

$$x = -\frac{b}{a}$$

Example (2)

$$ax^2 + bx + c = 0, \quad a \neq 0$$

$$\Longrightarrow$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Imaginary Number

$$x^2 + 1 = 0$$

$$\Longrightarrow$$

$$x = \pm\sqrt{-1}$$

Real Numbers

Real Numbers and Decimal System

Real Numbers	Decimal System	Classification
Natural Numbers	Positive Integers	Rational
Integers	Integers	Rational
Fractional Numbers	Finite Decimal	Rational
Fractional Numbers	Recurring Decimal	Rational
Non-Fractional Numbers	Non-Recurring Decimal	Irrational

Finite Decimal (1)

$$\frac{1}{4} = 0.25$$

$$\frac{118}{25} = 4.72$$

Finite Decimal (2)

$$0.0625 = \frac{625}{10000}$$

$$= \frac{1}{16}$$

Recurring Decimal (1)

$$\frac{83}{74} = 1.121621621216\cdots$$

$$= 1.1\dot{2}\dot{1}\dot{6}$$

$$\frac{89}{13} = 6.846153846153\cdots$$

$$= 6.\dot{8}\dot{4}\dot{6}\dot{1}\dot{5}\dot{3}$$

Recurring Decimal (2)

$$1.1\dot{2}1\dot{6} = 1.1216216216\cdots$$

$$= 1.1 + 0.0216 + 0.0000216 + \cdots$$

$$= \frac{11}{10} + 216 \times \frac{1}{10^4} + 216 \times \frac{1}{10^7} + \cdots$$

$$= \frac{11}{10} + 216 \times \frac{1}{10^4}\left(1 + \frac{1}{10^3} + \cdots\right)$$

$$= \frac{11}{10} + 216 \times \frac{1}{10^4} \times \frac{1}{1 - \frac{1}{10^3}}$$

$$= \frac{11205}{9990} = \frac{83}{74}$$

Non-Recurring Decimal

$$\sqrt{2} = 1.41421356\cdots$$

$$e = 2.718281828459045904\cdots$$

The square root of a prime number is irrational (1)

Let p be a prime number.

Assume that $\sqrt{p}$ is rational.

(*) $$\sqrt{p} = \frac{n}{m}$$

Here the right−hand side is irreducible.

The square root of a prime number is irrational (2)

(*) $\Rightarrow$

(**) $\quad n^2 = pm^2$

p **is a prime number**

n^2 **is a multiple of** $p \Leftrightarrow$

n **is a multiple of** p

$n = pa$ **+** (**) $\Rightarrow$

$pm^2 = n^2 = p^2 a^2 \Rightarrow$

$m^2 = pa^2$

The square root of a prime number is irrational (3)

$$m^2 = pa^2$$

implies that

m is a multiple of p:

$$m = pb$$

$$\Longrightarrow$$

$$\sqrt{p} = \frac{n}{m} = \frac{pa}{pb} = \frac{a}{b}$$

(contradiction)

Theory
of
Real Numbers

Main Theme

How do we characterize irrational numbers ?

What is the convergence of sequences ?

Four Fundamental Theorems in Real Numbers

(Ⅰ) Dedekind Cut
(Ⅱ) Supremum and Infimum
(Ⅲ) Convergence of bounded monotone sequences
(Ⅳ) Cantor's Nested-Interval Property

Dedekind Cut

Cut of Real Numbers

A cut of real numbers : (A, B)

$$\overset{\text{def}}{\Longleftrightarrow}$$

$$\mathbb{R} = A \cup B$$

$$\forall a \in A, \forall b \in B \Rightarrow a < b$$

Examples

$$A = \{x \in \mathbf{R} : x \leq 0\}, \quad B = \{x \in \mathbf{R} : 0 < x\}$$

$$A = \{x \in \mathbf{N} : 0 \leq x \leq 5\}, \quad B = \{x \in \mathbf{N} : 6 \leq x\}$$

$$A = \{x \in \mathbf{N} : 0 \leq x < \tfrac{1}{2}\}, \quad B = \{x \in \mathbf{N} : \tfrac{1}{2} < x\}$$

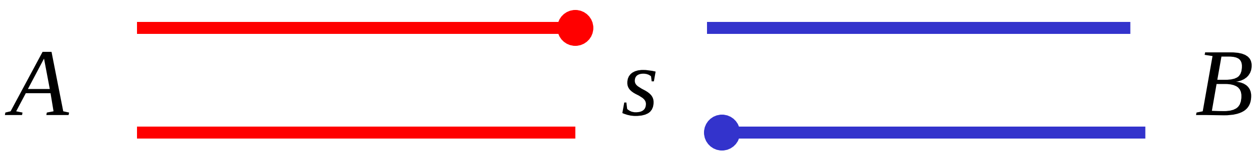
$$A = \{x \in \mathbf{Z} : -\infty < x < 0\}, \quad B = \{x \in \mathbf{Z} : 0 \leq x < \infty\}$$

Dedekind Cut

A cut (A, B) **defines a number** s **such that :**

(1) s **is the maximum of** A**, but** B **has no minimun.**

(2) s **is the minimum of** B**, but** A **has no maximun.**



Supremum and Infimum

Upper Bound and Lower Bound

(1) The set S is bounded from above

$$\overset{\text{def}}{\Longleftrightarrow}$$

$$\exists M : a \leq M \quad \forall a \in S$$

M **is called a upper bound**

(2) The set S is bounded from below

$$\overset{\text{def}}{\Longleftrightarrow}$$

$$\exists L : L \leq a \quad \forall a \in S$$

L **is called a lower bound**

Example (1)

(1) $I = (-\infty, 1)$

$\Rightarrow$

I **is bounded from above**

Example (2)

$(2)\ J = (-1, \infty)$

$\Rightarrow$

J **is bounded from below**

Example (3)

$(3)\ K = [-1, 1]$

$\Longrightarrow$

K **is bounded from below and from above**

Supremum and Infimum (1)

(I) a : **supremum of** S**:**

(I-1) a **is a upper bound of** S.

(I-2) a **is the least upper bound.**

(II) b : **infimum of** S**:**

(II-1) b **is a lower bound of** S.

(II-2) b **is the greatest lower bound.**

Supremum and Infimum (2)

(I) Supremum of $S: \ a = \sup S$

(II) Infimum of $S: b = \inf S$

(I) Supremum = Generalization of Maximum
(II) Infimum = Generalization of Minimum

Example (1)

$$(1)\ I = (-1, 1)$$

$$\Longrightarrow$$

$$\sup I = 1$$

$$\inf\ I = -1$$

Example (2)

$$(2)\ J = (-1, 1]$$

$$\Rightarrow$$

$$\sup J = \max J = 1$$

$$\inf J = -1$$

Example (3)

$$(3)\ K=[-1,1]$$

$$\Rightarrow$$

$$\sup K=\max K=1$$

$$\inf\ K=\min K=-1$$

Example (4)

$$a_n = (-1)^n + \frac{1}{n}$$

$$a_1 = 0,\ a_3 = -\frac{2}{3},\ a_5 = -\frac{4}{5}$$

$$a_2 = \frac{3}{2},\ a_4 = \frac{5}{4},\ a_6 = \frac{7}{6}$$

$$\Longrightarrow$$

$$\begin{cases} \sup a_n = \max a_n = \dfrac{3}{2} \\ \inf a_n = -1 \end{cases}$$

Weierstrass' Theorem

Weierstrass（1815—1897）

Existence of Supremum and Infimum

(I) A set S **of** **bounded from above** **has the** **supremum** $a : \exists a = \sup S$

(II) A set S **of** **bounded from below** **has the** **infimum** $b : \exists b = \inf S$

Sequences

Sequences versus Functions

	Domain of Definition	Range
Sequence	Natural Numbers	Real Numbers
Functions	Real Numbers	Real Numbers

Definition

The sequnce $\{a_n\}$ **converges to** a

$$\overset{\text{def}}{\Longleftrightarrow}$$

$\forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbf{N}$ **such that**

$\forall n \geq N \Rightarrow |a_n - a| < \varepsilon$

Notation : $\lim_{n \to \infty} a_n = a$

Fundamental Example

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

Proof

$$\forall \varepsilon > 0,\ \exists N = N(\varepsilon) \in \mathbf{N} \text{ such that}$$

$$\forall n \geq N \Rightarrow \frac{1}{n} < \varepsilon$$

Archimedes' Principle

Archimedes' Principle

$$\forall\ a, b > 0,\ \ \exists n \in \mathbf{N} \text{ such that}$$

$$\boxed{na > b}$$

$\forall a, b > 0,\ \exists n \in \mathbf{N}$ such that

$$\boxed{na > b}$$

$$\Updownarrow$$

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

Examples (1)

$$(1)\ \lim_{n\to\infty}\frac{n+1}{n}=1$$

$$(2)\ \lim_{n\to\infty}\frac{n+1}{n^2}=0$$

$$(3)\ \lim_{n\to\infty}\left(\sqrt{n^2+1}-n\right)=0$$

Example (2)

$$\lim_{n\to\infty} a^n = \begin{cases} 0 & \textbf{if} \quad 0 < a < 1 \\ 1 & \textbf{if} \quad a = 1 \\ +\infty & \textbf{if} \quad a > 1 \end{cases}$$

Proof

$$a > 1 \Rightarrow a = 1 + h,\ \ h > 0$$

$$\Rightarrow a^n = (1 + h)^n$$

$$= 1 + nh + \frac{n(n-1)}{2} h^2 + \cdots$$

$$> 1 + nh$$

$$\Rightarrow \lim_{n \to \infty} a^n = +\infty$$

Examples (3)

$$(1)\ \lim_{n\to\infty} a^{\frac{1}{n}} = 1 \quad \textbf{for} \quad a > 0$$

$$(2)\ \lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

Binomial Theorem

$$\forall a, b \in \mathbf{R}, n \in \mathbf{N}$$

$$(a+b)^n = \sum_{r=0}^{n} {}_nC_r \, a^{n-r} b^r$$

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

Examples

$$(1)\ (a+b)^2 = a^2 + 2ab + b^2$$

$$(2)\ (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(3)\ (a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Bounded Sequences

Fact

A convergent sequence is bounded.

Bolzano-Weierstrass Theorem

Bolzano（1781—1848）

Weierstrass（1815—1897）

Bolzano-Weierstrass Theorem

Every bounded sequence has a convergent subsequence.

Example (1)

$$a_n = (-1)^n$$

$$\Rightarrow \begin{cases} a_{2k} = 1 \to 1 \\ a_{2k+1} = -1 \to -1 \end{cases}$$

Example (2)

$$a_n = (-1)^n + \frac{1}{n}$$

$$\Rightarrow \begin{cases} a_{2k} = 1 + \dfrac{1}{2k} \to 1 \\ a_{2k+1} = -1 + \dfrac{1}{2k+1} \to -1 \end{cases}$$

Bounded Monotone Sequence

Fundamental Theorem

Every bounded, monotone increasing sequence itself converges.

$a_n \leq \exists M$ （**Bounded**）

$a_n \leq a_{n+1}$ （**Monotone increasing**）

Example 1 (Golden Ratio)

$$a_1 = 1, \quad a_2 = \sqrt{1 + a_1}$$

$$a_n = \sqrt{1 + a_{n-1}} \quad (n \geq 3)$$

$$\Rightarrow$$

$$(1)\, 1 \leq a_n < 3$$

$$(2)\, 1 \leq a_n < a_{n+1}$$

$$(3)\, \lim_{n \to \infty} a_n = \frac{1 + \sqrt{5}}{2}$$

Example 2 (Napier's Number)

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

$$e = {}^{\exists}\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Proof (1)

$0 < a_n < a_{n+1}$ **(Monotone increasing)**

$\because$

$$a_n = 1 + \frac{n}{1!}\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \cdots + \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1-\frac{1}{n}}{2!} + \frac{(1-\frac{1}{n})(1-\frac{2}{n})}{3!} + \cdots + \frac{(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{n-1}{n})}{n!}$$

Proof (2)

$0 < a_n < 3$ **(Boundedness)**

$\because$

$$a_n < 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$$

$$= 1 + 2\left(1 - \left(\frac{1}{2}\right)^n\right)$$

$$< 3$$

Geometric Series

$$\forall b, r \in \mathbf{R}, n \in \mathbf{N}$$

$$\sum_{k=1}^{n} br^{k-1}$$

$$= b + br + br^2 + \cdots + br^{n-2} + br^{n-1}$$

$$= \begin{cases} nb & \textbf{if} \;\; r = 1 \\ \dfrac{b(1-r^n)}{1-r} & \textbf{if} \;\; r \neq 1 \end{cases}$$

Example 3 (Euler's Number)

$$b_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$$

$$\gamma = {}^{\exists}\lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right)$$

Numerical Analysis

Numerical Computing
with
BASIC

Napier's Number

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

$$= 2.718281828459045 \cdots$$

Napier's Number
(Sequence Version)

$$e = \lim_{n \to \infty} a_n = \lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n$$

$$a_1 = 2$$
$$a_2 = 2.25$$
$$a_3 = 2.37037037037037$$
$$a_4 = 2.44140625$$
$$a_5 = 2.48832$$
$$a_6 = 2.52162637174211$$
$$a_7 = 2.54649969704071$$
$$a_8 = 2.56578451395034$$
$$a_9 = 2.58117479171319$$
$$a_{10} = 2.5937424601$$

Napier's Number
(Series Version)

$$e = \lim_{n\to\infty} A_n$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots$$

$$A_1 = 1$$
$$A_2 = 2$$
$$A_3 = 2.5$$
$$A_4 = 2.666666666666666665$$
$$A_5 = 2.7083333333333333$$
$$A_6 = 2.716666666666666663$$
$$A_7 = 2.718055555555555554$$
$$A_8 = 2.718253968253968 4$$
$$A_9 = 2.71827876984127$$
$$A_{10} = 2.7182815255731922$$

Euler's Number

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right)$$

$$= 0.57721\cdots$$

$$b_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$$

$$b_n = 0.577632273697698 \ (n = 1200)$$

$$b_n = 0.577465644068048 \ (n = 2000)$$

$$b_n = 0.577265664067827 \ (n = 10000)$$

Square Root of 2

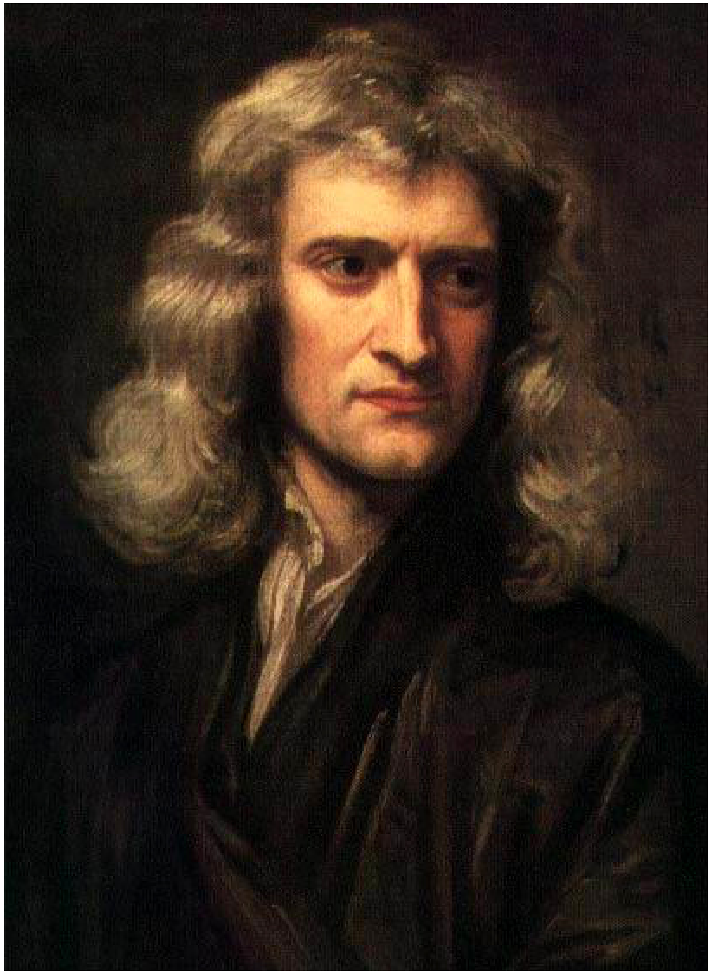
$$\sqrt{2} = 1.41421356\cdots$$

Newton's Method versus Bisection Method

Method	Newton's Method	Bisection Method
Hypotheses	Differentiability Monotonicity	Continuity
Merits Demerits	Strong Hypotheses Rapid Convergence	Weak Hypotheses Slow Convergence
Background	Convergence of Monotone Sequences	Intermediate Value Theorem

Newton's Approximation Method

Isaac Newton (1642-1727)



Fundamental Theorem

Every bounded, monotone increasing sequence itself converges.

$a_n \leq \exists M$ （**Bounded**）

$a_n \leq a_{n+1}$ （**Monotone increasing**）

Newton's Approximation Method

$$r > 0,\ a_0 > 0$$

$$a_{n+1} := \frac{1}{2}\left(a_n + \frac{r}{a_n} \right), \quad n = 0, 1, 2, \ldots$$

$$\Rightarrow$$

$$a_n \downarrow \sqrt{r} \quad (n \to \infty)$$

Example (Square root of 2)

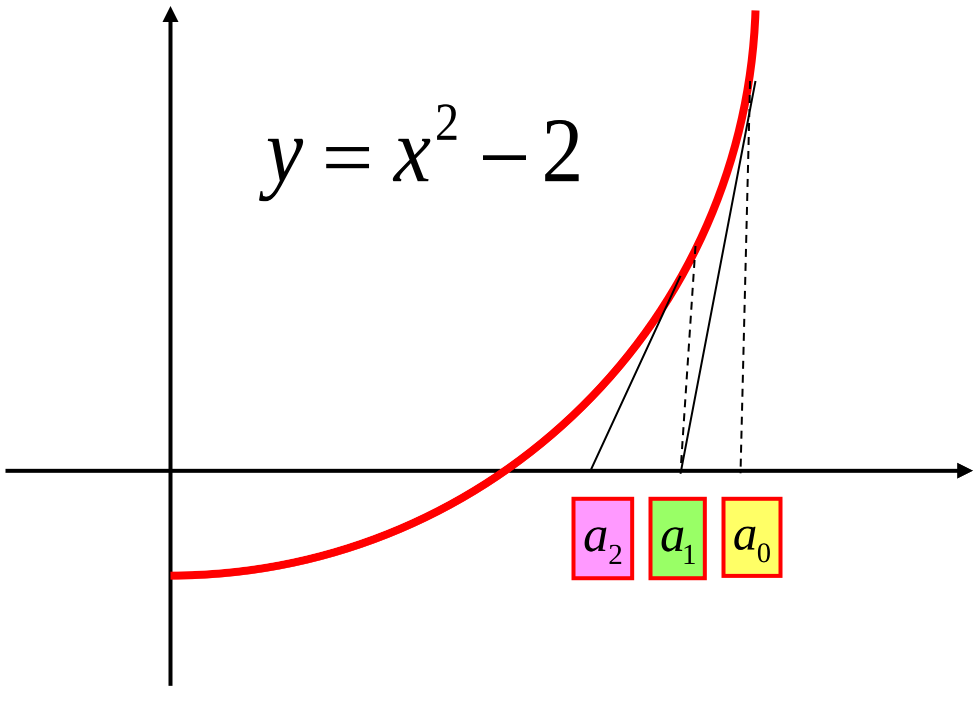
$$a_0 = 2, \quad a_1 = \frac{3}{2}$$

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n} \right)$$

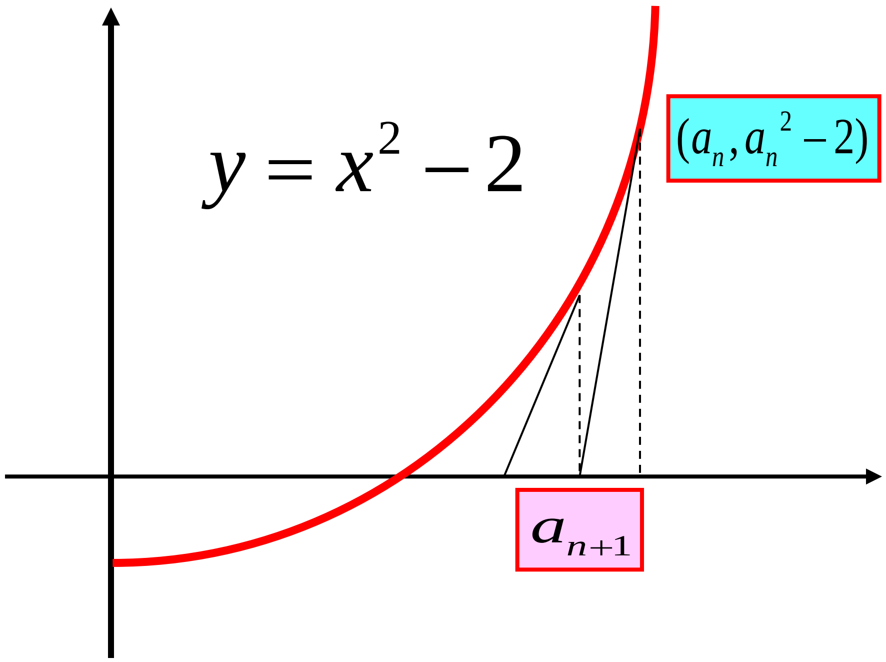
$$\Longrightarrow$$

$$\lim_{n \to \infty} = \sqrt{2}$$

Newton's Method (1)



Newton's Method (2)



Tangent Line at $(a_n, a_n^2 - 2)$:

$$y = 2a_n(x - a_n) + a_n^2 - 2 = 2a_n x - a_n^2 - 2$$

Numerical Computing with BASIC

Newton's Approximation Method

$$a_1 = 1.5$$

$$a_2 = 1.416666666666667$$

$$a_3 = 1.414215686274509 9$$

$$a_4 = 1.41421356237468 99$$

$$a_5 = 1.41421356237309 51$$

$$a_6 = 1.41421356237309 50$$

Bisection Method

Square Root of 2（1）

$$(1)\ \ 1^2 < 2 < 2^2 \Rightarrow 1 < \sqrt{2} < 2$$

$$\sqrt{2} \in I_1 = [1, 2]$$

$$(2)\ (1.4)^2 = 1.96 < 2 < (1.5)^2 = 2.25$$

$$\Rightarrow 1.4 < \sqrt{2} < 1.5$$

$$\sqrt{2} \in I_2 = [1.4, 1.5]$$

$$(3)\ (1.41)^2 = 1.9881 < 2 < (1.42)^2 = 2.0164$$

$$\Rightarrow 1.41 < \sqrt{2} < 1.42$$

$$\sqrt{2} \in I_3 = [1.41, 1.42]$$

Square Root of 2（2）

$(n)\quad a_n^{\ 2} < 2 < b_n^{\ 2} \Longrightarrow a_n < \sqrt{2} < b_n$

$$b_n - a_n = \frac{1}{10^n}$$

$\sqrt{2} \in I_n = [a_n, b_n]$

$\Longrightarrow$

$$\begin{cases} a_n \uparrow \alpha \\ b_n \downarrow \alpha \end{cases}$$

$\alpha = \sqrt{2}$

Principle of Successive Subdivision

Cantor (1845—1918)

Cantor's Nested-Interval Property

$\{I_n\}$ **Sequence of closed intervals**

(1) $I_{n+1} \subset I_n$

(2) $|I_n| \to 0$

$\Longrightarrow$

$$\bigcap_{n=1}^{\infty} I_n = \{\textbf{One Point}\}$$

Sequence Version

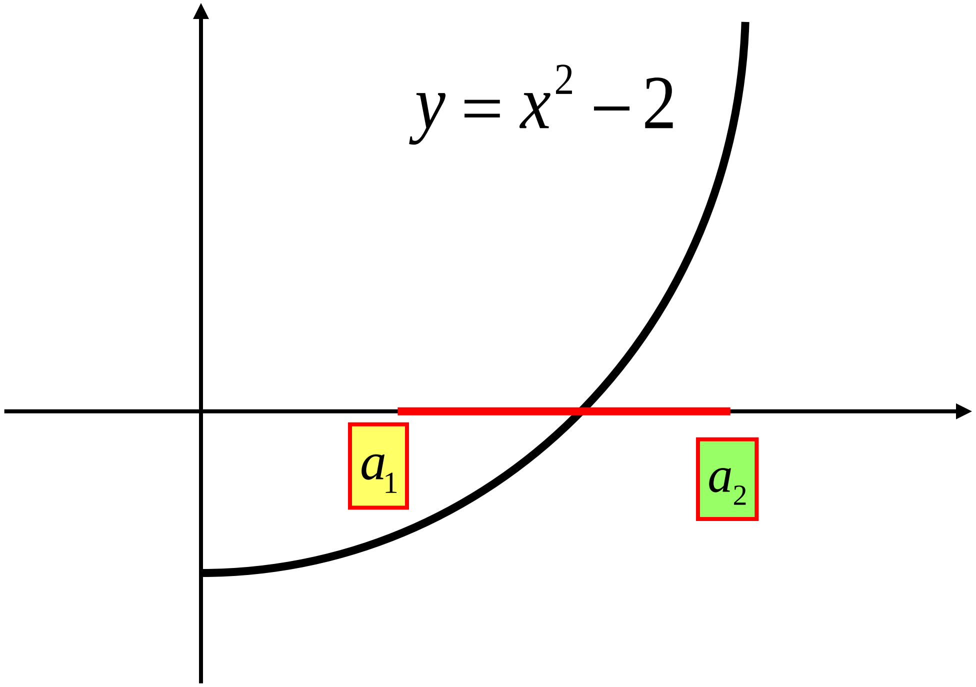
(1) $a_1 \le a_2 \le \cdots \le a_n \le a_{n+1} \le \cdots \le b_{n+1} \le b_n \le b_2 \le b_1$

(2) $b_n - a_n \to 0$ as $n \to \infty$

$\Rightarrow$

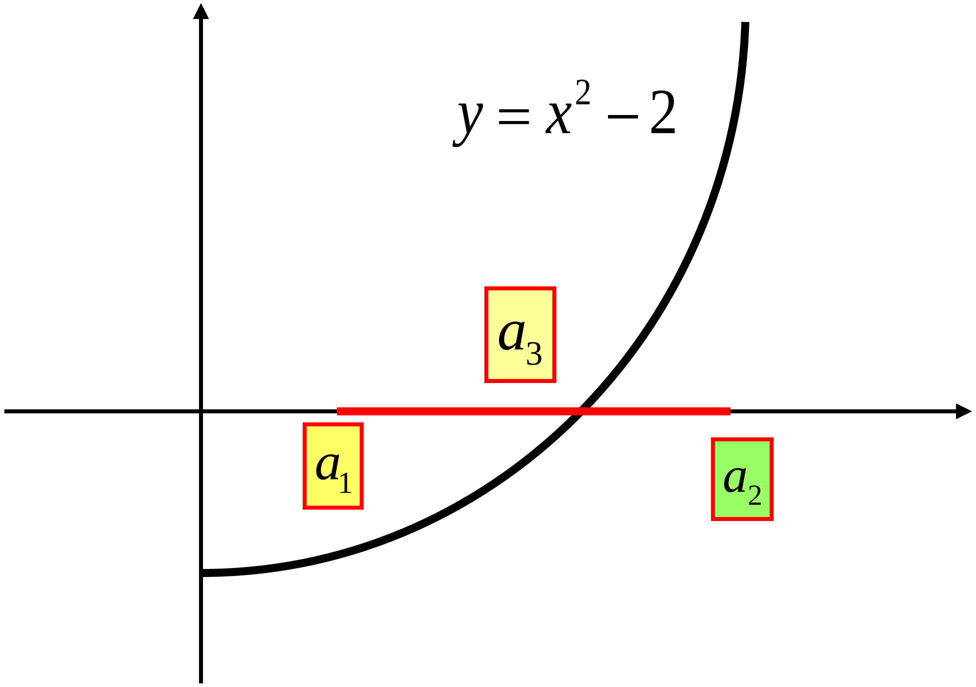
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

Bisection Method（1）

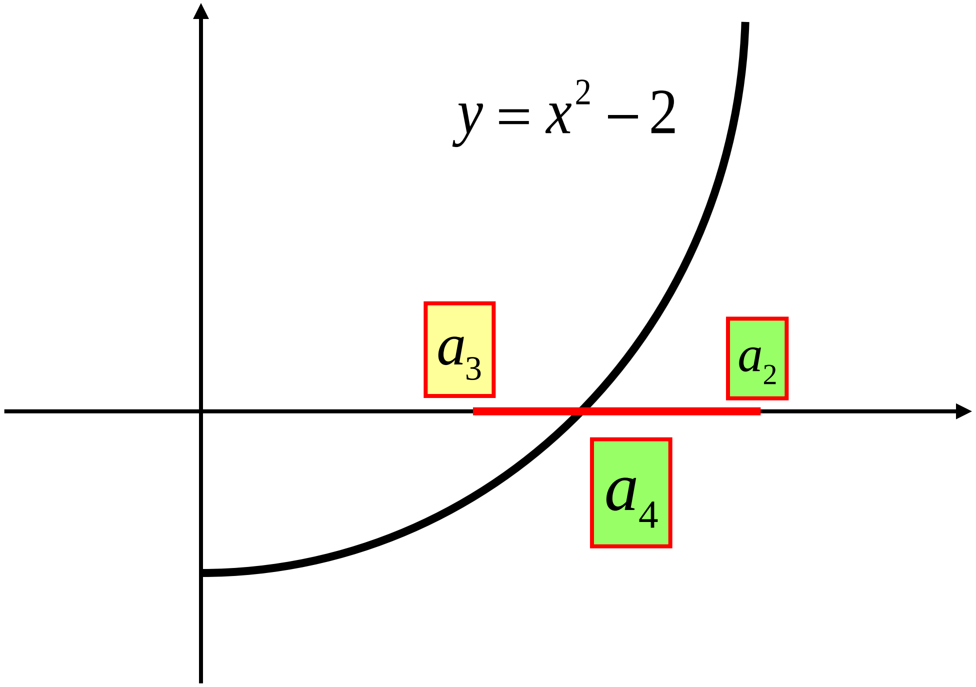


$\sqrt{2}$: **Square Root of** 2

Bisection Method（2）



Bisection Method（3）



Numerical Computing
with
BASIC

Bisection Method

$$a_1 = 1.5$$
$$a_2 = 1.25$$
$$a_3 = 1.375$$
$$a_4 = 1.4375$$
$$a_5 = 1.40625$$
$$a_6 = 1.421875$$
$$a_7 = 1.4140625$$
$$a_8 = 1.41796875$$
$$a_9 = 1.416015625$$
$$a_{10} = 1.4150390625$$

Number Pi

$$\pi = 3.14159265\cdots$$

$$\frac{\pi}{4} = 0.785398163397459\cdots$$

Taylor Series Version

Taylor Series

$$\tan^{-1} x$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$+ (-1)^{n-1} \frac{x^{2n-1}}{2n-1} \cdots$$

Abel's Theorem

$$A = \sum_{n=1}^{\infty} a_n$$ **converges**

$$\Rightarrow$$

$$f(x) = \sum_{n=1}^{\infty} a_n x^n \to A \text{ as } x \uparrow 1$$

Leibniz's Series

$$\frac{\pi}{4} = \tan^{-1} 1$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$$

$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Numerical Computing
with
BASIC

Leibniz's Series

$$a_1 = 1$$
$$a_2 = 0.66666666666666667$$
$$a_3 = 0.86666666666666667$$
$$a_4 = 0.72380952380952524$$
$$a_5 = 0.83492063492063492$$
$$a_6 = 0.74401154401154401$$
$$a_7 = 0.82093462093462093$$
$$a_8 = 0.75426795426795427$$
$$a_9 = 0.81309148367971919$$
$$a_{10} = 0.76045990473235190$$

Machin’s Series

Machin's Series

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

$$= 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{5}\right)^{2n+1} - \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{239}\right)^{2n+1}$$

Proof (1)

$$\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{2\tan A}{1 - \tan^2 A}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Proof (2)

$$\tan A = \frac{1}{5}$$

$$\Rightarrow$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A} = \frac{5}{12}$$

$$\tan 4A = \frac{2\tan 2A}{1 - \tan^2 2A} = \frac{120}{119}$$

Proof (3)

$$\tan\left(4A-\frac{\pi}{4}\right)=\frac{\tan 4A-\tan\frac{\pi}{4}}{1+\tan 4A\tan\frac{\pi}{4}}$$

$$=\frac{\tan 4A-1}{1+\tan 4A}=\frac{1}{239}$$

Proof (4)

$$\tan^{-1}\left(\frac{1}{239}\right) = 4A - \frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}$$

$$\Rightarrow$$

$$\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

Taylor Series

Example

$$\tan^{-1} x$$

$$= x - \frac{x^3}{3} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$$

$$(-1 < x \leq 1)$$

Numerical Computing with BASIC

Machin's Series

$$A_1 = 0.7595815899581590$$

$$A_2 = 0.7851492573315150$$

$$A_3 = 0.7854052573312590$$

$$A_4 = 0.7853979430455440$$

$$A_5 = 0.7853981706011000$$

$$A_6 = 0.7853981631538270$$

$$A_7 = 0.7853981634058990$$

$$A_8 = 0.7853981633971510$$

$$A_9 = 0.7853981633974590$$

$$A_{10} = 0.7853981633974480$$

Series

Series of Positive Terms

Series of Positive Terms

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$$

$$a_n \geq 0$$

Geometric Series

$$a + ar + ar^2 + \cdots + ar^n + \cdots$$

$$= \begin{cases} +\infty & \text{if } r = 1 \\ \dfrac{a}{1-r} & \text{if } 0 < r < 1 \end{cases}$$

Proof

$$0 < r < 1$$

$$\Rightarrow S_n = a + ar + \cdots + ar^{n-1}$$

$$= \frac{a(1-r^n)}{1-r} \rightarrow \frac{a}{1-r}$$

$$\left(\because \lim_{n\to\infty} r^n = 0 \right)$$

Example (1)

$$1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots$$

$$= \lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n = e$$

Example (2)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6}$$

Example (3)

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \textbf{converges for } p > 1 \\ \textbf{diverges for } 0 < p \leq 1 \end{cases}$$

Cauchy's Root Test

$\sum_{n=1}^{\infty} a_n$: **Series of positive terms**

$$\exists \lim_{n\to\infty} \sqrt[n]{a_n} = r$$

$$\Rightarrow$$

$$(1)\ \ r < 1 \Rightarrow \sum_{n=1}^{\infty} a_n < \infty$$

$$(2)\ \ r > 1 \Rightarrow \sum_{n=1}^{\infty} a_n = \infty$$

D'Alembert's Test

$\sum_{n=1}^{\infty} a_n$: **Series of positive terms**

$$\exists \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$$

$\Longrightarrow$

$$(1)\ \ r < 1 \Rightarrow \sum_{n=1}^{\infty} a_n < \infty$$

$$(2)\ \ r > 1 \Rightarrow \sum_{n=1}^{\infty} a_n = \infty$$

Numerical Computing with BASIC

Square of Number Pi

$$\frac{\pi^2}{6} = 1.644934066848 23 \cdots$$

$$A_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$$

$$A_1 = 1$$

$$A_2 = 1.25$$

$$A_{50} = 1.6251372733621 52$$

$$A_{100} = 1.6349839001849$$

$$A_{300} = 1.6416062828976 3$$

$$A_{600} = 1.6432687882988 7$$

$$A_{700} = 1.6435065153419 4$$

$$A_{800} = 1.6436848477727 5$$

$$A_{900} = 1.6438235727925 2$$

$$A_{1000} = 1.6439345666816 1$$

$$A_{1500} = 1.6442676223544$$

Alternating Series

Alternating Series

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 \cdots + a_{2k-1} - a_{2k} + \cdots$$

$$a_n > 0$$

Leibniz's Theorem

$(1)\ a_n > a_{n+1}$ **(monotone decreasing)**

$(2)\ \lim_{n\to\infty} a_n = 0$

$\Rightarrow$

$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ **converges**

Examples

(1) $$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\log_e 2$$

(2) $$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$$

Continuity of Functions

Definition of Continuity

Let $f(x)$ **be a function defined on an interval** I.

$f(x)$ **is continuous at** $a \in I$

$$\overset{\text{def}}{\Leftrightarrow}$$

$\forall \varepsilon > 0,\ \exists \delta = \delta(a, \varepsilon) > 0$ **such that**

$x, y \in I,\ |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

Example (1)

$$f(x) = x^2$$

$$I = [0, \infty)$$

Proof

$$(1)\ x > a:$$

$$\delta_1(a,\varepsilon) = \sqrt{a^2+\varepsilon} - a$$

$$(2)\ 0 \le x < a:$$

$$\delta_2(a,\varepsilon) = a - \sqrt{a^2-\varepsilon}$$

$$\Rightarrow$$

$$\delta(a,\varepsilon) = \min\{\delta_1(a,\varepsilon), \delta_2(a,\varepsilon)\}$$

Example (2)

$$g(x) = \frac{1}{x}$$

$$I = (0, \infty)$$

Proof

(1) $x > a:$

$$\delta_1(a,\varepsilon) = \frac{\varepsilon a^2}{1 - \varepsilon a}$$

(2) $0 \leq x < a:$

$$\delta_2(a,\varepsilon) = \frac{\varepsilon a^2}{1 + \varepsilon a}$$

$\Rightarrow$

$$\delta(a,\varepsilon) = \min\{\delta_1(a,\varepsilon), \delta_2(a,\varepsilon)\}$$

$$= \frac{\varepsilon a^2}{1 + \varepsilon a}$$

Criterion of Continuity (Sequence Version)

$f(x)$ **is continuous** at $x = a$

$\Leftrightarrow$

$x_n \to a \Rightarrow \lim\limits_{n\to\infty} f(x_n) = f(a)$

Example (1)

$$\sin x$$

$$\because \sin(a+h)$$

$$= \sin a \cos h + \cos a \sin h$$

$$\to \sin a \quad \textbf{as} \quad h \to 0$$

Example (2)

$$f(x) = \begin{cases} x \sin \dfrac{1}{x} & \textbf{if } x \neq 0, \\ 0 & \textbf{if } x = 0 \end{cases}$$

$$\because |f(h) - f(0)|$$

$$= \left| h \sin \frac{1}{h} \right|$$

$$\leq |h| \longrightarrow 0 \quad \textbf{as } h \longrightarrow 0$$

Example of a Discontinuous Function

$$g(x) = \begin{cases} \sin\dfrac{1}{x} & \textbf{if } x \neq 0, \\ 0 & \textbf{if } x = 0 \end{cases}$$

Proof

$$(1)\ x_n = \frac{1}{\frac{\pi}{2} + 2n\pi} \longrightarrow 0 \quad \textbf{as } n \longrightarrow \infty$$

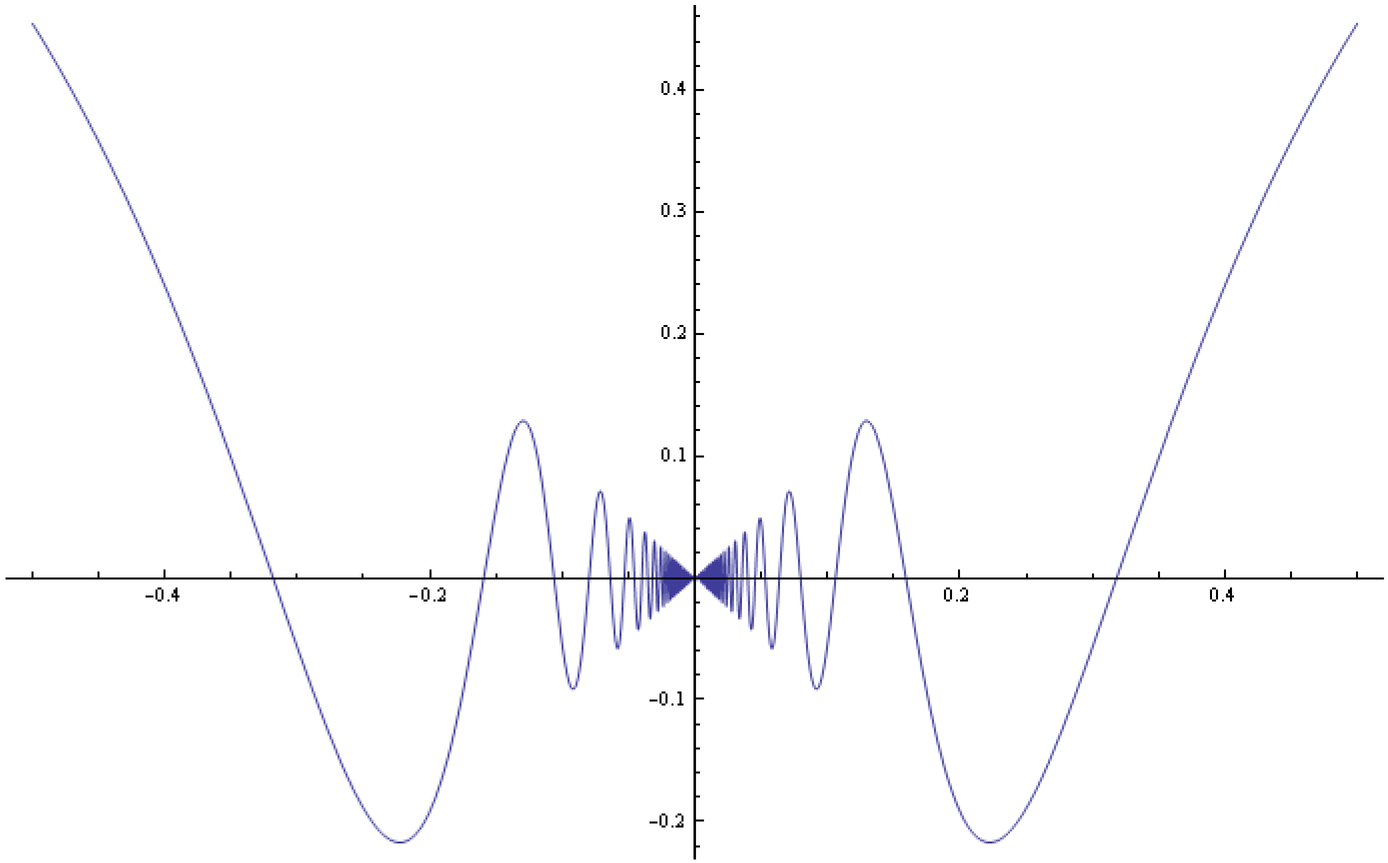
$$g(x_n) = \sin\frac{1}{x_n} = \sin\left(\frac{\pi}{2} + 2n\pi\right) = 1$$

$$(2)\ y_n = \frac{1}{\frac{3\pi}{2} + 2n\pi} \longrightarrow 0 \quad \textbf{as } n \longrightarrow \infty$$

$$g(y_n) = \sin\frac{1}{y_n} = \sin\left(\frac{3\pi}{2} + 2n\pi\right) = -1$$

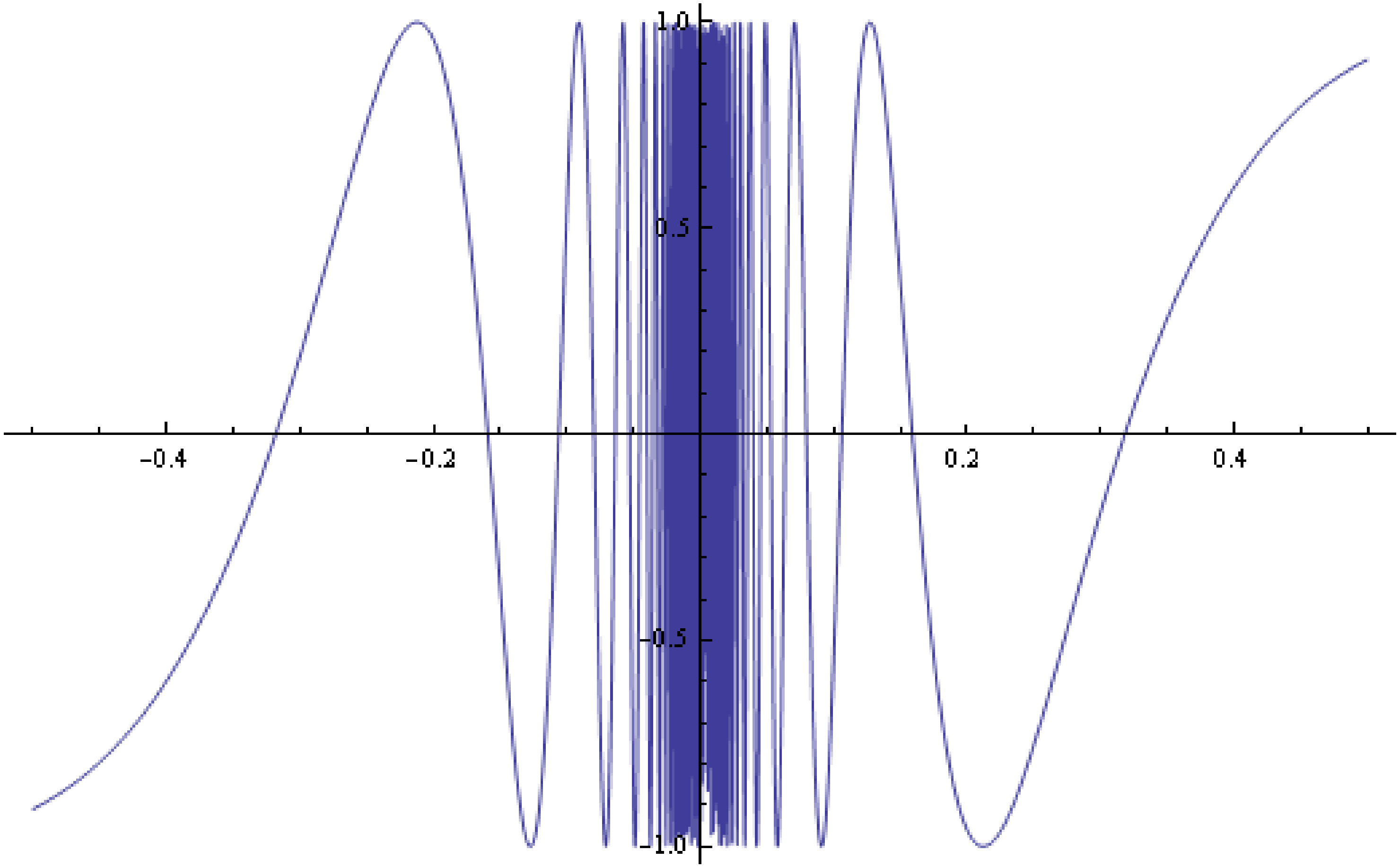
Numerical Computing with BASIC

Graph of $f(x)$



Numerical Computing with BASIC

Graph of $g(x)$



Operations of Continuous Functions

$f(x), g(x)$ **are continuous**

$\Longrightarrow$

(1) $f(x) \pm g(x)$ **is continuous**

(2) $kf(x)$ **is continuous**

(3) $f(x)g(x)$ **is continuous**

(4) $\dfrac{f(x)}{g(x)}$ $(g(x) \neq 0)$ **is continuous**

Continuity of Composite functions

$f(u)$ **is continuous**

$u = g(x)$ **is continuous**

$\Longrightarrow$

$f(g(x))$ **is continuous**

Weierstrass' Theorem

Maximum Value Theorem

$f(x)$ **is continuous on** $I = [a, b]$

$\Rightarrow$

$f(x)$ **takes its maximum**

Proof (1)

$$\exists \alpha = \sup \{ f(x) \mid a \leq x \leq b \}$$

$$\Rightarrow$$

$$\forall n \in \mathbf{N}, \ \exists x_n \in [a,b]$$

such that

$$\alpha - \frac{1}{n} < f(x_n) \leq \alpha$$

Proof (2)

$$\exists\{x_{n'}\}\subset\{x_n\},\ \exists c\in[a,b]$$

such that

$$x_{n'}\longrightarrow c$$

(Bolzano - Weierstrass)

Proof (3)

$$(a)\ \alpha - \frac{1}{n'} < f(x_{n'}) \le \alpha$$

$$(b)\ x_{n'} \to c$$

$$\Rightarrow$$

$$f(c) = \lim_{n' \to \infty} f(x_{n'}) = \alpha$$

Minimum Value Theorem

$f(x)$ **is continuous on** $I = [a, b]$

$\Rightarrow$

$f(x)$ **takes its minimum**

Proof (1)

$$\exists \beta = \inf \{ f(x) \mid a \le x \le b \}$$

$$\Rightarrow$$

$$\forall n \in \mathbf{N}, \ \exists y_n \in [a, b]$$

such that

$$\beta \le f(y_n) < \beta + \frac{1}{n}$$

Proof (2)

$$\exists\{y_{n'}\}\subset\{y_n\},\ \exists d\in[a,b]$$

such that

$$y_{n'}\to d$$

(Bolzano - Weierstrass)

Proof (3)

$$(a)\ \beta \le f(y_{n'}) < \beta + \frac{1}{n'}$$

$$(b)\ y_{n'} \longrightarrow d$$

$$\Rightarrow$$

$$f(d) = \lim_{n' \to \infty} f(y_{n'}) = \beta$$

Intermediate Value Theorem

$f(x)$ **is continuous on** $I=[a,b]$

$f(a)<0,\ f(b)>0$

$\Rightarrow$

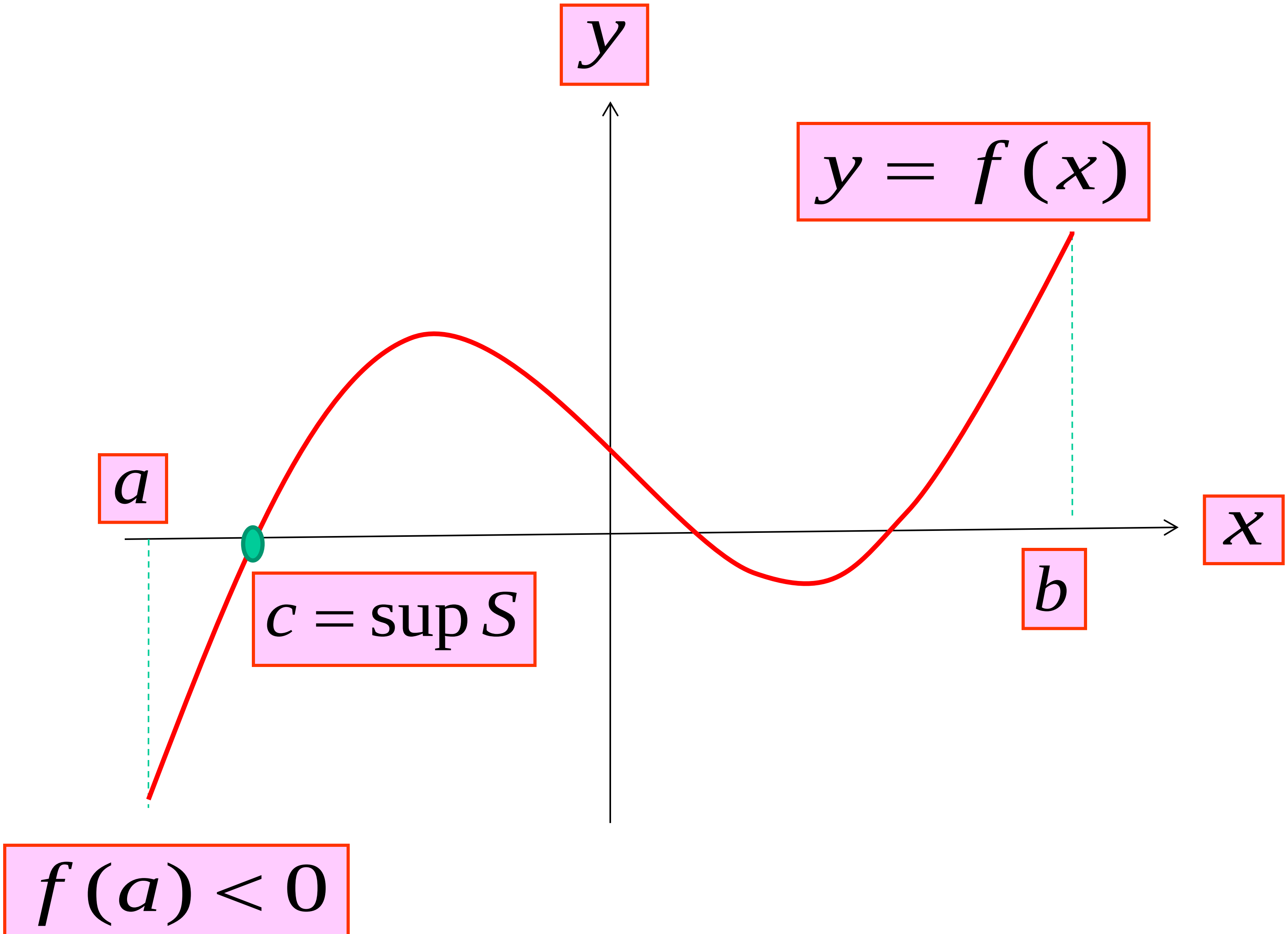
$a<\exists c<b$ **such that** $f(c)=0$

Proof (1)

$$S := \{ d \mid f(x) < 0,\ a \le \forall x < d \}$$

$$\Rightarrow$$

$$\exists c = \sup S$$



y
y = f(x)
a
x
b
c = sup S
f(a) < 0

Proof (2)

$$S = \{d \mid f(x) < 0,\ a \leq \forall x < d\}$$

$$c = \sup S$$

$$\Rightarrow$$

$$f(c) = 0$$

Corollary

$f(x)$ **is continuous on** $I=[a,b]$

$\Rightarrow$

$\{f(x) \mid a \leq x \leq b\} = [\alpha, \beta]$

$\alpha = \inf\{f(x) \mid a \leq x \leq b\}$

$\beta = \sup\{f(x) \mid a \leq x \leq b\}$

Fixed-Point Theorem

$f(x)$ **is continuous on** $I = [a, b]$

$f(I) \subseteq I$

$\Rightarrow$

$\exists c \in I$ **such that** $f(c) = c$

Proof (1)

$$\varphi(x) := x - f(x)$$

$$\Rightarrow$$

$$\varphi(a) = a - f(a) \leq 0$$

$$(i)\ \varphi(a) = 0 \Rightarrow c = a$$

$$(ii)\ \varphi(a) < 0:$$

$$\varphi(b) = b - f(b) \geq 0$$

Proof (2)

(*ii*-1) $\varphi(b) = 0 \Rightarrow c = b$

(*ii*-2) $\varphi(b) > 0 \;\; (\varphi(a) < 0)$

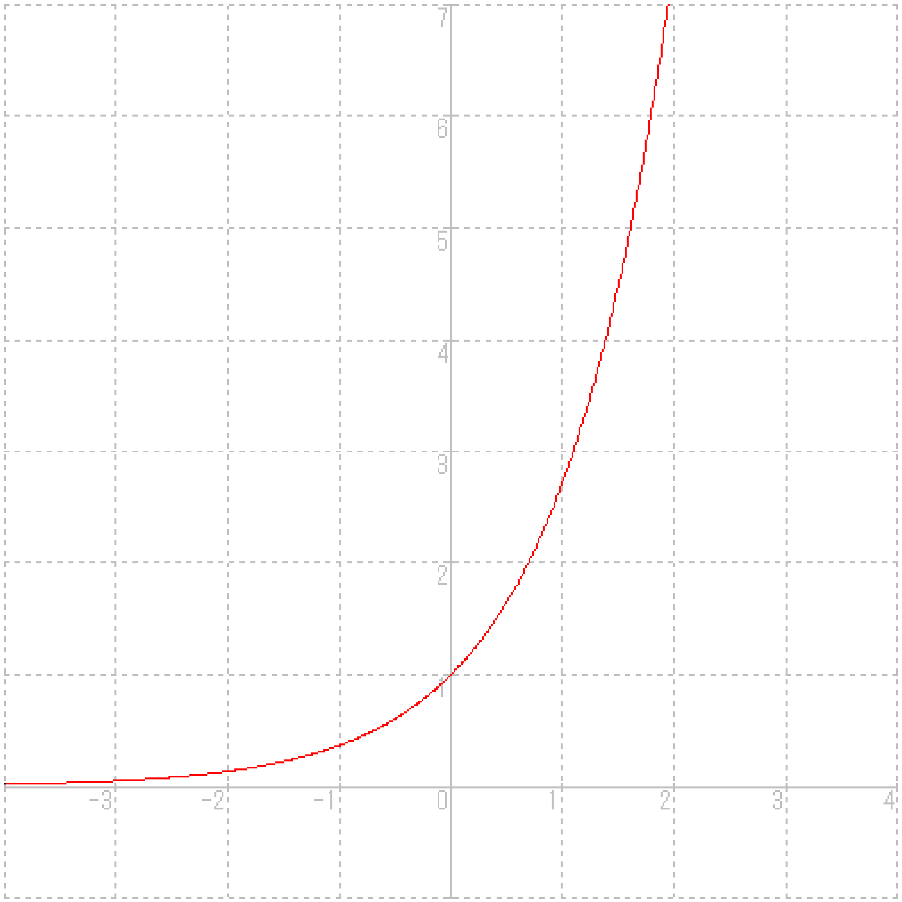
$\Rightarrow$

$a < \exists c < b$ **such that**

$\varphi(c) = c - f(c) = 0$

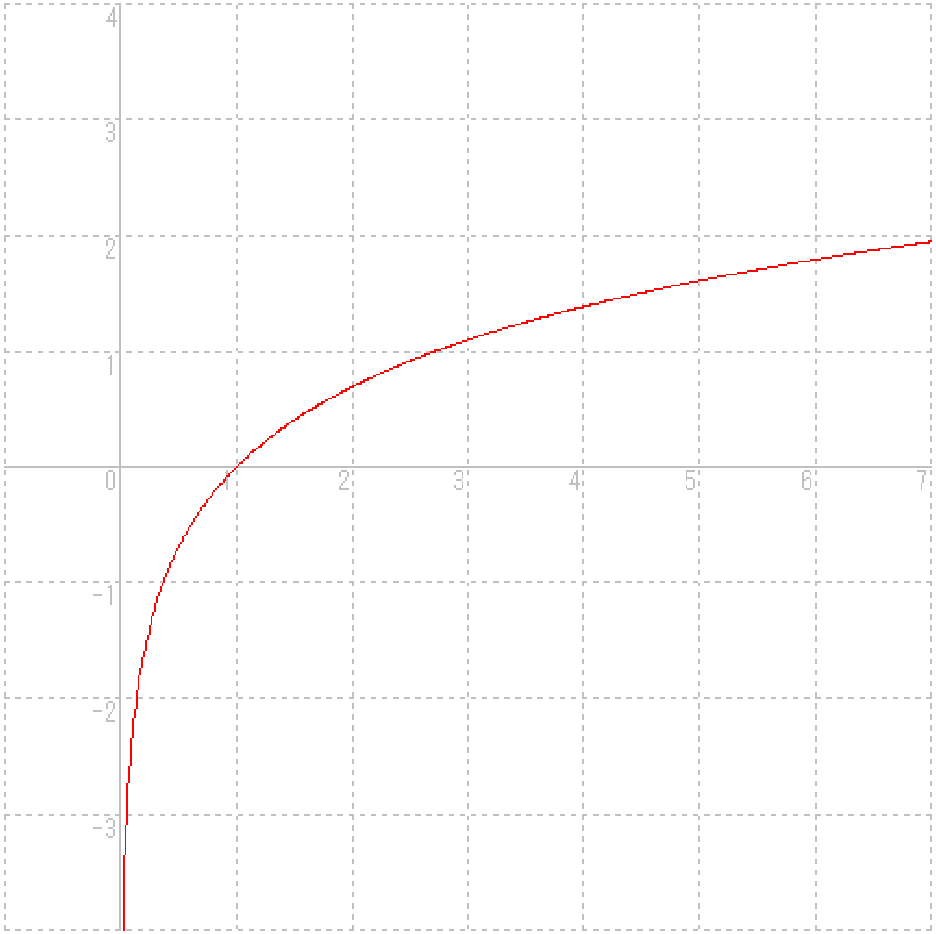
(Intermediate Value Theorem)

Exponential Function



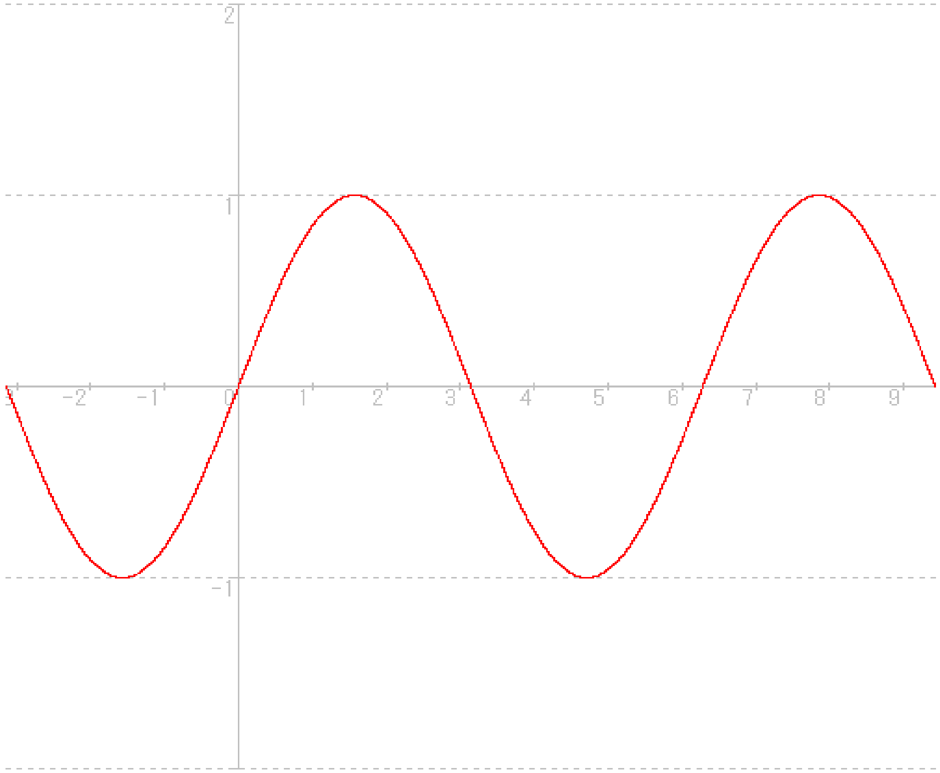
$$y = e^x, \; -\infty < x < \infty$$

Logarithm Function



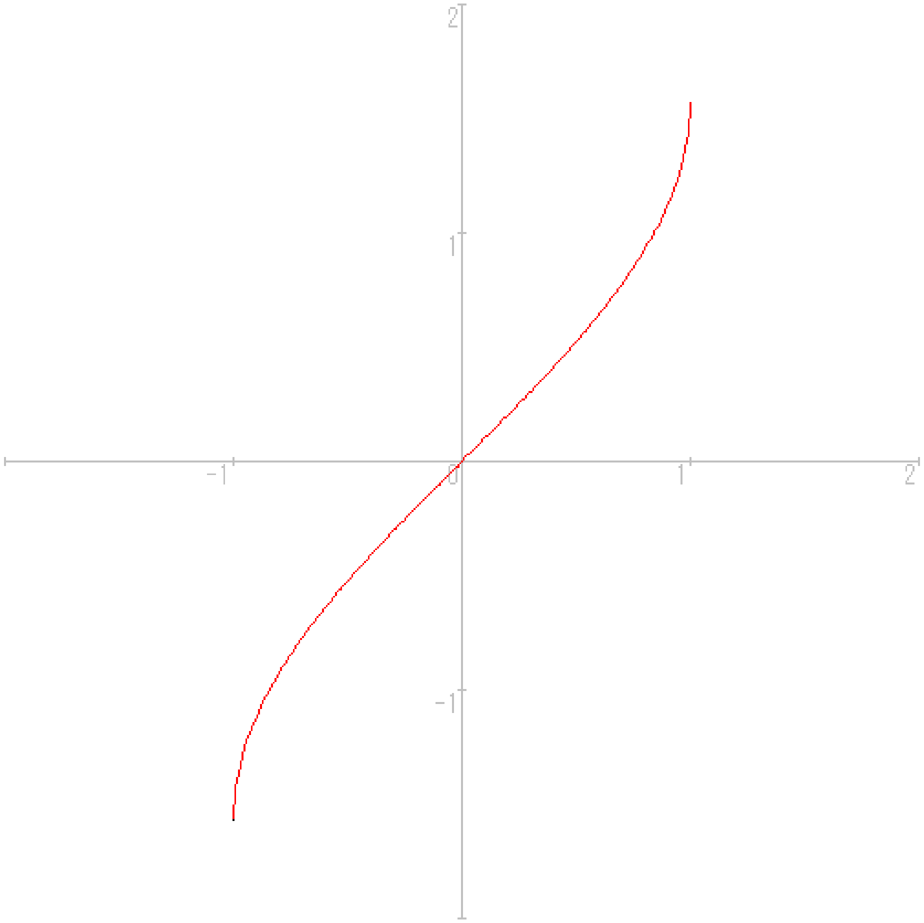
$$y = \log_e x, \;\; 0 < x < \infty$$

Sine Function (1)



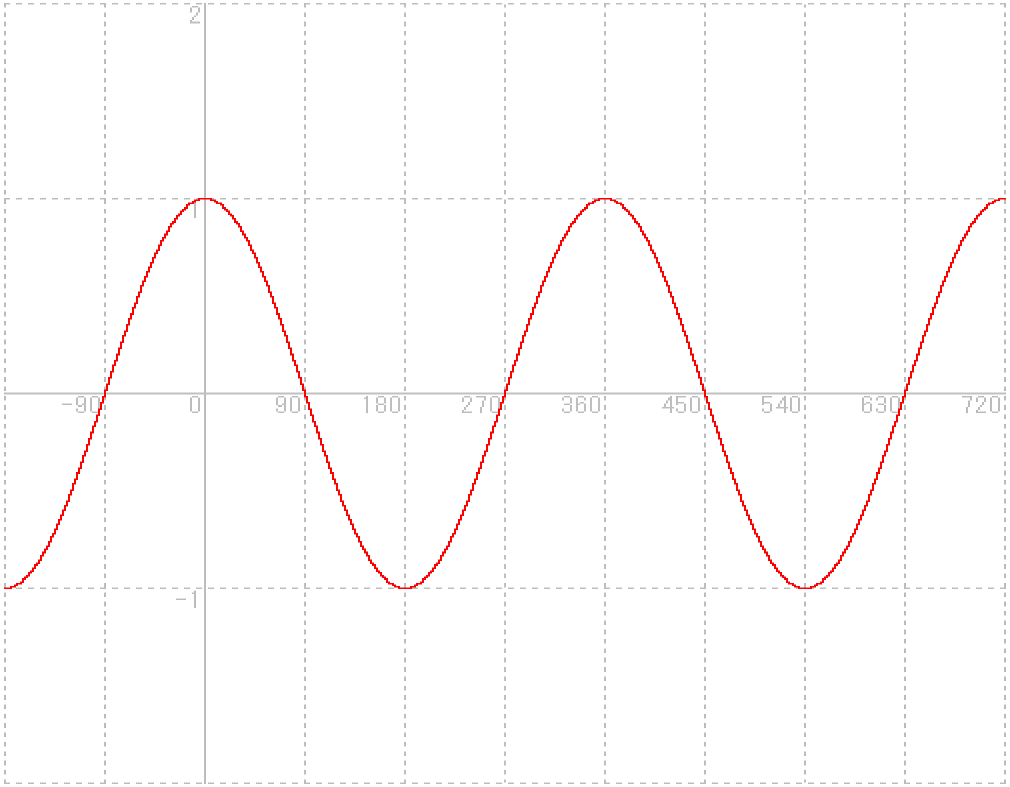
$$y = \sin x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

Arcsine Function (1)



$$y = \sin^{-1} x, \ -1 \le x \le 1$$

Cosine Function (2)

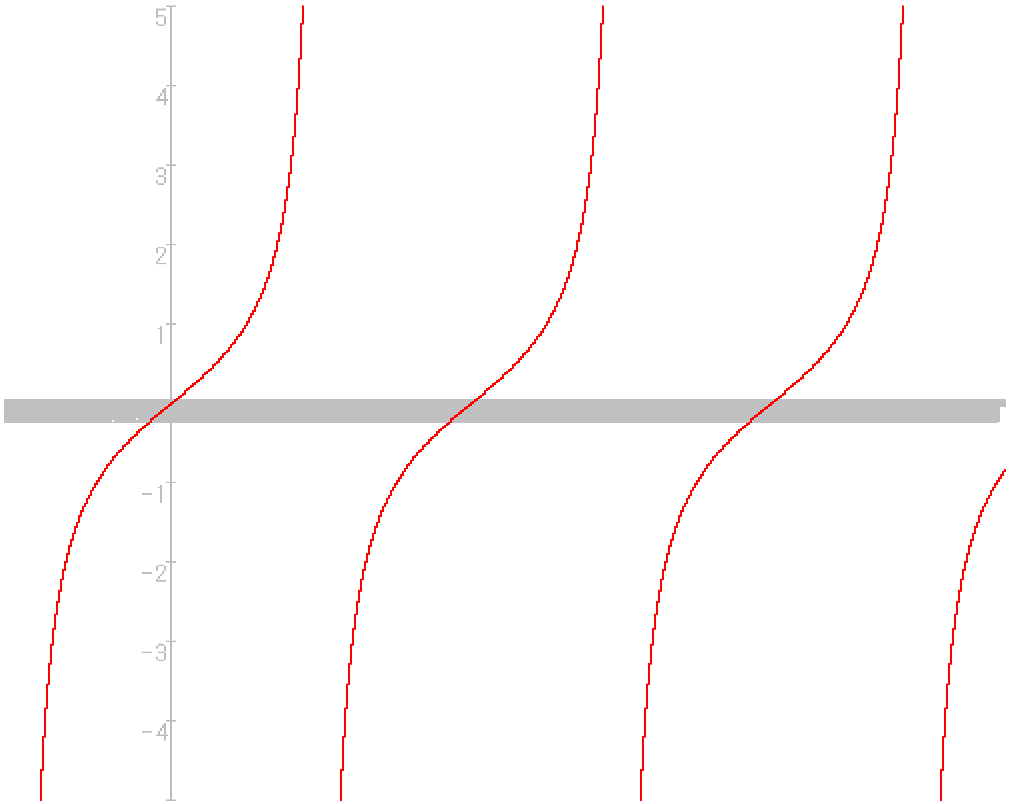


$$y = \cos x, \quad 0 \le x \le \pi$$

Arccosine (2)

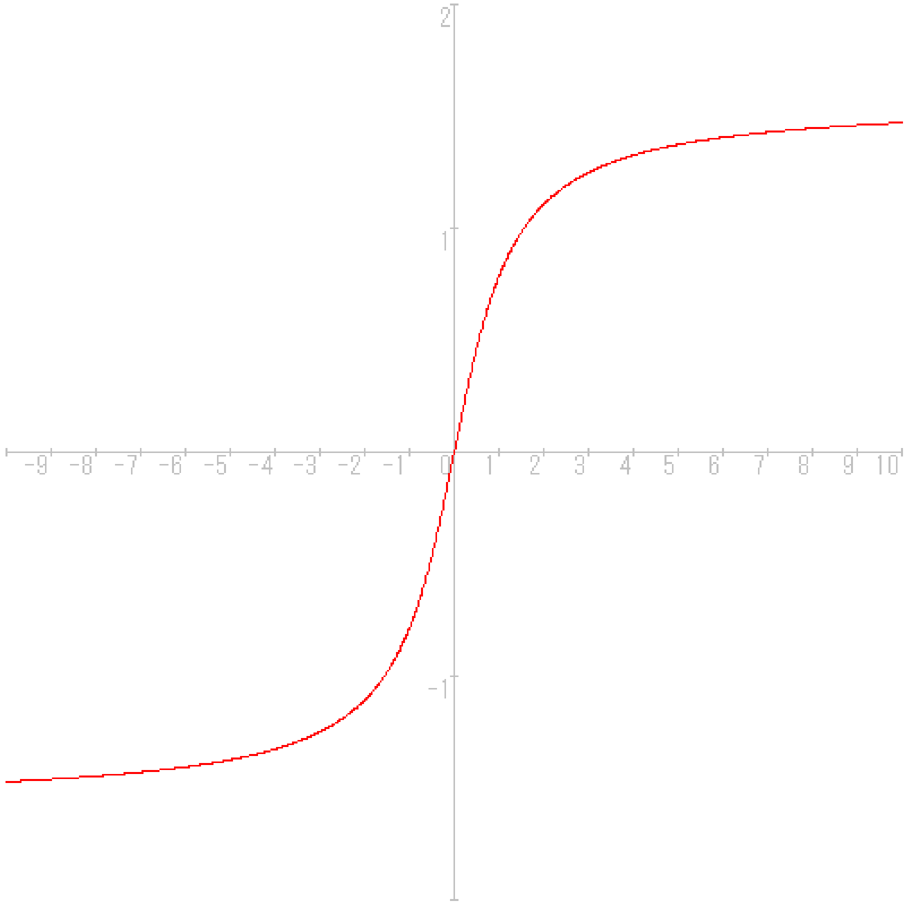
$$y = \cos^{-1} x, \; -1 \le x \le 1$$

Tangent Function (3)



$$y = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Arctangent Function (3)



$$y = \tan^{-1} x, \ -\infty < x < \infty$$

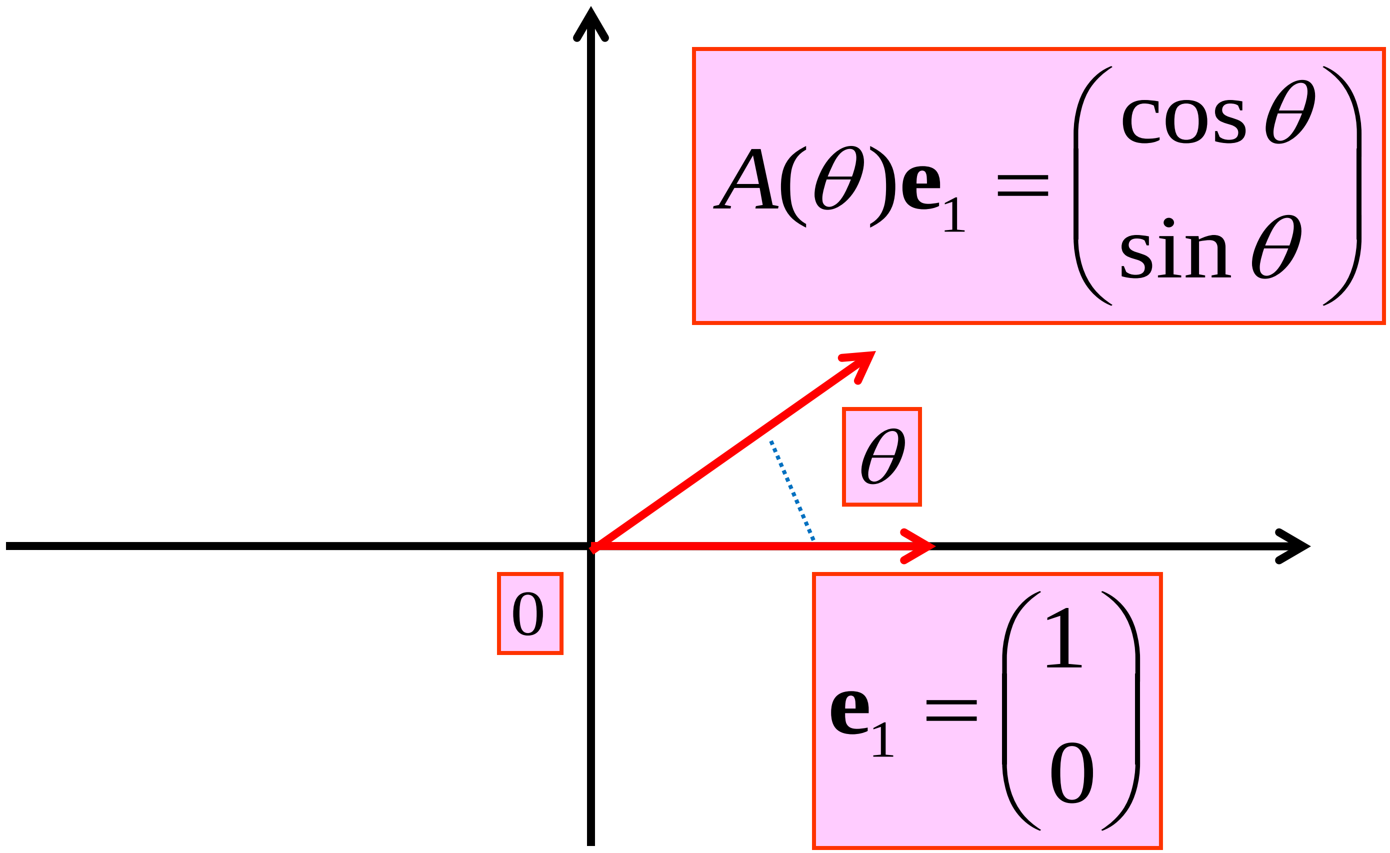
Napier's Number
(Continuous Version)

$$(1)\ e=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x}$$

$$(2)\ e=\lim_{x\to 0}\left(1+x\right)^{\frac{1}{x}}$$

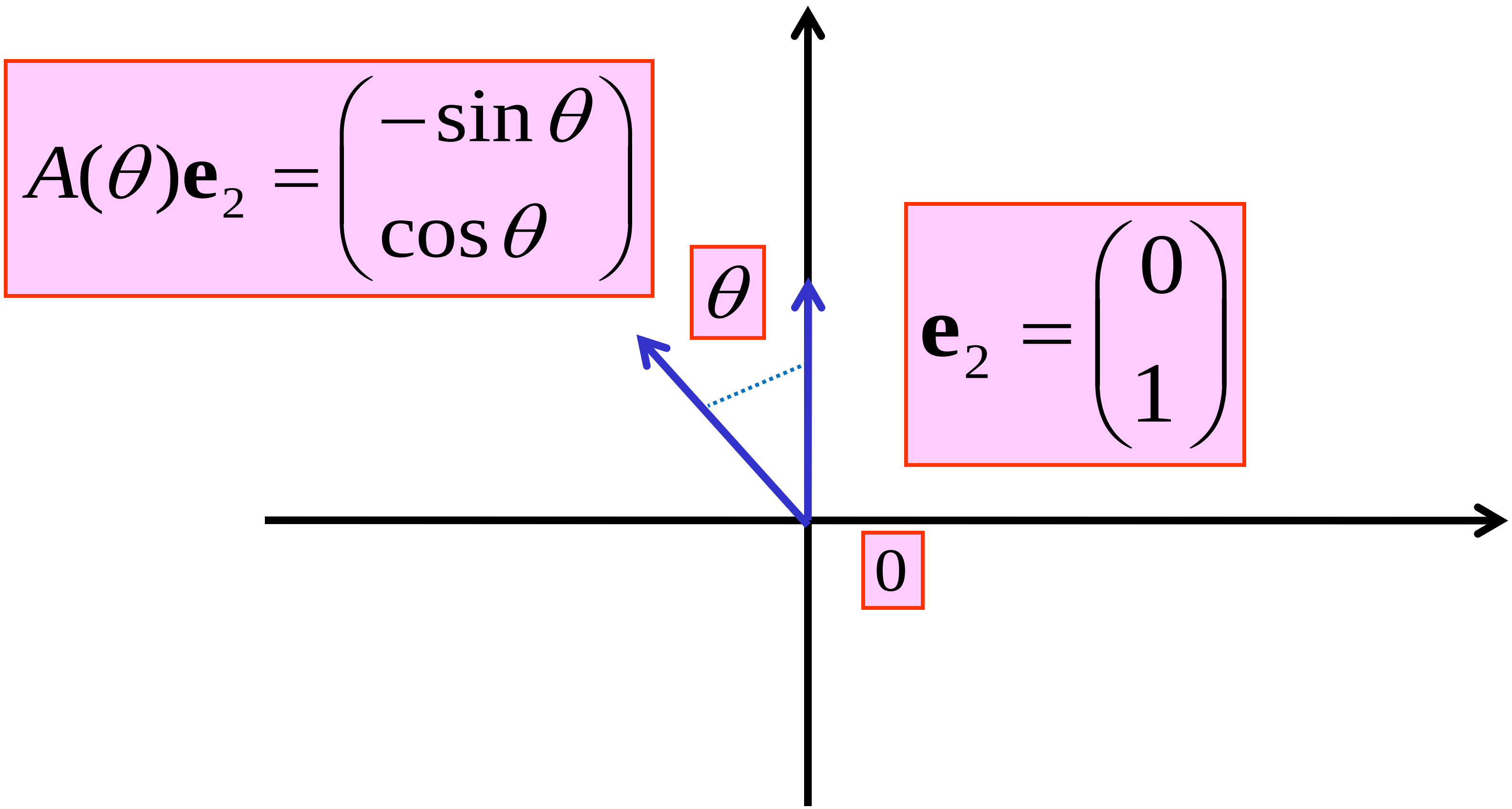
Addition Theorem of Trigonometric Functions

Rotation (1)



$A(\theta)\mathbf{e}_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$
θ
0
$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Rotation (2)



$A(\theta)\mathbf{e}_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$
θ
$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
0

Matrix of Rotation (1)

$$A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Rotation of θ

Matrix of Rotation (2)

$$A(-\theta) = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = A(\theta)^{-1}$$

Rotation of $-\theta$

Composition of Rotations (1)

$$A(\alpha)\mathbf{e}_1 = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

Composition of Rotations (2)

$$A(\beta)\left(A(\alpha)\mathbf{e}_1\right)$$

$$= \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha \\ \sin\beta\cos\alpha + \cos\beta\sin\alpha \end{pmatrix}$$

Composition of Rotations (3)

$$A(\alpha)\mathbf{e}_2 = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}$$

Composition of Rotations (4)

$$A(\beta)\left(A(\alpha)\mathbf{e}_2\right)$$

$$= \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} -\cos\beta\sin\alpha - \sin\beta\cos\alpha \\ -\sin\beta\sin\alpha + \cos\beta\cos\alpha \end{pmatrix}$$

Composition of Rotations (5)

$$A(\beta)A(\alpha)$$

$$=\begin{pmatrix}\cos\beta & -\sin\beta \\ \sin\beta & \cos\beta\end{pmatrix}\begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix}$$

$$=\begin{pmatrix}\cos\beta\cos\alpha-\sin\beta\sin\alpha & -\cos\beta\sin\alpha-\sin\beta\cos\alpha \\ \sin\beta\cos\alpha+\cos\beta\sin\alpha & -\sin\beta\sin\alpha+\cos\beta\cos\alpha\end{pmatrix}$$

Composition of Rotations (6)

$$A(\beta)\left(A(\alpha)\mathbf{e}_1\right) = A(\beta)A(\alpha)\mathbf{e}_1$$

$$A(\beta)\left(A(\alpha)\mathbf{e}_2\right) = A(\beta)A(\alpha)\mathbf{e}_2$$

Composition of Rotations (7)

$$A(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$A(\beta) = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

$$\Rightarrow$$

$$A(\alpha)A(\beta) = A(\beta)A(\alpha) = A(\alpha + \beta)$$

Composition of Rotations (8)

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

$$= A(\alpha+\beta)$$

$$= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$$

Addition Theorem (1)

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

Addition Theorem (2)

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

Addition Theorem (3)

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Addition Theorem (4)

$$\sin A\ \sin B = \frac{1}{2}\big(\cos(A - B) - \cos(A + B)\big)$$

$$\cos A \cos B = \frac{1}{2}\big(\cos(A - B) + \cos(A + B)\big)$$

$$\sin A \cos B = \frac{1}{2}\big(\sin(A + B) + \sin(A - B)\big)$$

Addition Theorem (5)

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$$

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$$

$$\sin 2A = 2 \sin A \cos A$$

Addition Theorem (6)

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$1+\tan^2 A=\frac{1}{\cos^2 A}$$

Uniform Continuity of Functions

Uniform Continuity

Let $f(x)$ **be a function defined on an interval** I.

$f(x)$ **is uniformly continuous on** I

$$\overset{\text{def}}{\Longleftrightarrow}$$

$\forall \varepsilon > 0,\ \exists \delta = \delta(\varepsilon) > 0$ **such that**

$$\forall x, y \in I,\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Fundamental Theorem

Every continuous function defined on a bounded, closed interval is uniformly continuous.

Proof (1)

Assume, to the contrary, that

$$\exists \varepsilon_0 > 0,\ \forall n \in \mathbf{N}$$

$$\begin{cases} \exists x_n, y_n \in I,\ |x_n - y_n| < \dfrac{1}{n} \\ |f(x_n) - f(y_n)| \geq \varepsilon_0 \end{cases}$$

Proof (2)

Bolzano - Weierstrass

$$\begin{cases} \exists x_{n'}, y_{n'} \in I, \ \ |x_{n'} - y_{n'}| < \dfrac{1}{n'} \\ x_{n'} \to \exists c, \ \ y_{n'} \to \exists d \end{cases}$$

$$\Rightarrow c = d$$

Proof (3)

$$\begin{cases} x_{n'}, y_{n'} \in I \\ x_{n'} \to c, \ \ y_{n'} \to c \end{cases}$$

$$\Rightarrow$$

$$0 = \lim_{n' \to \infty} \left| f(x_{n'}) - f(y_{n'}) \right| \geq \varepsilon_0 > 0$$

(Contradiction)

Example (1)

$$f(x) = x^2$$

$$I = [0, K], \quad K > 0$$

Proof

$$(1)\ x > a:$$

$$\delta_1(K,\varepsilon) = \sqrt{K^2 + \varepsilon} - K$$

$$(2)\ 0 \le x < a:$$

$$\delta_2(K,\varepsilon) = K - \sqrt{K^2 - \varepsilon}$$

$$\Rightarrow$$

$$\delta(K,\varepsilon) = \min\{\delta_1(K,\varepsilon), \delta_2(K,\varepsilon)\}$$

Example (2)

$$g(x) = \frac{1}{x}$$

$$I = [\alpha, \infty), \ \alpha > 0$$

Proof

$$(1)\ x > a:$$

$$\delta_1(\alpha,\varepsilon) = \frac{\varepsilon\alpha^2}{1-\varepsilon\alpha}$$

$$(2)\ 0 \le x < a:$$

$$\delta_2(\alpha,\varepsilon) = \frac{\varepsilon\alpha^2}{1+\varepsilon\alpha}$$

$$\Rightarrow$$

$$\delta(\alpha,\varepsilon) = \min\{\delta_1(\alpha,\varepsilon),\delta_2(\alpha,\varepsilon)\}$$

$$= \frac{\varepsilon\alpha^2}{1+\varepsilon\alpha}$$

Lipschitz Continuity

Let $f(x)$ **be a function defined on an interval** I.

$f(x)$ **is Lipschitz continuous on** I

$$\overset{\text{def}}{\Leftrightarrow}$$

$\exists L > 0$ **such that**

$$\left| f(x) - f(y) \right| \le L \left| x - y \right|, \quad \forall x, y \in I$$

Example

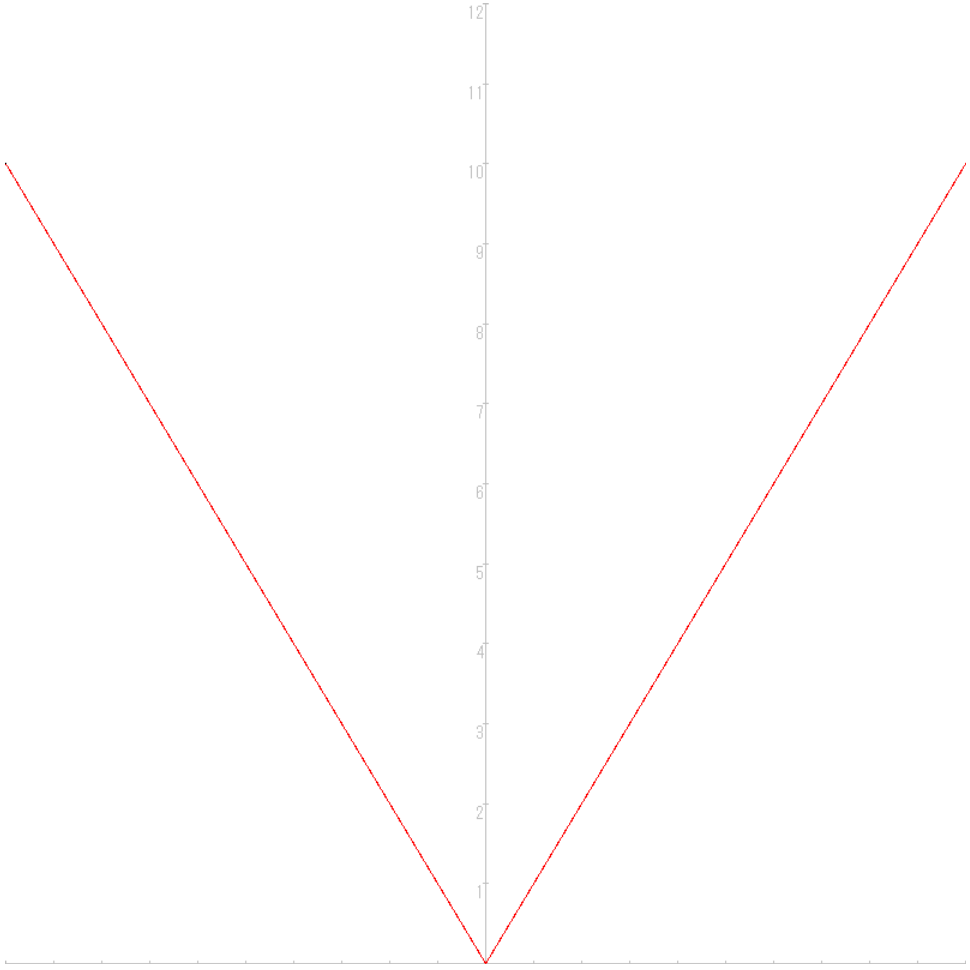
$$f(x) = |x|, \quad I = (-\infty, \infty)$$

$\because$

$$\big||x| - |y|\big| \leq |x - y|$$

Numerical Computing with BASIC

Piecewise Smooth Curve



$$y = |x|$$

Differentiation

Differentiability of Functions

Definition of Differentiability

Let $f(x)$ **be a function defined on an open interval** I.

$f(x)$ **is differentiable at** $a \in I$

$$\overset{\text{def}}{\Longleftrightarrow}$$

$$\exists \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \alpha$$

Notation : $\alpha = f'(a) = \frac{df}{dx}(a)$

Criterion for Differentiability

$$\begin{cases} f(x) = f(a) + \alpha(x-a) + R(x)(x-a) \\ \lim\limits_{x\to a} R(x) = 0 \end{cases}$$

$$\begin{cases} f(a+h) = f(a) + \alpha h + R(a+h)h \\ \lim\limits_{h\to 0} R(a+h) = 0 \end{cases}$$

Geometrical Meaning

Differentiability

Existence of Tangent Lines

Differentiability implies Continuity

$$\begin{cases} f(x) = f(a) + \alpha(x-a) + R(x)(x-a) \\ \lim_{x \to a} R(x) = 0 \end{cases}$$

$$\Rightarrow$$

$$|f(x) - f(a)| \le |\alpha + R(x)||x-a|$$

$$\to 0 \;\; \textbf{as } x \to a$$

Examples

Examples (1)

$$(1)\left(x^{\alpha}\right)' = \alpha x^{\alpha-1}$$

$$(2)\left(e^{x}\right)' = e^{x}$$

$$(3)\left(a^{x}\right)' = a^{x}\log_e a \quad (a > 0)$$

$$(4)\left(\log_e |x|\right)' = \frac{1}{x}$$

Examples (2)

$$(1)\ (\sin x)' = \cos x$$

$$(2)\ (\cos x)' = -\sin x$$

$$(3)\ (\tan x)' = \frac{1}{\cos^2 x}$$

$$(4)\ \left(\sin^{-1} x\right)' = \frac{1}{\sqrt{1 - x^2}}$$

$$(5)\ \left(\cos^{-1} x\right)' = -\frac{1}{\sqrt{1 - x^2}}$$

$$(6)\ \left(\tan^{-1} x\right)' = \frac{1}{1 + x^2}$$

Example (1)

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}, \quad x > 0$$

$$\frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2}$$

Proof (1)

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$=\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

Proof (2)

$$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

Example (2)

$$\frac{d}{dx}\left(\log_e x\right) = \frac{1}{x}$$

Proof (1)

$$(a)\ \lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^{x} = e$$

$$(b)\ \lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^{x} = e$$

$$(c)\ \lim_{h \to 0} (1 + h)^{\frac{1}{h}} = e$$

Proof (2)

$$\frac{\log_e(x+h) - \log_e x}{h}$$

$$= \log_e\left(\frac{x+h}{x}\right)^{\frac{1}{h}}$$

$$= \log_e\left(1 + \frac{h}{x}\right)^{\frac{1}{h}}$$

Proof (3)

$$\log_e\left(1+\frac{h}{x}\right)^{\frac{1}{h}}$$

$$=\log_e(1+y)^{\frac{1}{xy}}$$

$$\left(y=\frac{h}{x}\right)$$

$$=\frac{1}{x}\log_e(1+y)^{\frac{1}{y}}$$

Proof (4)

$$\lim_{h \to 0} \frac{\log_e(x+h) - \log_e x}{h}$$

$$= \frac{1}{x} \lim_{y \to 0} \log_e (1+y)^{\frac{1}{y}}$$

$$= \frac{1}{x}$$

Example (3)

$$\frac{d}{dx}\left(e^{x}\right) = e^{x}$$

Proof

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

Example (3)

$$\frac{d}{dx}(\sin x) = \cos x$$

Proof (1)

$$\frac{1}{h}\left(\sin(x+h)-\sin x\right)$$

$$=\frac{2}{h}\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}$$

$$=\frac{2}{h}\sin\frac{h}{2}\cos\left(x+\frac{h}{2}\right)$$

Proof (2)

$$\cos\left(x+\frac{h}{2}\right)$$

$$=\cos x\cos\frac{h}{2}-\sin x\sin\frac{h}{2}$$

$$\longrightarrow \cos x \quad \textbf{as} \ h \longrightarrow 0$$

Proof (3)

$$\frac{2}{h}\sin\frac{h}{2} = \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$\longrightarrow 1 \quad \textbf{as} \quad h \longrightarrow 0$$

Proof (4)

$$\frac{1}{h}\left(\sin(x+h)-\sin x\right)$$

$$=\frac{2}{h}\sin\frac{h}{2}\cos\left(x+\frac{h}{2}\right)$$

$$\longrightarrow \cos x \quad \textbf{as} \quad h \longrightarrow 0$$

Example (4)

$$\frac{d}{dx}(\cos x) = -\sin x$$

Proof (1)

$$\frac{1}{h}\left(\cos(x+h)-\cos x\right)$$

$$=-\frac{2}{h}\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}$$

$$=-\frac{2}{h}\sin\frac{h}{2}\sin\left(x+\frac{h}{2}\right)$$

Proof (2)

$$\sin\left(x+\frac{h}{2}\right)$$

$$=\sin x\cos\frac{h}{2}+\cos x\sin\frac{h}{2}$$

$$\longrightarrow \sin x \quad \textbf{as } h\longrightarrow 0$$

Proof (3)

$$\frac{2}{h}\sin\frac{h}{2} = \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$\to 1 \quad \textbf{as} \quad h \to 0$$

Proof (4)

$$\frac{1}{h}\left(\cos(x+h)-\cos x\right)$$

$$=-\frac{2}{h}\sin\frac{h}{2}\sin\left(x+\frac{h}{2}\right)$$

$$\longrightarrow -\sin x \quad \textbf{as} \quad h\longrightarrow 0$$

Operations of Differentiable Functions

$f(x), g(x)$ **are differentiable**

$\Longrightarrow$

(1) $f(x) \pm g(x)$ **are differentiable**

(2) $kf(x)$ **is differentiable**

(3) $f(x)g(x)$ **is differentiable**

(4) $\dfrac{f(x)}{g(x)}$ $(g(x) \neq 0)$ **is differentiable**

Proof of (3-1)

$$f(a+h) = f(a) + f'(a)h + R(h)h$$

$$\lim_{h \to 0} R(h) = 0$$

$$g(a+h) = g(a) + g'(a)h + S(h)h$$

$$\lim_{h \to 0} S(h) = 0$$

Proof of (3-2)

$$
\begin{aligned}
&f(a+h)g(a+h) \\
&= (f(a) + f'(a)h + R(h)h) \\
&\times (g(a) + g'(a)h + S(h)h) \\
&= f(a)g(a) + (f'(a)g(a) + f(a)g'(a))h \\
&+ (f(a)S(h) + g(a)R(h) + f'(a)g'(a)h)h
\end{aligned}
$$

Proof of (3-3)

$$f(a+h)g(a+h)$$

$$= f(a)g(a) + (f'(a)g(a) + f(a)g'(a))h$$

$$+(f(a)S(h) + g(a)R(h) + f'(a)g'(a)h)h$$

Here :

$$\lim_{h\to 0}(f(a)S(h) + g(a)R(h) + f'(a)g'(a)h) = 0$$

Chain Rule

Chain Rule

$y = f(u)$ **is differentiable at** $u = g(a)$

$u = g(x)$ **is differentiable at** $x = a$

$\Rightarrow$

The composite function $f(g(x))$

is differentiable at $x = a$**:**

$$\frac{d}{dx} f(g(x))\Big|_{x=a} = \frac{dy}{du}(g(a)) \cdot \frac{du}{dx}(a)$$

Proof (1)

$$g(a+h) = g(a) + g'(a)h + S(h)h$$

$$\lim_{h \to 0} S(h) = 0$$

$$f(g(a)+k) = f(g(a)) + f'(g(a))k + R(k)k$$

$$\lim_{k \to 0} R(k) = 0$$

Proof (2)

$$f(g(a+h))$$
$$= f(g(a)+g(a)h+S(h)h)$$
$$= f(g(a))+f'(g(a))(g'(a)h+S(h)h)$$
$$+R(g'(a)h+S(h)h)(g'(a)h+S(h)h)$$

Proof (3)

$$f(g(a+h))$$
$$= f(g(a)) + f'(g(a))g'(a)h$$
$$+ f'(g(a))S(h)h$$
$$+R(g'(a)h + S(h)h)(g'(a) + S(h))h$$

Proof (4)

$$\lim_{h\to 0} R(g'(a)h + S(h)h)(g'(a) + S(h))$$

$$= \lim_{h\to 0} R(g'(a)h + S(h)h) \bullet g'(a)$$

$$= 0$$

Proof (5)

$$f(g(a+h))$$

$$= f(g(a)) + f'(g(a))g'(a)h + T(h)h$$

$$\lim_{h \to 0} T(h) = 0$$

Proof (6)

$$f(g(a+h))$$

$$= f(g(a)) + f'(g(a))g'(a)h + T(h)h$$

$$\Rightarrow$$

$$\frac{d}{dx} f(g(x))\Big|_{x=a} = f'(g(a))g'(a)$$

$$= \frac{dy}{du}(g(a)) \cdot \frac{du}{dx}(a)$$

Example (1)

$$\frac{d}{dx}\left(\log_e\left(x+\sqrt{x^2+1}\right)\right)$$

$$=\frac{\left(x+\sqrt{x^2+1}\right)'}{x+\sqrt{x^2+1}}$$

$$=\frac{1}{\sqrt{x^2+1}}$$

Example (2)

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \textbf{if } x \neq 0, \\ 0 & \textbf{if } x = 0 \end{cases}$$

Remark

The derivative

$$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & \textbf{if } x \neq 0, \\ 0 & \textbf{if } x = 0 \end{cases}$$

is not continuous.

Mean Value Theorem

Maximum Value Theorem

Minimum Value Theorem

↓

Rolle's Theorem

↓

Mean Value Theorem

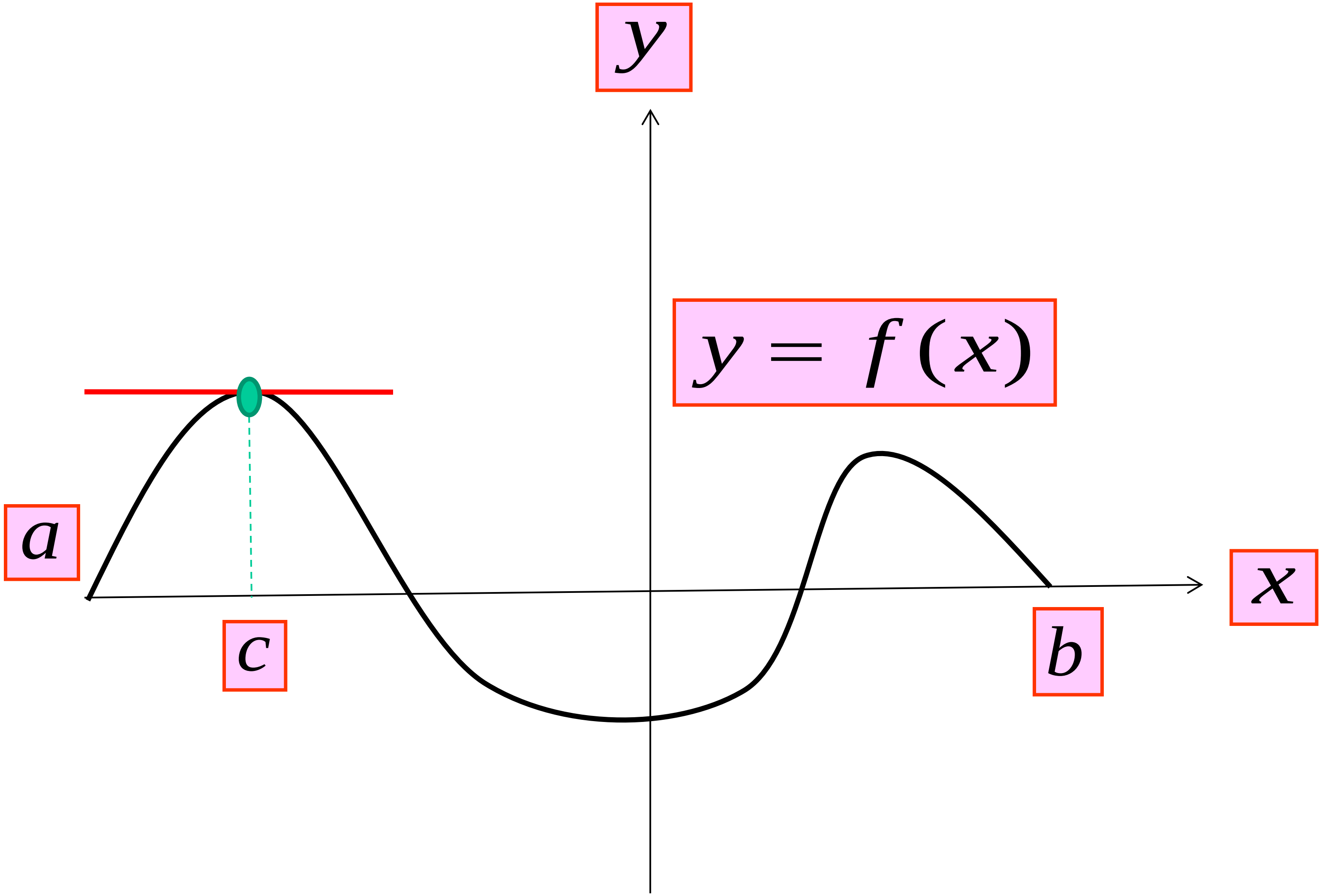
Rolle's Theorem

$f(x)$ **is continuous on** $[a,b]$ **and is differentiable in** (a,b)

$f(a) = f(b)$

$\Rightarrow$

$a < \exists c < b$ **such that** $f'(c) = 0$



y
$y = f(x)$
a
c
b
x

Mean Value Theorem

$f(x)$ is continuous on $[a,b]$ and is differentiable in (a,b)

$\Rightarrow$

$a < \exists c < b$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Proof

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

$$\Rightarrow$$

$$F(a) = F(b) = 0$$

$$\Rightarrow$$

$$a < \exists c < b \textbf{ such that}$$

$$F'(c) = 0$$

Behavior of Functions

Let $f(x)$ **be continuous on** $[a, b]$ **and differentiable in** (a, b).

On (a, b)

(1) $f'(x) = 0 \Rightarrow f(x)$ **is constant**

(2) $f'(x) \geq 0 \Rightarrow f(x)$ **is monotone increasing**

(3) $f'(x) \leq 0 \Rightarrow f(x)$ **is monotone decreasing**

Maximal and Minimal (1)

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)$$

$$+\frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

(1) $f'(a) \neq 0$: $f(x)$ **is not extremal**

(2) $f'(a) = 0$:

(2-1) $f''(a) > 0 \Rightarrow f(x)$ **is minimal**

(2-2) $f''(a) < 0 \Rightarrow f(x)$ **is maximal**

Maximal and Minimal (2)

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)$$
$$+\frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$
$$+\frac{f''''(a)}{4!}(x-a)^4 + \cdots$$

(3) $f'(a) = f''(a) = 0$:

$f'''(a) \neq 0 \Rightarrow f(x)$ **is not extremal**

(4) $f'(a) = f''(a) = f'''(a) = 0$:

(4-1) $f''''(a) > 0 \Rightarrow f(x)$ **is minimal**

(4-2) $f''''(a) < 0 \Rightarrow f(x)$ **is maximal**

Mean Value Theorem

Cauchy's Mean Value Theorem

$\Rightarrow$ de l'Hospital's Theorem

Taylor's Theorem $\Rightarrow$ Polynomial Approximation

Cauchy's Mean Value Theorem

$f(x)$ is continuous on $[a,b]$ and
is differentiable in (a,b)
$g(x)$ is continuous on $[a,b]$ and
is differentiable in (a,b) with $g'(x) \neq 0$

$\Rightarrow$

$a < \exists c < b$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

de l'Hospital's Theorem

Let $f(x), g(x)$ **be continuous near** a **and differentiable except for** a.

$$f(a) = g(a) = 0, \quad g'(x) \neq 0$$

Then :

$$\exists \lim_{x \to a} \frac{f'(x)}{g'(x)} = \alpha \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \alpha$$

Taylor Series

Taylor's Theorem (Lagrange)

$f(x)$ is of class C^{n-1} on $[a,b]$ and

is of class C^{n} in (a,b) where $n \geq 2$

$\Rightarrow$

$a < \exists c < b$ such that

$$f(b) = f(a) + f'(a)(b-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1}$$

$$+ \frac{f^{(n)}(c)}{n!}(b-a)^{n}$$

Taylor's Theorem (Cauchy)

$f(x)$ is of class C^{n-1} on $[a,b]$ and

is of class C^n in (a,b) where $n \geq 2$

$\Rightarrow$

$a < \exists c < b$ such that

$$f(b) = f(a) + f'(a)(b-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1}$$

$$+ \frac{f^{(n)}(c)}{(n-1)!}(b-c)^{n-1}(b-a)$$

Taylor's Theorem

(1) $f(x)$ **is infinitely differentiable in** $(-R, R)$

(2) $\sup\limits_{|x|<R} \left| f^{(n)}(x) \right| \le \exists M_R, \quad n = 1, 2, \cdots$

$\Rightarrow$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \quad |x| < R$$

Examples

Example (1)

$$e^{x} = 1 + \frac{x}{1!} + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \cdots + \frac{x^{n}}{n!} + \cdots$$

Remark (Napier's Number)

$$e = e^1$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots$$

$$(x = 1)$$

Example (2)

$$(\sin x)^{(n)} = \sin\left(x + \frac{n}{2}\pi\right)$$

$$\Rightarrow$$

$$(\sin x)^{(n)}\Big|_{x=0} = \sin\left(\frac{n}{2}\pi\right)$$

$$= \begin{cases} 0 & \textbf{for } n = 2k \\ (-1)^k & \textbf{for } n = 2k+1 \end{cases}$$

Example (2)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \cdots$$

Example (3)

$$(\cos x)^{(n)} = \cos\left(x + \frac{n}{2}\pi\right)$$

$$\Rightarrow$$

$$(\cos x)^{(n)}\Big|_{x=0} = \cos\left(\frac{n}{2}\pi\right)$$

$$= \begin{cases} (-1)^k \ \textbf{for} \ \ n = 2k \\ 0 \quad \textbf{for} \ n = 2k+1 \end{cases}$$

Example (3)

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots$$

Example (4)

$$\log_e(1+x)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$+ \frac{x^{2n-1}}{2n-1} - \frac{x^{2n}}{2n} + \cdots$$

$$(-1 < x \leq 1)$$

Example (5)

$$\tan^{-1} x$$

$$= x - \frac{x^3}{3} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$$

$$(-1 < x \leq 1)$$

Abel's Theorem

$$A = \sum_{n=1}^{\infty} a_n \text{ } \textbf{converges}$$

$$\Rightarrow$$

$$f(x) = \sum_{n=1}^{\infty} a_n x^n \to A \text{ as } x \uparrow 1$$

Examples

$$(1)\ 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\log(1+1)=\log_e 2$$

$$(2)\ 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\tan^{-1}1=\frac{\pi}{4}$$

Computational Approach

Numerical Computing
with
BASIC

Alternating Series Version

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$
$$= 0.69314718055994 5$$

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

$$0.693147180554981 \ (n = 10)$$

$$0.6931471805599944 \ (n = 100)$$

$$0.\ 6931471805599944 \ (n = 1000)$$

Taylor Series Version

Taylor Series Version (1)

$$\log_e(1+x)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$+ \frac{x^{2n-1}}{2n-1} - \frac{x^{2n}}{2n} + \cdots$$

$$(-1 < x \le 1)$$

Taylor Series Version (2)

$$\log_e(1-x)$$

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

$$- \frac{x^{2n-1}}{2n-1} - \frac{x^{2n}}{2n} - \cdots$$

$$(-1 \leq x < 1)$$

Taylor Series Version (3)

$$\log_e \frac{1+x}{1-x} = \log_e(1+x) - \log_e(1-x)$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n-1}}{2n-1} + \cdots \right)$$

$$(-1 < x < 1)$$

$$\log_e 2 = \log_e \frac{1+\frac{1}{3}}{1-\frac{1}{3}} \qquad \left(x = \frac{1}{3}\right)$$

$$= 2\left(\frac{1}{3} + \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 + \cdots + \frac{1}{2n-1}\left(\frac{1}{3}\right)^{2n-1} + \cdots\right)$$

$$= 0.69314718055994 5\cdots$$

Computational Approach

Newton's Iteration Method

$$I = [a, b]$$

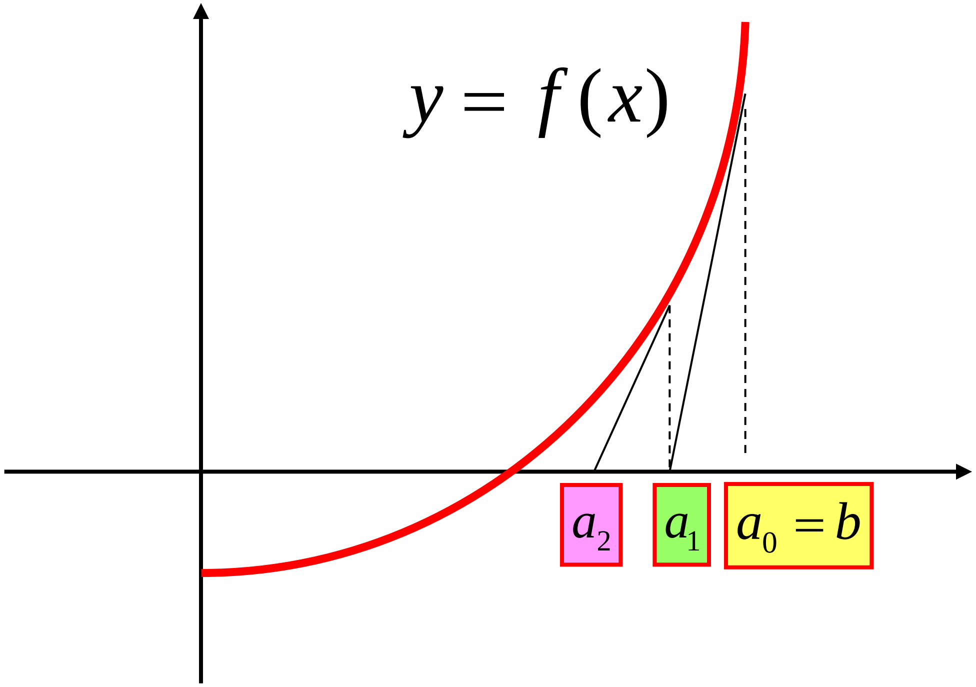
$$\begin{cases} f(a) < 0, \ f(b) > 0 \\ f''(x) > 0 \end{cases}$$

$$\Longrightarrow$$

$$\begin{cases} a_0 = b, \\ a_n = a_{n-1} - \dfrac{f(a_{n-1})}{f'(a_{n-1})}, \quad n = 1, 2, \ldots \end{cases}$$

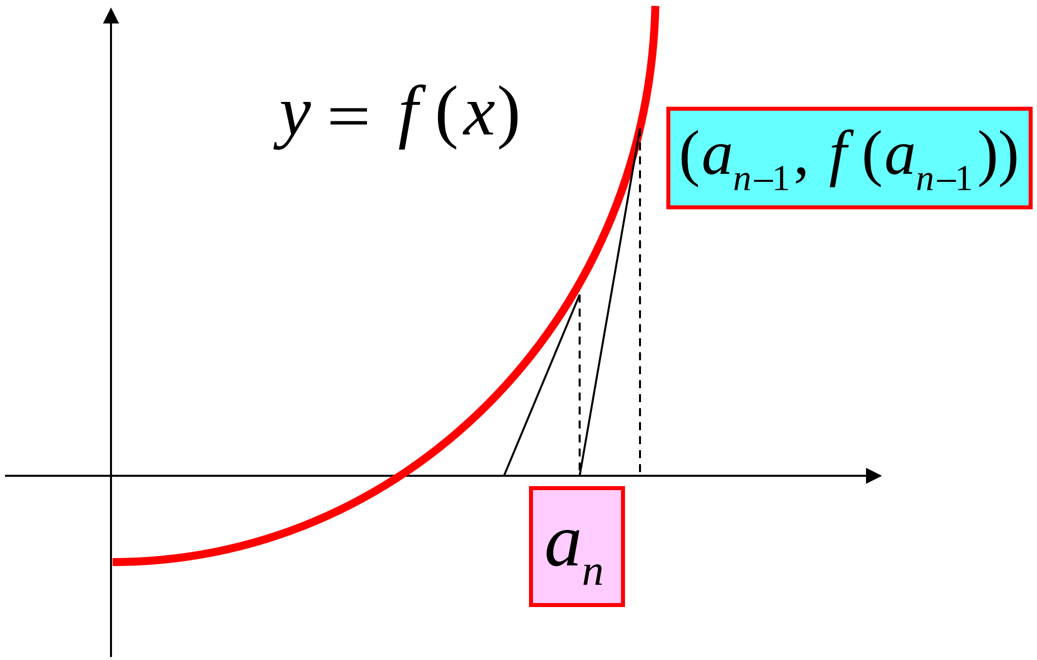
$\{a_n\}$ **converges to the solution** ξ **of the equation** $f(x) = 0$

Newton's Method (1)



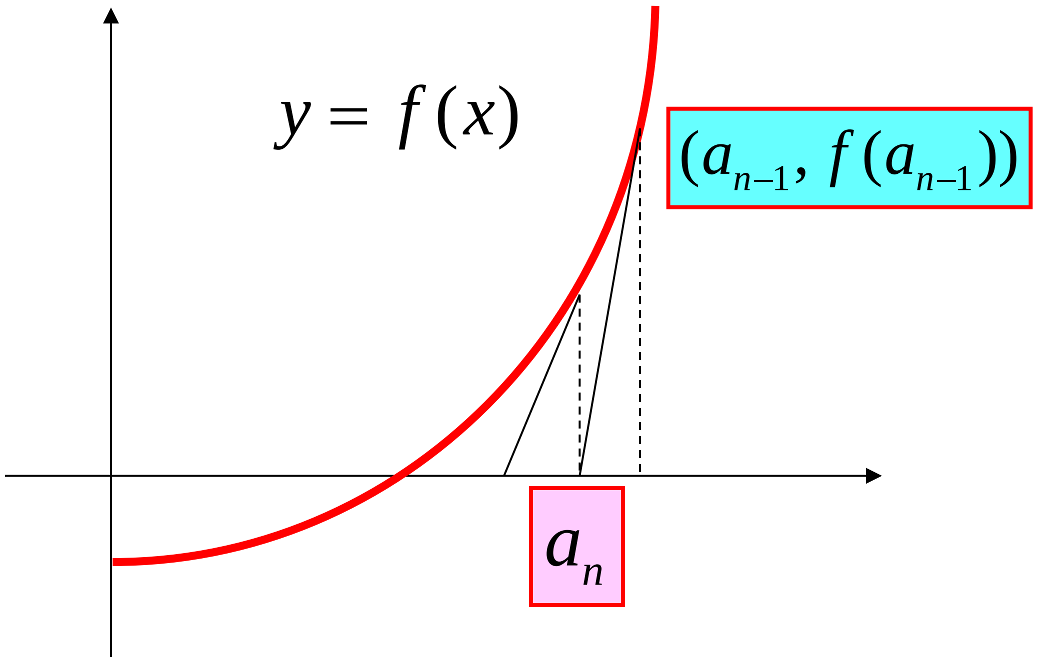
$$f(x) = 0$$

Newton's Method (2)



$$y = f'(a_{n-1})(x - a_{n-1}) + f(a_{n-1})$$

Newton's Method (3)

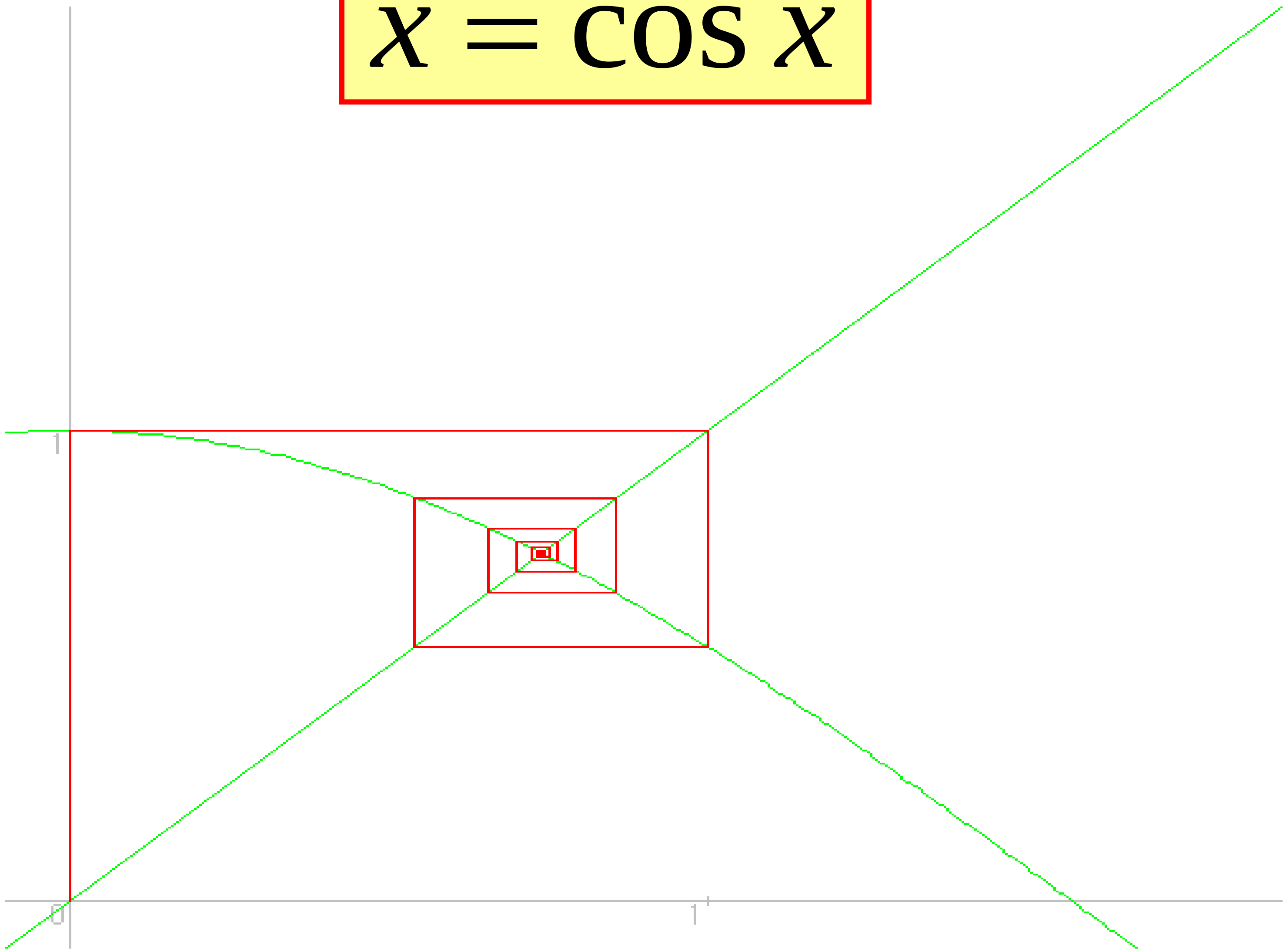


Definition of $\{a_n\}$

$$0 = f'(a_{n-1})(a_n - a_{n-1}) + f(a_{n-1})$$

Numerical Computing
with
BASIC

$$x = \cos x$$



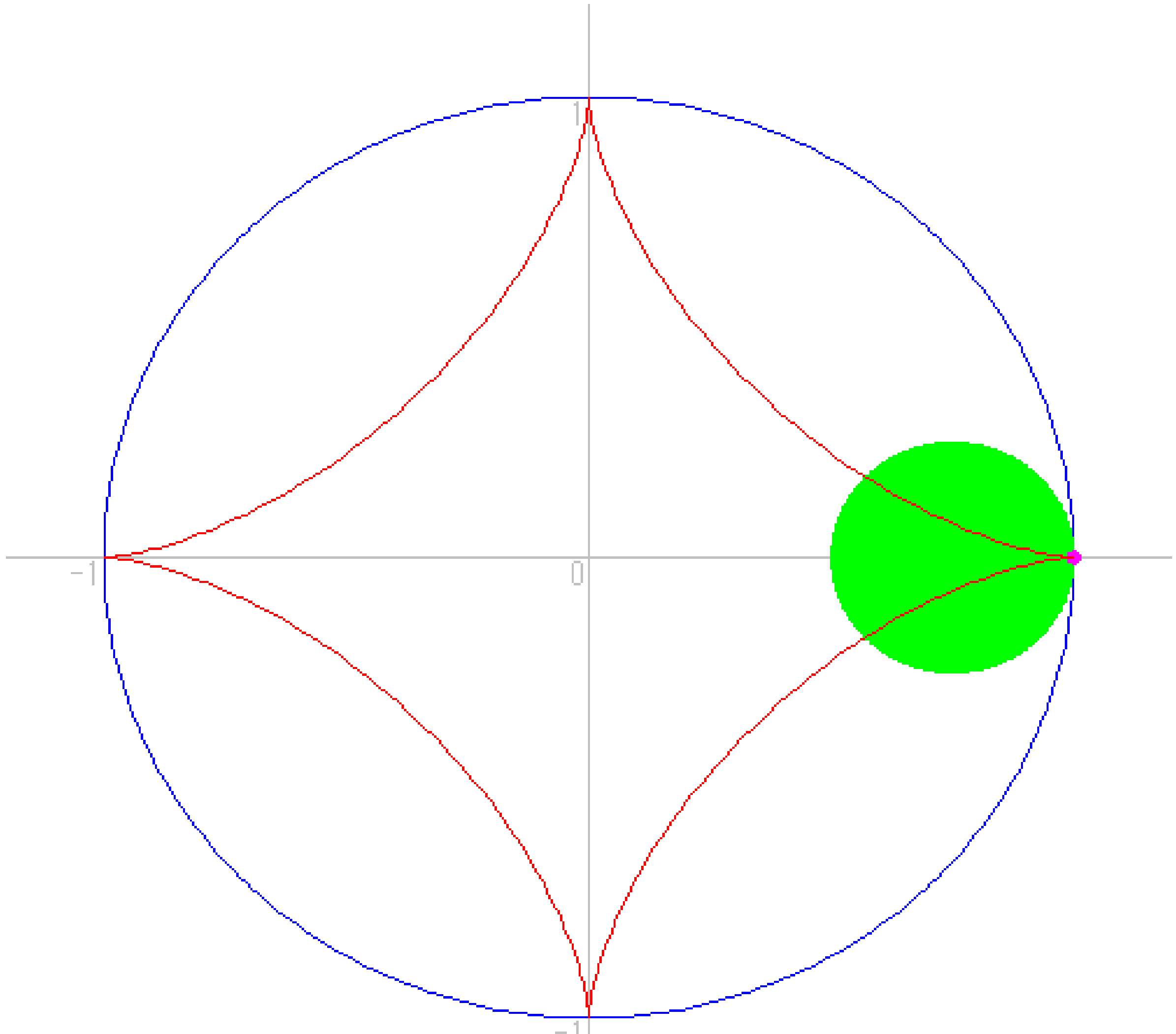
$$x = \cos x$$

$$x \div 0.7390851332151 66$$

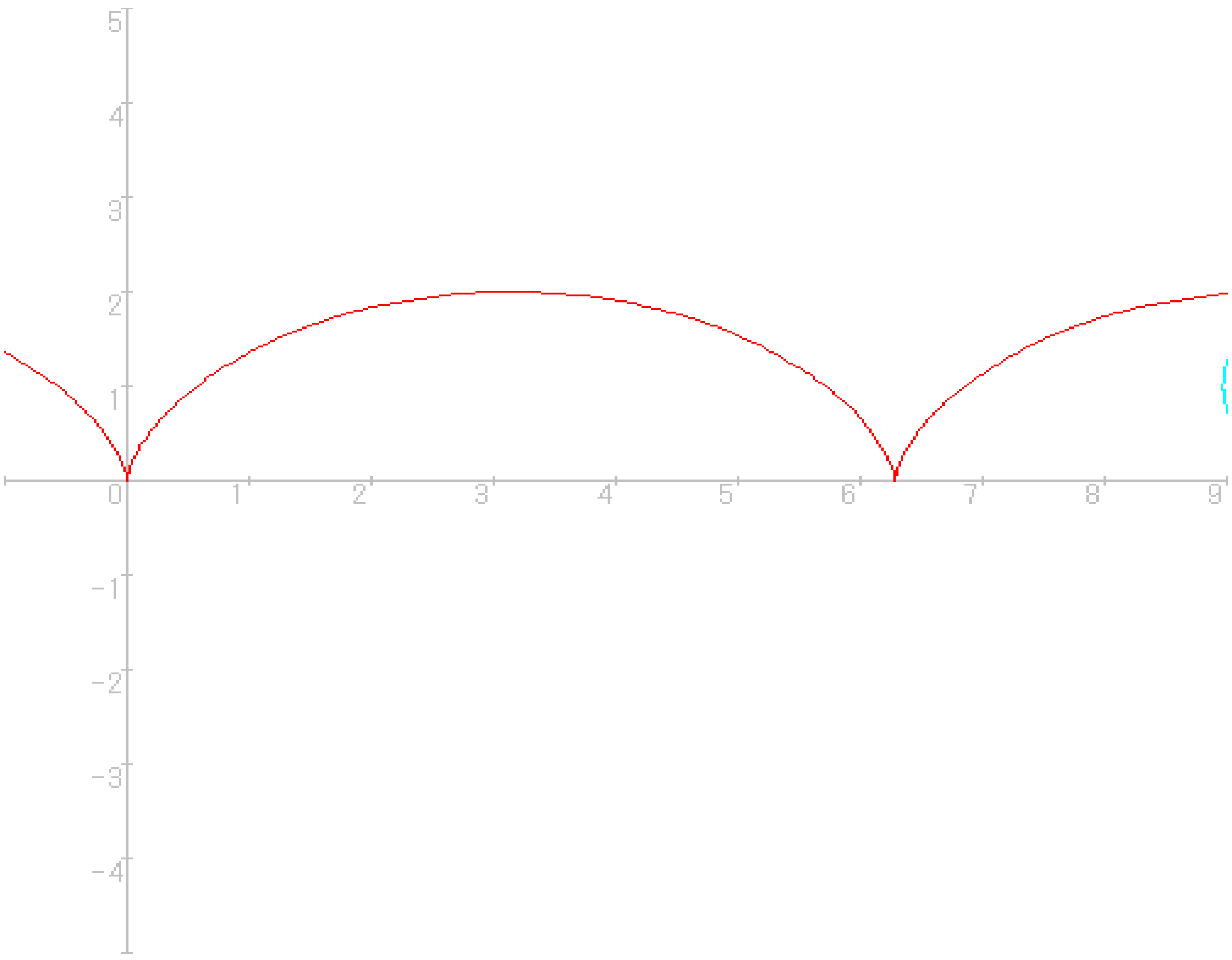
Miscellaneous Curves

Numerical Computing with BASIC

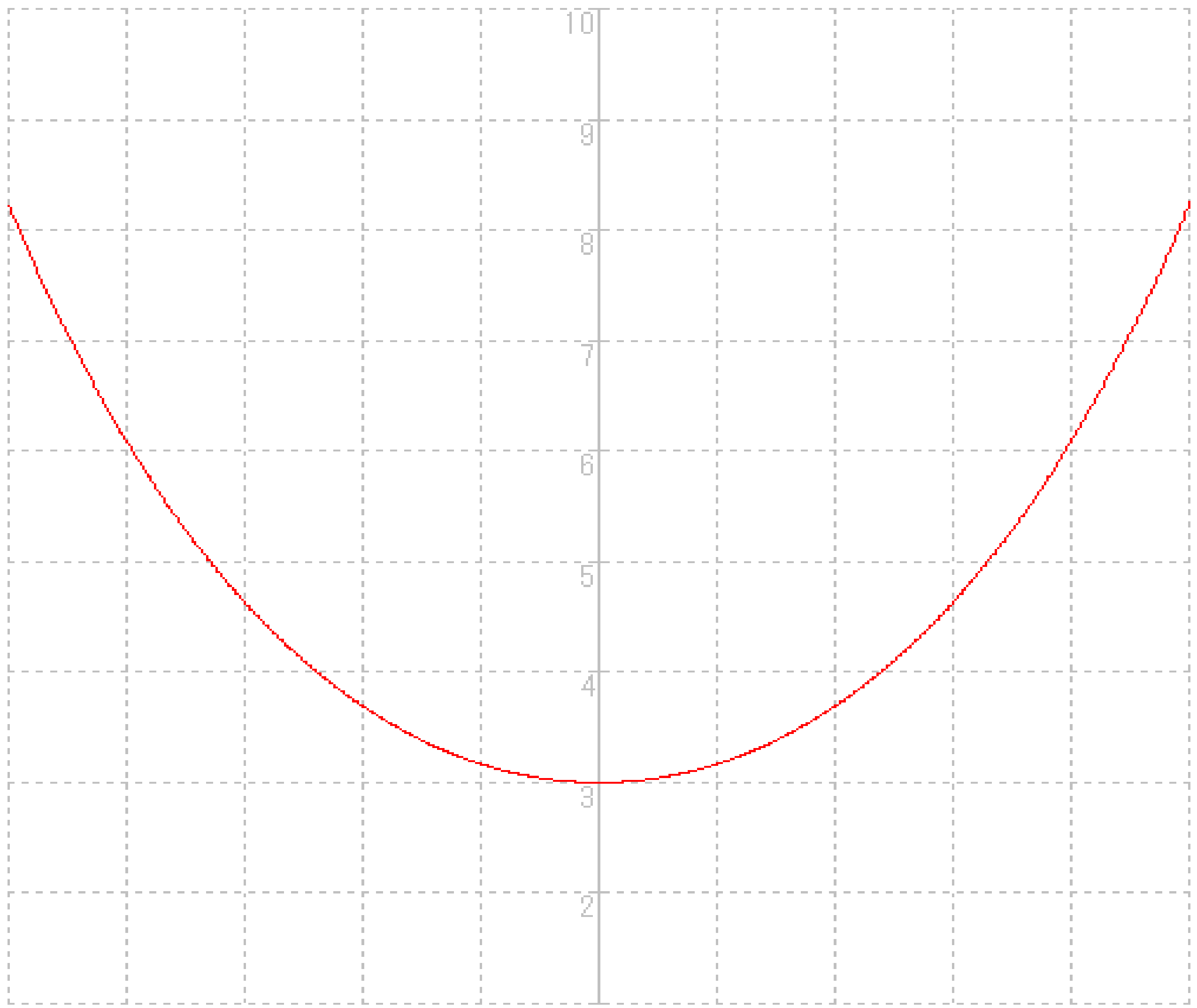
Asteroid



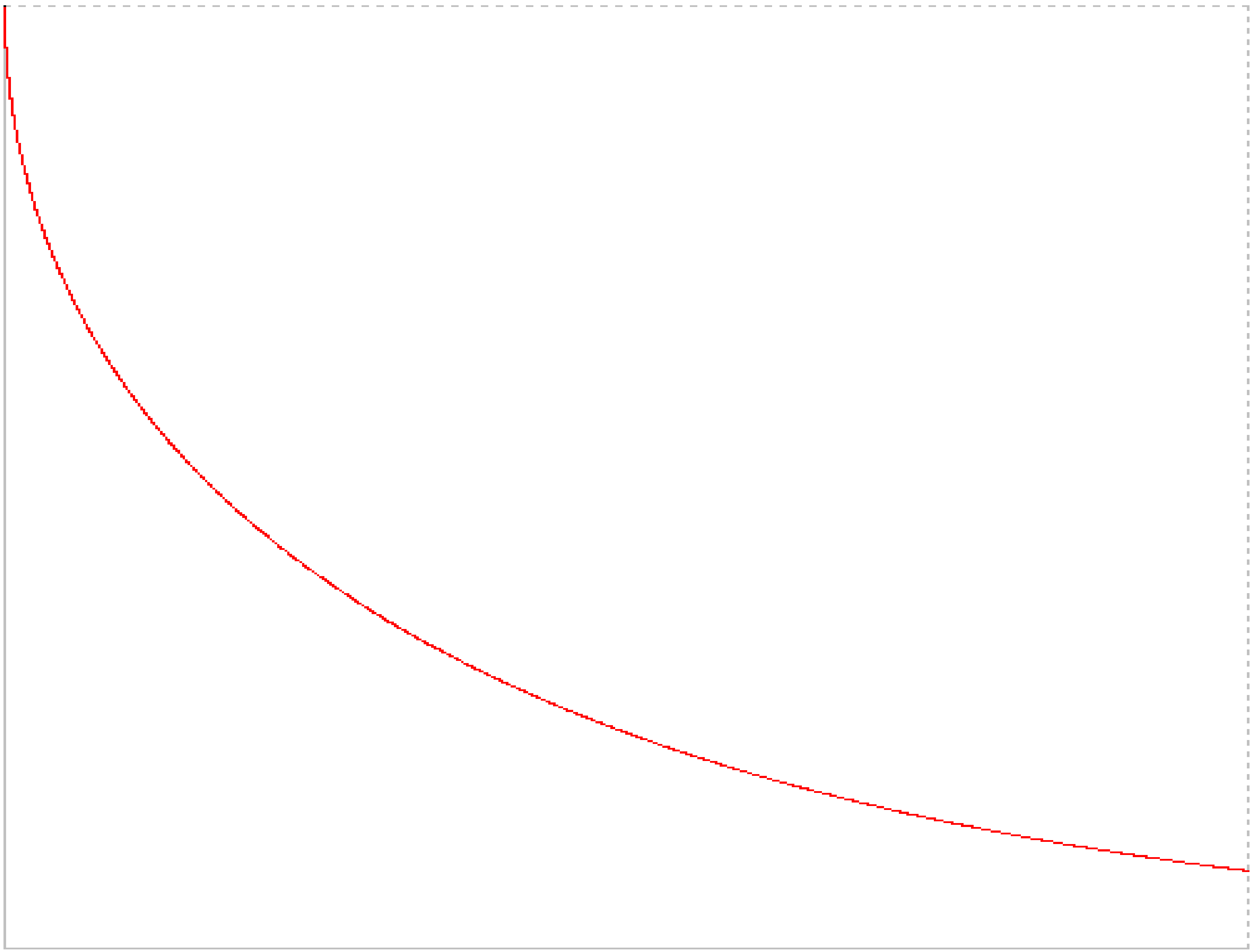
Cycloid



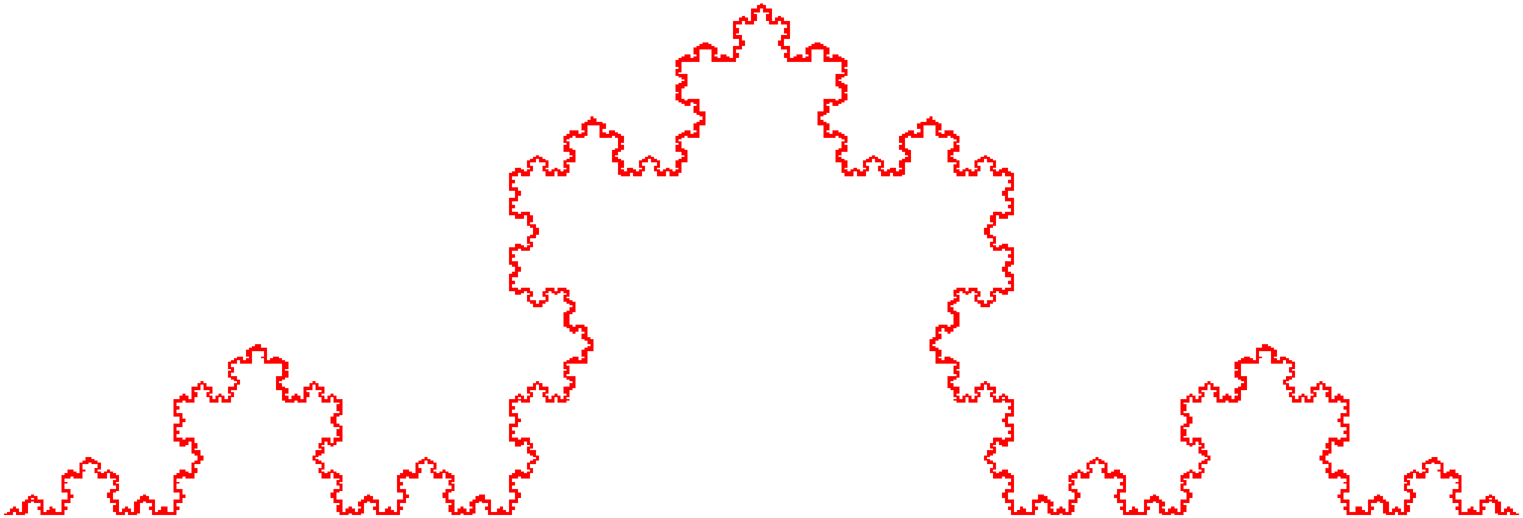
Catenary



Tractrix



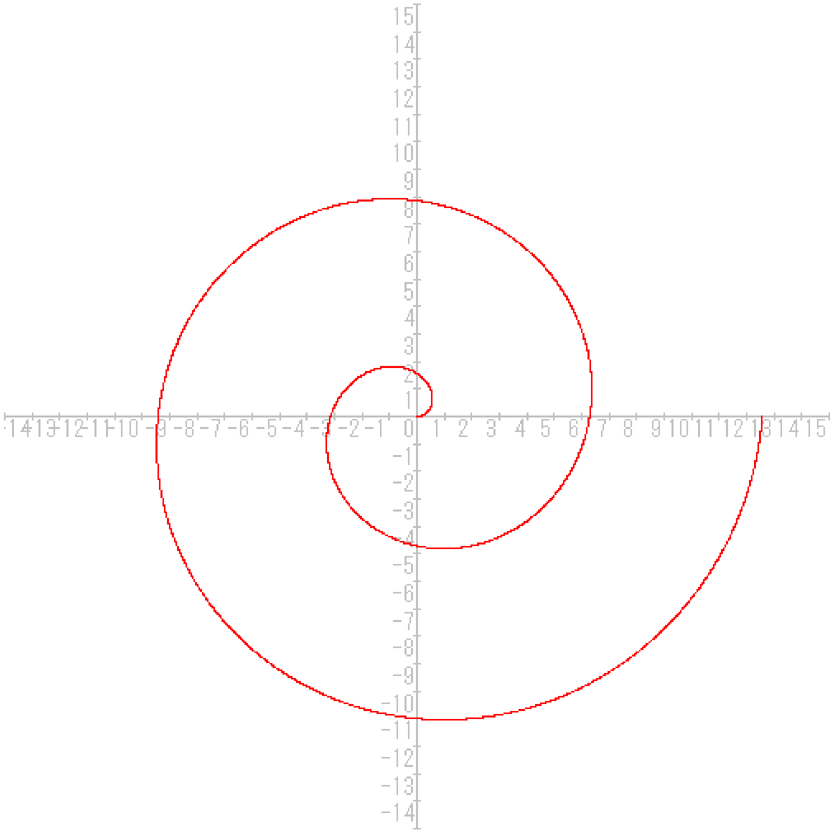
von Koch's Curve



Curves defined by Polar Coordinates

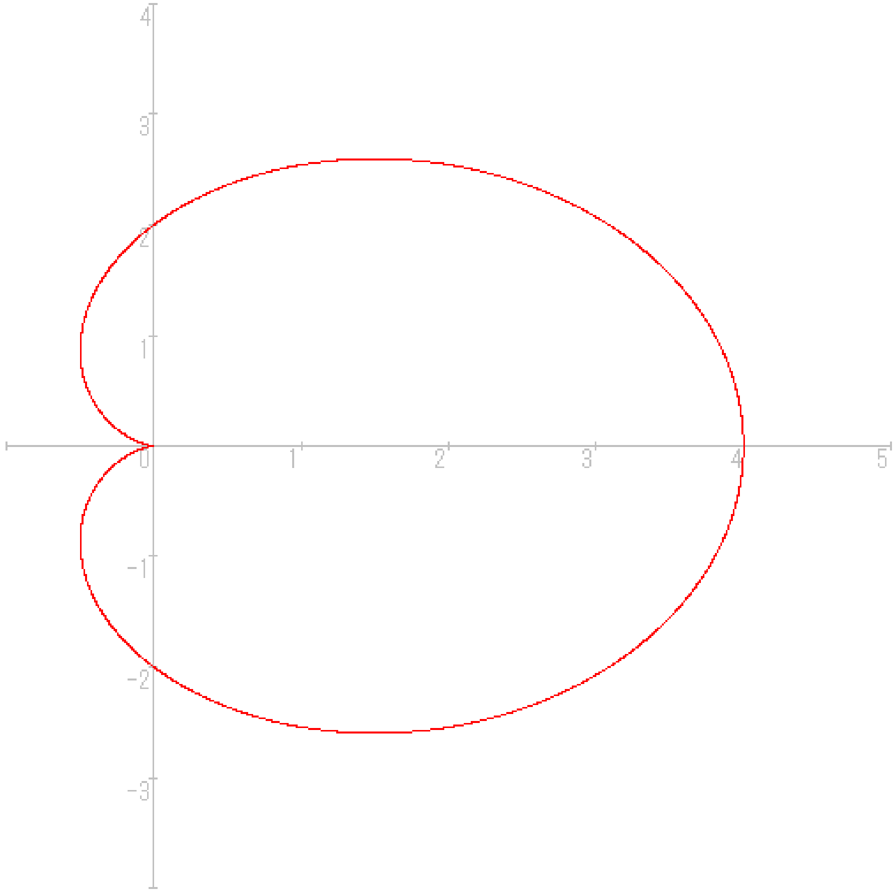
Numerical Computing with BASIC

Archimedes' Spiral



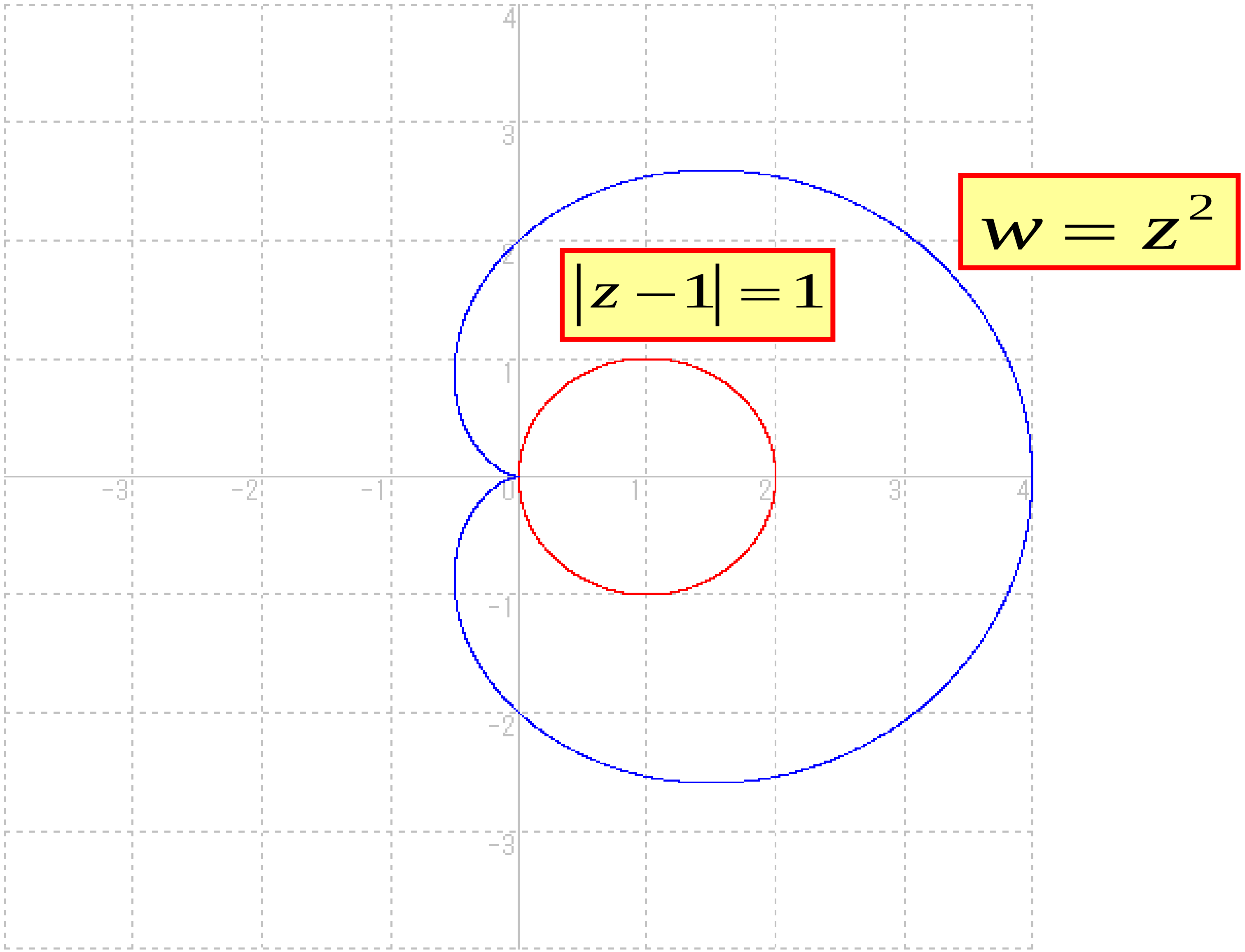
$$r = \theta$$

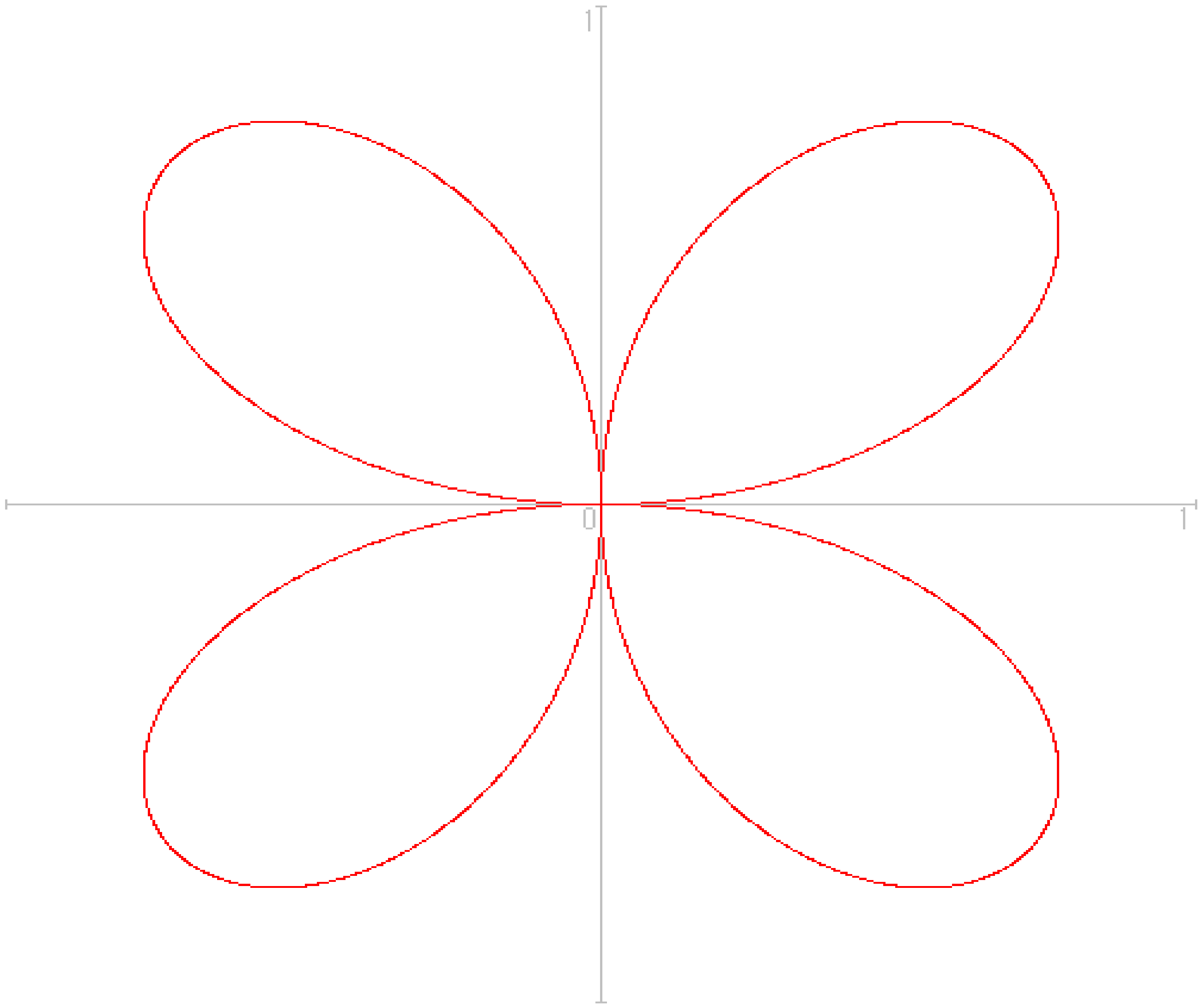
Cardioid



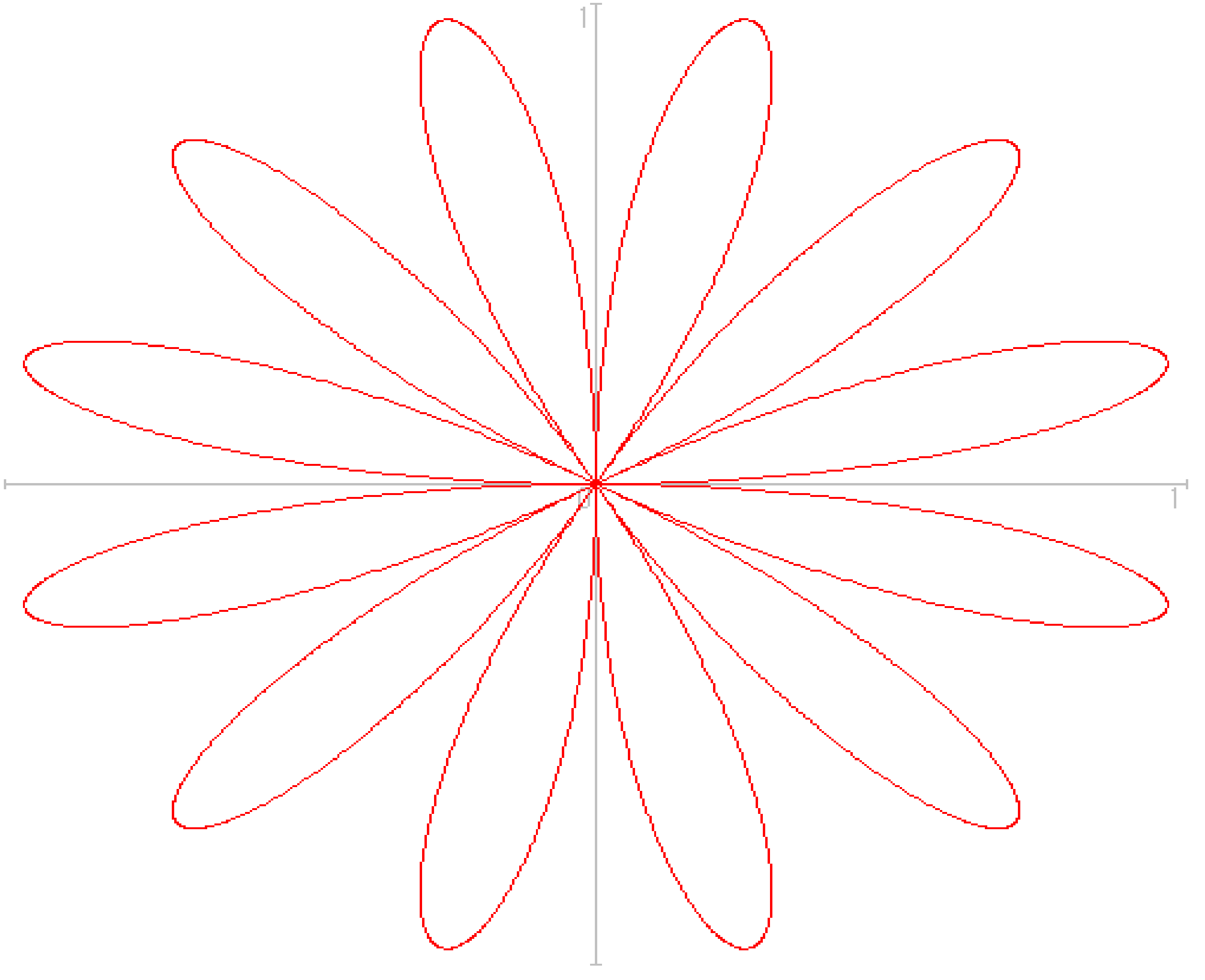
$$r = 2(1 + \cos\theta)$$

Cardioid (Complex Version)





$$r = \sin 2\theta$$



$r = \sin 6\theta$

Conic Section

$$r(\theta) = \frac{\ell}{1+\varepsilon\cos\theta}$$

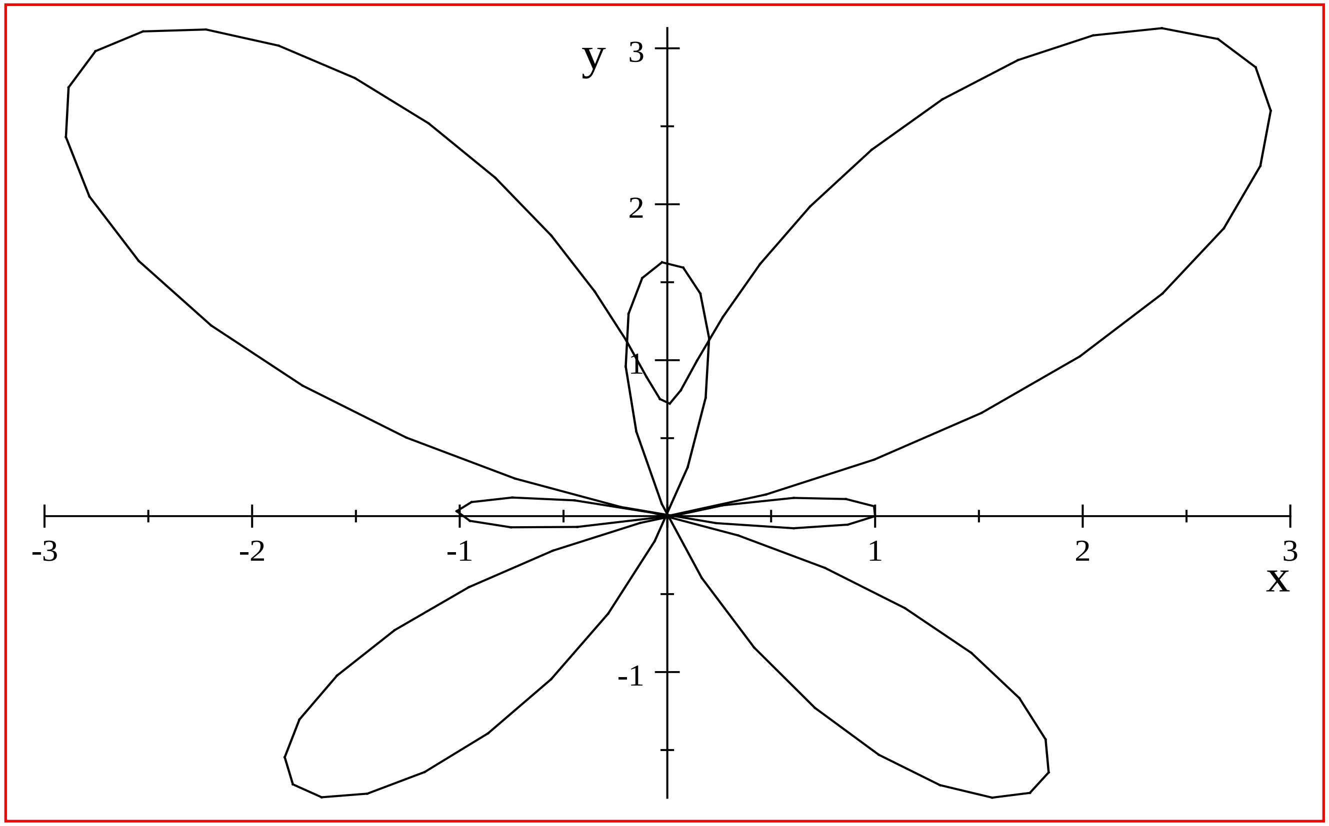
$$0 < \varepsilon < 1\,(\text{Ellipse}): \quad r + r' = \frac{2\ell}{1-\varepsilon^2}$$

$$\varepsilon > 1\,(\text{Hyperbola}): \quad r' - r = \frac{2\ell}{\varepsilon^2 - 1}$$

$$\varepsilon = 1\,(\text{Parabola}): \quad x = -\frac{1}{2\ell}y^2 + \frac{\ell}{2}$$

Numerical Computing with MuPAD

Butterfly



Compactness

Real Numbers	Bolzano-Weierstrass' Theorem (Sequences)
Calculus	Ascoli-Arzela's Theorem (Continuous Functions)

Bolzano-Weierstrass Theorem

Every bounded sequence has a convergent subsequence.

Ascoli-Arzela Theorem

If a sequence of continuous functions is uniformly bounded and equicontinuous, then it has a subsequence which converges in the uniform topology.

(1) $\exists M > 0: \ |f_n(x)| \le M$

(**Uniformly Bounded**)

(2) $\forall \varepsilon > 0, \ \exists \delta = \delta(\varepsilon) > 0:$

$|x - y| < \delta \Longrightarrow |f_n(x) - f_n(y)| < \varepsilon$

(**Equicontinuous**)

Indefinite Integrals

Antiderivative

A function $F(x)$ is called an **antiderivative** of $f(x)$ on an interval I if it satisfies the condition

$$F'(x) = f(x), \ \forall x \in I$$

Fundamental Theorem

If $F(x)$ is an antiderivative of $f(x)$ on an interval $I,$ then the most general **antiderivative** of f on I is of the form $F(x)+C,$ where C is an arbitrary constant.

Examples (1)

$$(1)\int x^{\alpha}dx=\frac{1}{\alpha+1}x^{\alpha+1},\quad \alpha\neq -1$$

$$(2)\int \frac{1}{x}dx=\log_e|x| \quad (\alpha=-1)$$

$$(3)\int a^{x}dx=\frac{a^{x}}{\log_e a},\quad a>0$$

$$\int e^{x}dx=e^{x}\quad (a=e)$$

Examples (2)

$$(4)\int \frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a\neq 0$$

$$\int \frac{1}{x^2+1}dx=\tan^{-1}x \quad (a=1)$$

$$(5)\int \frac{1}{x^2-a^2}dx=\frac{1}{2a}\log_e\left|\frac{x-a}{x+a}\right|, \quad a\neq 0$$

Examples (3)

$$(6)\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\frac{x}{a}, \quad a > 0$$

$$\int \frac{1}{\sqrt{1 - x^2}}dx = \sin^{-1} x \quad (a = 1)$$

$$(7)\int \log_e x\,dx = x\left(\log_e x - 1\right)$$

Examples (4)

$$(8) \int \sqrt{a^2 - x^2}\, dx$$

$$= \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a} \right), \quad a > 0$$

$$(9) \int \frac{1}{\sqrt{x^2 + A}}\, dx = \log_e \left| x + \sqrt{x^2 + A} \right|, A > 0$$

Examples (5)

$$(10)\int \sqrt{x^2 + A}\,dx$$

$$= \frac{1}{2}\left(x\sqrt{x^2 + A} + A\log_e \left| x + \sqrt{x^2 + A} \right| \right),\ A > 0$$

Examples

$$(1)\ \int \sin x\, dx = -\cos x$$

$$(2)\ \int \cos x\, dx = \sin x$$

$$(3)\ \int \frac{1}{\cos^2 x} dx = \tan x$$

$$(4)\ \int \frac{1}{\sin^2 x} dx = -\cot x$$

$$(5)\ \int \tan x\, dx = -\log|\cos x|$$

$$(6)\ \int \cot x\, dx = \log|\sin x|$$

Indefinite Integrals of Rational Functions

Example

$$\frac{1}{x\left(x^2+1\right)^2}$$

$$=\frac{1}{x}-\frac{x}{x^2+1}-\frac{x}{(x^2+1)^2}$$

Formula (5)

$$(1)\int \frac{1}{x-a}dx = \log_e |x-a|$$

$$(2)\int \frac{1}{(x-a)^l}dx$$

$$= -\frac{1}{(l-1)(x-a)^{l-1}}, \quad l \neq 1$$

Formula (6)

$$\int \frac{x}{x^2 + a^2} dx$$

$$= \frac{1}{2} \log\left(x^2 + a^2 \right), \quad a \neq 0$$

Formula (7)

$$\int \frac{x}{\left(x^2+a^2\right)^m}dx$$

$$=\frac{1}{2(m-1)\left(x^2+a^2\right)^{m-1}},$$

$$a \neq 0, \ \ m \neq 1$$

Example 1-1

$$\int \frac{1}{x\left(x^2+1\right)^2}dx$$

$$=\int \frac{1}{x}dx-\int \frac{x}{x^2+1}dx$$

$$-\int \frac{x}{\left(x^2+1\right)^2}dx$$

Example 1-2

$$\int \frac{1}{x\left(x^2+1\right)^2}dx$$

$$= \log_e \frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2\left(x^2+1\right)}$$

Formula (8)

$$I_m = \int \frac{1}{\left(x^2 + a^2\right)^m} dx,$$

$$a \neq 0, \quad m \geq 1$$

Formula (9)

$$I_1 = \int \frac{1}{x^2 + a^2} dx$$

$$= \frac{1}{a} \tan^{-1} \frac{x}{a}$$

Formula (10)

$$I_2 = \int \frac{1}{\left(x^2 + a^2\right)^2} dx$$

$$= \frac{1}{2a^2}\left(\frac{x}{2\left(x^2 + a^2\right)} + \frac{1}{a}\tan^{-1}\frac{x}{a}\right)$$

Formula (11)

$$I_m = \int \frac{1}{\left(x^2 + a^2\right)^m} dx \qquad (m \geq 2)$$

$$= \frac{1}{a^2}\left(\frac{x}{2(m-1)\left(x^2 + a^2\right)^{m-1}} + \frac{2m-3}{2m-2} I_{m-1} \right)$$

Definite Integrals

Series and Integrals

$$\sum_{n=1}^{\infty} a_n$$

Discrete Case

Continuous Case

$$\int_a^b f(x)dx$$

Riemann Integral

Georg Friedrich Bernhard Riemann (1826-1866)

Henri Lebesgue (1875—1941)

Definition of Riemann Integral

Let $f(x)$ **be a bounded function defined on an interval** $I=[a,b]$

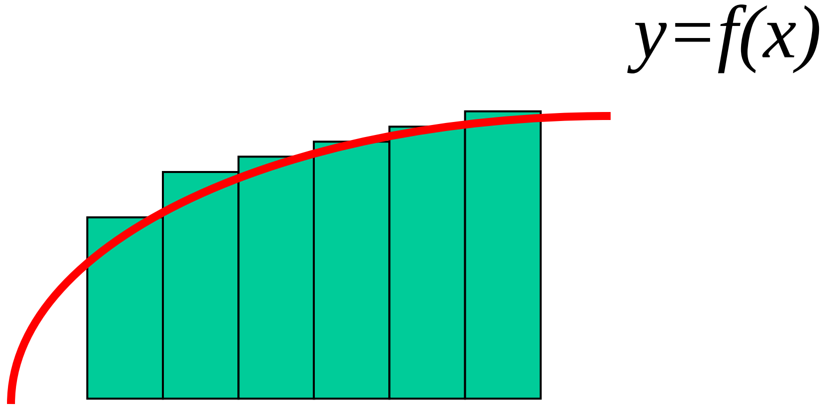
$\Delta: x_0 = a < x_1 < \cdots < x_n = b$ (**partition of** I)

$$|\Delta| = \max_{1\le i\le n} (x_i - x_{i-1})$$

$$S(\Delta, f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}),\ \ M_i = \sup_{x_{i-1}\le t\le x_i} f(t)$$

$$s(\Delta, f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}),\ \ m_i = \inf_{x_{i-1}\le t\le x_i} f(t)$$

Upper Integral

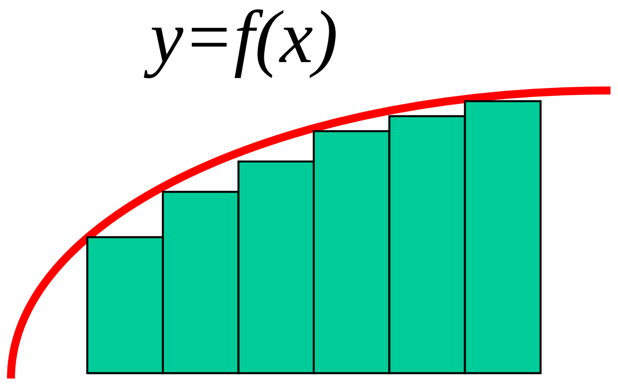


Approximation area from outside

Upper Integral

$$\overline{\lim}\int_a^b f(x)\,dx = \inf_{\Delta} S(\Delta, f)$$

Lower Integral



Approximation area from inside

Lower Integral

$$\underline{\lim}\int_a^b f(x)\,dx = \sup_{\Delta} s(\Delta, f)$$

Definition of Riemannian Integrability

$$\underline{\lim}\int_a^b f(x)\,dx = \overline{\lim}\int_a^b f(x)\,dx$$

Examples

(1) Continuous functions

(2) Monotone increasing (decreasing) functions

(3) Bounded variation functions

Darboux's Theorem

Darboux's Theorem

$$(1)\ \overline{\lim}\int_a^b f(x)\,dx = \lim_{|\Delta|\to 0} S(\Delta, f)$$

$$(2)\ \underline{\lim}\int_a^b f(x)\,dx = \lim_{|\Delta|\to 0} s(\Delta, f)$$

Mensuration by Parts

Riemannian Sum

Let $f(x)$ **be a Riemann integrable function defined on an interval** $I=[a,b]$

$$\Delta : x_0 = a < x_1 < \cdots < x_n = b$$

$$\left|\Delta\right| = \max_{1\le i\le n} (x_i - x_{i-1})$$

$$\Rightarrow$$

$$\int_a^b f(x)dx = \lim_{|\Delta|\to 0} \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}), \quad x_{i-1} \le \forall t \le x_i$$

Example

Let $f(x)$ **be a continuous function defined on the interval** $I = [0,1]$

$\Rightarrow$

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n} = \int_0^1 f(x)dx$$

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{i-1}{n}\right)\frac{1}{n} = \int_0^1 f(x)dx$$

Example

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{n^2+i^2}}=\int_0^1\frac{1}{\sqrt{x^2+1}}dx$$

$$=\log_e\left|x+\sqrt{x^2+1}\right|_0^1$$

$$=\log_e\left(1+\sqrt{2}\right)$$

Fundamental Theorem

of

Calculus

Fundamental Theorem of Calculus
Part 1

If $f(x)$ is continuous on $[a, b]$, then

$$g(x) = \int_a^x f(t)dt, \quad a \leq x \leq b,$$

is continuous on $[a, b]$ and differentiable on (a, b), and

$$g'(x) = f(x).$$

Fundamental Theorem of Calculus Part 2

If $f(x)$ is continuous on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$.

Notation

Because of the relation given by
the fundamental theorem of calculus
between antiderivatives and integrals,
the notation

$$\int f(x)dx$$

is traditionally used for an antiderivative
of $f(x)$, and is called an **indefinite integral**.

Strategy for Integration

Formula for Integration by Parts

If $f(x)$ and $g(x)$ are

C^1 functions on $[a,b]$, then

$$\int_a^b f'(x)g(x)dx$$

$$=\left[f(x)g(x)\right]_{x=a}^{x=b}-\int_a^b f(x)g'(x)dx$$

Substitution Rule for Definite Integrals

If $g(x)$ is a C^1 function on $[a,b]$ and

if $f(x)$ is continuous on the range of g,

then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Areas between Curves

If a region D is bounded by the curves with equations

$$y = f(x),\ y = g(x)$$

and the lines

$$x = a,\ x = b,$$

where

$$f(x) \geq g(x), \quad a \leq x \leq b,$$

then the area A of D is

$$A = \int_a^b [f(x) - g(x)]\,dx$$

Center of Mass

If a region D is bounded by the curves with equations

$y = f(x),\ y = g(x)$

and the lines

$x = a,\ x = b,$

where

$f(x) \geq g(x), \quad a \leq x \leq b,$

then the center of mass the D is located at

$$\overline{x} = \frac{1}{A}\int_a^b x\left[f(x) - g(x)\right]dx$$

$$\overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[f(x)^2 - g(x)^2\right]dx$$

Volume of Revolution

If S is the solid generated when the region bounded by the curves with equations

$y = f(x),\ y = g(x) \quad (f(x) \geq g(x))$

and the lines

$x = a,\ x = b,$

is rotated about the x-axis,

then the volume V of S is

$$V = \pi \int_a^b \left[f(x)^2 - g(x)^2 \right] dx$$

Improper Integrals

Improper Integral of Type 1

(I) If the definite integral

$$\int_a^t f(x)dx$$

exists for every $t \geq a,$ then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$$

provided the limit exists

as a finite number.

Example 1

$$\int_1^{\infty} \frac{1}{x^{\alpha}} dx = \lim_{t \to \infty} \int_1^{t} \frac{1}{x^{\alpha}} dx$$

$$= \lim_{t \to \infty} \left(\frac{1}{\alpha - 1} - \frac{1}{\alpha - 1} \frac{1}{t^{\alpha - 1}} \right)$$

$$= \frac{1}{\alpha - 1}, \quad \alpha > 1$$

Improper Integral of Type 1

(II) If the definite integral

$$\int_t^b f(x)dx$$

exists for every $t \leq b$, then

$$\int_{-\infty}^b f(x)dx = \lim_{t \to -\infty} \int_t^b f(x)dx$$

provided the limit exists as a finite number.

Improper Integral of Type 1

(III) If the improper integrals

$$\int_a^{\infty} f(x)dx, \quad \int_{-\infty}^{a} f(x)dx$$

exist, then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{\infty} f(x)dx$$

Improper Integral of Type 2

(I) If $f(x)$ is continuous on $[a,b)$
and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t \uparrow b} \int_a^t f(x)dx$$

provided the limit exists
as a finite number.

Improper Integral of Type 2

(II) If $f(x)$ is continuous on $(a,b]$

and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t \downarrow a} \int_t^b f(x)dx$$

provided the limit exists

as a finite number.

Example 2

$$\int_0^1 \frac{1}{x^\alpha} dx = \lim_{t \to 0} \int_t^1 \frac{1}{x^\alpha} dx$$

$$= \lim_{t \to 0} \left(\frac{1}{1-\alpha} - \frac{1}{1-\alpha} t^{1-\alpha} \right)$$

$$= \frac{1}{1-\alpha}, \quad 0 < \alpha < 1$$

Improper Integral of Type 2

(III) If $f(x)$ is discontinuous at $c \in (a,b)$,
and the improper integrals

$$\int_a^c f(x)dx, \quad \int_c^b f(x)dx$$

exist, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Application to Taylor Series

$$\tan^{-1} x$$

$$= x - \frac{x^3}{3} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$$

$$(-1 < x \leq 1)$$

Proof (1)

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{x^2+1}$$

$$\int_0^x \frac{1}{t^2+1}\,dt = \tan^{-1} x$$

Proof (2)

$$\frac{1}{1+t^2} = \frac{1}{1-(-t^2)}$$

$$= 1 + (-t^2) + \cdots + (-1)^n t^{2n} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n t^{2n}$$ **(Geometric Series)**

$$(-1 < t < 1)$$

Proof (3)

$$\tan^{-1} x$$

$$= \int_0^x \frac{1}{1+t^2} dt$$

$$= \sum_{n=0}^{\infty} \int_0^x (-1)^n t^{2n} dt$$

$$= x - \frac{x^3}{3} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$$

$$(-1 < x < 1)$$

Numerical Analysis

Trapezoidal Rule

$$\int_a^b f(x)dx \doteq \frac{b-a}{n}$$

$$\times \frac{1}{2}\left\{ f(x_0) + 2\left(f(x_1) + \cdots + f(x_{n-1}) \right) + f(x_n) \right\}$$

Simpson's Rule

$$\int_a^b f(x)dx \doteq \frac{b-a}{2n}$$

$$\times\frac{1}{3}\left\{ f(x_0)+2\Big(f(x_2)+\cdots+ f(x_{2n-2})\Big)+ f(x_{2n})\right\}$$

$$+\frac{b-a}{2n}\times\frac{1}{3}\left\{4\Big(f(x_1)+\cdots+ f(x_{2n-1})\Big)\right\}$$

Computational Approach

Example

$$\int_0^1 \sqrt{x}\, dx = \frac{2}{3} = 0.66666666666\cdots$$

Numerical Computing
with
BASIC

Trapezoidal Rule

```
REM 台形公式による近似積分
REM 関数 sqr(x) 積分区間 [0,1]
PRINT "台形公式により積分の近似計算をします"
PRINT "何等分しますか？"
INPUT PROMPT "n=": n
LET s=0
LET h = 1/n
FOR k = 0 TO n-1
   LET  x = k*h
   LET  y = (k+1)*h
   LET  s = s + (SQR(x) + SQR(y))*h/2
NEXT k
PRINT s
PRINT 0.6666666666666666
END
```

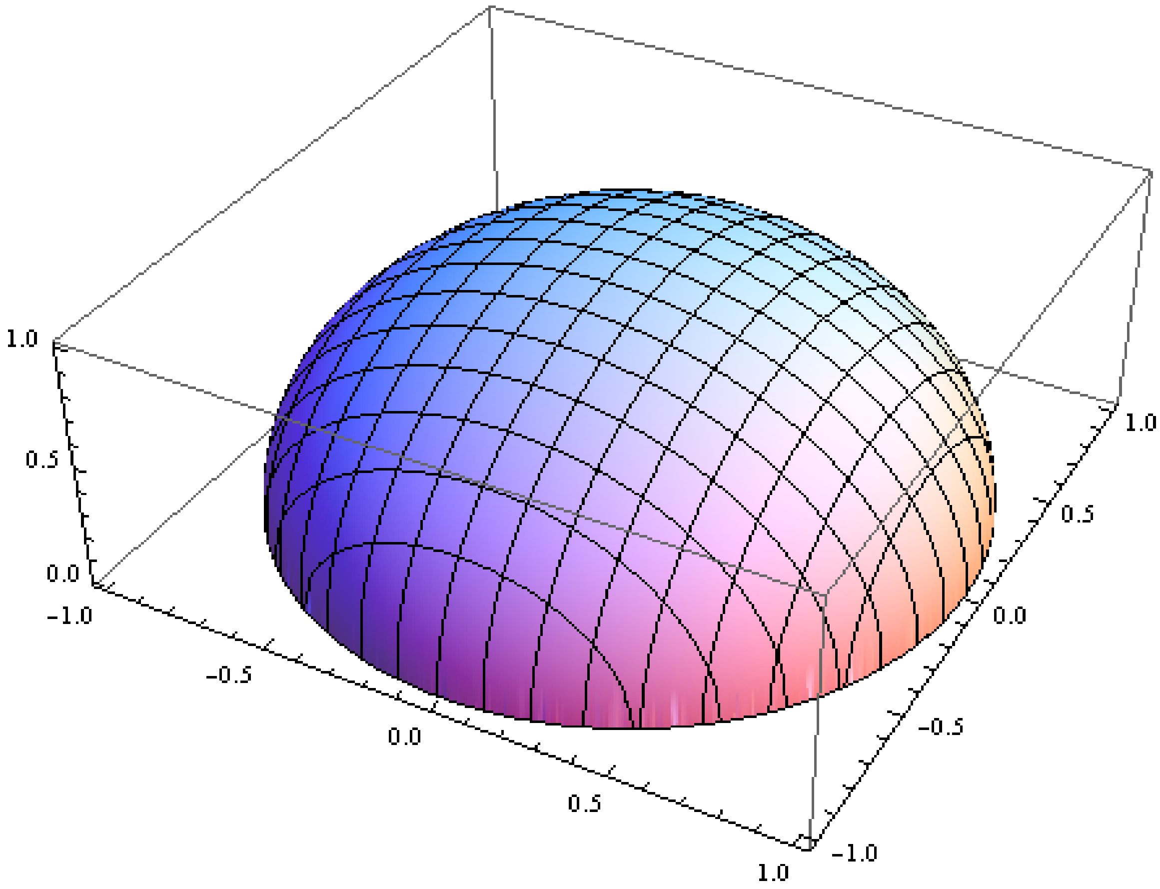
Simpson's Rule

```
REM シンプソン公式による近似積分
REM 関数 sqrt(x) 積分区間 [0,1]
PRINT "シンプソン公式による近似面積"
PRINT "何等分しますか？"
INPUT PROMPT "n=": n
LET h = 1/(2*n)
FOR k = 0 TO n-1
   LET  x = 2*k*h
   LET  m = (2*k+1)*h
   LET  y = (2*k+2)*h
   LET  s = s + (SQR(x)+4*SQR(m)+SQR(y))*(h/3)
NEXT k
PRINT s
PRINT 0.6666666666666666666666
END
```

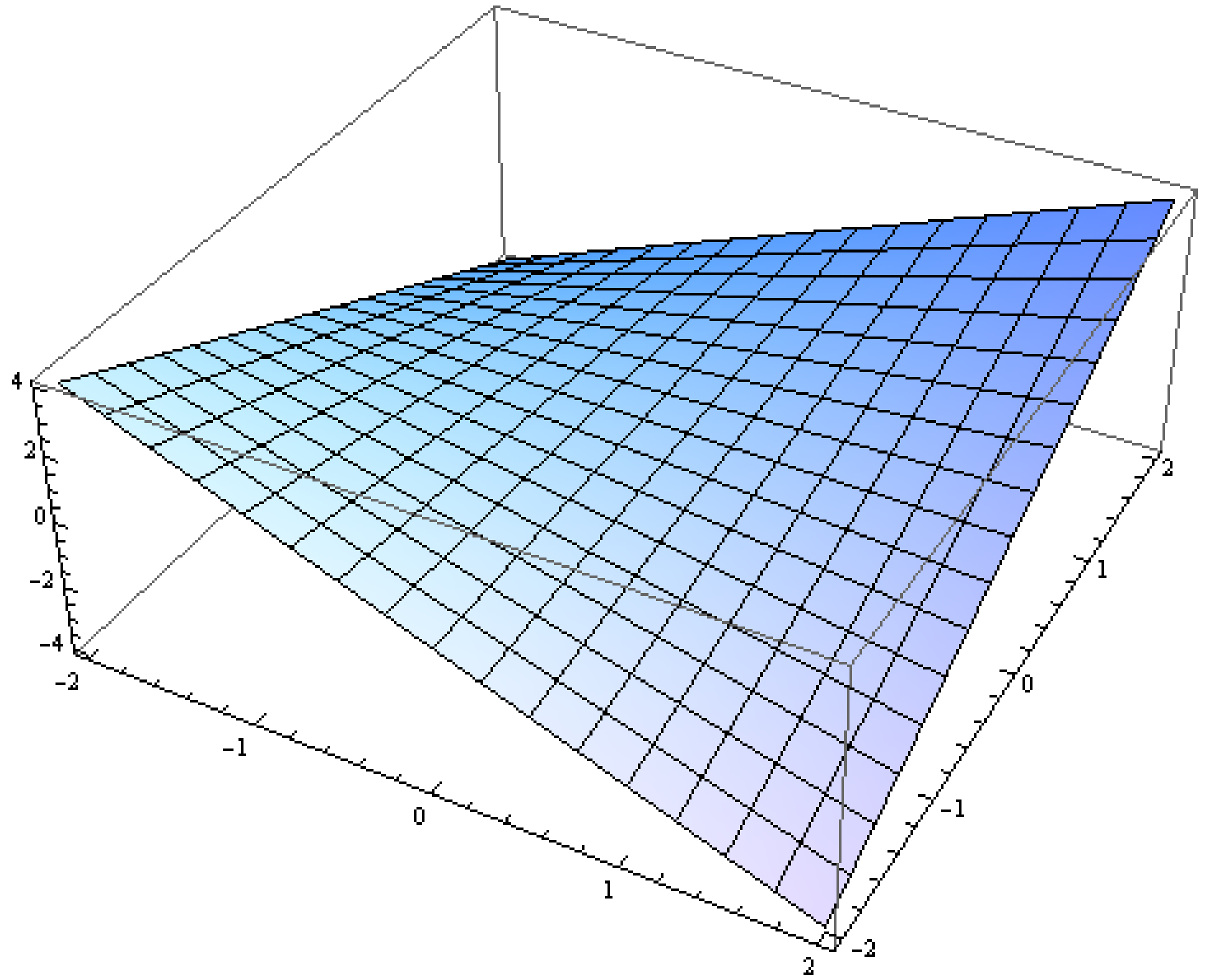
Mesh	Trapezoidal Rule	Simpson's rule
n=10	.660509341706818	.664099589757422
n=100	.666462947103147	.666585482066722
n=1000	.666660134393675	.666664099383542
n=10000	.666666459197103	.666666585482054

Calculus of Two Variables

$$z = \sqrt{1 - x^2 - y^2}$$



$$z = xy$$



Continuity of Functions

Continuity

Let D be a domain in $\mathbf{R}^2$

A function $f(x, y)$ defined in D

is **continuous** at $(a,b) \in D$

$$\Leftrightarrow$$

$\forall \varepsilon > 0, \exists \delta = \delta((a,b),\varepsilon) > 0$ such that

$$|x-a| < \delta,\ |y-b| < \delta \Rightarrow |f(x,y) - f(a,b)| < \varepsilon$$

Differentiation

Partial Differentiability of Functions

Partial Differentiability

Let D be a domain in $\mathbf{R}^2$

A function $f(x,y)$ defined in D

is **partially differentiable** at $(a,b)\in D$

$$\Leftrightarrow$$

$$\exists \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}=\frac{\partial f}{\partial x}(a,b)$$

$$\exists \lim_{k\to 0}\frac{f(a,b+k)-f(a,b)}{k}=\frac{\partial f}{\partial y}(a,b)$$

Total Differentiability of Functions

Total Differentiability

Let D be a domain in $\mathbf{R}^2$

A function $f(x, y)$ defined in D

is **totally differentiable** at $(a, b) \in D$

$$\Leftrightarrow$$

$\exists \alpha \in \mathbf{R}, \exists \beta \in \mathbf{R}$ such that

$$f(a+h, b+k)$$

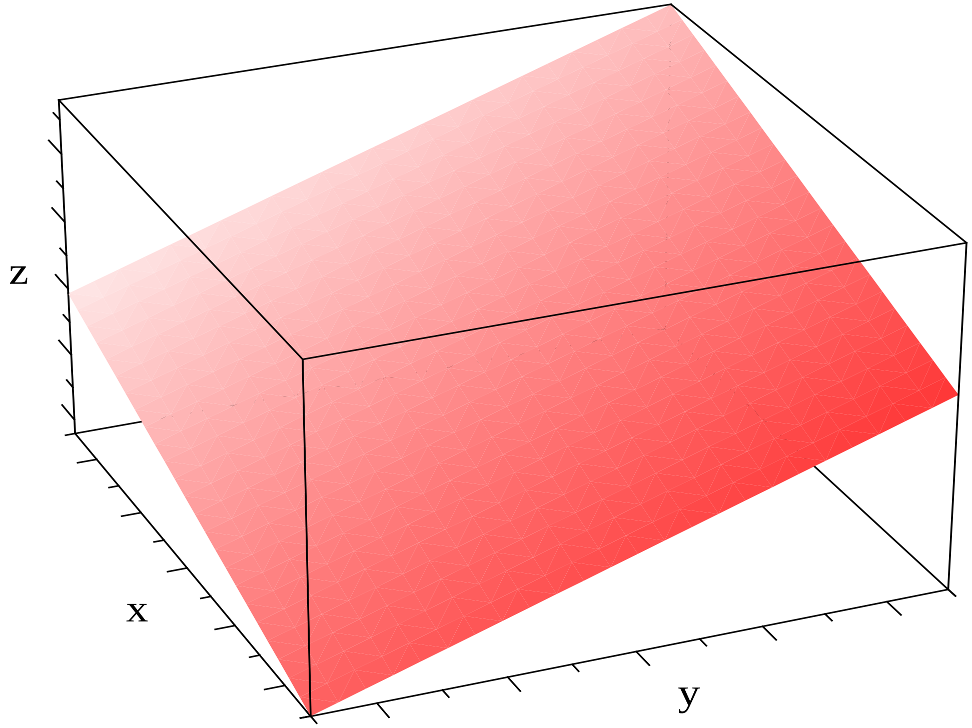
$$= f(a,b) + \alpha h + \beta k + o\left(\sqrt{h^2 + k^2}\right)$$

Geometric Meaning
of
Total Differentiability

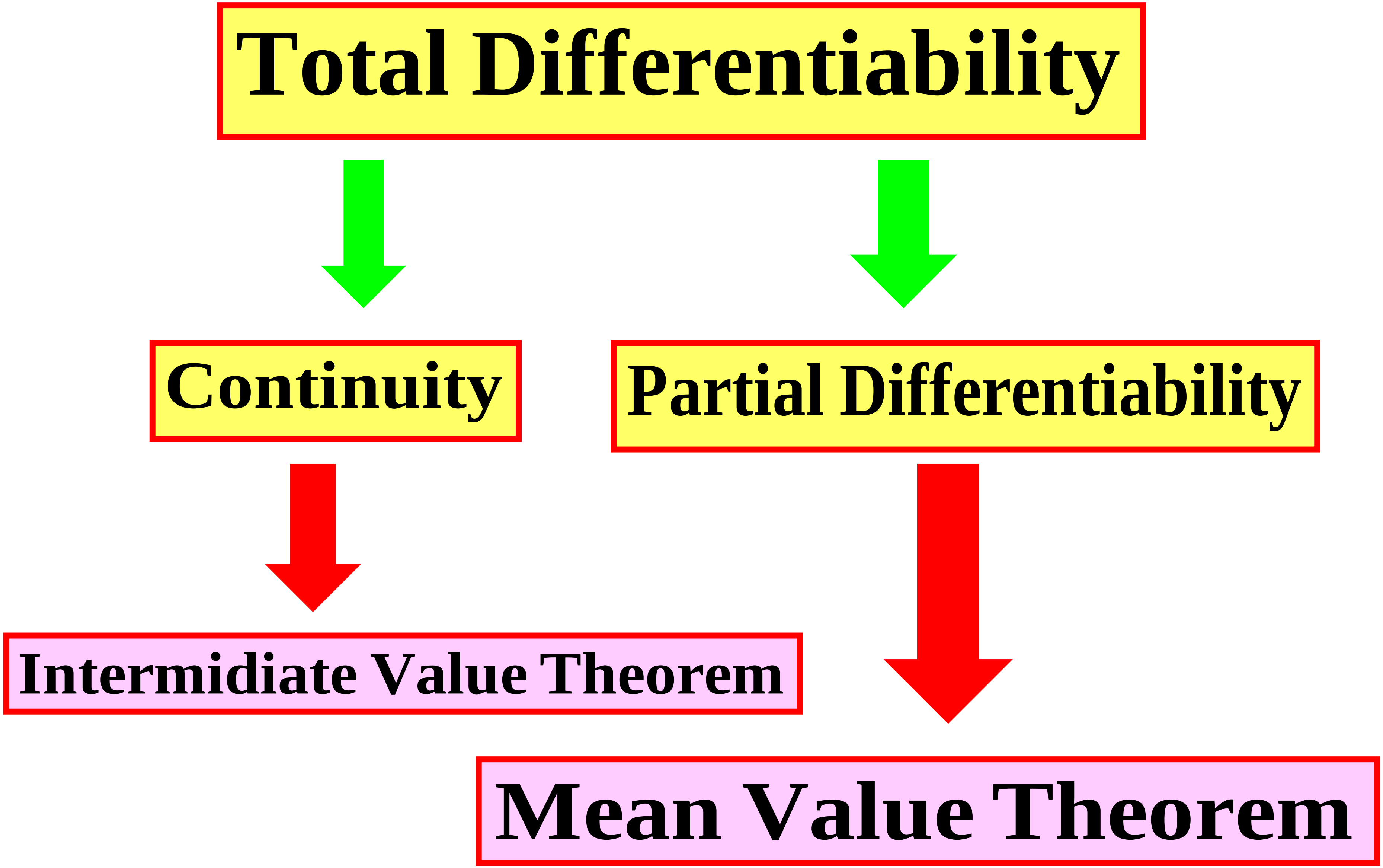
Tangent Plane

$$\begin{pmatrix} x \\ y \\ f(a,b) + \alpha(x-a) + \beta(y-b) \end{pmatrix} = \begin{pmatrix} a \\ b \\ f(a,b) \end{pmatrix} + (x-a)\begin{pmatrix} 1 \\ 0 \\ \alpha \end{pmatrix} + (y-b)\begin{pmatrix} 0 \\ 1 \\ \beta \end{pmatrix}$$

Equation of a Plane



$$-\alpha x - \beta y + z = d$$

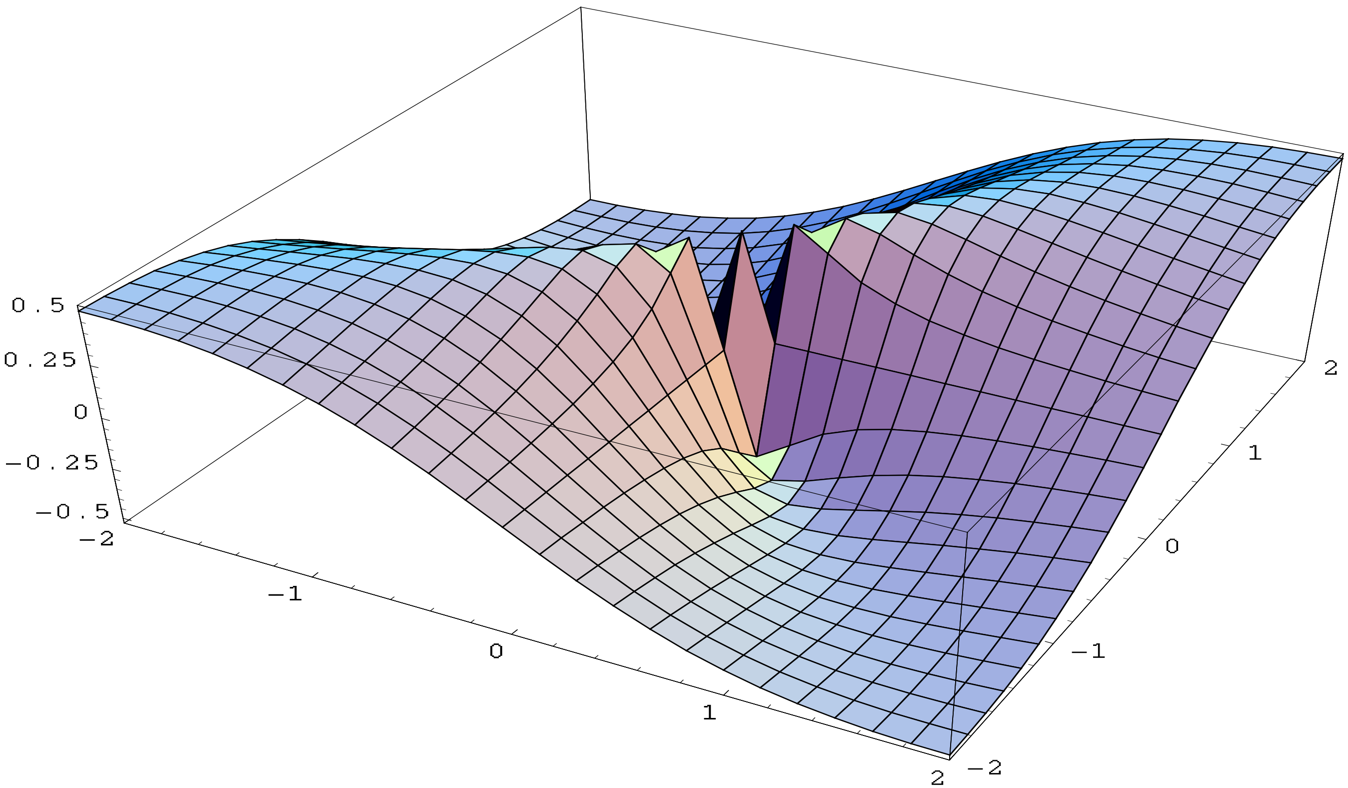

Total Differentiability
Continuity
Partial Differentiability
Intermidiate Value Theorem
Mean Value Theorem

Example of Functions (Surfaces)

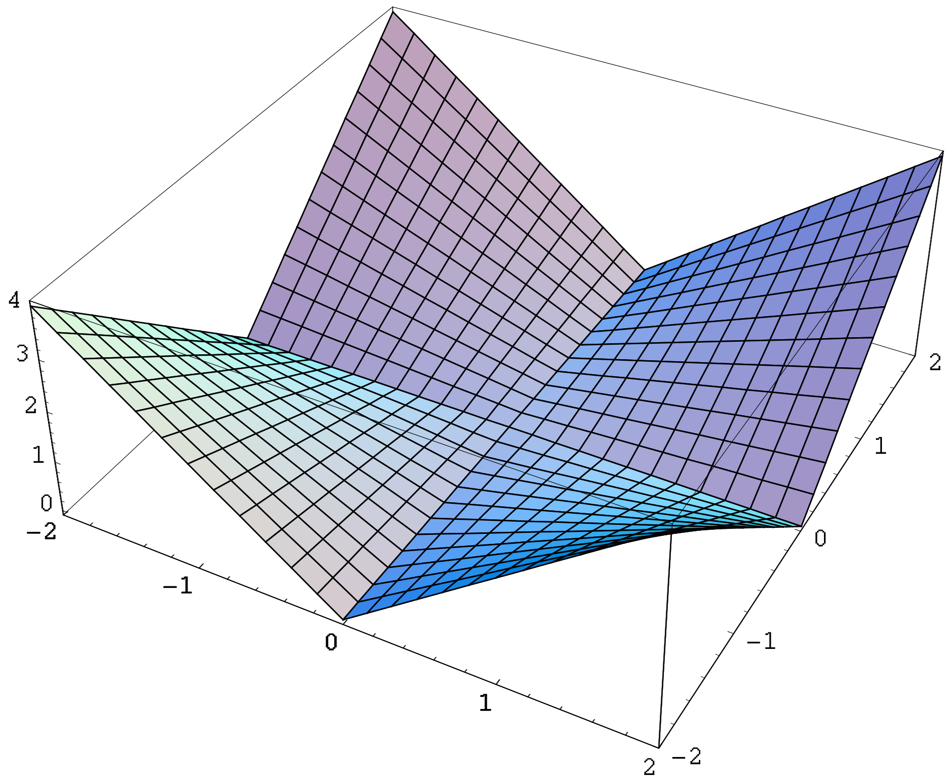
Function	Continuity	Partial Differentiability	Total Differentiability
$\frac{xy}{x^2+y^2}$	×	○	×
$\lvert xy \rvert$	○	×	×
$xy\,e^{-\frac{x^2+y^2}{2}}$	○	○	○

Numerical Computing with MATHEMATICA

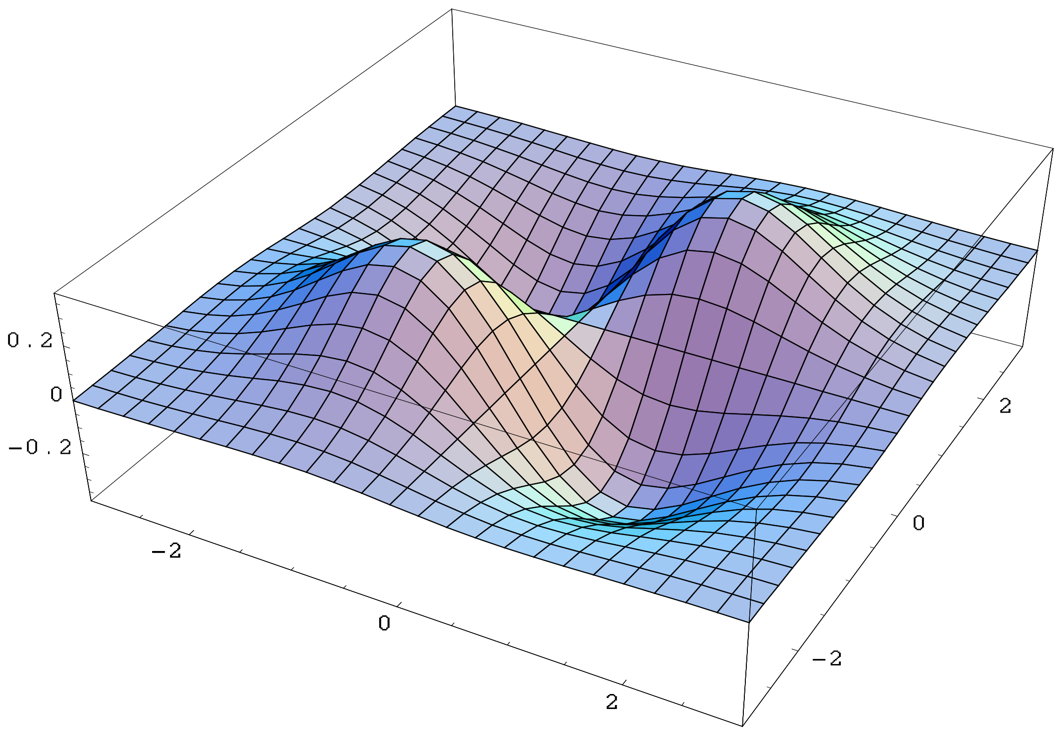
$$z = \frac{xy}{x^2 + y^2}$$



$$z = |xy|$$



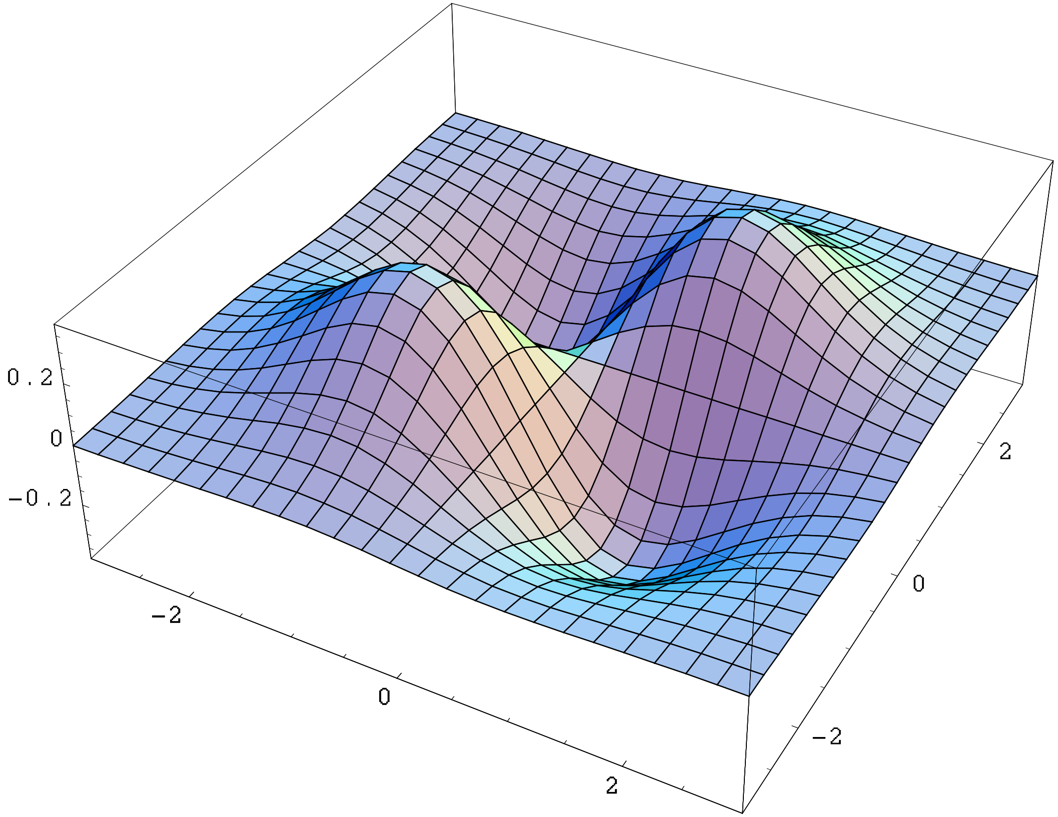
$$z = xy\,e^{-\frac{x^2+y^2}{2}}$$



Extremes of Functions

Point (x, y)	$(0,0)$	$(1,1)$	$(-1,-1)$	$(1,-1)$	$(-1,1)$
$xy\,e^{-\frac{x^2+y^2}{2}}$	0	$\frac{1}{e}$	$\frac{1}{e}$	$-\frac{1}{e}$	$-\frac{1}{e}$
Behavior	**Saddle Point**	**Maximal Value**	**Maximal Value**	**Minimal Value**	**Minimal Value**

$$z = xy\, e^{-\frac{x^2+y^2}{2}}$$



Canonical Forms
of
Polynomials of second-order

Mean Value Theorem

$$\Downarrow$$

Taylor's Theorem

$$\Downarrow$$

Polynomial Approximation

Polynomial

$$z = f(x, y)$$
$$= ax^2 + 2bxy + cy^2$$

Matrix Form

$$z = f(x, y)$$

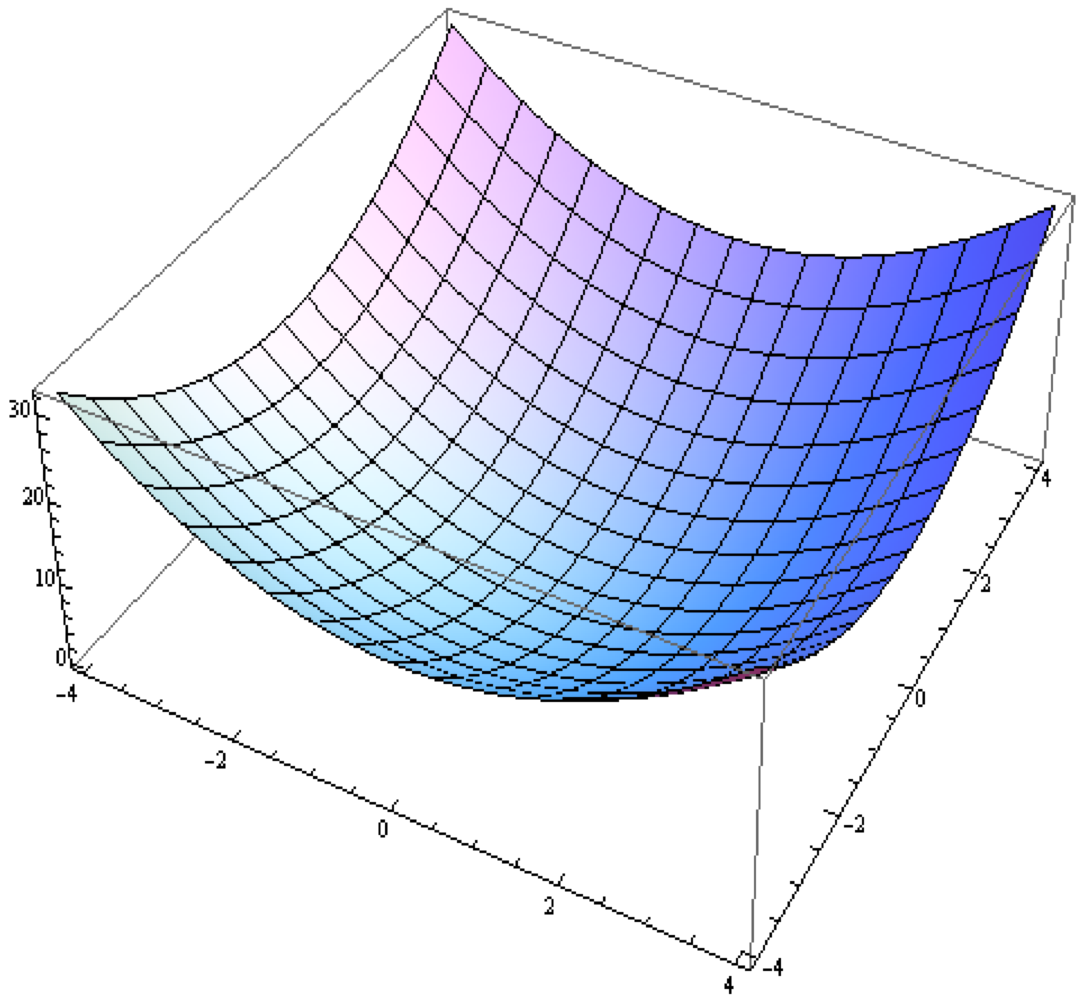
$$= ax^2 + 2bxy + cy^2$$

$$\Longrightarrow$$

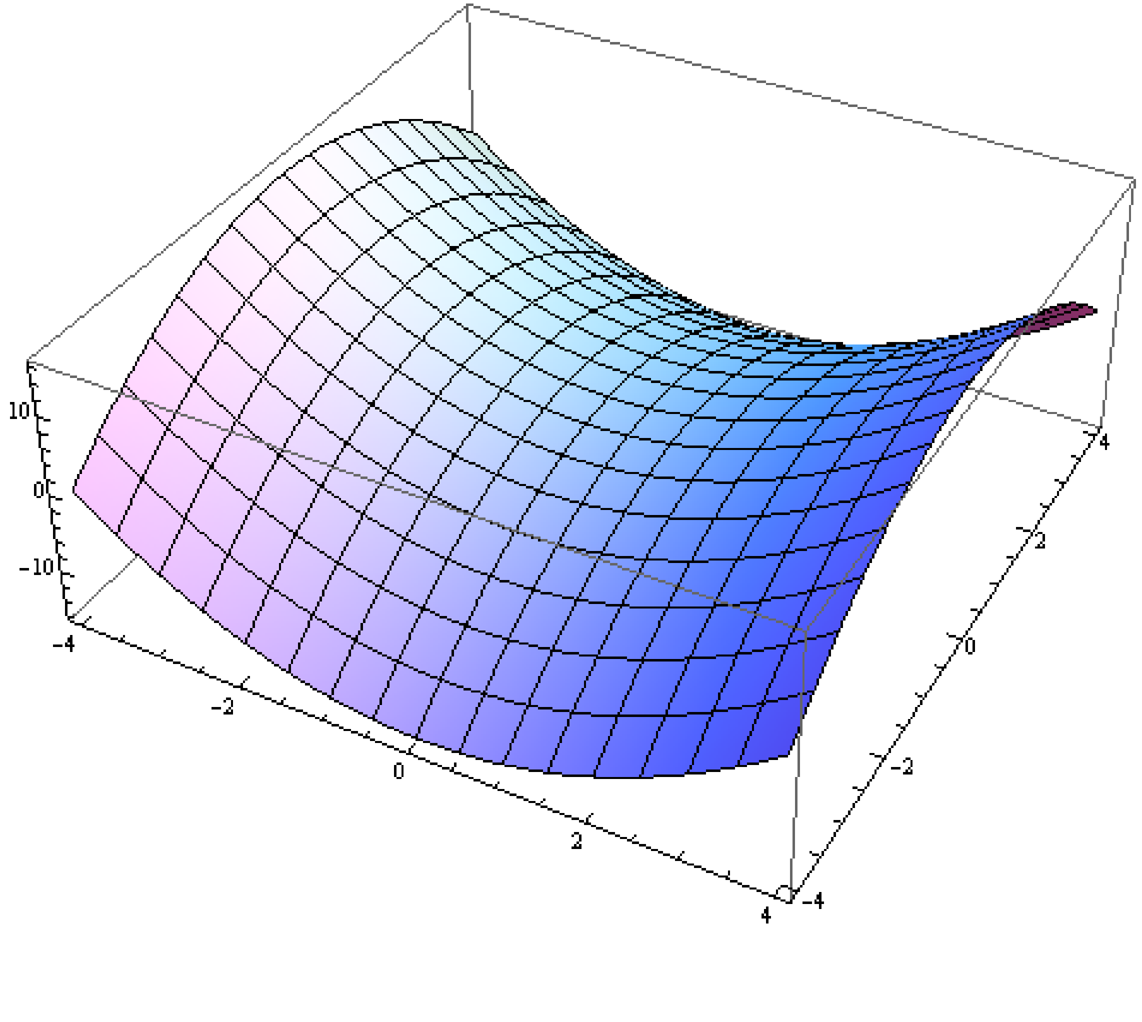
$$ax^2 + 2bxy + cy^2$$

$$= \left\langle \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle$$

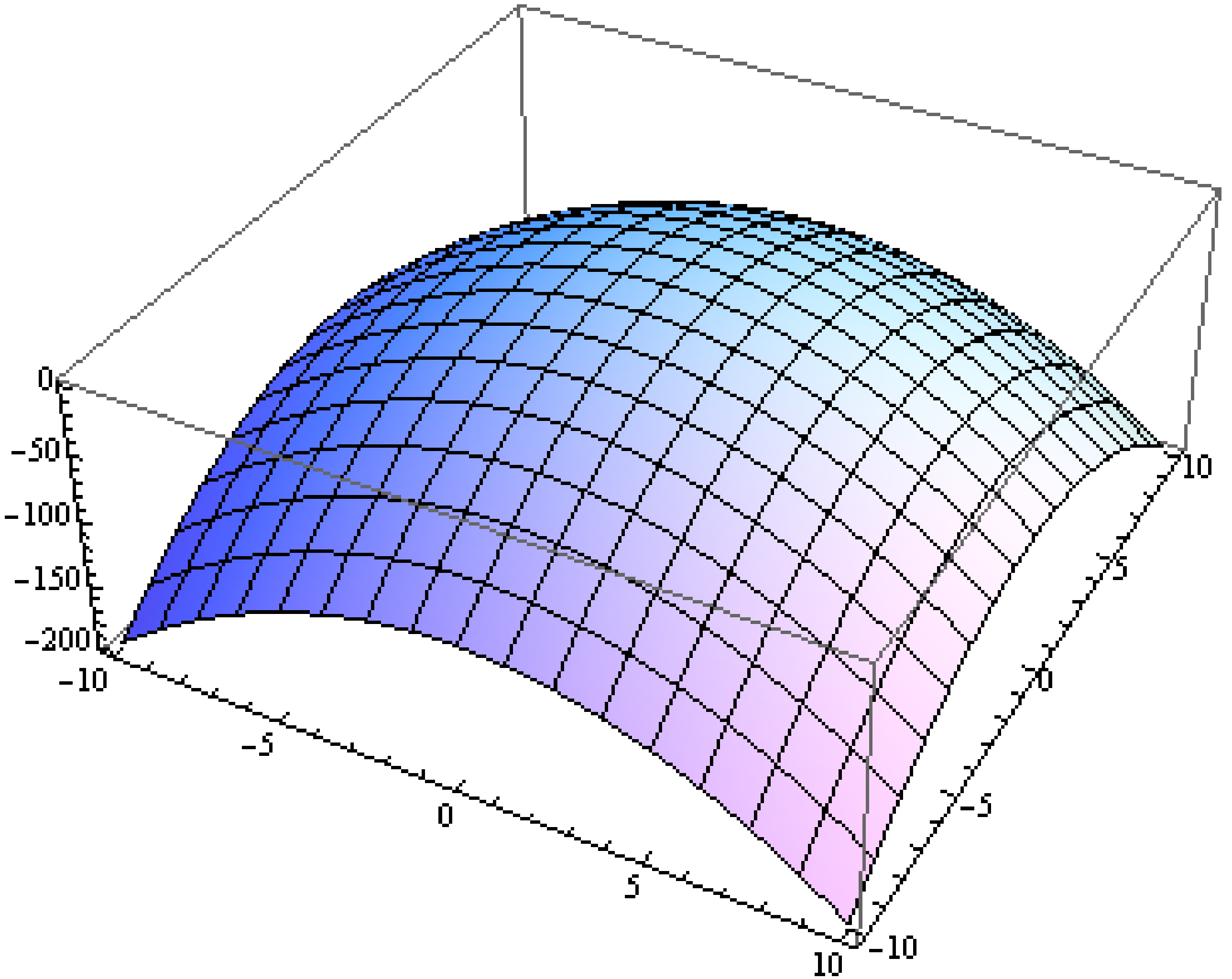
$z = x^2 + y^2$ （minimal point）



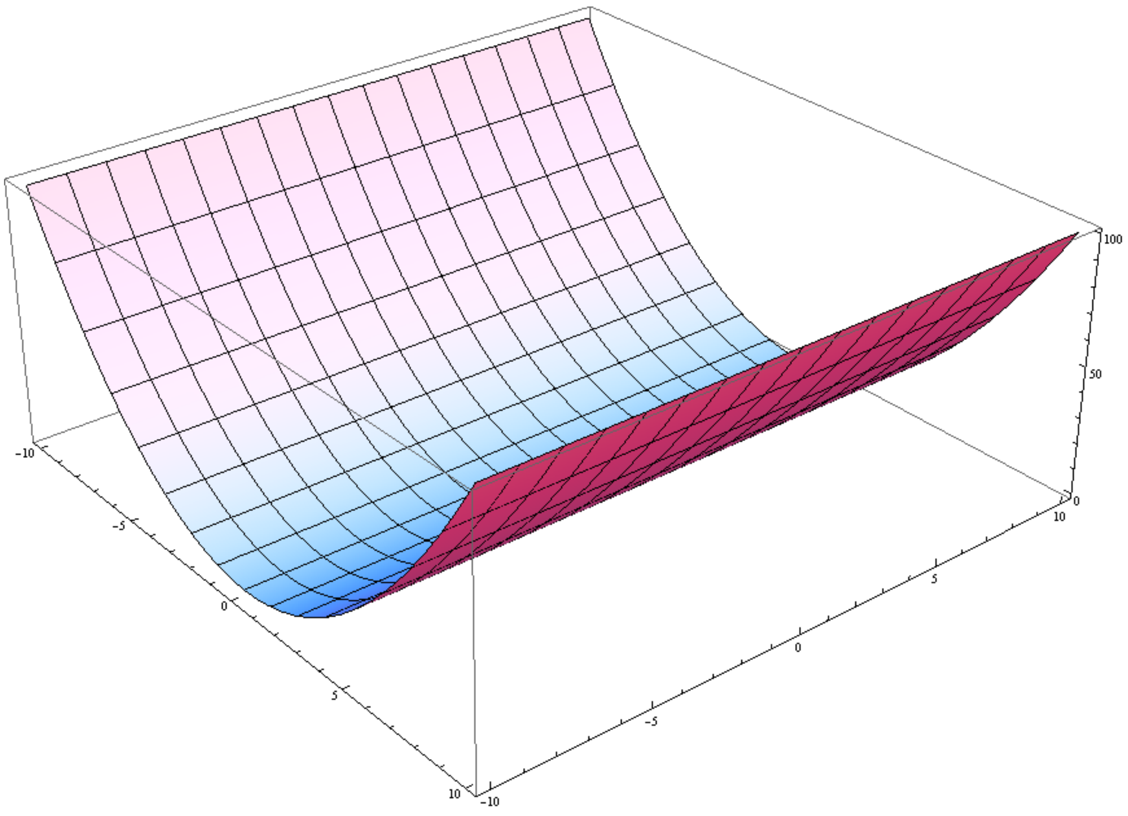
$z = x^2 - y^2$ **（saddle point）**



$z = -x^2 - y^2$ **(maximal point)**



$z = x^2$ （degenerate point）

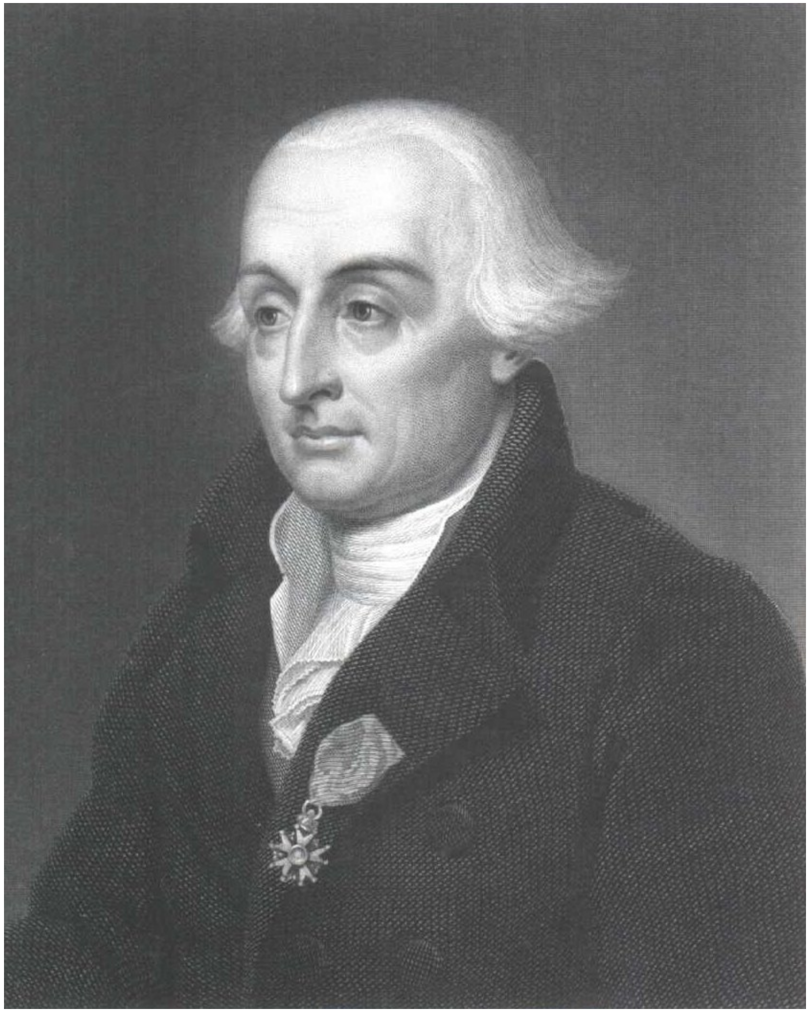


Lagrange's Method

of

Undetermined Multipliers

Joseph Louis Lagrange (1736-1813)



Extreme Problem (1)

$f(x,y),\ \varphi(x,y): C^1$ **functions**

Under the condition

$\varphi(x,y)=0$

find the extremes of the function

$f(x,y)$

Idea (1)

Introduce the function

$$F(x, y, \lambda) = f(x, y) - \lambda\varphi(x, y)$$

λ : **parameter**

Idea (2)

Assume that $f(x, y)$ **takes an extreme at** (a, b)

Then either (1) or (2) holds true :

(1) $dF(x, y, \lambda) = 0$:

$$\begin{cases} f_x(a,b) - \lambda\varphi_x(a,b) = 0 \\ f_y(a,b) - \lambda\varphi_y(a,b) = 0 \\ \varphi(a,b) = 0 \end{cases}$$

(2) $d\varphi(x, y) = 0$:

$$\varphi_x(a,b) = \varphi_y(a,b) = 0$$

Example 1

Under the condition

$$p_1 x + p_2 y = C$$

find the maximum of the function

$$z = a x^s y^t, \quad s + t = 1, s > 0, t > 0$$

Example 2

Under the condition

$$4x + 5y - 6 = 0, \quad x > 0, \; y > 0$$

find the maximum of the function

$$z = x^{1/3} y^{2/3}$$

Extremes Problem (2)

$f(x, y), \;\; g(x, y) : C^1$ **functions**

Under the condition

$g(x, y) \geq 0$

find the extremes of the function

$f(x, y)$

Idea

Introduce the function

$$F(x,y,z,\lambda) = f(x,y) - \lambda\left(g(x,y) - z\right)$$

Consider the extremes of the function

$$F(x,y,z,\lambda)$$

in the half space

$$\mathbf{R}_+^3 = \left\{(x,y,z) \in \mathbf{R}^3 : z \geq 0\right\}$$

Example

$$\Omega = \left\{ (x, y) \in \mathbf{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \right\}$$

Find the maximum and minimum of the function $x + y$ in Ω.

Extreme Problem (3)

$f(x, y), \; g(x, y), h(x, y) : C^1$ **functions**

Under the two conditions

$g(x, y) \geq 0, h(x, y) \geq 0$

find the extremes of the function

$f(x, y)$

Idea

Introduce the function

$$F(x, y, z, w, \lambda, \mu)$$

$$= f(x, y) - \lambda\left(g(x, y) - z\right) - \mu\left(h(x, y) - w\right)$$

Find the extremes of the function

$$F(x, y, z, w, \lambda, \mu)$$

in the domain

$$\left\{(x, y, z, w) \in \mathbf{R}^4 : z \geq 0, w \geq 0\right\}$$

Example

$$\Omega = \left\{ (x, y) \in \mathbf{R}^2 : y \geq 0,\ y \leq 1 - x^2 \right\}$$

Find the maximum and minimum of the function $2x + y$ in Ω.

Differential Equations

Bird's-Eye View

Theme	Mathematics	Mechanics
Differential Equation	Second-order Differential Equation	Newtonian Equation of Motion

List of Mathematicians

List of Mathematicians

- Isaac Newton (1642-1727) England
- Leonhard Euler (1707-1783) Switzerland
- Jean-Baptiste Fourier (1768-1830) France
- Joseph Louis Lagrange (1736-1813) Italy, France
- Augustin Louis Cauchy (1789-1857) France
- Thomas Robert Malthus (1766-1834) England
- Pierre Francois Verhulst (1804-1849) Belgium

Method of Quadrature

Linear Case

First-Order Case

$$\frac{dx}{dt} + p(t)x = q(t)$$

Variables Separable Form

Homogeneous Case

$$\frac{dx}{dt} + p(t)x = 0$$

General Solution

$$\frac{dx}{dt} + p(t)x = 0$$

$$\Longrightarrow$$

$$x(t) = Ce^{-\int_{t_0}^{t} p(s)ds}$$

C : **Constant**

Example 1

$$\frac{dx}{dt} - ax = 0 \quad (a \in \mathbf{R})$$

$$\Rightarrow$$

$$x(t) = Ce^{at}$$

Structure Theorem (1)

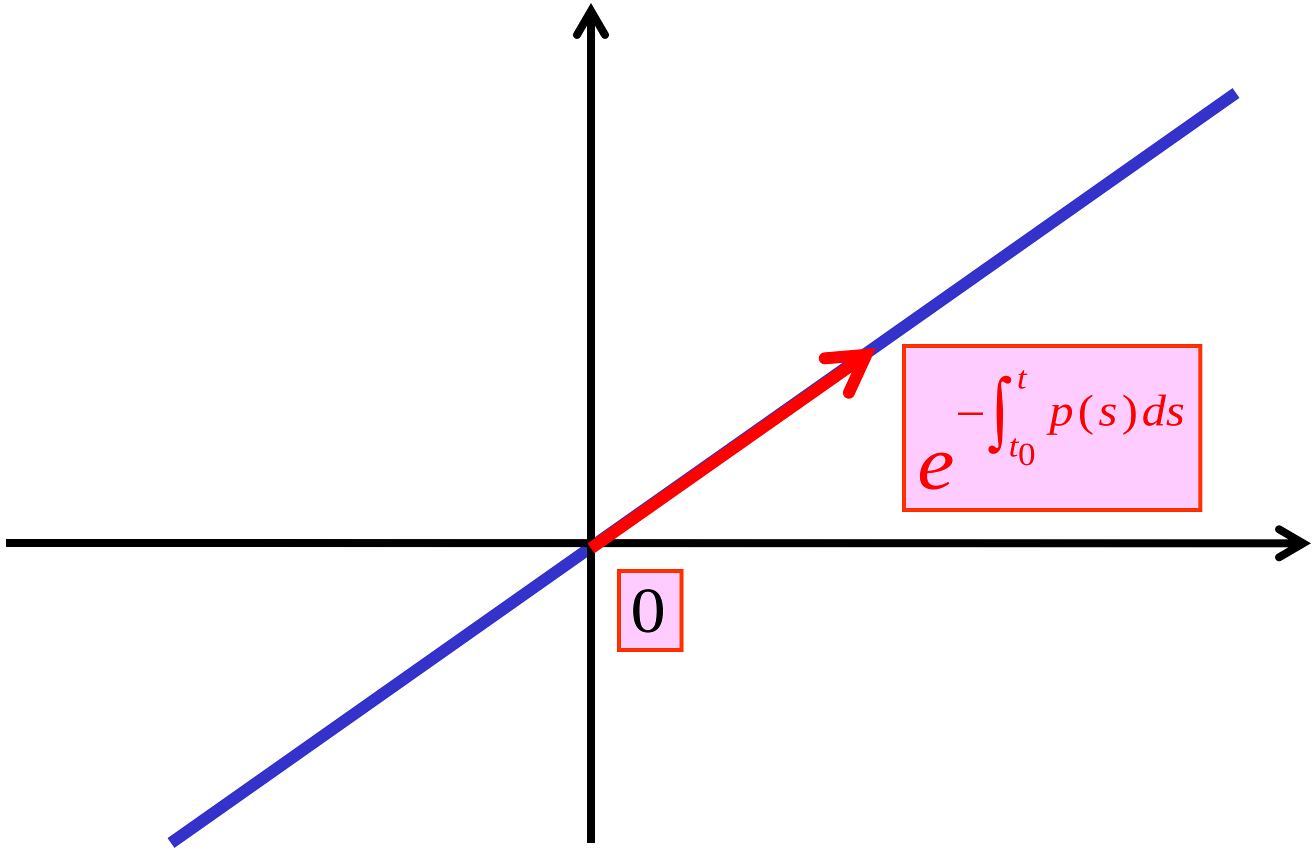
The solution space of the homogeneous equation

$$\frac{dx}{dt} + p(t)x = 0$$

forms a one dimensional vector space spanned by a solution

$$e^{-\int_{t_0}^{t} p(s)ds}$$

Image (1)



Proof (1)

$$\frac{dx}{dt} + p(t)x = 0$$

$$\Rightarrow$$

$$\frac{dx}{x} = -p(t)dt$$

Proof (2)

$$\frac{dx}{x} = -p(t)dt$$

$$\Longrightarrow$$

$$\int_{x_0}^{x} \frac{dy}{y} = -\int_{t_0}^{t} p(s)ds$$

$$\Longrightarrow$$

$$\log_e x = -\int_{t_0}^{t} p(s)ds + C_1$$

Proof (3)

$$\log_e x(t) = -\int_{t_0}^{t} p(s)ds + C_1$$

$$\Longrightarrow$$

$$x(t) = Ce^{-\int_{t_0}^{t} p(s)ds}$$

Non-Homogeneous Case

$$\frac{dx}{dt} + p(t)x = q(t)$$

General Solution

$$\frac{dx}{dt} + p(t)x = q(t)$$

$$\Longrightarrow$$

$$x(t) = \int_{t_0}^{t} q(s) \cdot e^{\int_{t_0}^{s} p(\tau)d\tau} ds \bullet e^{-\int_{t_0}^{t} p(s)ds}$$

$$+ Ce^{-\int_{t_0}^{t} p(s)ds}$$

C : **Constant**

Example 2

$$\frac{dx}{dt} - 2x = e^{3t}$$

$$\Rightarrow$$

$$x(t) = e^{3t} + Ce^{2t}$$

Structure Theorem (2)

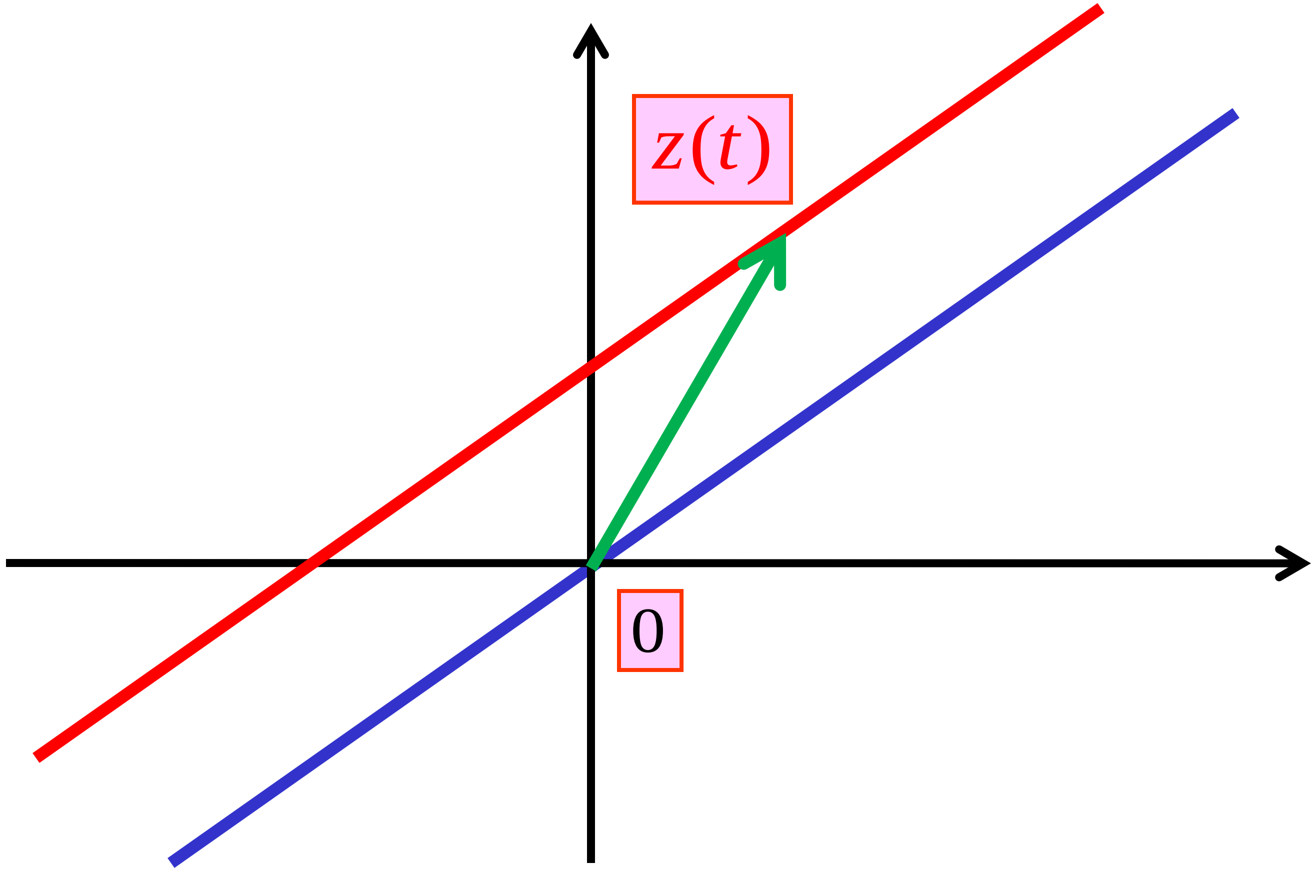
The solution space of the non-homogeneous equation

$$\frac{dx}{dt} + p(t)x = q(t)$$

forms a one dimensional affine space translated by a particular solution

$$z(t) = \int_{t_0}^{t} q(s) \cdot e^{\int_t^s p(\tau)d\tau} ds$$

Image (2)



Method

of

the Variation of Constants

Proof (1)

$$x(t) = C(t)e^{-\int_{t_0}^{t} p(s)ds}$$

$$\Rightarrow$$

$$\frac{dx}{dt} + p(t)x = \frac{dC}{dt}e^{-\int_{t_0}^{t} p(s)ds}$$

Proof (2)

$$\frac{dC}{dt} e^{-\int_{t_0}^{t} p(s)ds}$$

$$= \frac{dx}{dt} + p(t)x = q(t)$$

$$\Rightarrow$$

$$\boxed{\frac{dC}{dt} = q(t)e^{\int_{t_0}^{t} p(s)ds}}$$

Proof (3)

$$C(t) = \int_{t_0}^{t} q(s) \cdot e^{\int_{t_0}^{s} p(\tau)d\tau} ds + C$$

$$\Rightarrow$$

$$x(t) = C(t)\, e^{-\int_{t_0}^{t} p(s)ds}$$

$$= \left(\int_{t_0}^{t} q(s) \cdot e^{\int_{t_0}^{s} p(\tau)d\tau} ds + C \right) e^{-\int_{t_0}^{t} p(s)ds}$$

Non-Linear Case

Example

$$\begin{cases} \dfrac{dx}{dt} = r\left(1 - \dfrac{1}{K}x(t)\right)x(t) \\ x(0) = x_0 \end{cases}$$

Logistic Equation

$$\begin{cases} \dfrac{dx}{dt} = a\left(A - x\right)x \\ x(0) = C \end{cases}$$

General Solution

$$x(t) = \frac{CA}{C + (A - C)e^{-aAt}}$$

C : **Constant**

Variables Separable Form

Proof (1)

$$\frac{dx}{dt} = a(A - x)x$$

$$\Longrightarrow$$

$$adt = \frac{dx}{(A - x)x}$$

$$= \frac{1}{A}\left(\frac{dx}{x} + \frac{dx}{A - x}\right)$$

Proof (2)

$$\frac{1}{A}\left(\int \frac{dx}{x} - \int \frac{dx}{x-A}\right) = \int a\,dt$$

$$\Rightarrow$$

$$\frac{1}{A}\log_e \frac{x}{x-A} = at + C_1$$

Proof (3)

$$\frac{1}{A}\log_e \frac{x(t)}{x(t)-A} = at + C_1$$

$$\Rightarrow$$

$$x(t) = \frac{CA}{C+(A-C)e^{-aAt}}$$

$$x(0) = C$$

General Case

Initial-Value Problem

$$\frac{dx}{dt} = f(t, x(t))$$

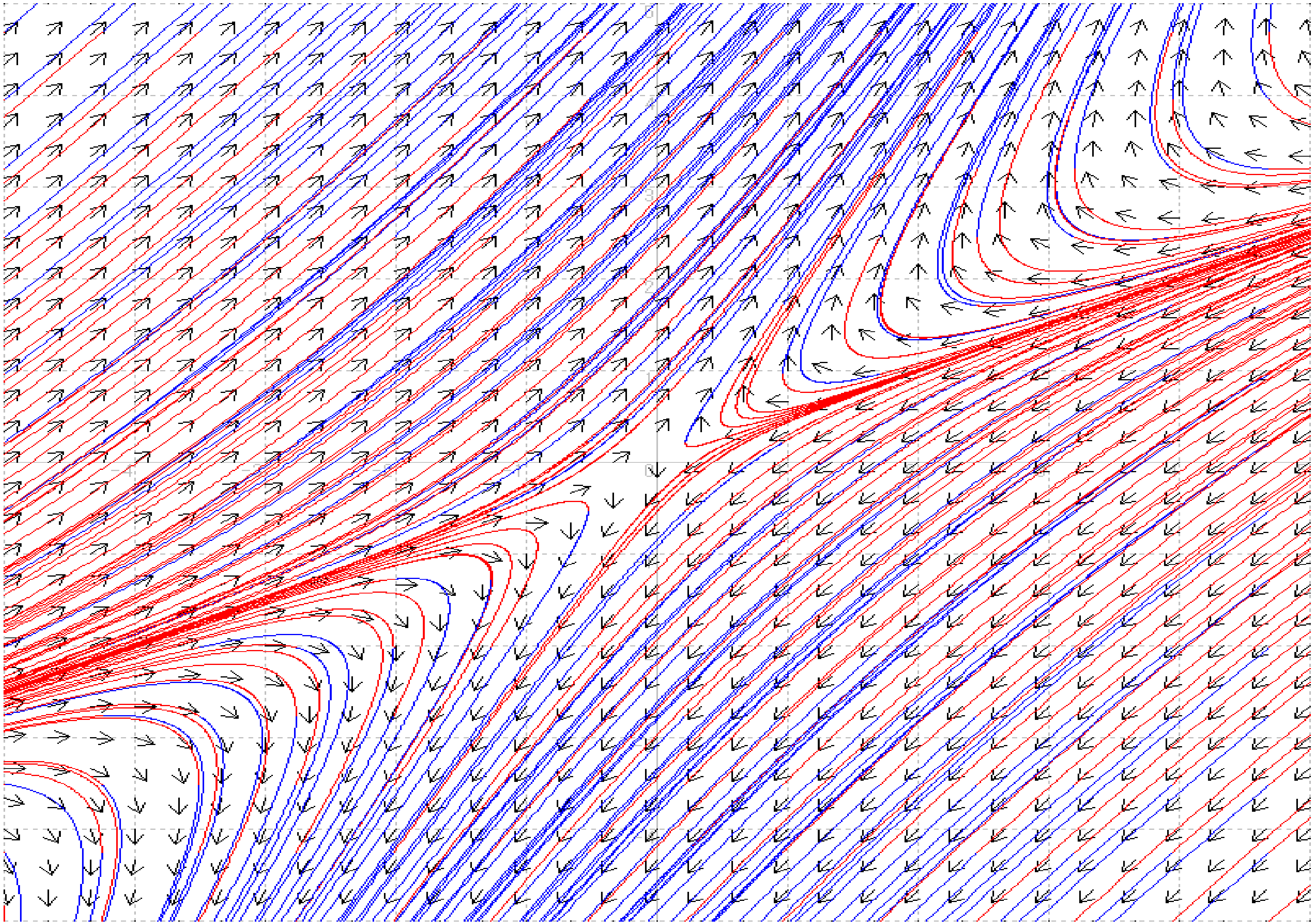
$$x(0) = x_0$$ **（Initial Condition）**

Examples of $f(t,x)$

Numerical Computing
with
BASIC

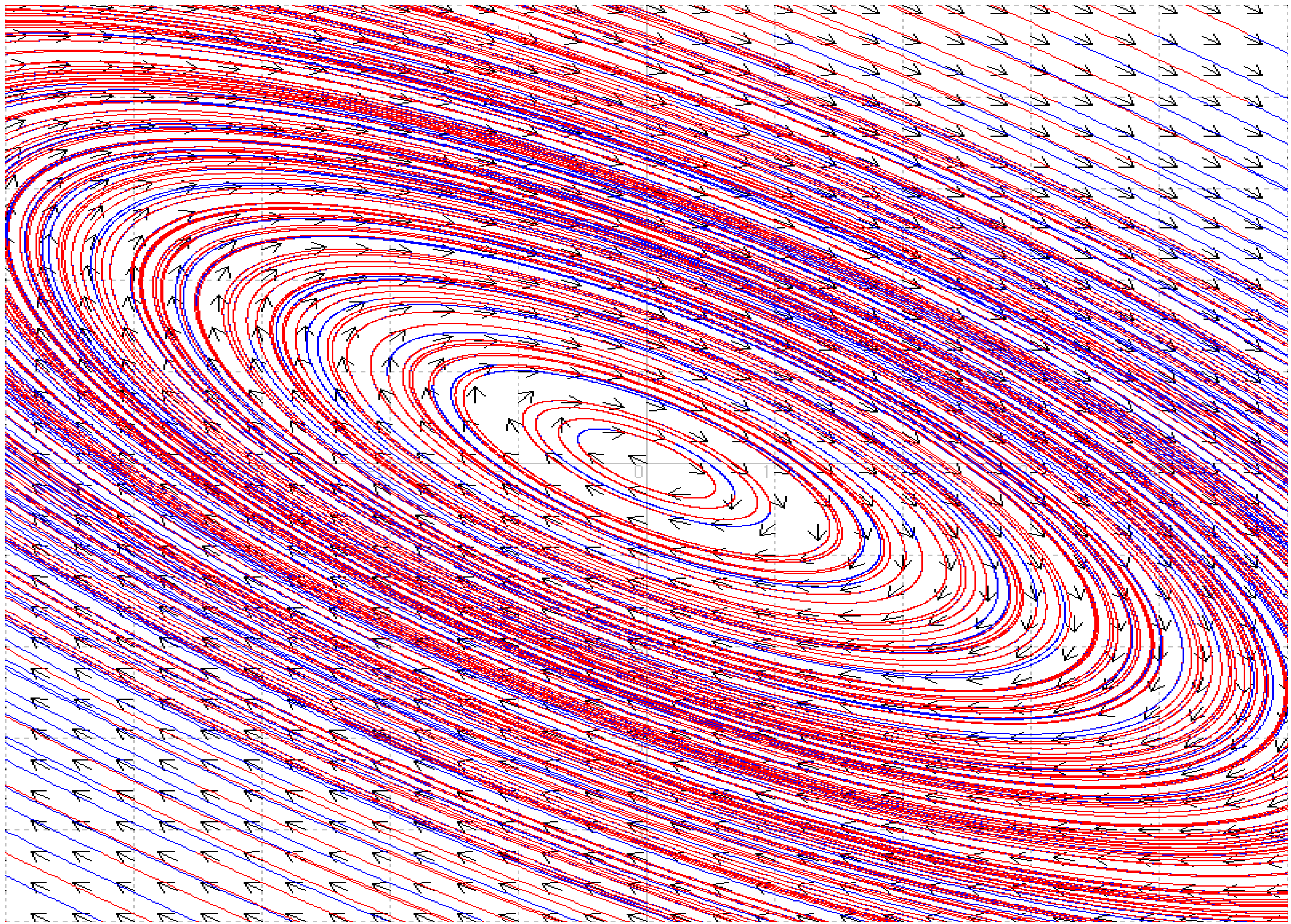
Example (1)

$$\begin{cases} \dfrac{dx_1}{dt} = -2x_1 + 2x_2 \\ \dfrac{dx_2}{dt} = -2x_1 + 3x_2 \end{cases}$$



Example (2)

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 + 2x_2 \\ \dfrac{dx_2}{dt} = -x_1 - x_2 \end{cases}$$



Picard's Method
of
Successive Approximation

Reduction to an Integral Equation

$$\begin{cases} \dfrac{dx}{dt} = f(t, x(t)) \\ x(0) = x_0 \end{cases}$$

Differentiation ↑ ↓ **Integration**

$$x(t) = x_0 + \int_0^t f(s, x(s))\,ds$$

Initial-Value Problem

$$\frac{dx}{dt} = f(t, x(t))$$

$$x(0) = x_0$$ **（Initial Condition）**

Algorithm

for

Successive Approximation

Successive Approximation (1)

$$\begin{cases} \dfrac{dx_n}{dt} = f(t, x_{n-1}(t)) \\ x_n(0) = x_0 \end{cases}$$

Successive Approximation (2)

$$x_1(t) = x_0 + \int_0^t f(s, x_0)ds$$

$$x_2(t) = x_0 + \int_0^t f(s, x_1(s))ds$$

$$\vdots$$

$$x_n(t) = x_0 + \int_0^t f(s, x_{n-1}(s))ds$$

(n-th **Approximation**)

Successive Approximation (3)

$x_n(t) \to \exists x(t)$ （**Uniform Convergence**）

$\Rightarrow$

$$x_n(t) = x_0 + \int_0^t f(s, x_{n-1}(s))ds$$

$\Rightarrow \quad (n \to \infty)$

$$x(t) = x_0 + \int_0^t f(s, x(s))ds$$

Example (1)

$$\begin{cases} \dfrac{dx}{dt} = ax \\ x(0) = 1 \end{cases}$$

(Solution : $x(t) = e^{at}$)

Successive Approximation (1)

$$x_1(t) = 1 + a\int_0^t 1\,ds = 1 + at$$

$$x_2(t) = 1 + a\int_0^t x_1(s)\,ds = 1 + at + \frac{(at)^2}{2!}$$

.

.

$$x_n(t) = 1 + a\int_0^t x_{n-1}(s)\,ds$$

$$= 1 + at + \cdots + \frac{(at)^n}{n!}$$

Successive Approximation (2)

$$x_n(t) = 1 + at + \cdots + \frac{(at)^n}{n!}$$

$$\to e^{at} \quad (n \to \infty)$$

Example (2)

$$\begin{cases} \dfrac{dx}{dt} = tx + \sqrt{t} & t > 0 \\ x(0) = 1 \end{cases}$$

Successive Approximation (1)

$$x_1(t) = 1 + \int_0^t (s + \sqrt{s})ds = 1 + \frac{t^2}{2} + \frac{2}{3}t^{\frac{3}{2}}$$

$$x_2(t) = 1 + \int_0^t (sx_1(s) + \sqrt{s})ds$$

$$= 1 + \frac{t^2}{2} + \frac{2}{4\cdot 2}t^4 + \frac{2}{3}t^{\frac{3}{2}} + \frac{2\cdot 2}{7\cdot 3}t^{\frac{7}{2}}$$

Successive Approximation (2)

$$x_n(t) = 1 + \int_0^t \left(s x_{n-1}(s) + \sqrt{s}\right) ds$$

$$= 1 + \sum_{k=0}^{n-1} \frac{1}{2^k (1+k)!} t^{2(k+1)}$$

$$+ \sum_{k=0}^{n-1} \frac{2^{k+1}}{3(3+4)\cdots(3+4k)} t^{\frac{3}{2}+2k}$$

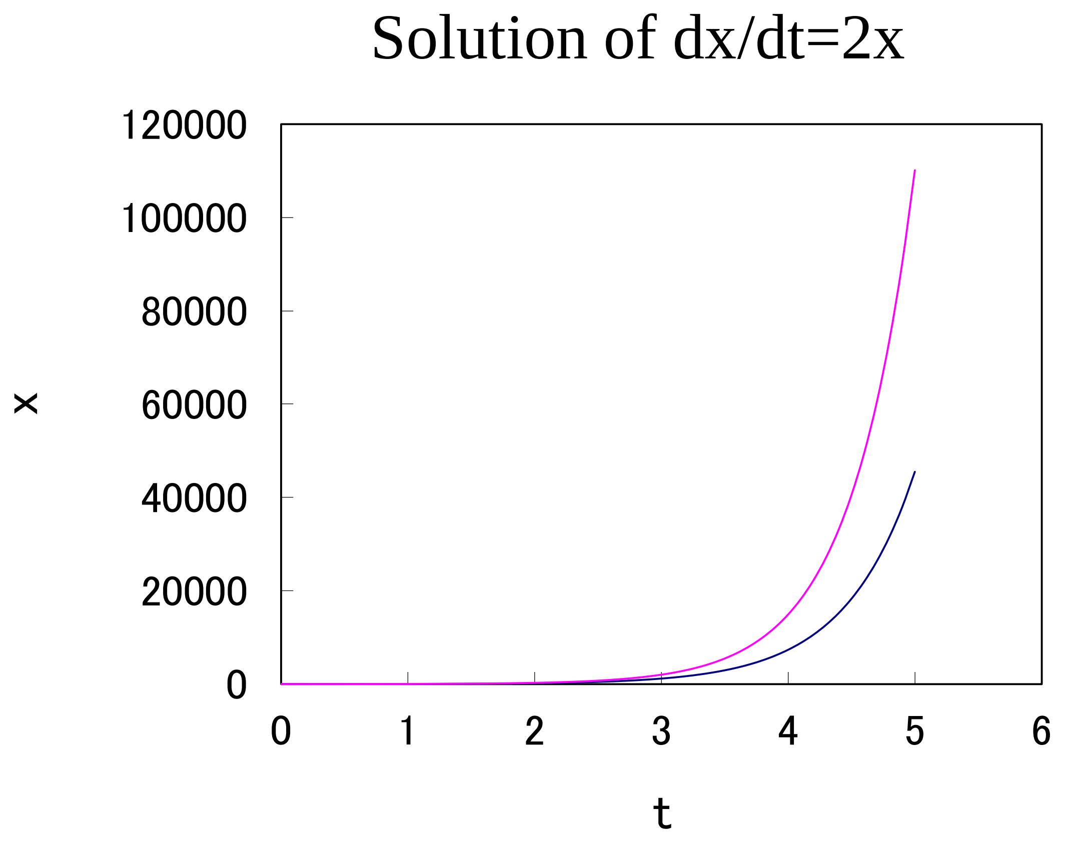
Numerical Computing
with
Excel (VBA)

Example

$$\begin{cases} \dfrac{dx}{dt} = 2x \\ x(0) = 5 \end{cases}$$

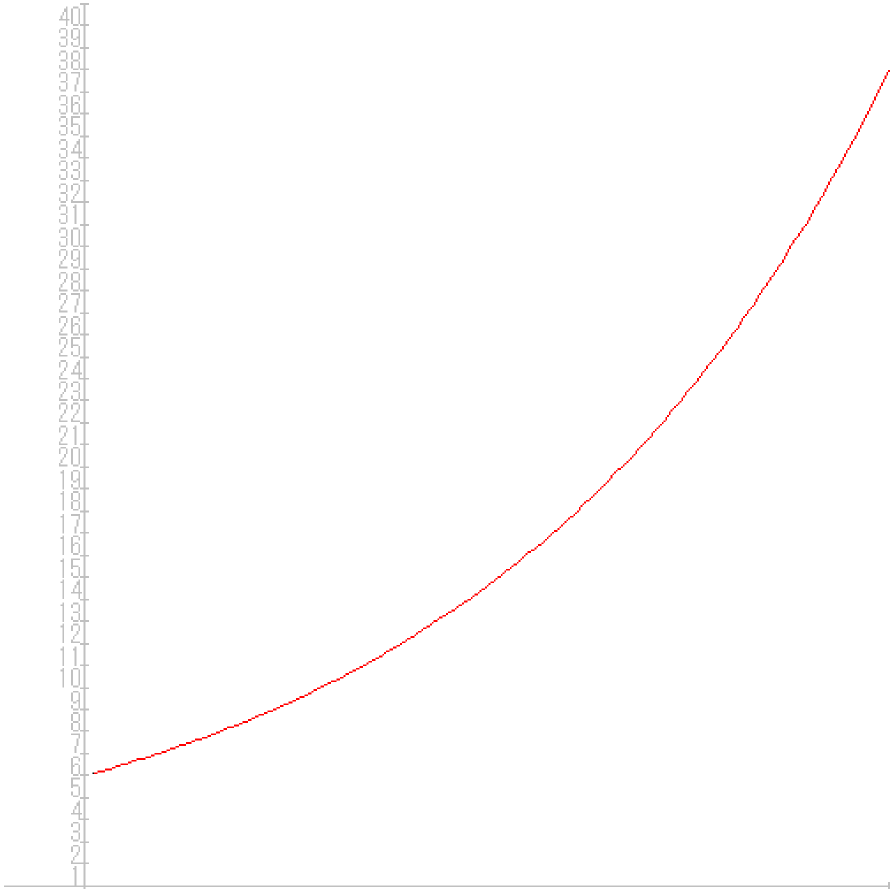
(Solution : $x(t) = 5e^{2t}$)

Euler's Method



Numerical Computing with BASIC

Runge-Kutta Method



$$x(t) = 5e^{2t}$$

Example

$$\begin{cases} \dfrac{dx}{dt} = x^2 + 1 \\ x(0) = 0 \end{cases}$$

(Solution : $x(t) = \tan t$ **)**

Runge-Kutta Method (1)

```
REM ルンゲ･クッタ法による正弦関数の計算
OPTION ANGLE RADIANS
DEF F(t,x) = x^2+1
SET WINDOW 0,PI/2,0,10
DRAW grid
DRAW axes
!'tの初期値
LET  t = 0
!'xの初期値
LET  x = 0
!'tの１ステップの変化量
LET  h = 0.25
!'何回計算するか
LET  N = 2
FOR i = 0 TO N-1
   LET  k1 = F(t, x)
   LET  k2 = F(t + h, x + h * k1)
   LET  x = x + h * (k1 + k2) / 2
   PLOT LINES: t,x;
   SET LINE COLOR 4
NEXT i
PRINT x
END
```

Runge-Kutta Method (2)

x=0.550499174772995

x-Tan(1/2)=-0.0632930624066609

Successive Approximation and Fixed-Point Theorem

Initial-Value Problem

$$\begin{cases} \dfrac{dx}{dt} = f(t, x(t)) \\ x(0) = x_0 \quad \textbf{(Initial Condition)} \end{cases}$$

Reduction to an Integral Equation

$$\begin{cases} \dfrac{dx}{dt} = f(t, x(t)) \\ x(0) = x_0 \end{cases}$$

Differentiation ↑ ↓ **Integration**

$$x(t) = x_0 + \int_0^t f(s, x(s))\,ds$$

Solution and Fixed-Point

$$x(t) = x_0 + \int_0^t f(s, x(s))ds$$

$$Fx(t) := x_0 + \int_0^t f(s, x(s))ds$$

$$\Rightarrow$$

$Fx = x$ (**Fixed - point of** F)

Stefan Banach

Banach's Fixed-Point Theorem

(X, d) **Complete Metric Space**

$F : X \to X$ **Contraction Map:**

$$\begin{cases} 0 < \exists k < 1 \\ d(F(x), F(y)) \le k\, d(x, y), \forall x, y \in X \end{cases}$$

$\Rightarrow$

$\exists! z \in X$ **such that** $F(z) = z$

Linear Case

Second-Order Case

$$\begin{cases} u''(t) + 2bu'(t) + cu(t) = 0, \\ u(0) = u_0, \\ u'(0) = u_1 \end{cases}$$

General Solutions

General Solution（1）

$$D/4 = b^2 - c > 0$$

$$u(t) = e^{-bt}\left(Ae^{t\sqrt{b^2-c}} + Be^{-t\sqrt{b^2-c}} \right)$$

A, B : **Constants**

Example

$$\begin{cases} x''(t) - x(t) = t \\ x(0) = x'(0) = 0 \end{cases}$$

Solution : $x(t) = \dfrac{1}{2}(e^{t} - e^{-t}) - t$

General Solution（2）

$$D/4 = b^2 - c < 0$$

$$u(t) = e^{-bt}\left(A\cos\sqrt{c-b^2}\,t + B\sin\sqrt{c-b^2}\,t\right)$$

A, B : **Constants**

General Solution（3）

$$D/4 = b^2 - c = 0$$

$$u(t) = e^{-bt}\left(At + B\right)$$

A, B : **Constants**

Linea Algebra
and
Differential Equations

Exponential Matrix

Main Idea

$$u''(t)+2bu'(t)+cu(t)=0$$

$$u''(t)+2bu'(t)+cu(t)=0$$

Matrix Representation

Original Form

$$\frac{dU(t)}{dt}=AU(t) \;\Rightarrow\; \textbf{Calculation of } e^{tA}$$

Solution (1)

$$\begin{cases} u_1(t) = u(t), \\ u_2(t) = u'(t) \end{cases}$$

$$\begin{cases} u_1'(t) = u'(t) = u_2(t), \\ u_2'(t) = u''(t) = -2bu'(t) - cu(t) \\ \qquad\quad = -2bu_2(t) - cu_1(t) \end{cases}$$

Solution (2)

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c & -2b \end{pmatrix}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \\ \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \end{cases}$$

Solution (3)

$$U(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ -c & -2b \end{pmatrix}$$

$$\begin{cases} \dfrac{d}{dt}U(t) = AU(t), \\ U(0) = U_0 \end{cases}$$

Solution (4)

$$U(t) = e^{tA} U_0$$

$$e^{tA} = I + tA + \frac{(tA)^2}{2!} + \cdots + \frac{(tA)^n}{n!} + \cdots$$

(**Exponential Matrix**)

Example
of
Exponential Matrices

Simple Eigenvalue Case

Calculation (1)

$$A = \begin{pmatrix} 0 & 1 \\ -c & -2b \end{pmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ c & \lambda + 2b \end{vmatrix} = \lambda^2 + 2b\lambda + c$$

Calculation (2)

$$\textbf{Case}: D/4 = b^2 - c \neq 0$$

$$\begin{cases} \lambda_1 = -b + \sqrt{b^2 - c}, \\ \lambda_2 = -b - \sqrt{b^2 - c} \end{cases}$$

Calculation (3)

$$P = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -b+\sqrt{b^2-c} & -b-\sqrt{b^2-c} \end{pmatrix}$$

$$P^{-1}AP = \Lambda$$ **(Diagonal)**

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -b+\sqrt{b^2-c} & 0 \\ 0 & -b-\sqrt{b^2-c} \end{pmatrix}$$

Calculation (4)

$$
\begin{aligned}
& P^{-1}e^{tA}P \\
&= P^{-1}\left(I + tA + \frac{(tA)^2}{2!} + \cdots + \frac{(tA)^n}{n!} + \cdots \right) P \\
&= P^{-1}P + t\left(P^{-1}AP\right) + \frac{t^2}{2!}\left(P^{-1}AP\right)\left(P^{-1}AP\right) + \cdots + \\
&\qquad + \frac{t^n}{n!}\underbrace{\left(P^{-1}AP\right)\left(P^{-1}AP\right)\cdots\left(P^{-1}AP\right)}_{n-\text{times}} + \cdots \\
&= I + t\Lambda + \frac{(t\Lambda)^2}{2!} + \cdots + \frac{(t\Lambda)^n}{n!} + \cdots \\
&= e^{t\Lambda}
\end{aligned}
$$

Calculation (5)

$$e^{t\Lambda} = I + t\Lambda + \frac{(t\Lambda)^2}{2!} + \cdots + \frac{(t\Lambda)^n}{n!} + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} {\lambda_1}^2 & 0 \\ 0 & {\lambda_2}^2 \end{pmatrix} + \cdots$$

$$+ \frac{t^n}{n!}\begin{pmatrix} {\lambda_1}^n & 0 \\ 0 & {\lambda_2}^n \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

Calculation (6)

$$e^{tA} = Pe^{t\Lambda}P^{-1}$$

$$= \frac{1}{\lambda_2 - \lambda_1}\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}\begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix}$$

$$= \frac{1}{\lambda_2 - \lambda_1}\begin{pmatrix} \lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} & -e^{\lambda_1 t} + e^{\lambda_2 t} \\ \lambda_1\lambda_2\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right) & -\lambda_1 e^{\lambda_1 t} + \lambda_2 e^{\lambda_2 t} \end{pmatrix}$$

Calculation (7)

$$\textbf{Case}: D/4 = b^2 - c \neq 0$$

$$U(t) = e^{tA} U_0,$$

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} & -e^{\lambda_1 t} + e^{\lambda_2 t} \\ \lambda_1 \lambda_2 \left(e^{\lambda_1 t} - e^{\lambda_2 t} \right) & -\lambda_1 e^{\lambda_1 t} + \lambda_2 e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$$

Double Eigenvalue Case

Jordan Canonical Form
of
Matrices

Marie Ennemond Camille Jordan

Jordan

- **Marie Ennemond Camille Jordan (1838-1922)**

French Mathematician

Jordan's Canonical Form

$$P^{-1}AP = \Lambda \quad \textbf{(Jordan Form)}$$

$$\Lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Calculation (1)

$$A = \begin{pmatrix} 0 & 1 \\ -c & -2b \end{pmatrix}$$

$$\left|\lambda I - A\right| = \begin{vmatrix} \lambda & -1 \\ c & \lambda + 2b \end{vmatrix} = \lambda^2 + 2b\lambda + c$$

Calculation (2)

$$\textbf{Case}: D/4 = b^2 - c = 0$$

$$\lambda = -b \quad \textbf{(Double Root)}$$

$$P = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}$$

Calculation (3)

$$P^{-1}AP = \Lambda \quad \textbf{(Jordan Form)}$$

$$\Lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} -b & 1 \\ 0 & -b \end{pmatrix}$$

Calculation (4)

$$P^{-1}e^{tA}P$$

$$= P^{-1}\left(I + tA + \frac{(tA)^2}{2!} + \cdots + \frac{(tA)^n}{n!} + \cdots \right)P$$

$$= P^{-1}P + t\left(P^{-1}AP\right) + \frac{t^2}{2!}\left(P^{-1}AP\right)\left(P^{-1}AP\right) + \cdots +$$

$$+ \frac{t^n}{n!}\underbrace{\left(P^{-1}AP\right)\left(P^{-1}AP\right)\cdots\left(P^{-1}AP\right)}_{n-\text{times}} + \cdots$$

$$= I + t\Lambda + \frac{(t\Lambda)^2}{2!} + \cdots + \frac{(t\Lambda)^n}{n!} + \cdots$$

$$= e^{t\Lambda}$$

Calculation (5)

$$e^{t\Lambda} = I + t\Lambda + \frac{(t\Lambda)^2}{2!} + \cdots + \frac{(t\Lambda)^n}{n!} + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix} + \cdots$$

$$+ \frac{t^n}{n!}\begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

Calculation (6)

$$e^{tA} = Pe^{t\Lambda}P^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda t} - \lambda te^{\lambda t} & te^{\lambda t} \\ -\lambda^2 + e^{\lambda t} & (\lambda t + 1)e^{\lambda t} \end{pmatrix}$$

Calculation (7)

Case : $D/4 = b^2 - c = 0$

$$U(t) = e^{tA} U_0,$$

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} e^{\lambda t} - \lambda t e^{\lambda t} & t e^{\lambda t} \\ -\lambda^2 + e^{\lambda t} & (\lambda t + 1) e^{\lambda t} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$$

2-dimensional Autonomous System

Linear Case

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$$

Matrix Form

$$U(t) := \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \mathbf{A} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Rightarrow$$

$$\frac{d}{dt}U(t) = \mathbf{A}U(t)$$

Stability of Solutions

Computational Approach

Numerical Computing
with
BASIC

Example 1 (Unstable Node)

$$\begin{cases} \dfrac{dx}{dt} = 2x \\ \dfrac{dy}{dt} = y \end{cases}$$

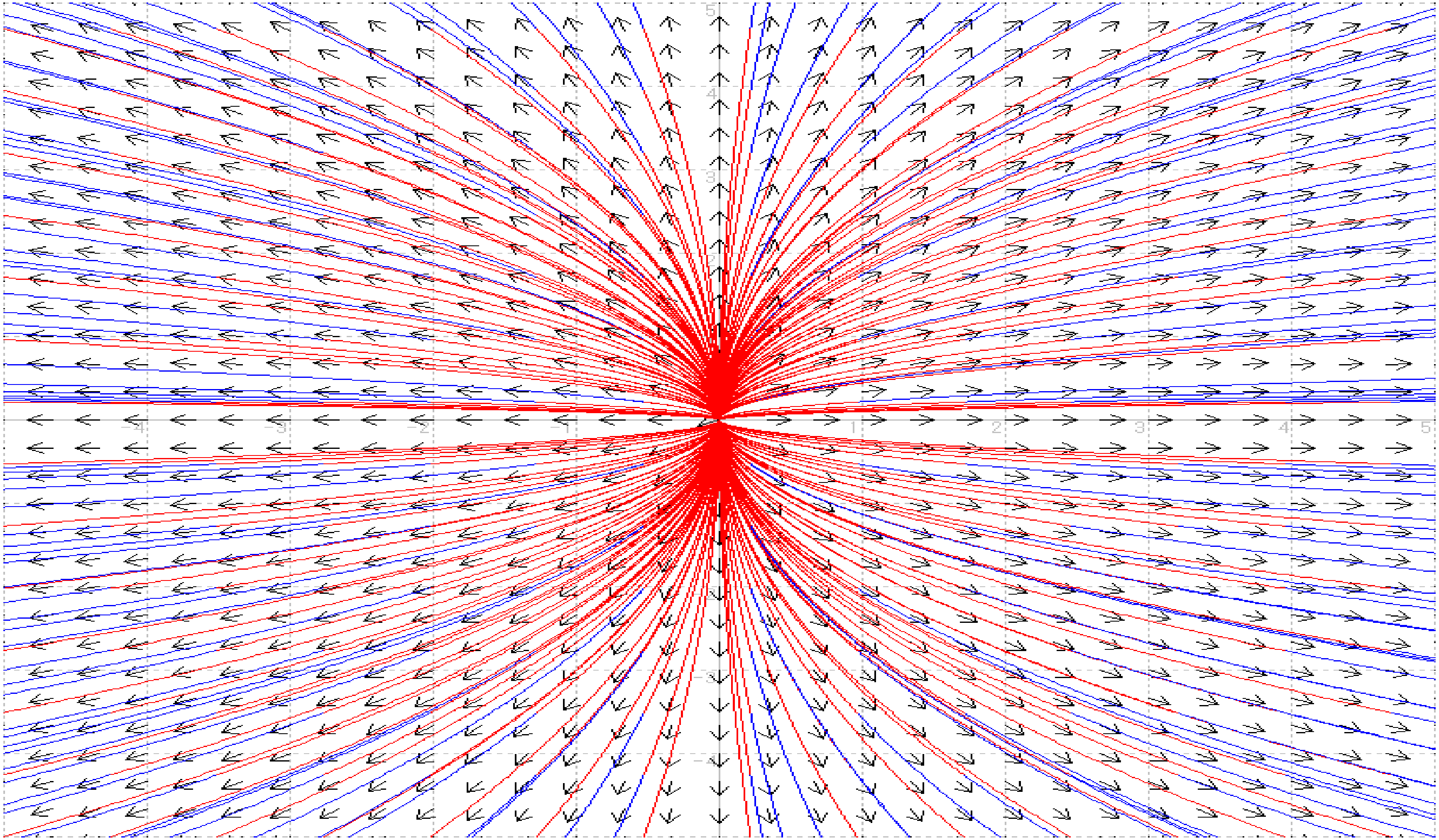
$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\textbf{Eigenvalues:}\ 2,\ 1$$

Unstable Node



Example 2 (Saddle Point)

$$\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = -y \end{cases}$$

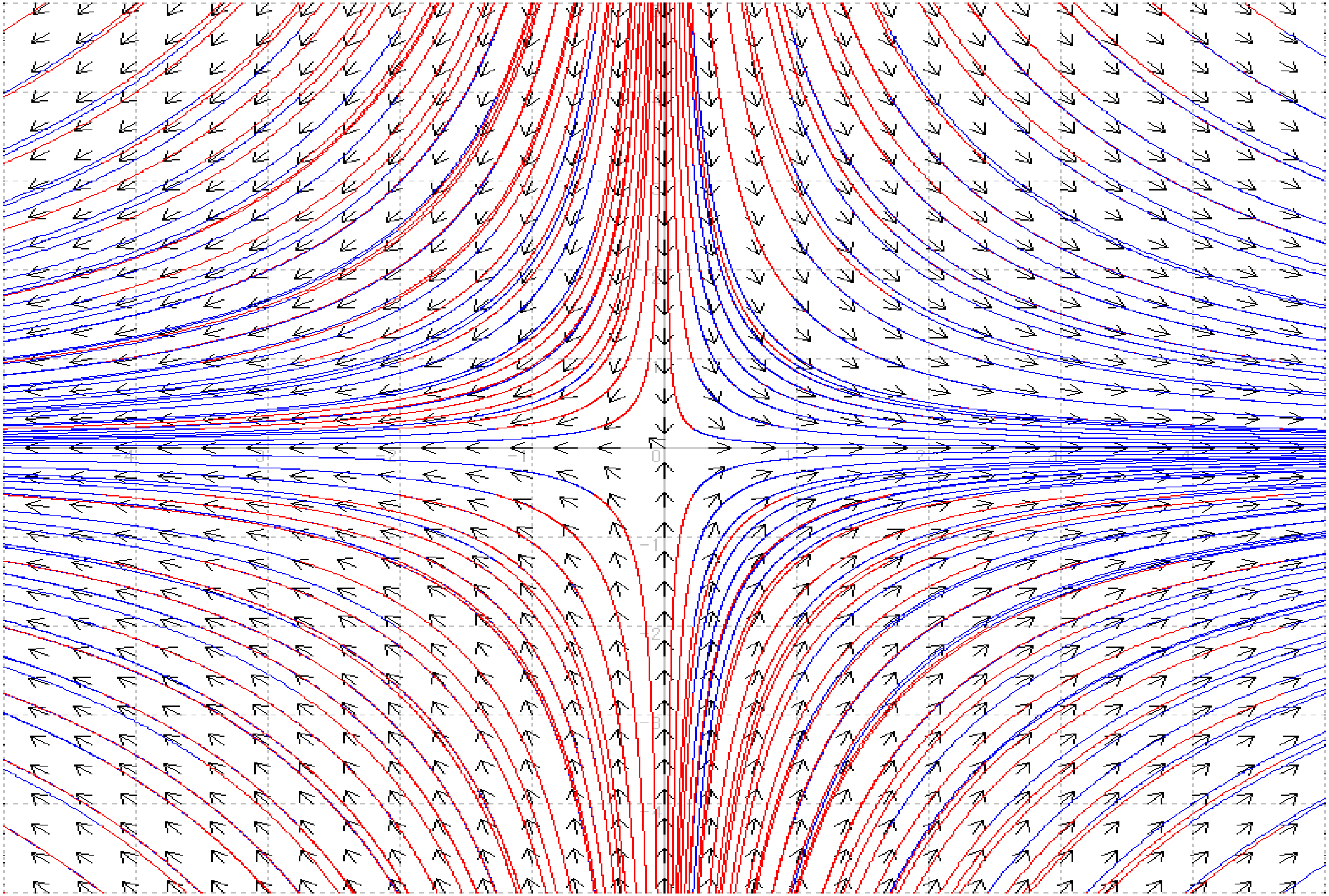
$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\textbf{Eigenvalues:}\ 1,\ -1$$

Saddle Point



Example 3 (Unstable Node)

$$\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = 3x + 2y \end{cases}$$

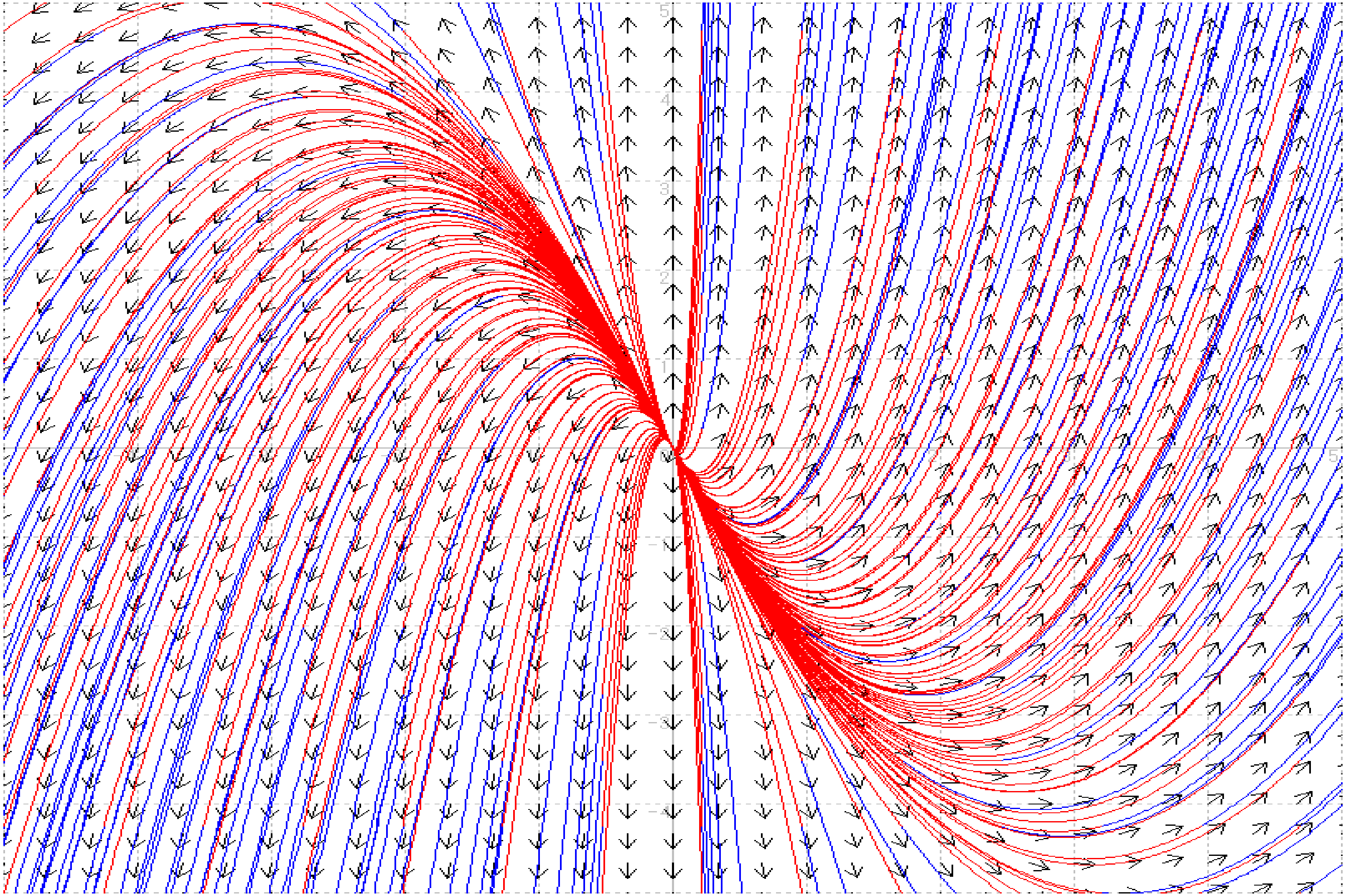
$$A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$$

Eigenvalues: 1, 2

Unstable Node



Example 4 (Stable Node)

$$\begin{cases} \dfrac{dx}{dt} = -2x - 1.5y \\ \dfrac{dy}{dt} = x - 5.5y \end{cases}$$

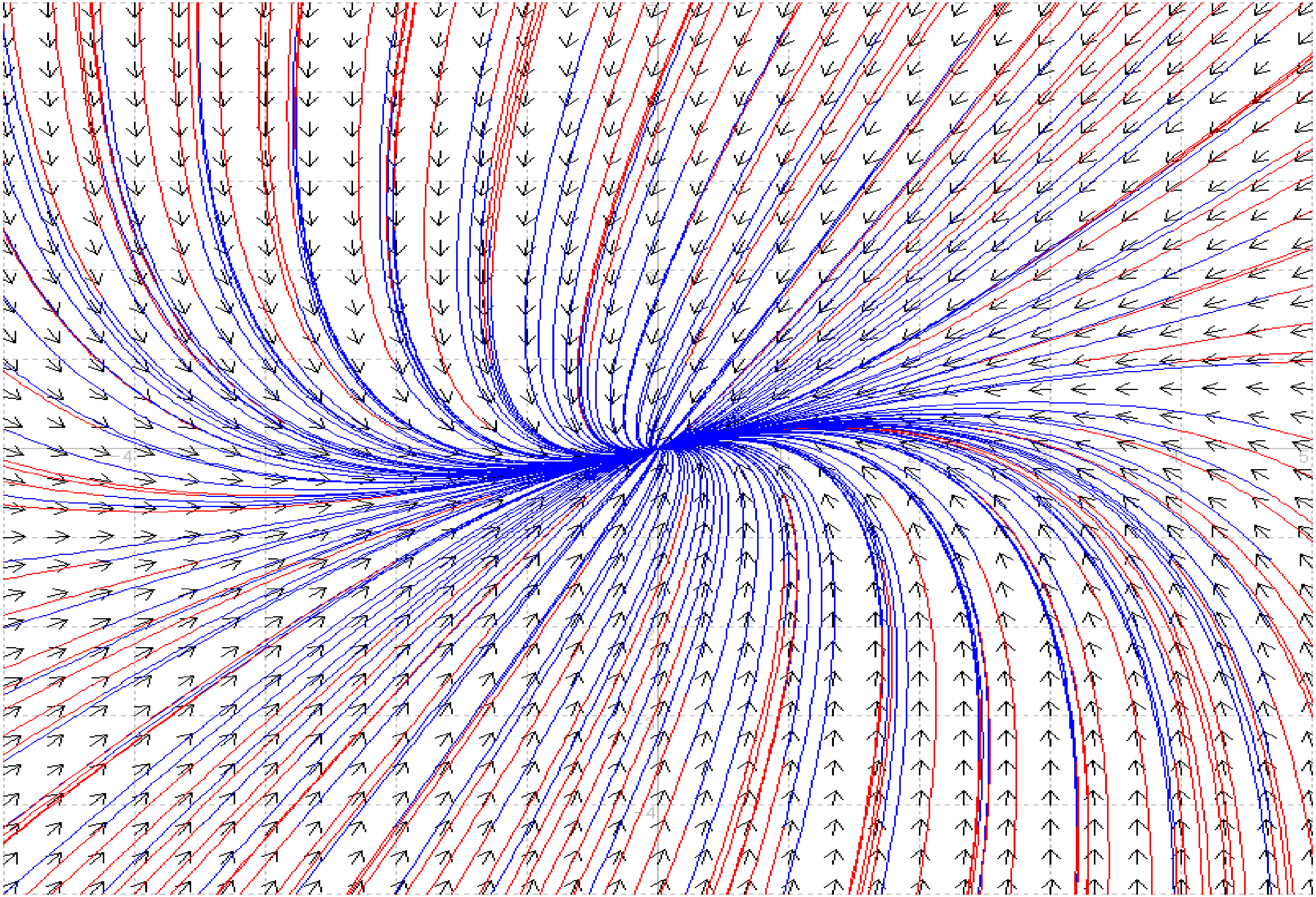
$$A = \begin{pmatrix} -2 & -1.5 \\ 1 & -5.5 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} -2 & -1.5 \\ 1 & -5.5 \end{pmatrix}$$

$$\textbf{Eigenvalues:} \ -2.5, \ -5$$

Stable Node



Example 5 (Saddle Point)

$$\begin{cases} \dfrac{dx}{dt} = -2x + 2y \\ \dfrac{dy}{dt} = -2x + 3y \end{cases}$$

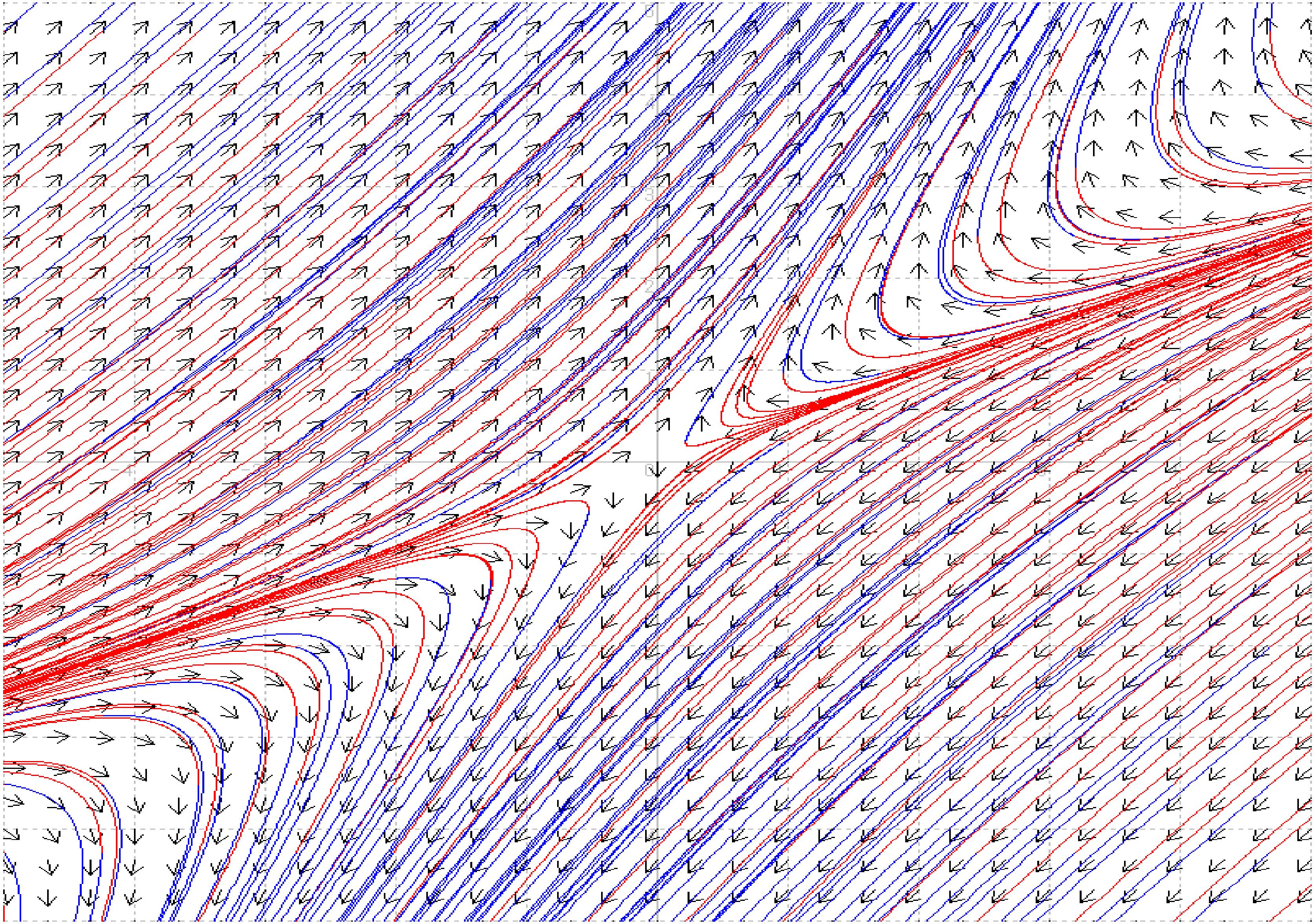
$$A = \begin{pmatrix} -2 & 2 \\ -2 & 3 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} -2 & 2 \\ -2 & 3 \end{pmatrix}$$

$$\textbf{Eigenvalues}: 2, \; -1$$

Saddle Point



Example 6 (Unstable Node)

$$\begin{cases} \dfrac{dx}{dt} = 2x + y \\ \dfrac{dy}{dt} = 2y \end{cases}$$

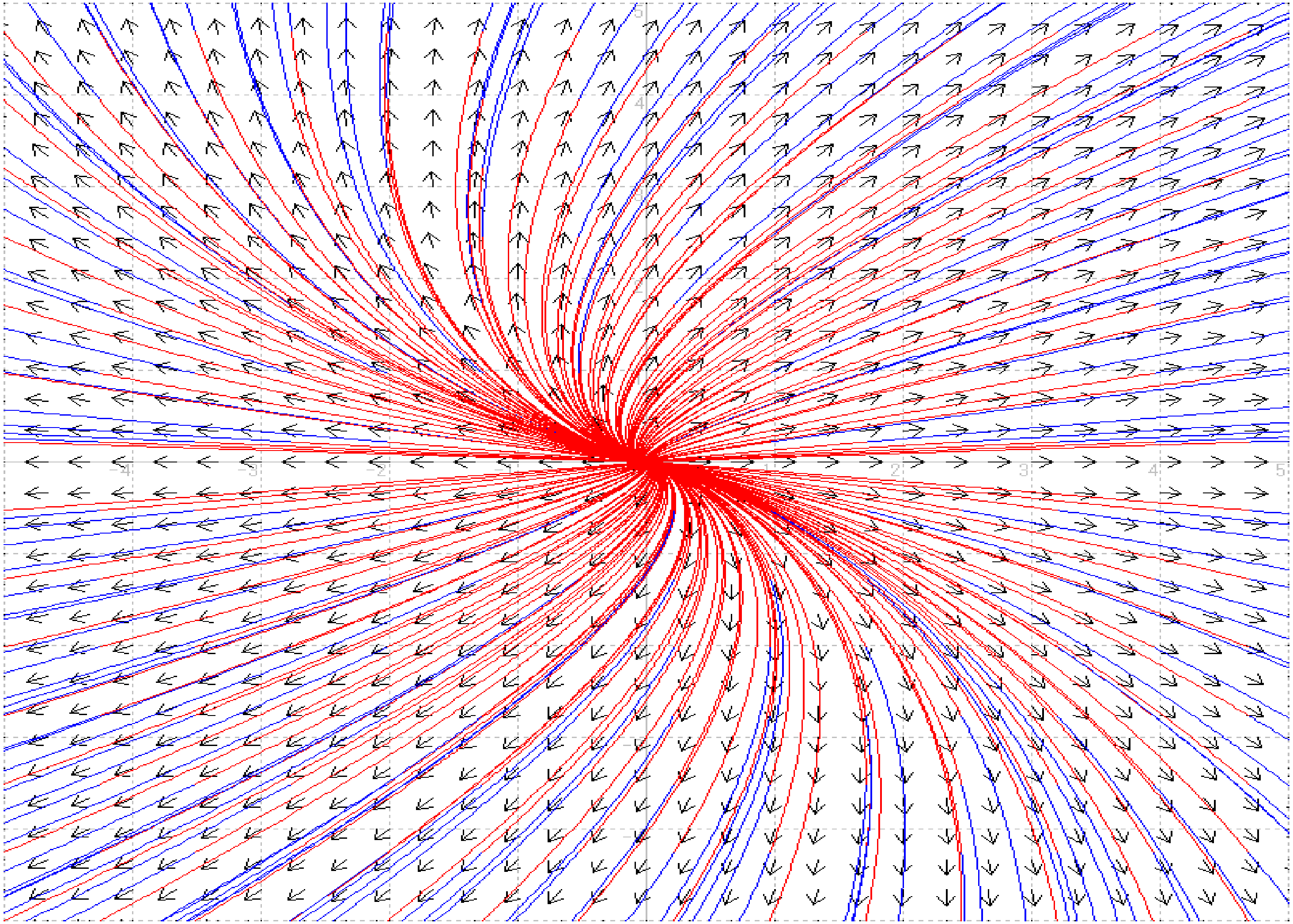
$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

$$\textbf{Eigenvalues}: 2, \quad 2$$

Unstable Node



Example 7 (Center)

$$\begin{cases} \dfrac{dx}{dt} = x + 2y \\ \dfrac{dy}{dt} = -x - y \end{cases}$$

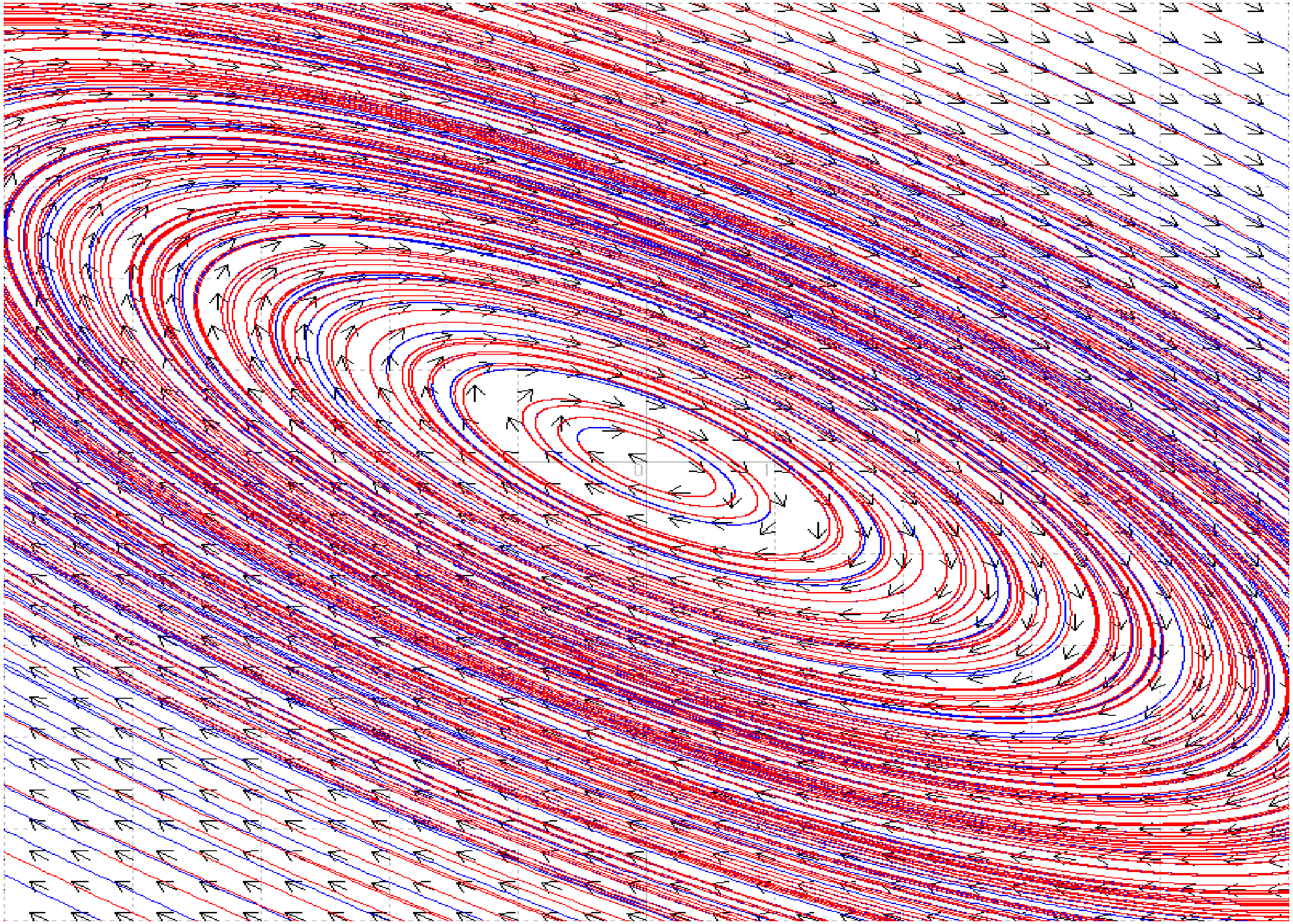
$$A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

$$\textbf{Eigenvalues}: \sqrt{-1},\ -\sqrt{-1}$$

Center



Example 8 (Unstable Focus)

$$\begin{cases} \dfrac{dx}{dt} = x + y \\ \dfrac{dy}{dt} = -2x + y \end{cases}$$

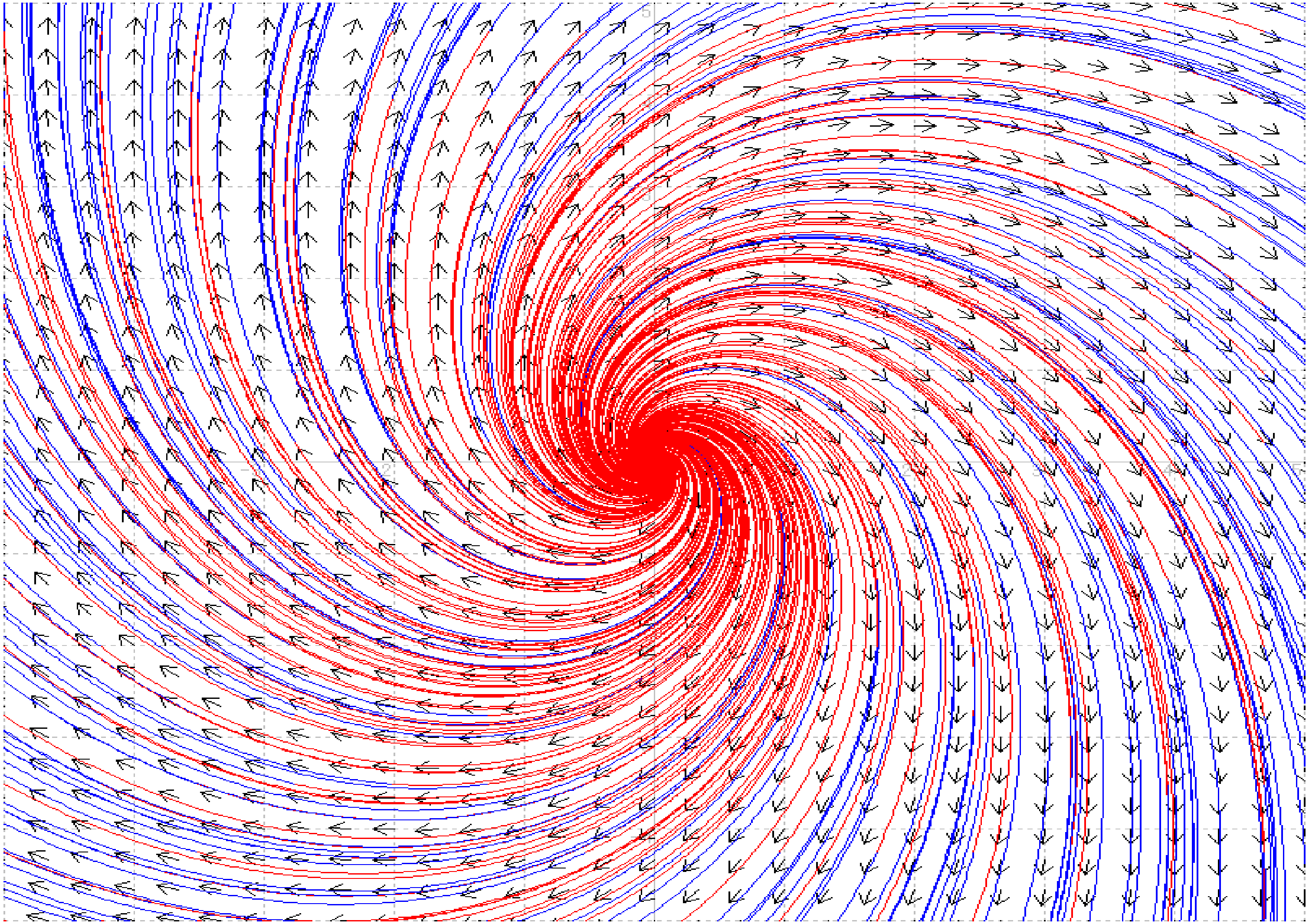
$$A = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

Eigenvalues: $1 + \sqrt{2}i,\ 1 - \sqrt{2}i$

Unstable Node



Example 9 (Degenerate Node)

$$\begin{cases} \dfrac{dx}{dt} = 2x + 2y \\ \dfrac{dy}{dt} = 3x + 3y \end{cases}$$

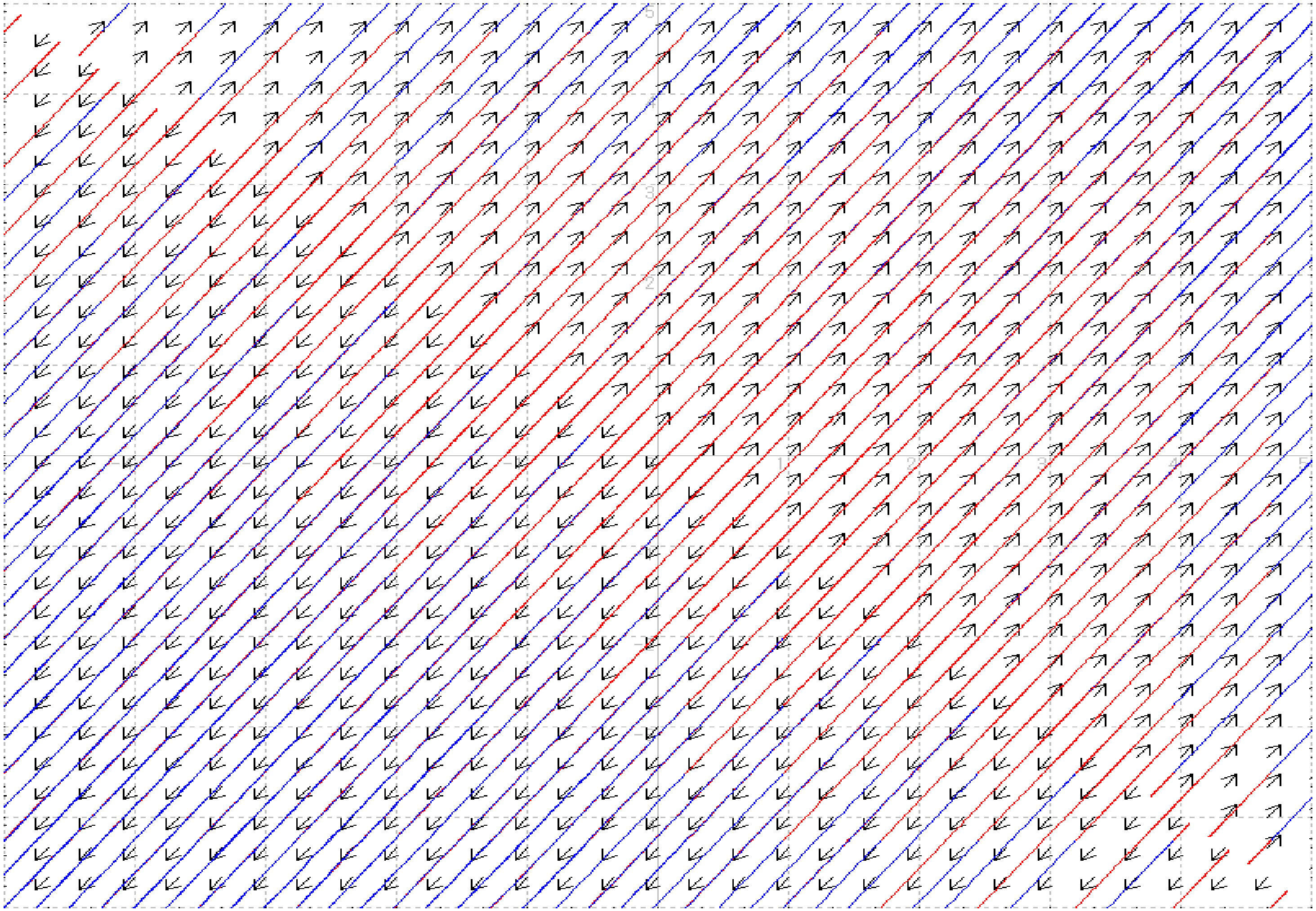
$$A = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$$

Signature of Eigenvalues

$$A = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$$

$$\textbf{Eigenvalues: } 0, \quad 5$$

Degenerate Node



Vector Analysis

Bird's- Eye View

Theme	Mathematics	Mechanics
Vector Analysis	Calculus on Surfaces	Continuum Mechanics

Line Integrals

Example (1)

$$\int_{x^2+y^2=1} x\,dx = \int_0^{2\pi} \cos\theta \left(d\cos\theta \right)$$

$$= -\frac{1}{2}\int_0^{2\pi} \sin 2\theta\, d\theta$$

$$= 0$$

Example (2)

$$\int_{x^2+y^2=1} xdy = \int_0^{2\pi} \cos\theta \left(d\sin\theta \right)$$

$$= \frac{1}{2}\int_0^{2\pi} \left(1+\cos 2\theta\right) d\theta$$

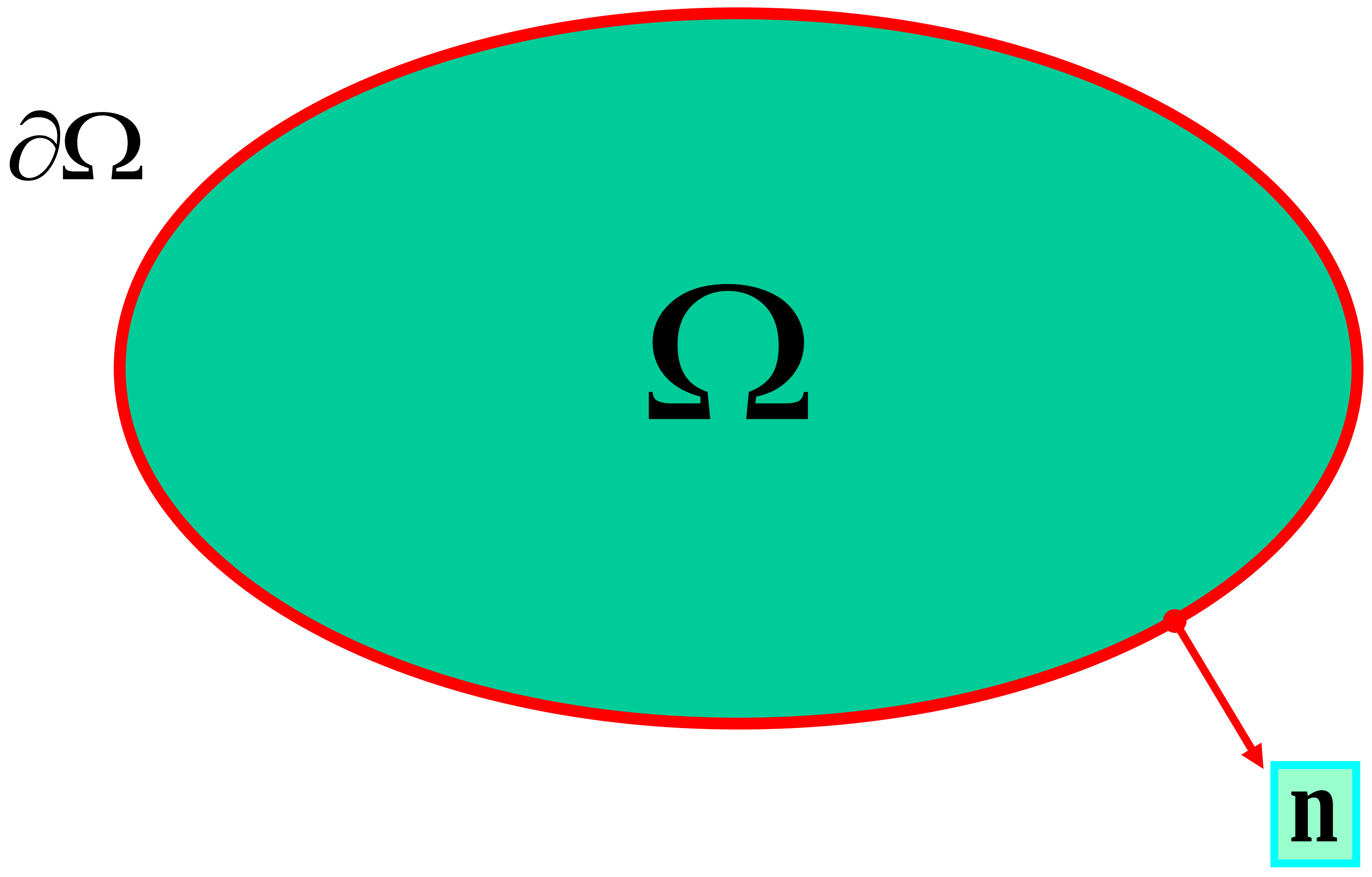
$$= \pi$$

Example (3)

$$\int_{x^2+y^2=1} ydx + xdy = 0$$

Green's Theorem

2-dimensional Domain



Green's Theorem (1)

$$\iint_{\Omega}\left(\frac{\partial f}{\partial x}-\frac{\partial g}{\partial y}\right)dxdy$$

$$=\int_{\partial\Omega} fdy+gdx$$

Example (1)

$$|\Omega| = \iint_{\Omega} 1\,dxdy$$

$$= \frac{1}{2}\int_{\partial\Omega} xdy - ydx$$

Green's Theorem (2)

$$\iint_{\Omega} \mathrm{div}\mathbf{F}\, dv = \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}\, ds$$

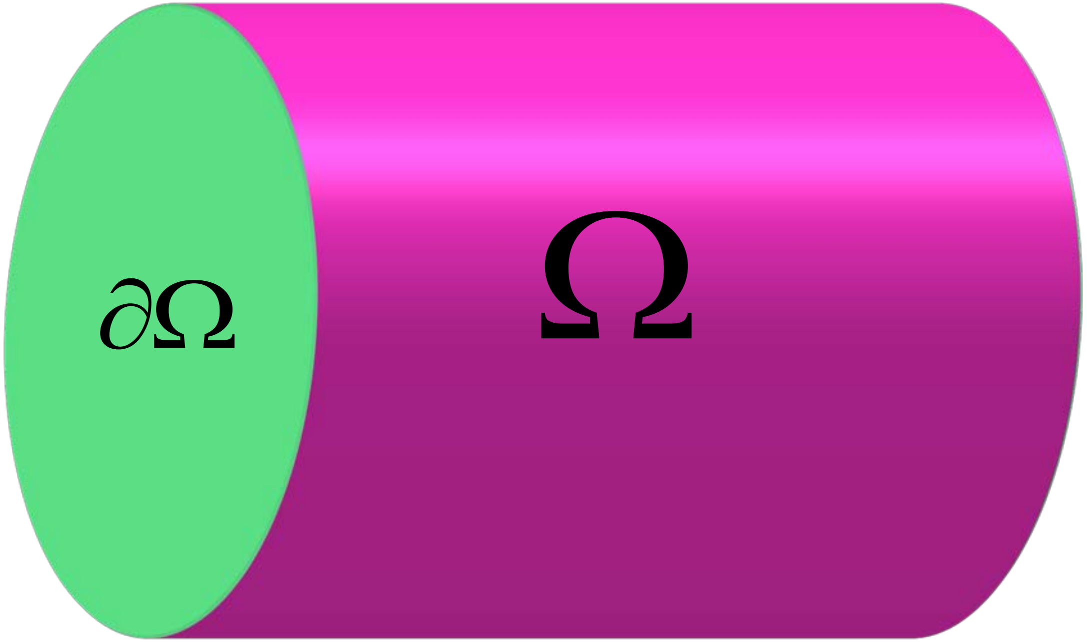
$$\mathbf{F} = (f, g)$$

Example (2)

$$\iint_{\Omega}\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)dxdy=\int_{\partial\Omega}\frac{\partial f}{\partial \mathbf{n}}\,ds$$

Gauss' Divergence Theorem

3-dimensional Domain



Gauss' Divergence Theorem (1)

$$\iiint_{\Omega}\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}\right)dxdydz$$

$$=\iint_{\partial\Omega} fdydz+gdzdx+hdxdy$$

Example (1)

$$|\Omega| = \iiint_{\Omega} 1\,dxdydz$$

$$= \frac{1}{3}\iint_{\partial\Omega} xdydz + ydzdx + zdxdy$$

Example (2)

$$\int_{x^2+y^2+z^2=1} x^3 dydz + y^3 dzdx + z^3 dxdy$$

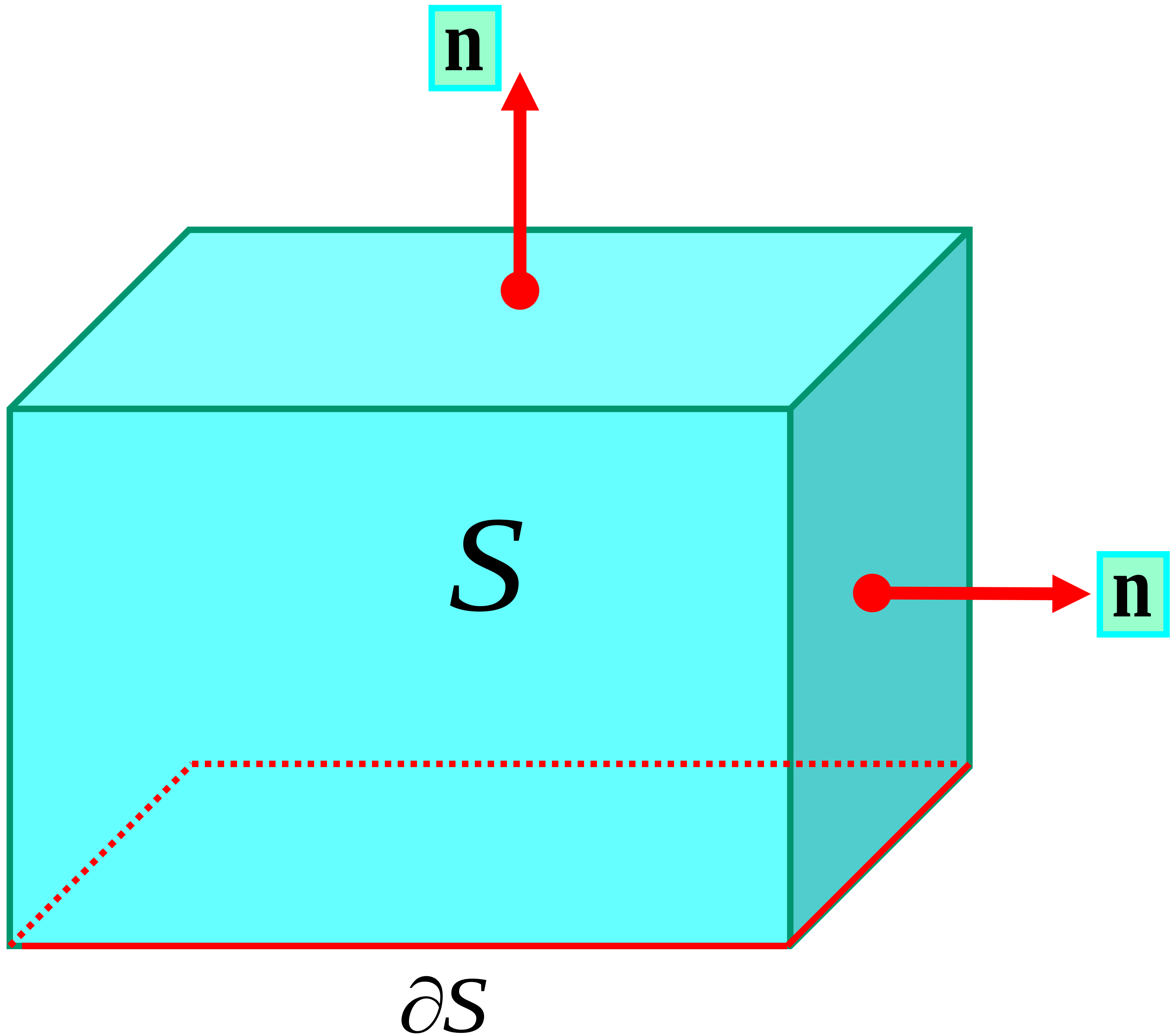
$$= \frac{12}{5}\pi$$

Gauss' Divergence Theorem (2)

$$\iiint_D \operatorname{div}\mathbf{F}\, dV = \iint_{\partial D} \mathbf{F}\cdot\mathbf{n}\, dS$$

$$\mathbf{F} = (f, g, h)$$

Stokes' Theorem



n
n
S
∂S

Stokes' Theorem (1)

$$\iint_S \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dydz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) dzdx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$$

$$= \int_{\partial S} f dx + g dy + h dz$$

Example

$$\int_{(1,2-1)}^{(2,3,1)} y^2 z^3 dx + 2xyz^3 dy + 3xy^2 z^2 dz = 22$$

Stokes' Theorem (2)

$$\iint_S \operatorname{rot}\mathbf{F}\cdot\mathbf{n}\,dS = \int_{\partial S}\mathbf{F}\cdot d\mathbf{s},$$

$$\mathbf{F} = (f, g, h)$$

Differential Forms
（Elie Cartan）

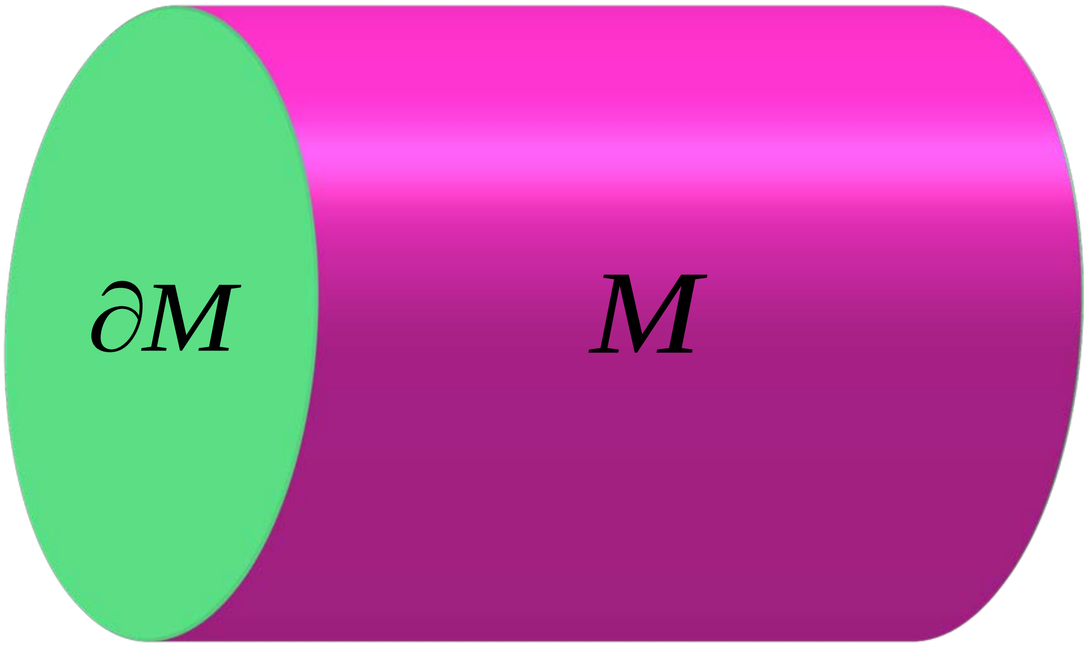
Differential Forms
and
Figures

Duality of Concepts

Degree	Differential Forms	Figures
0	Functions	Points
1	dx, dy, dz	Segments
2	$dxdy, dydz, dzdx$	Rectangles
3	$dxdydz$	Cubes

General Form
of
Stokes' Formula

Manifold with Boundary



Stokes' Formula

$$\int_M d\omega = \int_{\partial M} \omega$$

$$\langle M, d\omega \rangle = \langle \partial M, \omega \rangle$$

Examples of Stokes' Formula

Figure	Differential Form	Contents
Interval	Function	Fundamental Theorem of Calculus
2-Domain	One Form	Green's Theorem
3-Domain	Two Form	Gauss' Theorem
Surface	One Form	Stokes' Theorem

Exterior Derivation

Gradient

$$\mathrm{grad}\ f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

Rotation

$$\operatorname{rot}(f, g, h) = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)$$

Divergence

$$\operatorname{div}(f, g, h) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

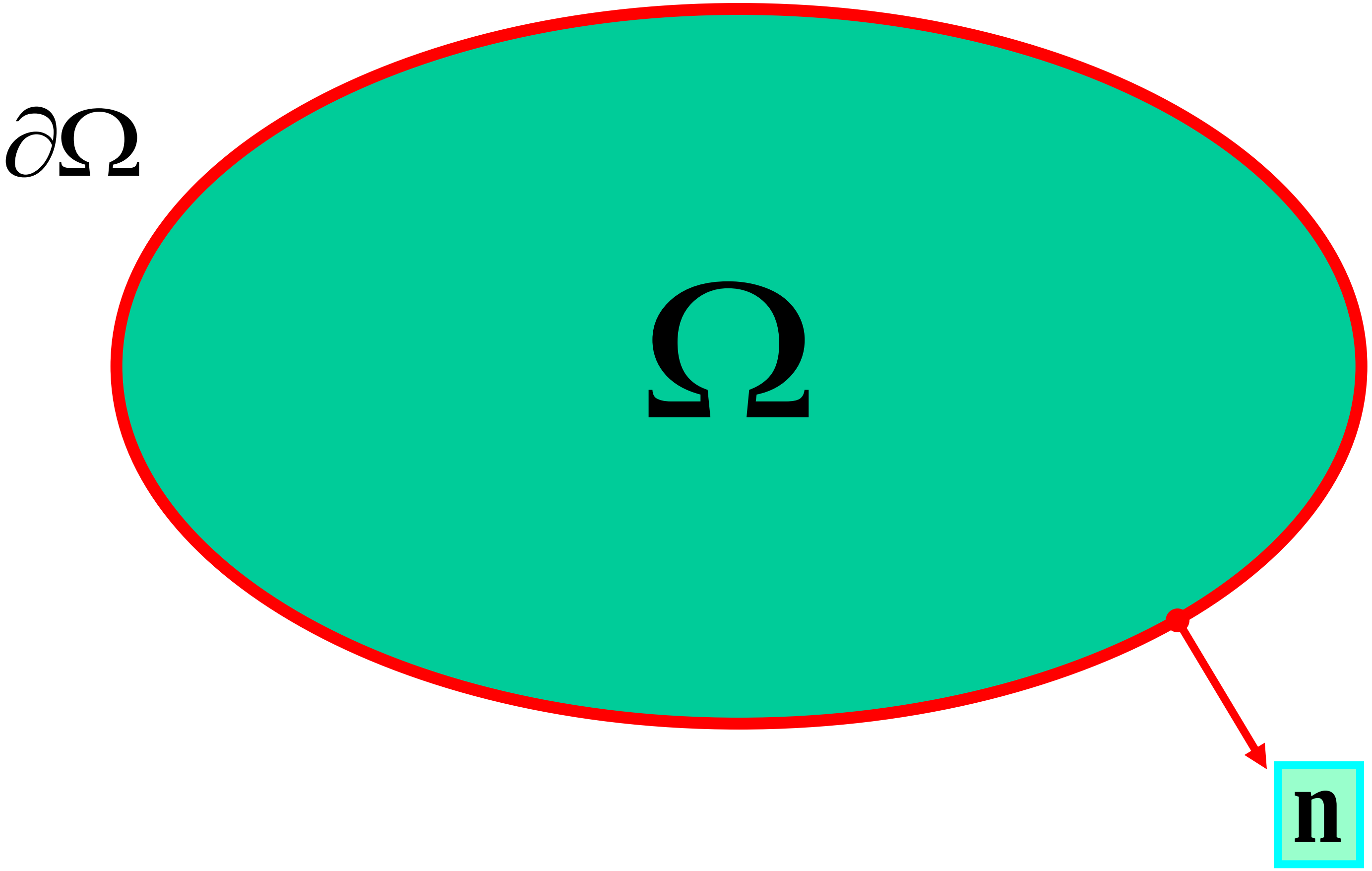
Well-known Formulas

$$\mathrm{rot} \circ \mathrm{grad}\ f = 0$$

$$\mathrm{div} \circ \mathrm{rot}\ \mathbf{v} = 0$$

Green's Theorem

2-dimensional Domain



Green's Theorem (1)

$$\iint_{\Omega} \left(\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y} \right) dxdy$$

$$= \int_{\partial\Omega} fdy + gdx$$

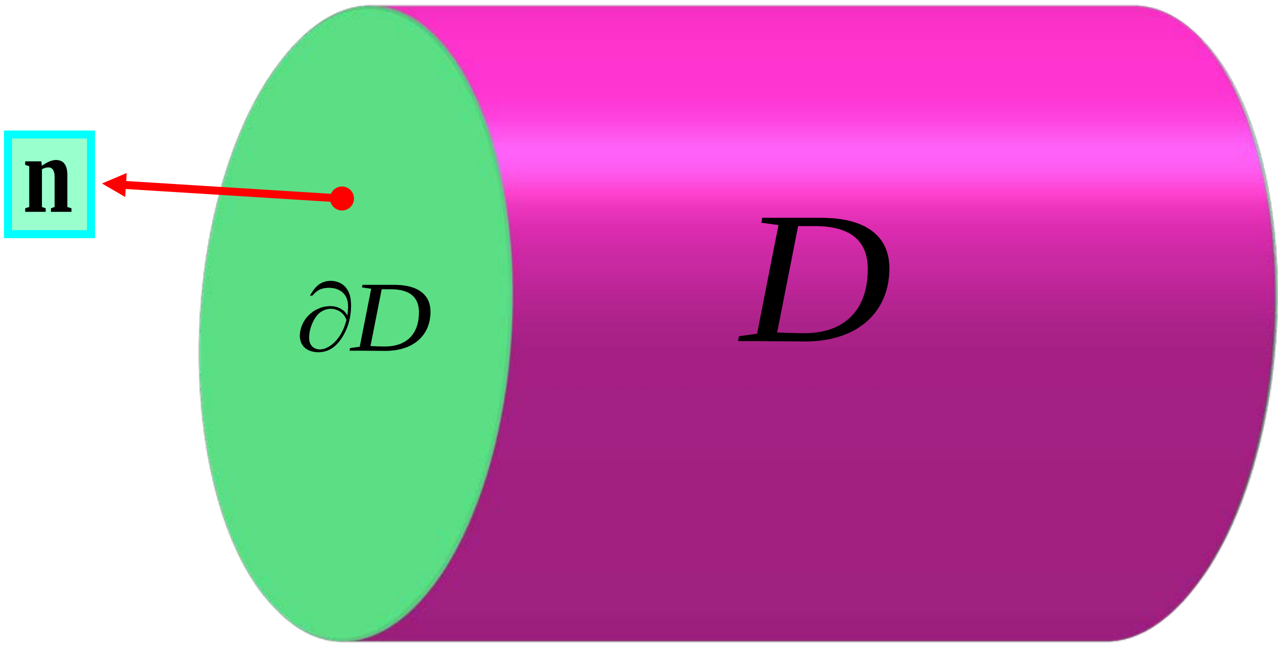
Green's Theorem (3)

$$\iint_{\Omega} d\omega = \int_{\partial\Omega} \omega$$

$$\omega = fdy + gdx$$

Gauss' Divergence Theorem

3-dimensional Domain



Gauss' Divergence Theorem (1)

$$\iiint_D \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \right) dxdydz$$

$$= \iint_{\partial D} fdydz + gdzdx + hdxdy$$

Gauss' Divergence Theorem (3)

$$\iint_D d\omega = \int_{\partial D} \omega$$

$$\omega = \ f dydz + g dzdx + h dxdy$$

Application to Electro-magnetism

Gauss' Theorem (Magnetic Field)

$$\iint_{\partial D} \mathbf{B}(x) \cdot \mathbf{n}\, dS = 0$$

$$\mathbf{B}(x) = \textbf{Magnetrostatics}$$

Gauss' Theorem (Electric Field)

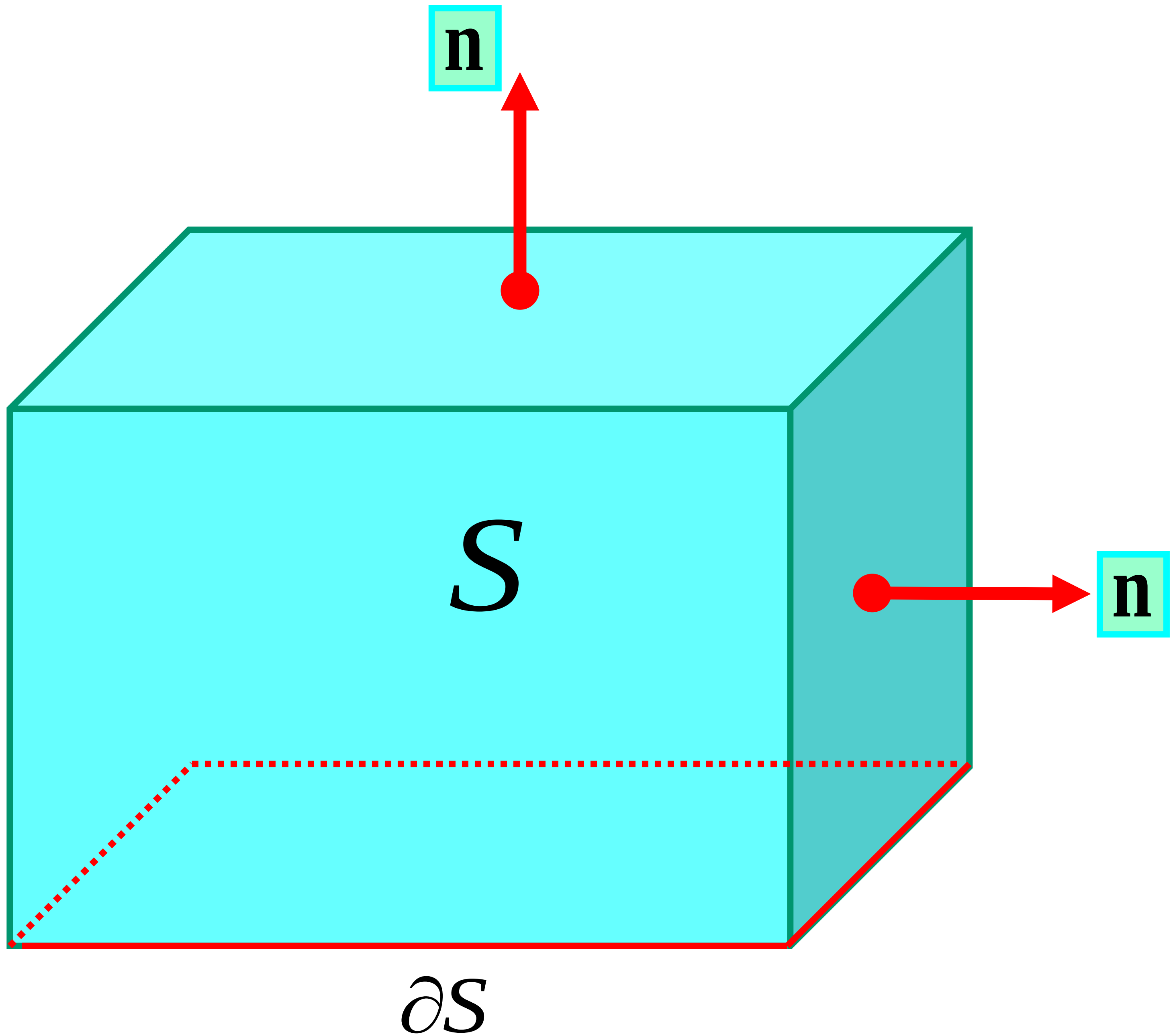
$$\iint_{\partial D} E(x)\cdot\mathbf{n}\,dS = \frac{1}{\varepsilon_0}\iiint_D \rho(x)\,dx$$

$E(x) =$ **Electrostatic Field**

$\rho(x) =$ **Electric Density**

$\varepsilon_0 =$ **Inductive Capacity in Free Space**

Stokes' Theorem



n
n
S
∂S

Stokes' Theorem (1)

$$\iint_S \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) dydz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) dzdx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy$$

$$= \int_{\partial S} fdx + gdy + hdz$$

Stokes' Theorem (3)

$$\iint_S d\omega = \int_{\partial S} \omega$$

$$\omega = f\,dx + g\,dy + h\,dz$$

Application to Electro-magnetism

Faraday's Law

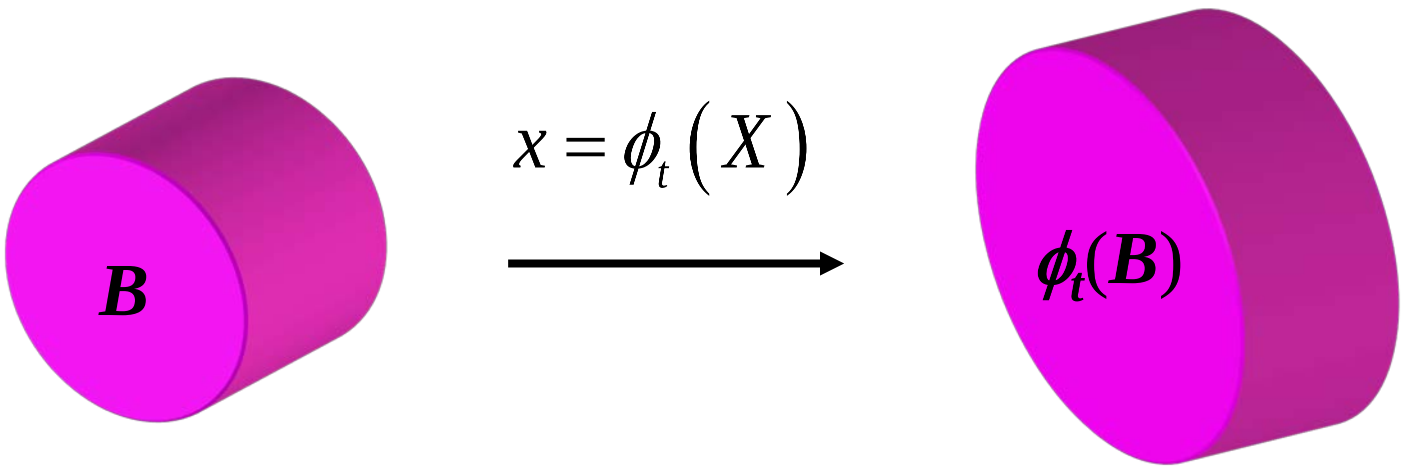
$$-\frac{d}{dt}\left(\iint_{S} \mathbf{B}(x,t)\cdot\mathbf{n}\,dS\right)=\int_{\partial S}\mathbf{E}(x,t)\cdot d\mathbf{r}\,,$$

$$d\mathbf{r}=(dx,dy,dz)$$

Mathematical Theory of Elasticity

Motions and Configurations

$$x = \phi_t(X)$$

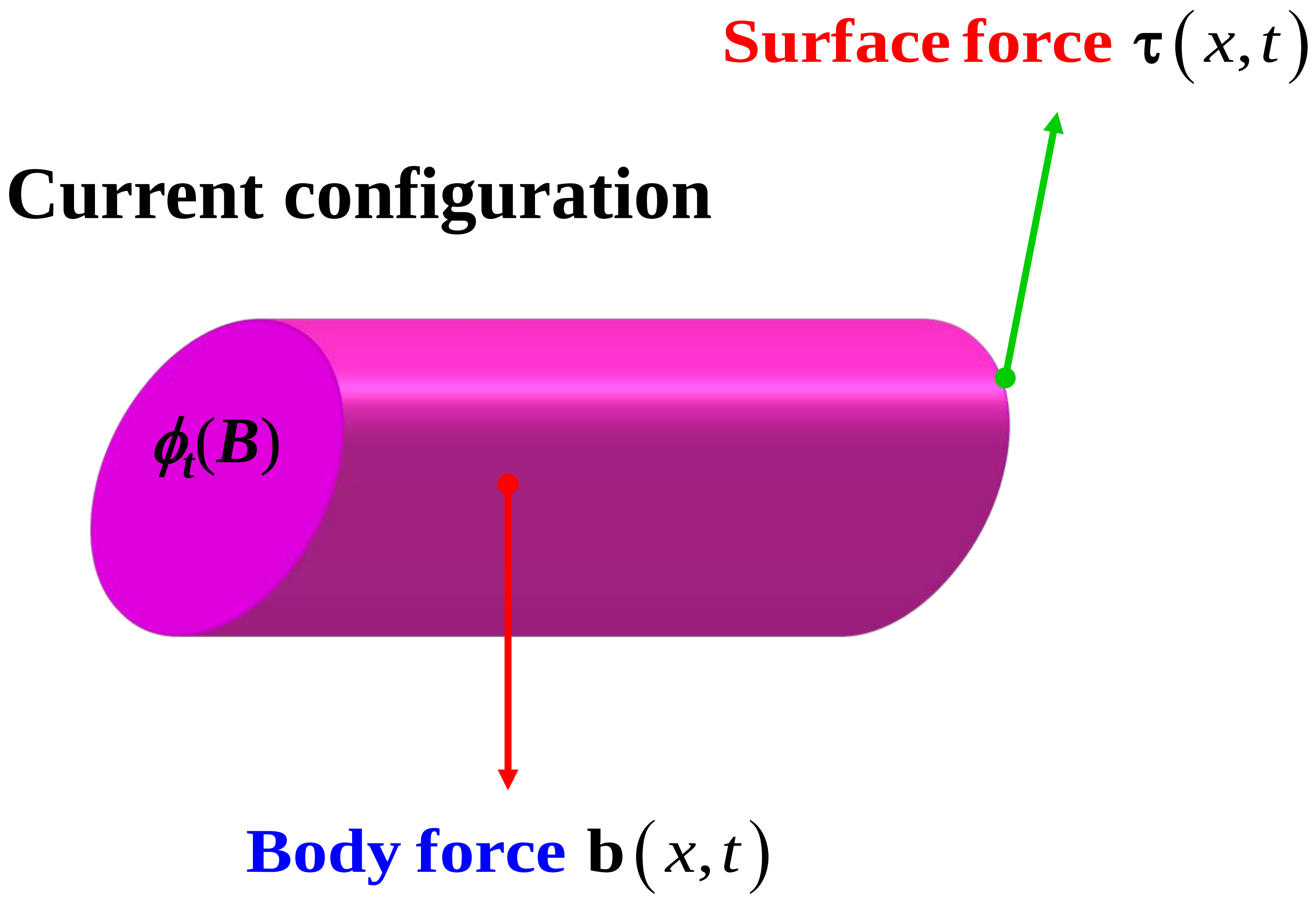


Reference configuration of a body

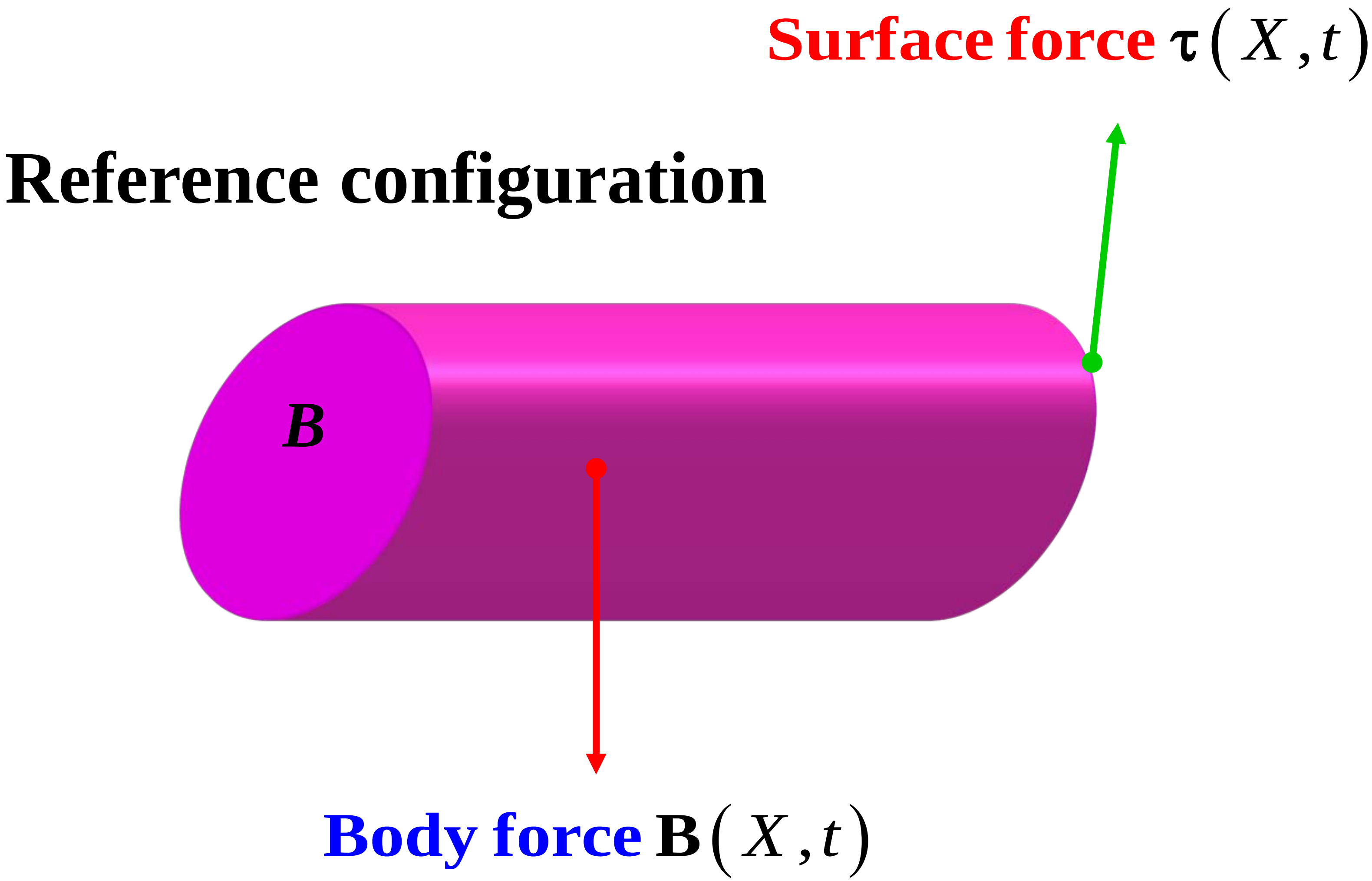
Body after time *t*

Two Descriptions in Elastodynamics

Euler's Description



Lagrange's Description



Continuum Mechanics (1)

Description	Conservation Law of Mass	Balance Law of Momentum
Euler	$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0$	$\rho \dot{\mathbf{v}} = \operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b}$
Lagrange	$\rho_0(X) = \rho(\phi_t(X), t) J(X, t)$	$\rho_0 \dfrac{\partial \mathbf{V}}{\partial t} = \operatorname{Div} \mathbf{P} + \rho_0 \mathbf{B}$

Continuum Mechanics (2)

Description	Balance Law of Angular Momentum	Balance Law of Energy
Euler	$\boldsymbol{\sigma} = {}^{t}\boldsymbol{\sigma}$	$\rho \dot{e} + \operatorname{div} \mathbf{q} = \operatorname{tr}(\boldsymbol{\sigma}\mathbf{d}) + \rho r$
Lagrange	$\mathbf{S} = {}^{t}\mathbf{S}$	$\rho_0 \dfrac{\partial E}{\partial t} + \operatorname{Div} \mathbf{Q} = \operatorname{tr}(\mathbf{S}\mathbf{D}) + \rho_0 R$

Probability and Calculus

Weierstrass' Polynomial Approximation Theorem

Weierstrass' Polynomial Approximation Theorem

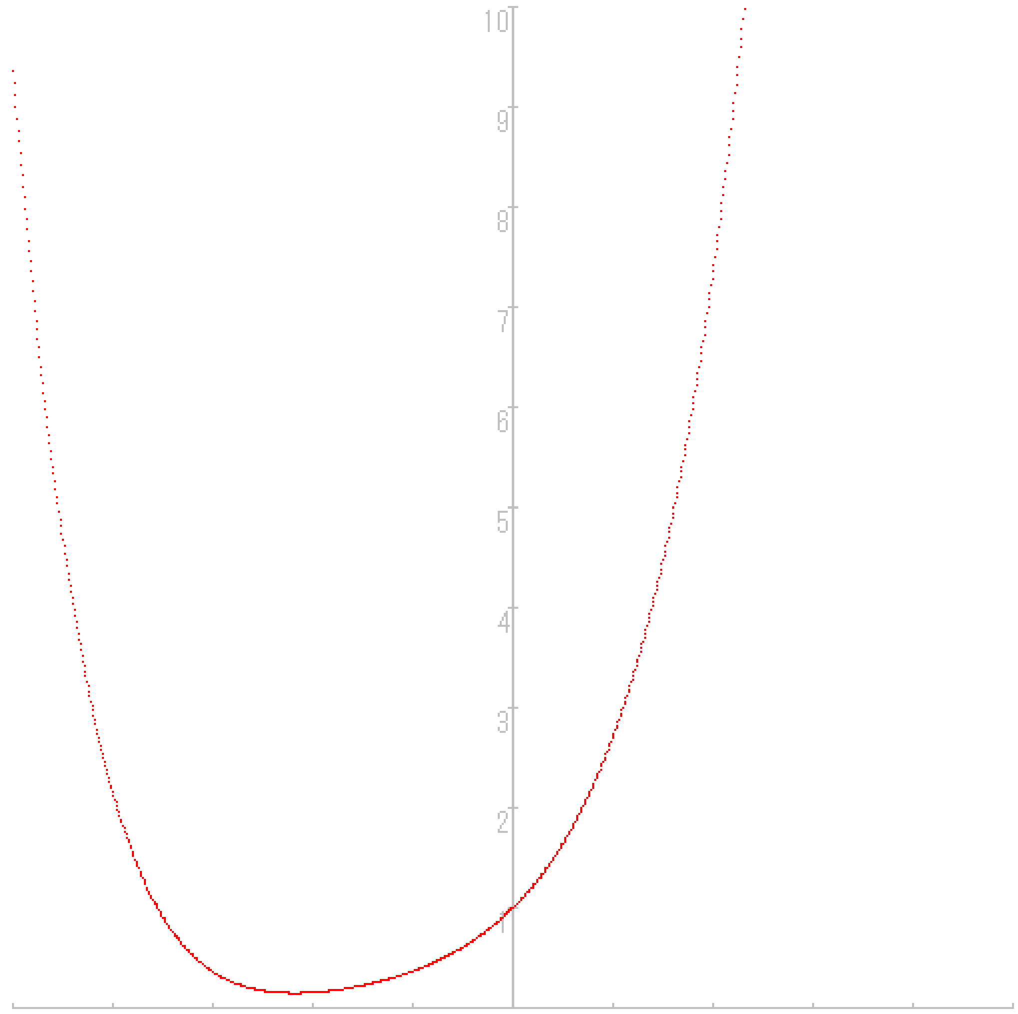
Any continuous function defined on a bounded closed interval may be approximated uniformly by polynomials.

Examples of Taylor's Expansion

Example 1

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Numerical Computing
with
BASIC

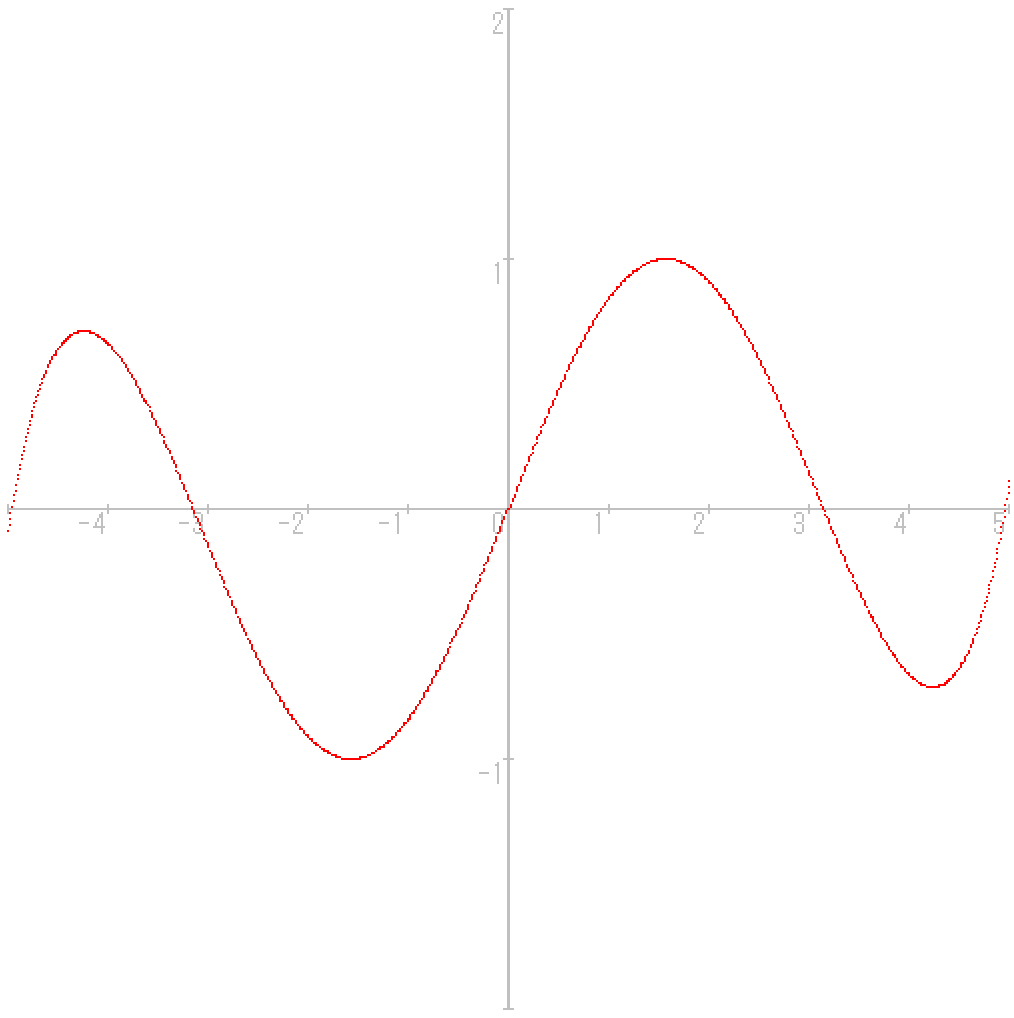


10
9
8
7
6
5
4
3
2
1

Example 2

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Numerical Computing
with
BASIC



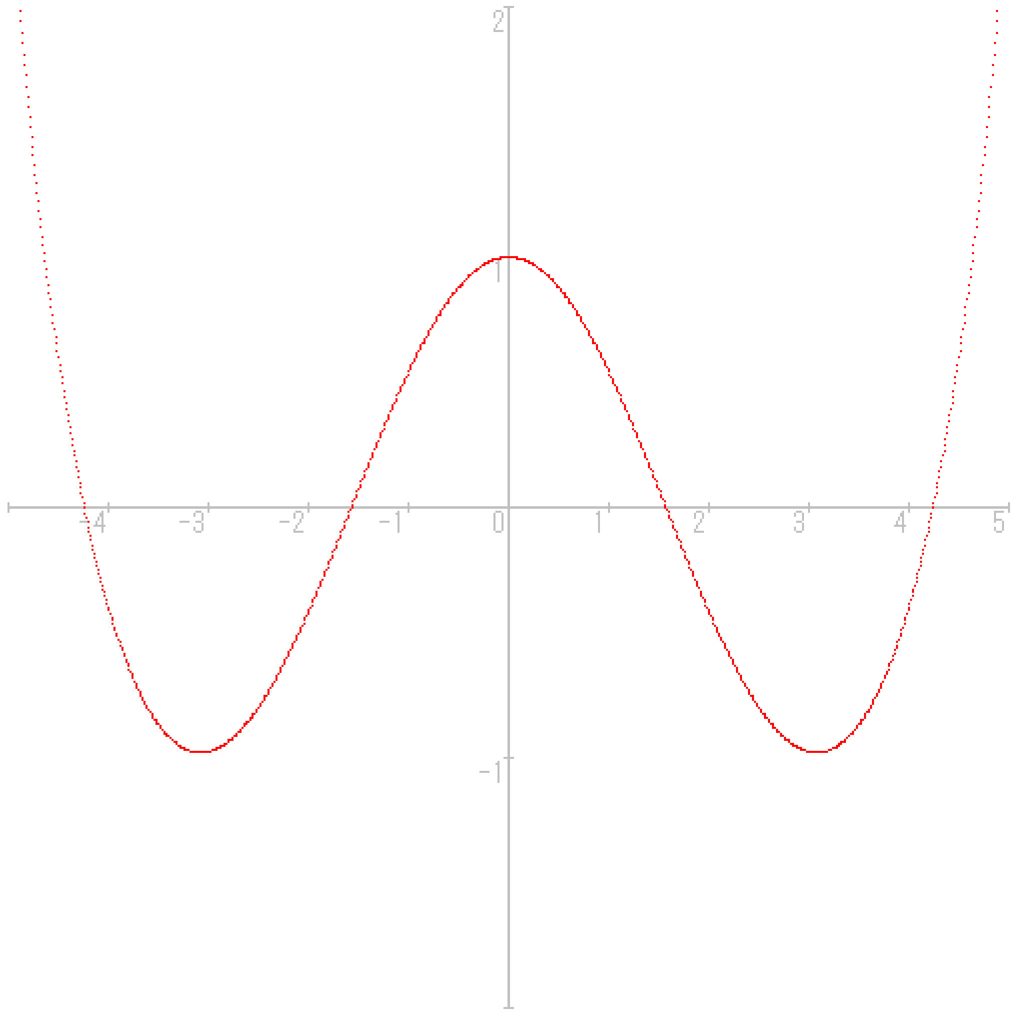
2
1
-1
-4
-3
-2
-1
0
1
2
3
4
5

Example 3

$$\cos x$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Numerical Computing
with
BASIC



2
1
-1
-4
-3
-2
-1
0
1
2
3
4
5

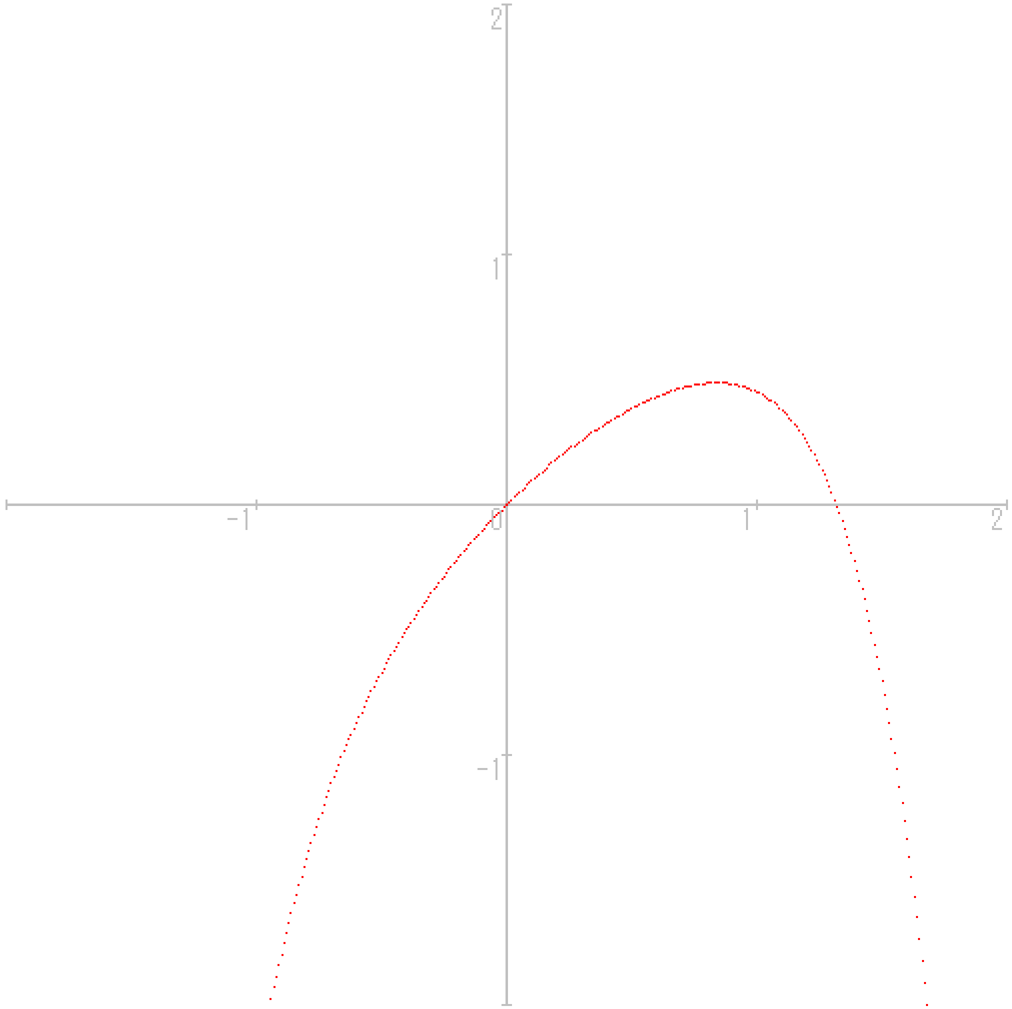
Example 4

$$\log_e(1+x)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$(-1 < x \leq 1)$$

Numerical Computing
with
BASIC



2
1
-1
0
1
2
-1

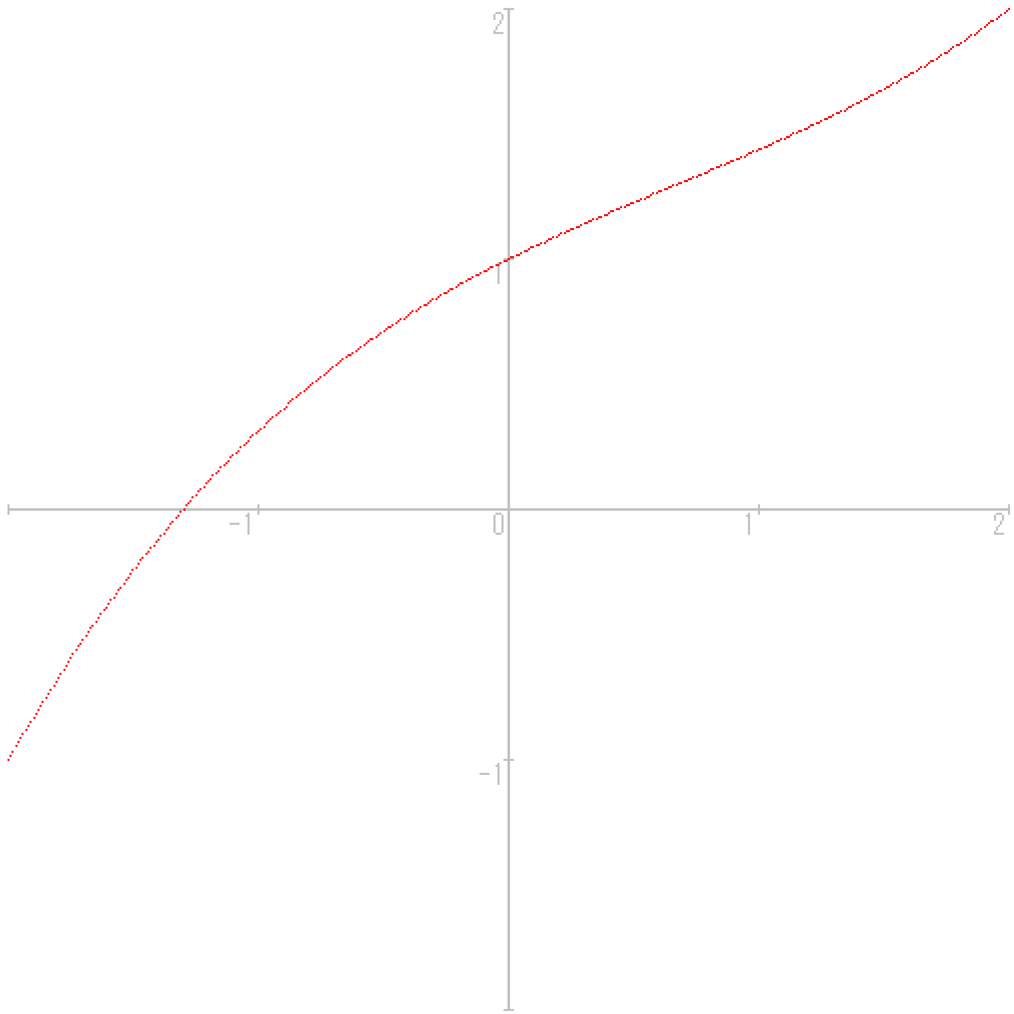
Example 5

$$\sqrt{1+x}$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots$$

$$(-1 < x < 1)$$

Numerical Computing with BASIC



2
1
0
−1
−1
1
2

Probabilistic Approach

Purpose

Any continuous function defined on the closed interval [0,1] may be approximated uniformly by Bernstein's polynomials.

Bernstein's Polynomial Approximation Theorem

Bernstein's Polynomial Approximation Theorem

$$f(x) \in C[0,1]$$

$$f_n(p) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} p^k (1-p)^{n-k}, \quad 0 \le p \le 1$$

($n-$**th Bernstein's polynomial**)

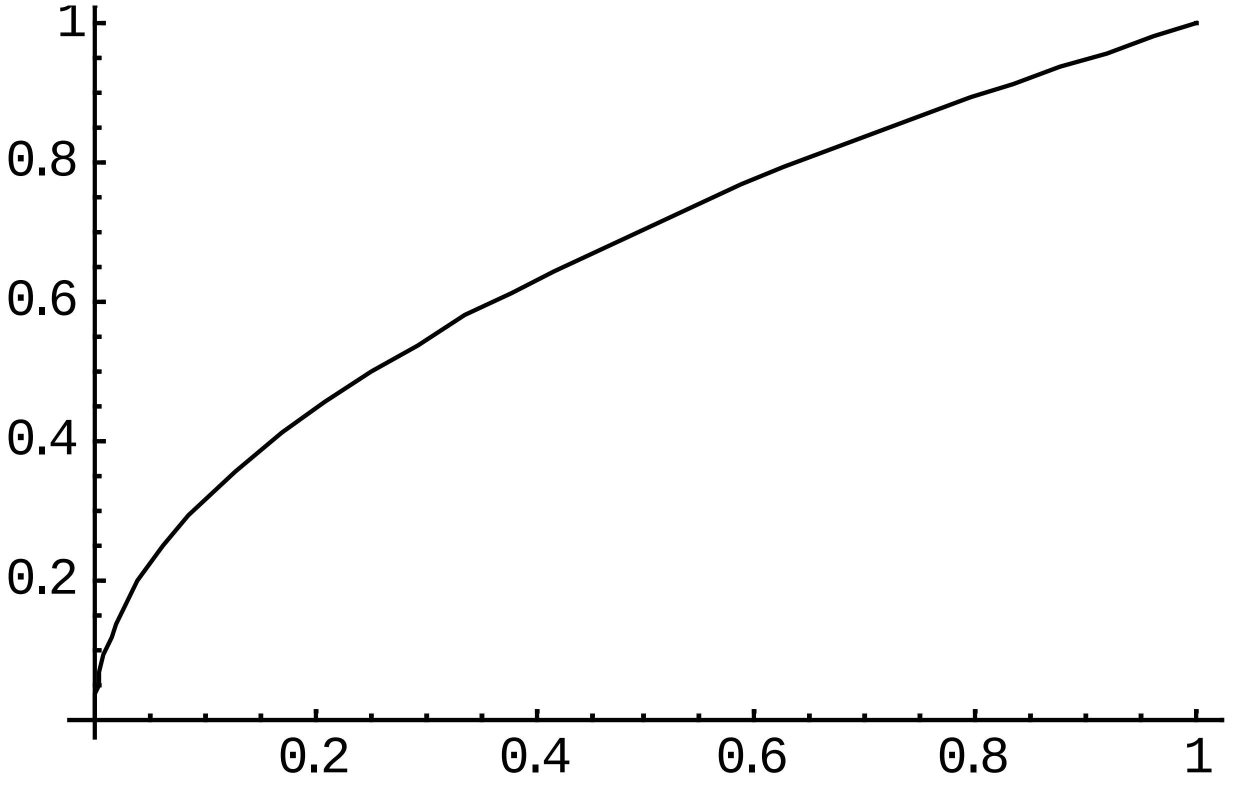
$\Rightarrow$

$\forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbf{N}$ **such that**

$$\forall n \ge N \Rightarrow \max_{0 \le p \le 1} \left| f(p) - f_n(p) \right| < \varepsilon$$

Numerical Computing With MATHEMATICA

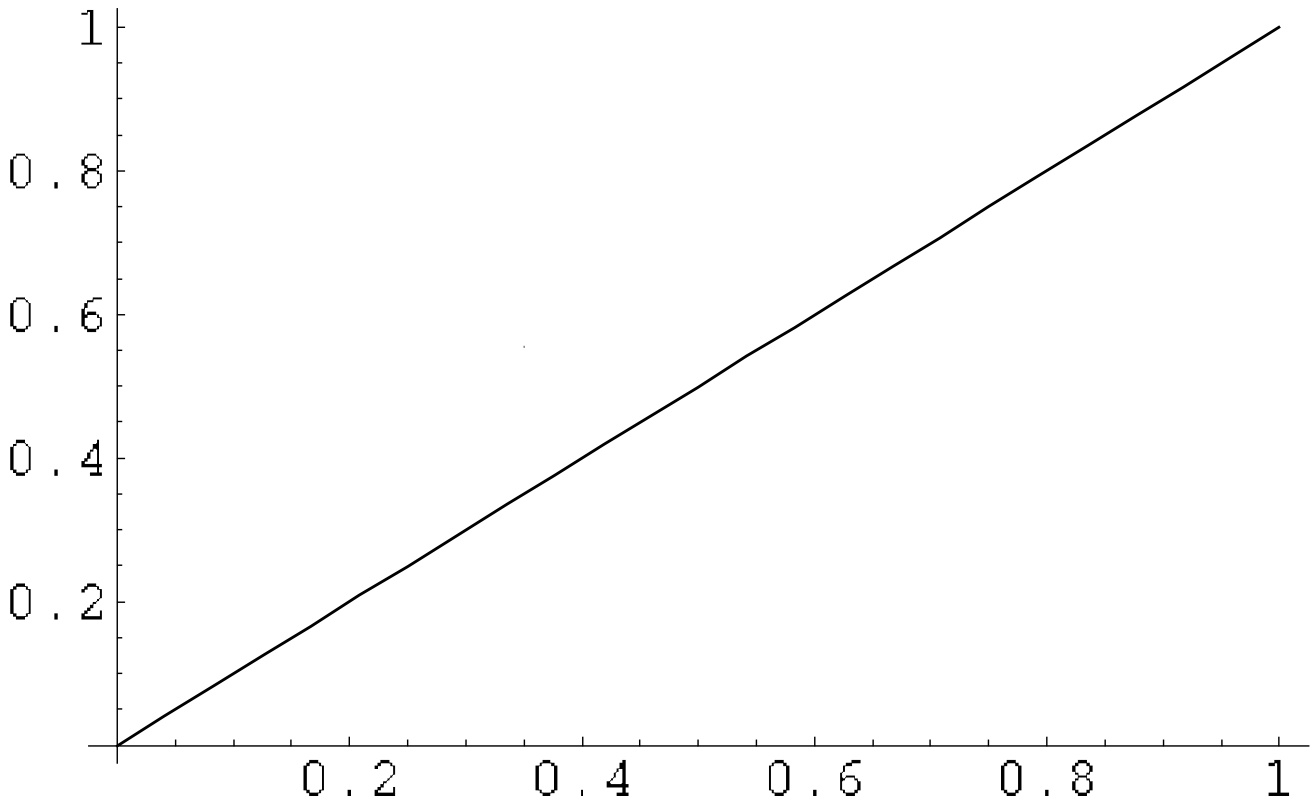
$$f(x) = \sqrt{x}$$



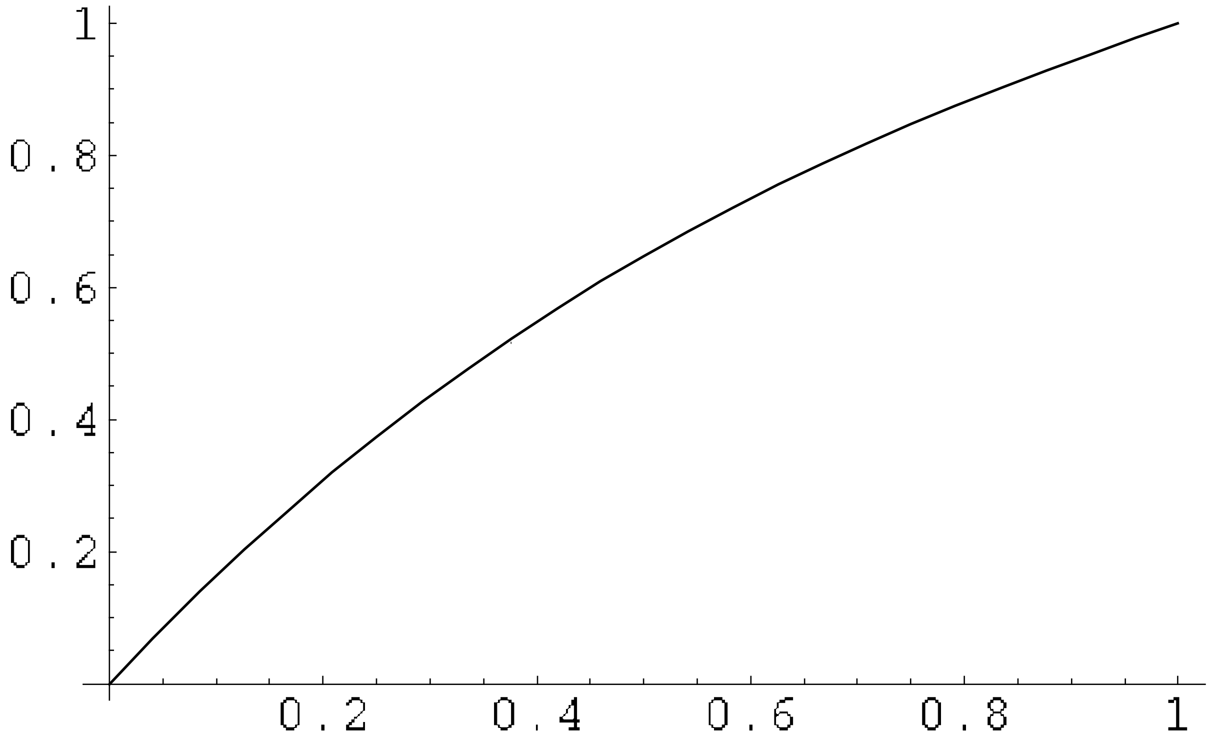
Bernstein's Polynomial

$$f_n(x) := \sum_{k=0}^{n} \sqrt{\frac{k}{n}} \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k}, \quad 0 \le x \le 1$$

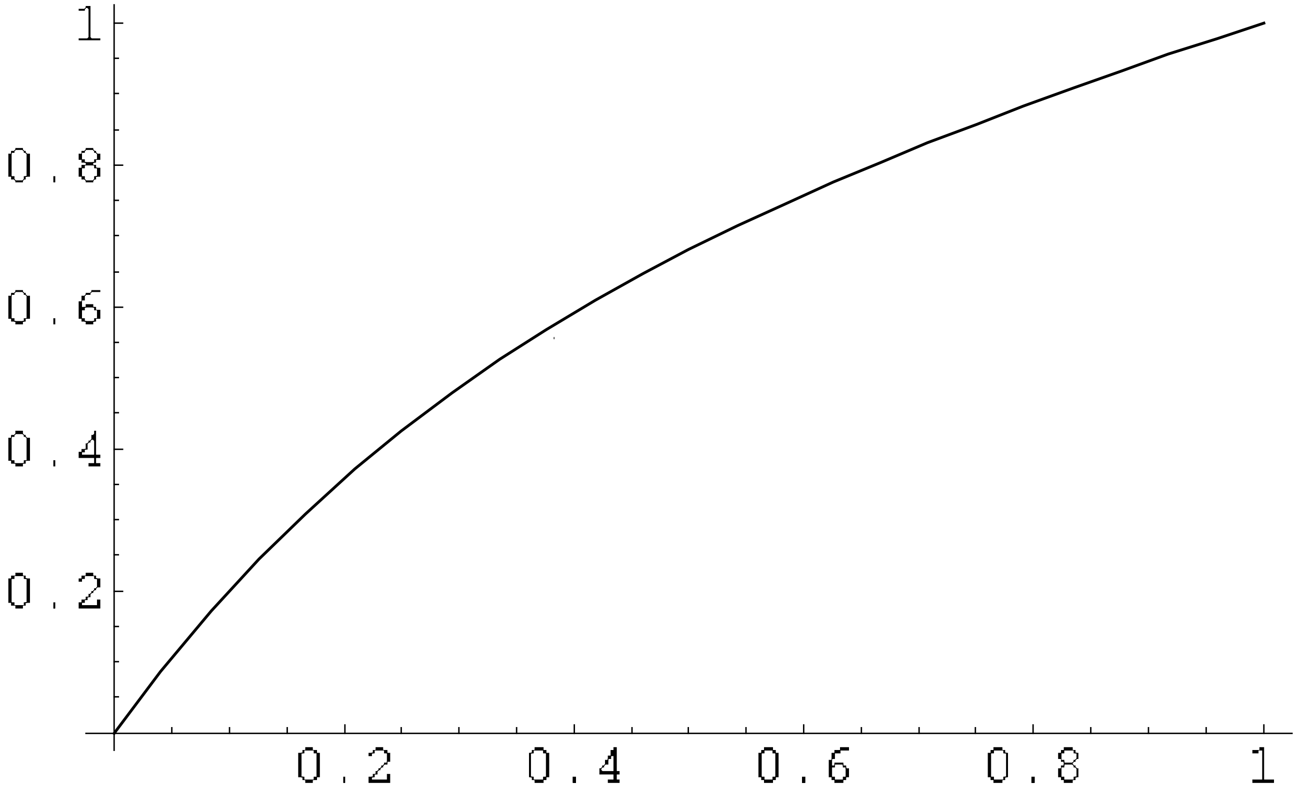
$f_1(x)$



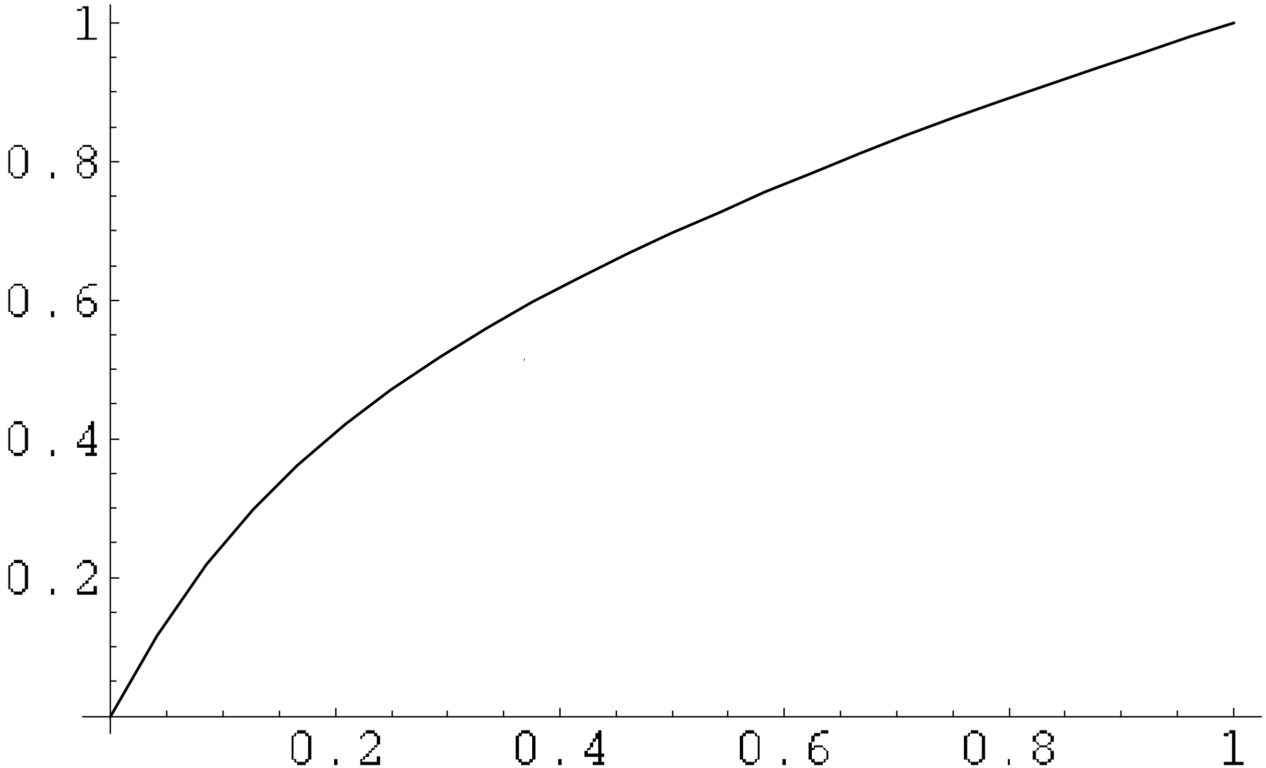
$f_3(x)$



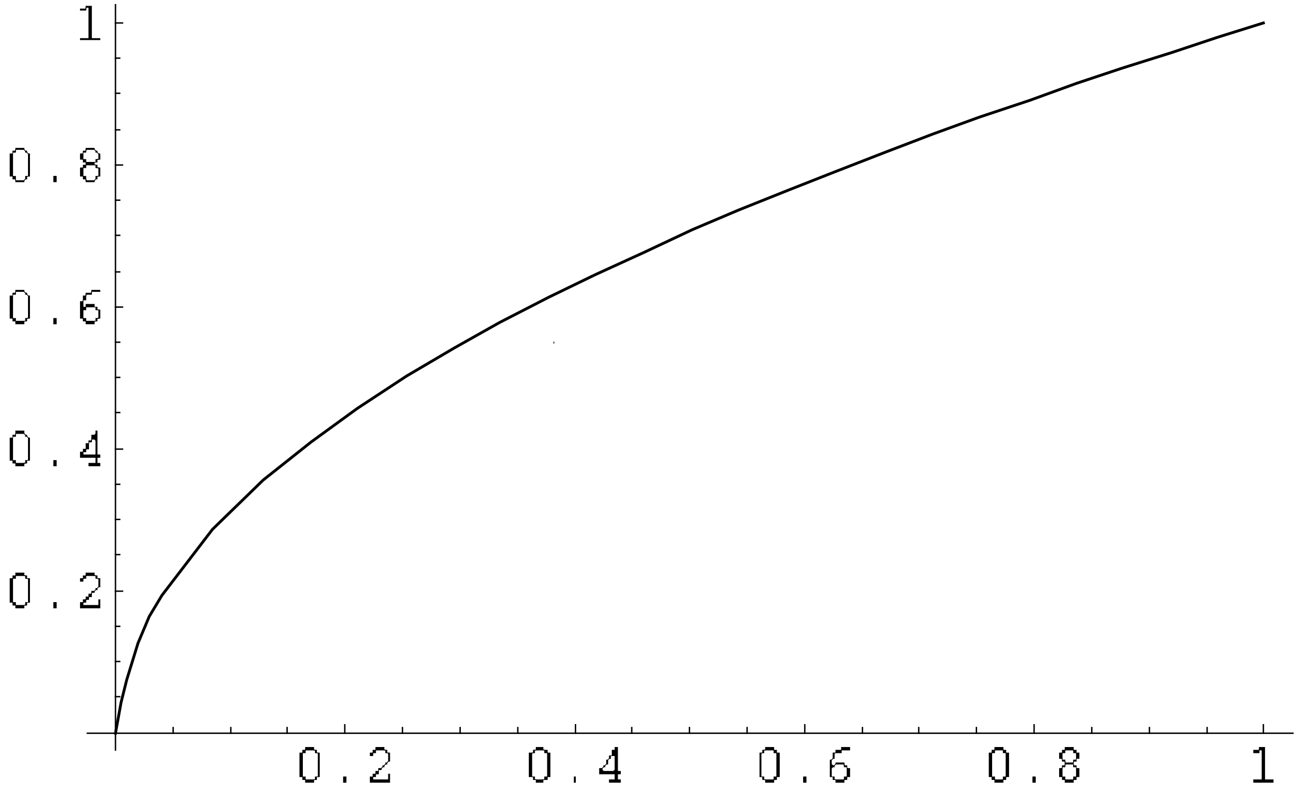
$f_5(x)$



$$f_{10}(x)$$



$$f_{100}(x)$$



Probability

Probability Space

Ω :**Sample Space**

B :**Completely Additive Class**

P :**Probability Measure**

$\left(\Omega, \mathrm{B}, \mathrm{P}\right)$: **Probability Space**

X_i: **Random Variable (Measurable)**

$E(X_i) = \int_\Omega X_i(\omega) d\mathrm{P}$: **Mean (Expectation)**

$V(X_i) = E\left((X_i - E(X_i))^2\right)$: **Variance**

Independent Variables

$\left(\Omega, \mathrm{B}, \mathrm{P}\right)$: **Probability Space**

X_1, X_2 **Independent**

$\underset{\text{def}}{\Longleftrightarrow}$

$\forall$ **Borel Sets** A_1, A_2

$$\mathrm{P}(\{\omega \in \Omega \mid X_1(\omega) \in A_1, X_2(\omega) \in A_2\})$$

$$= \mathrm{P}(\{\omega \in \Omega \mid X_1(\omega) \in A_1\})\, \mathrm{P}(\{\omega \in \Omega \mid X_2(\omega) \in A_2\})$$

Chebychev's Inequality

$$\begin{cases} (\Omega, \mathrm{B}, \mathrm{P}) \text{ **Probability Space** } \\ X : \text{ **Random Variable** } \\ E(X^2) < \infty \end{cases}$$

$$\Rightarrow$$

$$\forall \varepsilon > 0$$

$$\mathrm{P}\left(\left\{\omega \in \Omega \mid \left|X(\omega) - E(X)\right| \geq \varepsilon\right\}\right) \leq \frac{V(X)}{\varepsilon^2}$$

Coin Flipping

Bernoulli Distribution

$$\begin{cases} \textbf{probability of a head 1}: & p \quad (0 \le p \le 1) \\ \textbf{probability of a tail 0}: & 1-p \end{cases}$$

$$X_n : \textbf{n-th throw}$$

$$\Rightarrow$$

$$\begin{cases} \mathrm{P}(X_n = 1) = p \\ \mathrm{P}(X_n = 0) = 1 - p \end{cases}$$

$$\begin{cases} E(X_n) = 1 \cdot \mathrm{P}(X_n = 1) + 0 \cdot \mathrm{P}(X_n = 0) = p \\ V(X_n) = E(X_n^2) - E(X_n)^2 = p(1-p) \end{cases}$$

Bernstein's Polynomial Approximation Theorem

$f(x) \in C[0,1]$

$$f_n(p) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} p^k (1-p)^{n-k}, \quad 0 \le p \le 1$$

($n-$ **th Bernstein's polynomial**)

$\Rightarrow$

$\forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbf{N}$ **such that**

$$\forall n \ge N \Rightarrow \max_{0 \le p \le 1} \left| f(p) - f_n(p) \right| < \varepsilon$$

Mathematical Biology

Population Models

- **Two predator-prey type species residing in a common district.**
- **Two competing species residing in a common district.**

Predator-Prey Model (Lotka-Volterra)

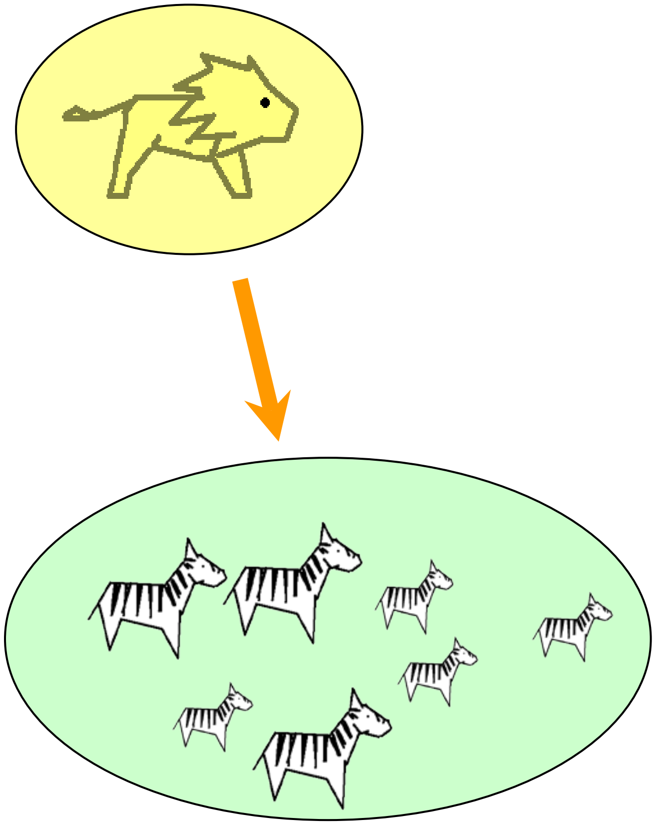
Vito Volterra

Volterra

◆**Vito Volterra (1860-1940)**

Italian Mathematician and Physicist

Predator-Prey Model



$$\begin{cases} \dfrac{dx}{dt} = (1 - y(t))\,x(t) \\[2ex] \dfrac{dy}{dt} = (-1 + x(t))\,y(t) \\[2ex] x(0) = 20 \\[1ex] y(0) = 2 \end{cases}$$

Model

Differential Equations

Predator-Prey Model

$$\begin{cases} \dfrac{dx}{dt} = r_1(1 - ay(t))x(t) \\ \dfrac{dy}{dt} = r_2(-1 + bx(t))y(t) \end{cases}$$

$x(t)$: **density of the Prey**

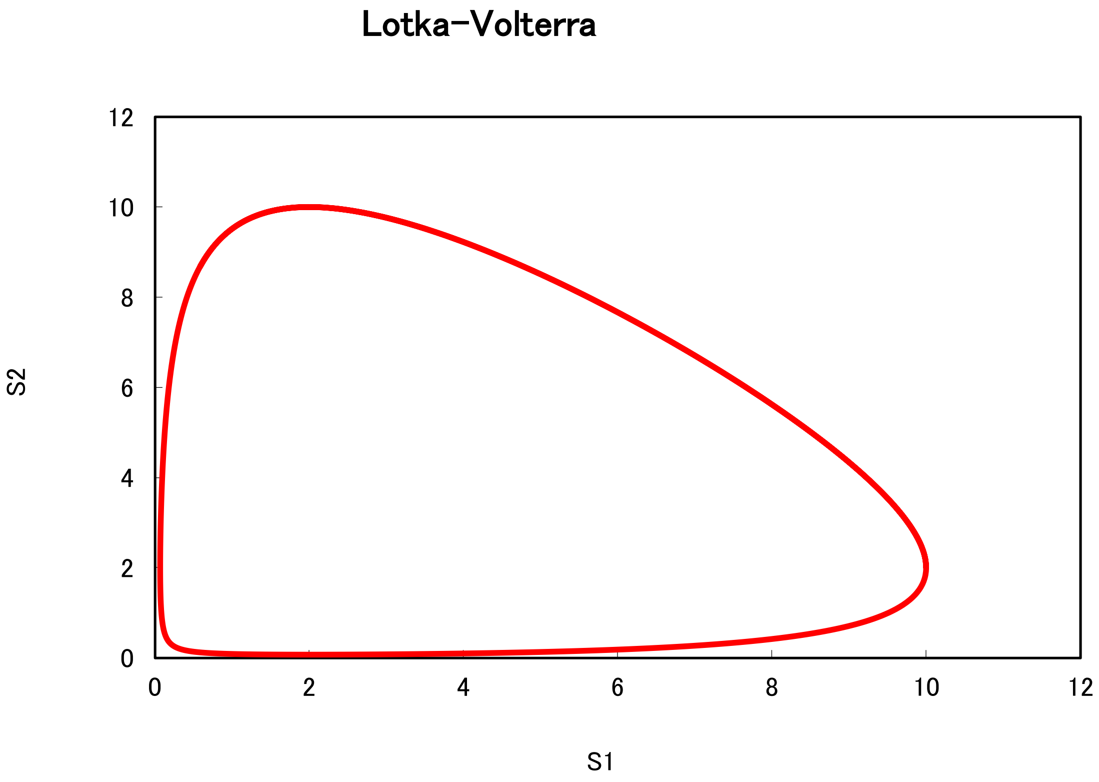
$y(t)$: **density of the Predator**

Numerical Computing
with
Excel (VBA)

Lotka-Volterra's Model

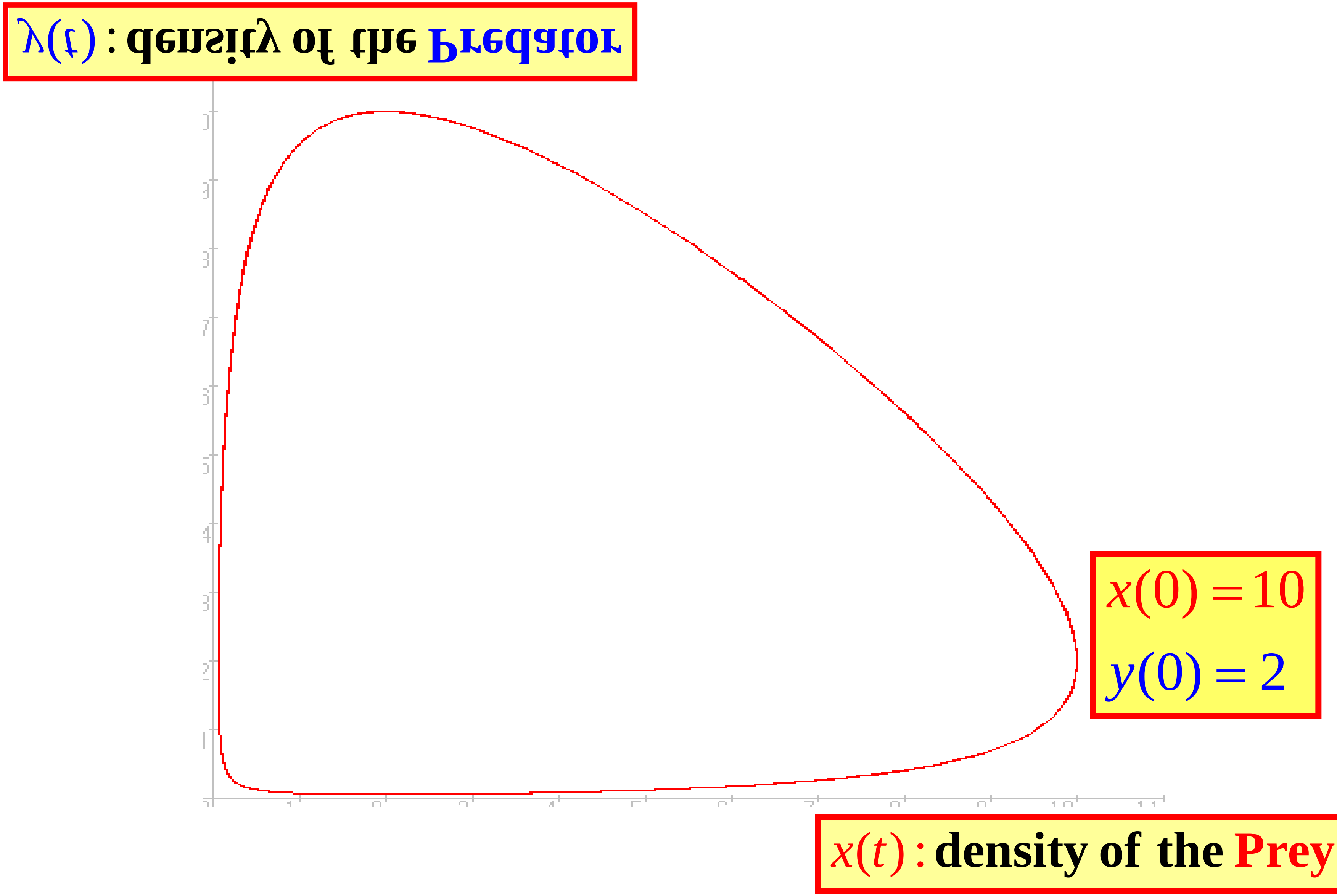
$$\begin{cases} \dfrac{dx}{dt} = (2 - y(t))x(t) \\ \dfrac{dy}{dt} = (-2 + x(t))y(t) \\ x(0) = 10 \\ y(0) = 2 \end{cases}$$

Runge-Kutta Method

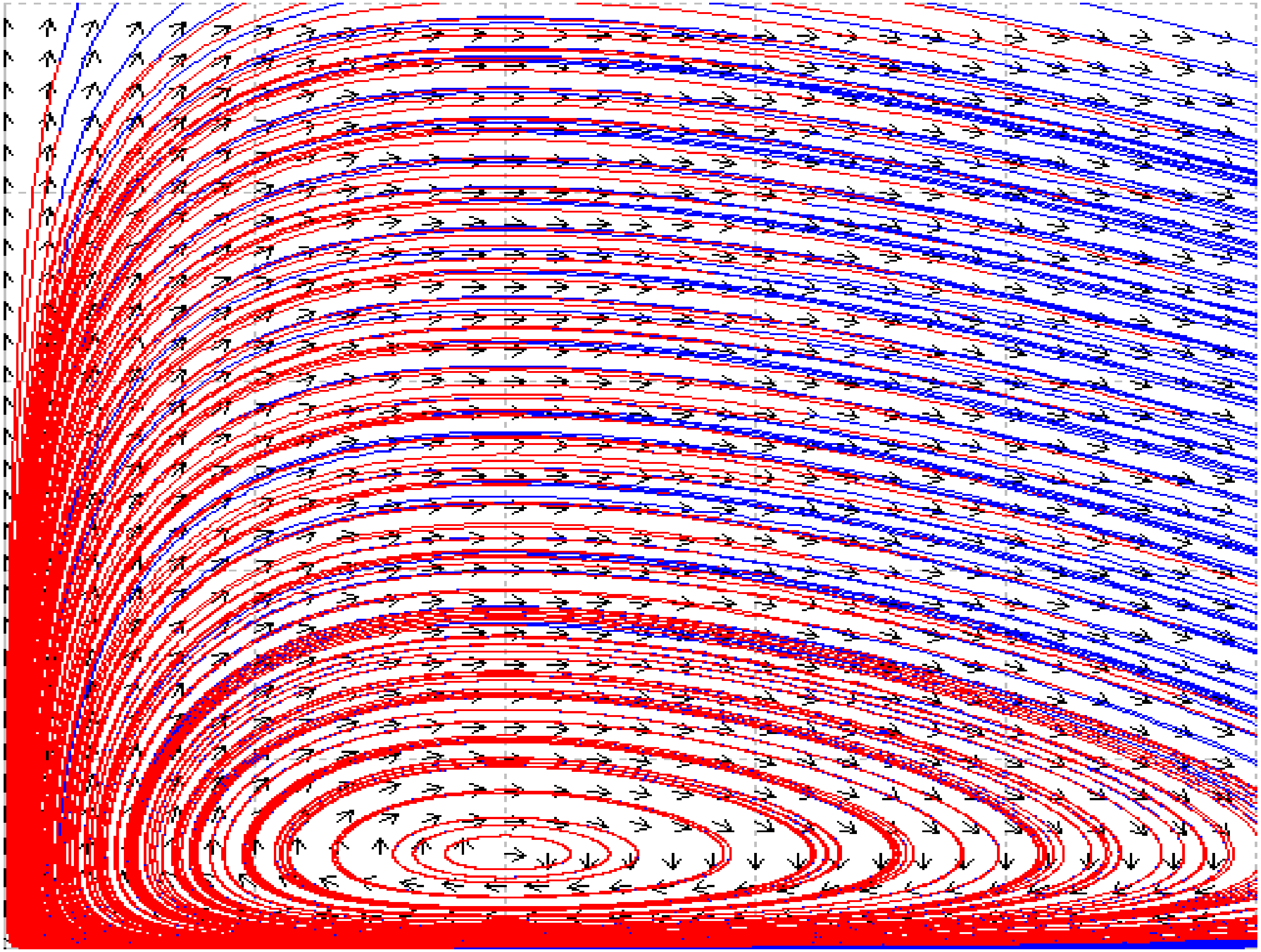


Numerical Computing
with
BASIC

Runge-Kutta Method



Trajectories depending on Initial Values



Competitive Model (Volterra)

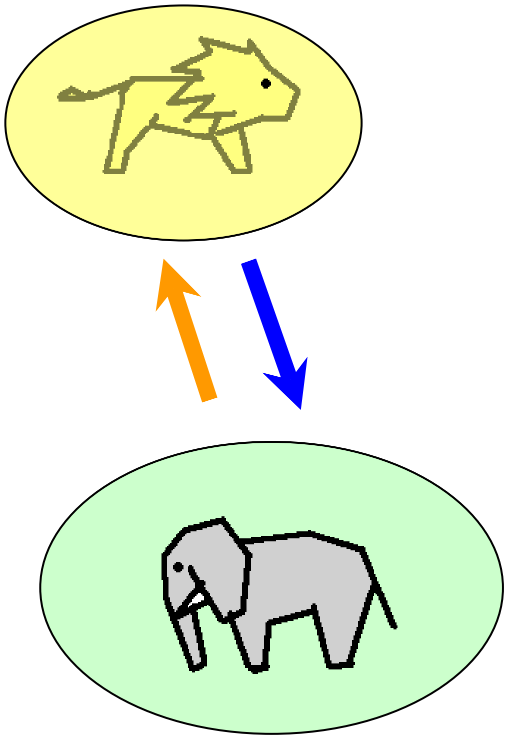
Competitive Model

$$\begin{cases} \dfrac{dx}{dt} = r_1(1 - ax(t) - by(t))x(t) \\ \dfrac{dy}{dt} = r_2(1 - cy(t) - dx(t))y(t) \end{cases}$$

$x(t)$: **density of species 1**

$y(t)$: **density of species 2**

Competitive Model



$$\begin{cases} \dfrac{dx}{dt} = r_1(1 - ax(t) - by(t))x(t) \\ \dfrac{dy}{dt} = r_2(1 - cy(t) - dx(t))y(t) \\ x(0) = \alpha \\ y(0) = \beta \end{cases}$$

Model

Differential Equations

Stability of Solutions

Numerical Computing with BASIC

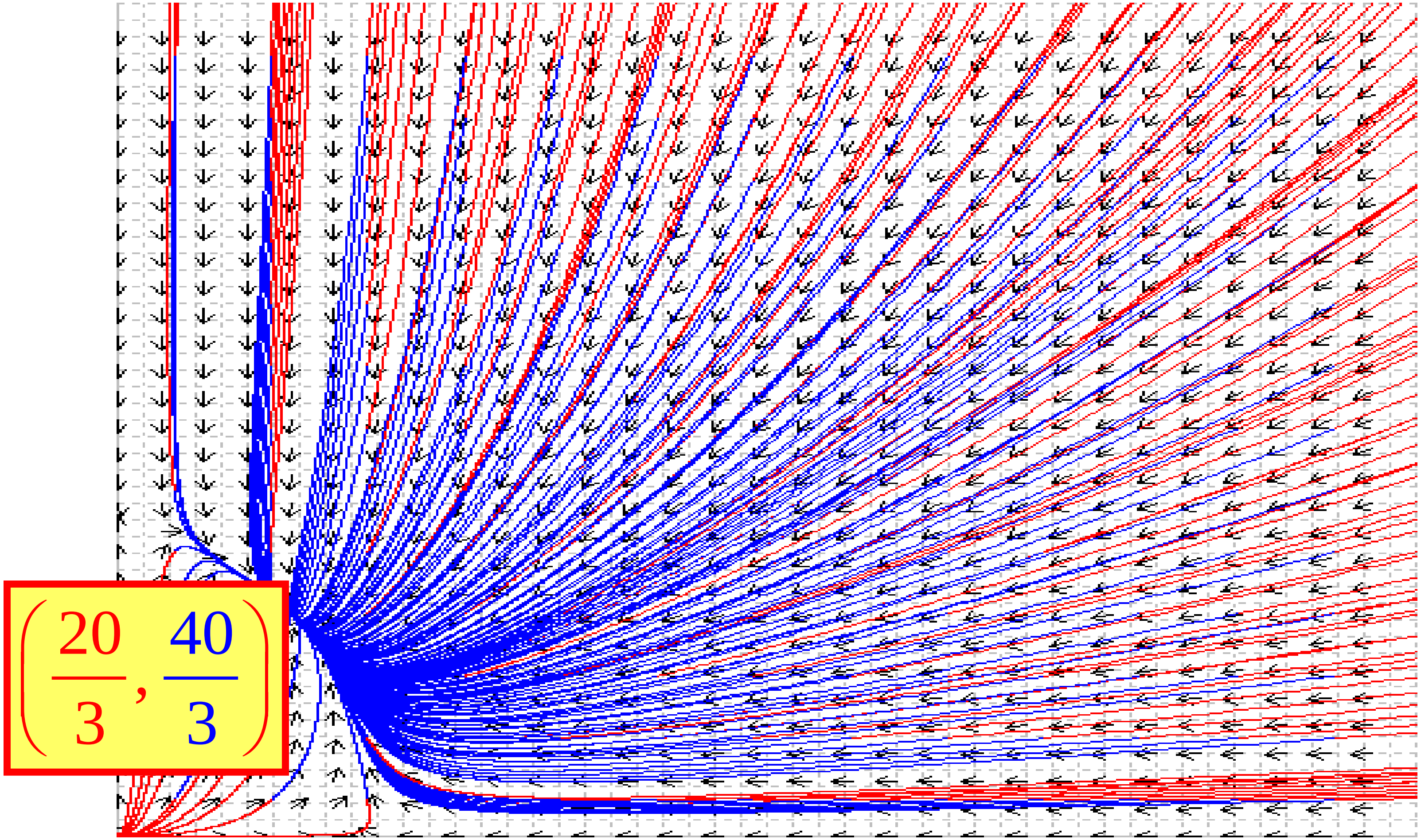
Coexistence Model

$$\begin{cases} \dfrac{dx}{dt} = (1 - 0.1x(t) - 0.025y(t))x(t) \\ \dfrac{dy}{dt} = 2(1 - 0.05y(t) - 0.05x(t))y(t) \end{cases}$$

Runge-Kutta Method

```
DEF F(X,Y)=(1-0.1*X-0.025*Y)*X
DEF G(X,y)=2*(1-0.05*Y-0.05*X)*Y
DEF FF(X,Y)=F(X,Y)/SQR(F(X,Y)^2+G(X,Y)^2)
DEF GG(X,Y)=G(X,Y)/SQR(F(X,Y)^2+G(X,Y)^2)
LET  h0=0.005
LET M=50
SET WINDOW 0,M,0,M
LET MS=M/30
DRAW Grid
FOR Y=-M TO M STEP MS
  FOR X=-M TO M STEP MS
    PLOT LINES
    PLOT LINES: X,Y;
    PLOT LINES : X+FF(X,Y)*MS/2,  Y+GG(X,Y)*MS/2
    PLOT LINES : X+(FF(X,Y)+GG(X,Y))*MS/5,  Y+(GG(X,Y)-FF(X,Y))*MS/5;
    PLOT LINES : X+FF(X,Y)*MS/2,  Y+GG(X,Y)*MS/2
    PLOT LINES : X+(FF(X,Y)-GG(X,Y))*MS/5,  Y+(GG(X,Y)+FF(X,Y))*MS/5;
    PLOT LINES : X+FF(X,Y)*MS/2,  Y+GG(X,Y)*MS/2
  NEXT X
NEXT Y
PAUSE
FOR Y00=-M TO M STEP MS*SQR(5)
  FOR X00=-M TO M STEP MS*SQR(5)
    FOR K=1 TO 2
      LET x0=X00
      LET Y0=Y00
      LET H=(-1)^(K-1)*H0
      SET COLOR 2^k
      PLOT LINES
      PLOT LINES: X0,Y0;
      LET COUNT=0
      DO WHILE ABS(X0)<2*M  AND ABS(Y0)<2*M  AND COUNT<1000
        LET COUNT=COUNT+1
        LET X=X0+F(X0,Y0)*H/2
        LET Y=Y0+G(X0,Y0)*H/2
        LET X1=X0+F(X,Y)*H
        LET Y1=Y0+G(X,Y)*H
        PLOT LINES: X1,Y1;
        LET X0=X1
        LET Y0=Y1
      LOOP
    NEXT K
  NEXT X00
NEXT Y00
END
```

One Stable Point



Stable Point

$$\boxed{\frac{dx_{\infty}}{dt} = 0, \quad \frac{dy_{\infty}}{dt} = 0}$$

$$\Longrightarrow$$

$$\begin{cases} 1 - 0.1x_{\infty} - 0.025y_{\infty} = 0 \\ 1 - 0.05y_{\infty} - 0.05x_{\infty} = 0 \end{cases}$$

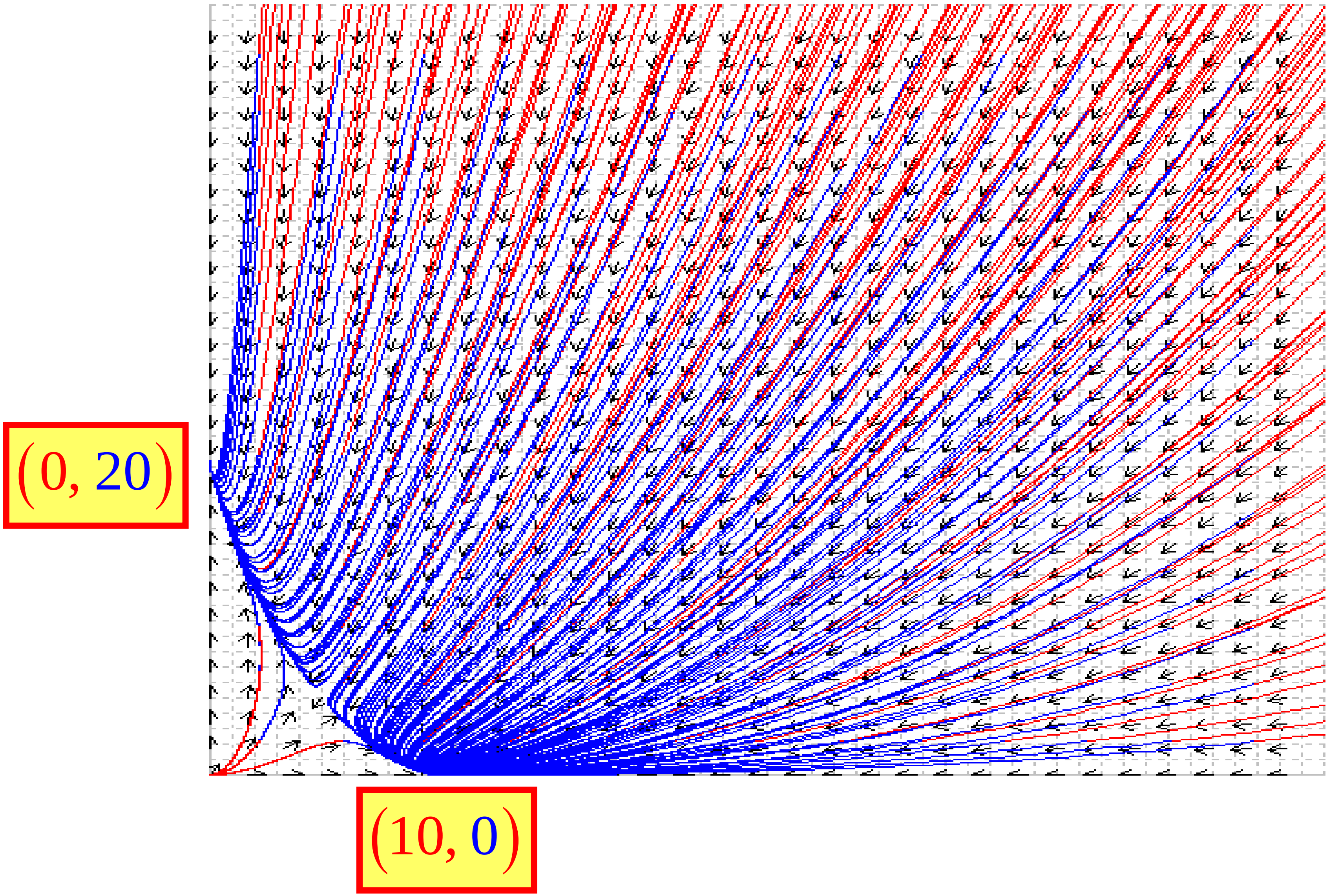
$$\Longrightarrow$$

$$\boxed{x_{\infty} = \frac{20}{3}, \quad y_{\infty} = \frac{40}{3}}$$

Bistable Model

$$\begin{cases} \dfrac{dx}{dt} = (1 - 0.1x(t) - 0.1y(t))x(t) \\ \dfrac{dy}{dt} = 2(1 - 0.05y(t) - 0.15x(t))y(t) \end{cases}$$

Two Stable Points



Stable Point (1)

$$\boxed{\frac{dx_\infty}{dt} = 0, \quad \frac{dy_\infty}{dt} = 0}$$

$$\Longrightarrow$$

$$\begin{cases} 1 - 0.1x_\infty - 0.1y_\infty = 0 \\ y_\infty = 0 \end{cases}$$

$$\Longrightarrow$$

$$\boxed{x_\infty = 10, \quad y_\infty = 0}$$

Stable Point (2)

$$\boxed{\frac{dx_\infty}{dt} = 0, \quad \frac{dy_\infty}{dt} = 0}$$

$$\Rightarrow$$

$$\begin{cases} x_\infty = 0 \\ 1 - 0.05 y_\infty - 0.15 x_\infty = 0 \end{cases}$$

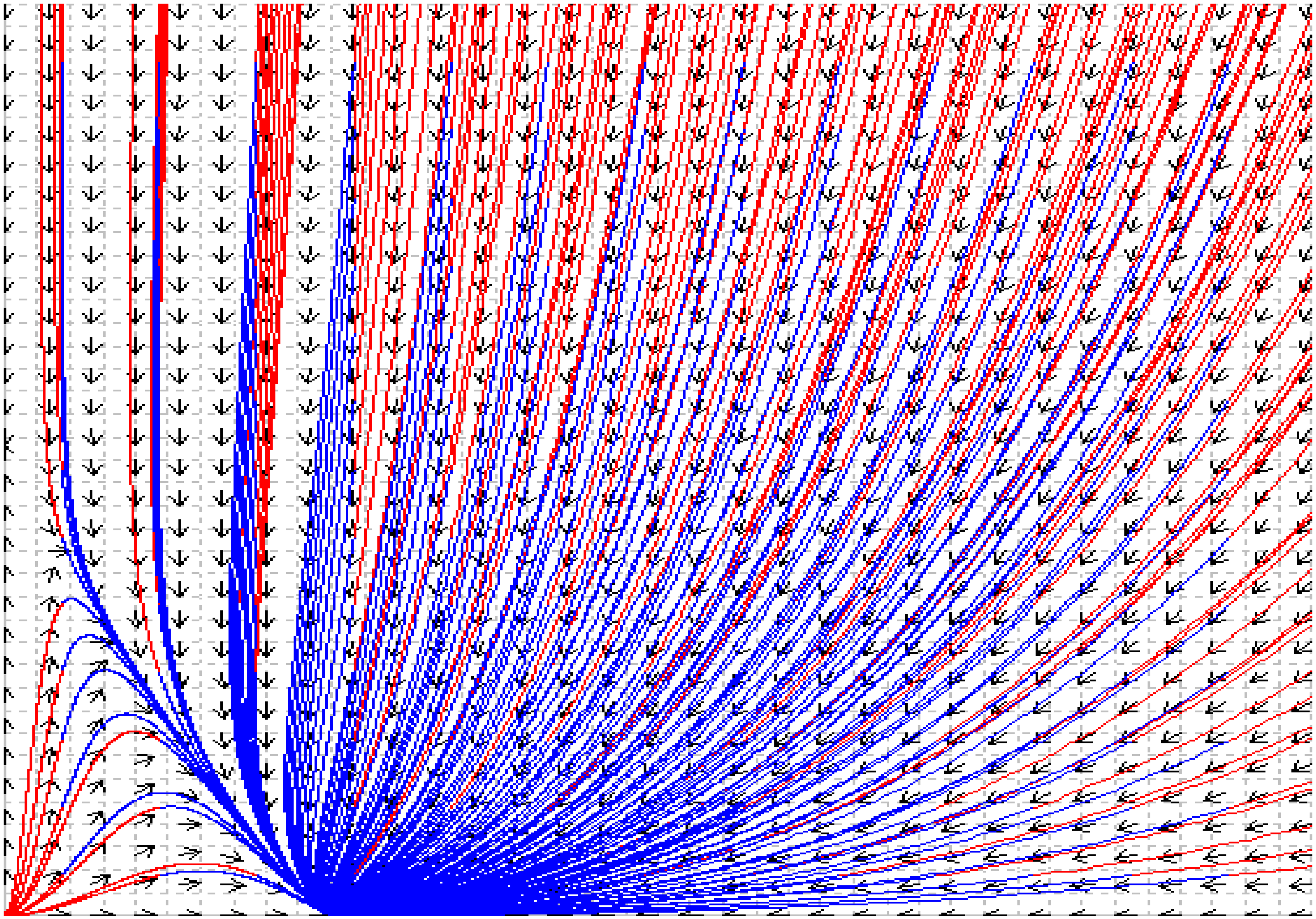
$$\Rightarrow$$

$$\boxed{x_\infty = 0, \quad y_\infty = 20}$$

Dominant Model (1)

$$\begin{cases} \dfrac{dx}{dt} = (1 - 0.1x(t) - 0.025y(t))x(t) \\ \dfrac{dy}{dt} = 2(1 - 0.05y(t) - 0.15x(t))y(t) \end{cases}$$

Dominant Model (1)



(10, 0)

Stable Point (1)

$$\boxed{\frac{dx_\infty}{dt} = 0, \quad \frac{dy_\infty}{dt} = 0}$$

$$\Longrightarrow$$

$$\begin{cases} 1 - 0.1x_\infty - 0.025y_\infty = 0 \\ y_\infty = 0 \end{cases}$$

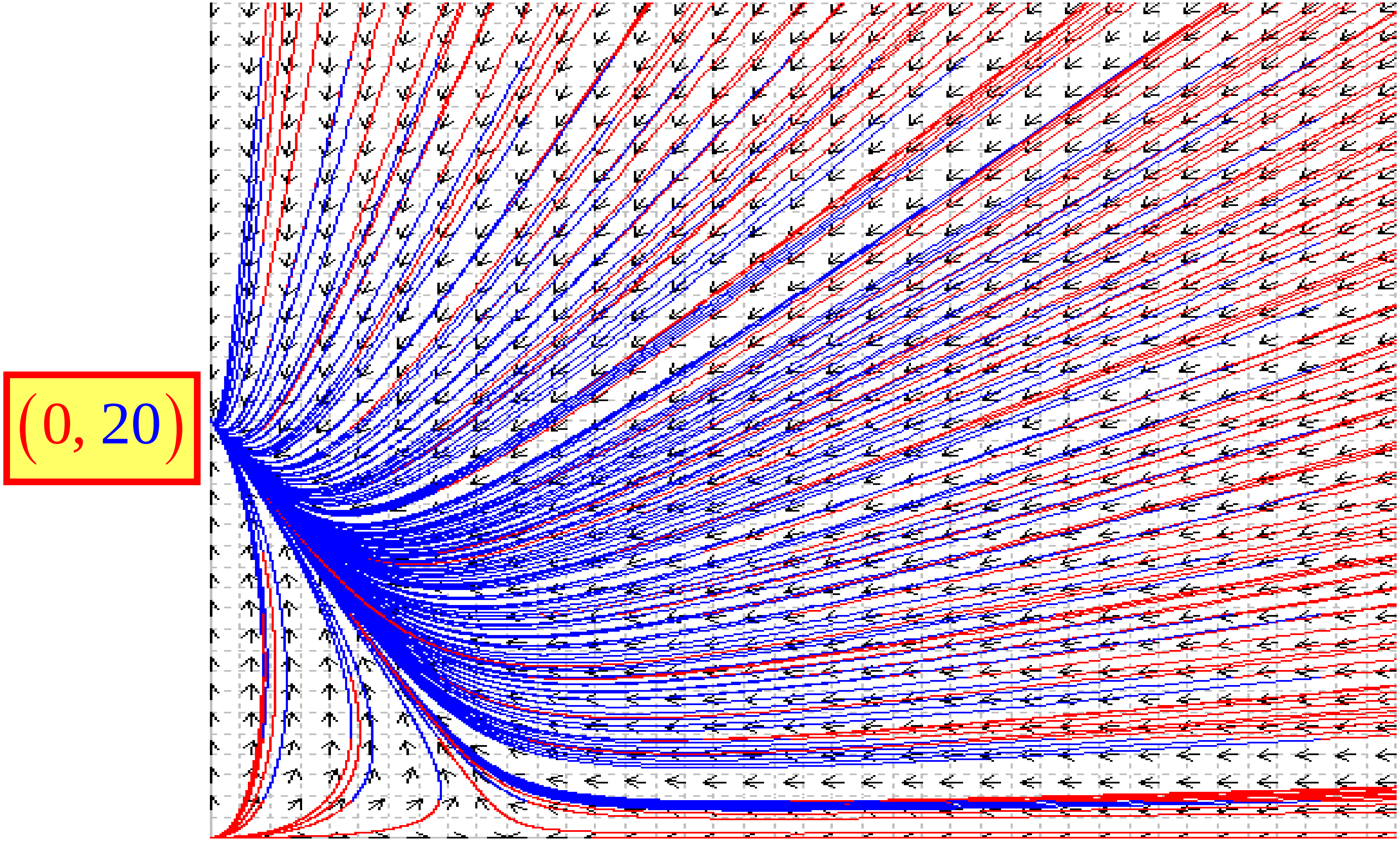
$$\Longrightarrow$$

$$\boxed{x_\infty = 10, \quad y_\infty = 0}$$

Dominant Model (2)

$$\begin{cases} \dfrac{dx}{dt} = (1 - 0.1x(t) - 0.1y(t))x(t) \\ \dfrac{dy}{dt} = 2(1 - 0.05y(t) - 0.05x(t))y(t) \end{cases}$$

Dominant Model (2)



Stable Point (2)

$$\boxed{\frac{dx_\infty}{dt} = 0, \quad \frac{dy_\infty}{dt} = 0}$$

$$\Rightarrow$$

$$\begin{cases} x_\infty = 0 \\ 1 - 0.05y_\infty - 0.05x_\infty = 0 \end{cases}$$

$$\Rightarrow$$

$$\boxed{x_\infty = 0, \quad y_\infty = 20}$$

Mathematical Study
of
Population Dynamics

Catchphrase

Theme	Real World
Population Dynamics	Problem of Population Growth

Thomas Robert Malthus

Malthus

- **Thomas Robert Malthus (1766-1834)**

English Economist

An Essay on the Principle of Population (1798)

Idea Credited to Malthus

- **A population will grow exponentially until limited by lack of available resources.**

Malthus Model

$$\begin{cases} \dfrac{dx}{dt} = ax \\ x(0) = x_0 \quad \textbf{(Initial Condition)} \end{cases}$$

$x(t)$: **Population Density**

a : **Growth Rate**

Computational Approach

Numerical Computing with BASIC

Example of Malthus

$$\begin{cases} \dfrac{dx}{dt} = 2x \\ x(0) = 5 \end{cases}$$

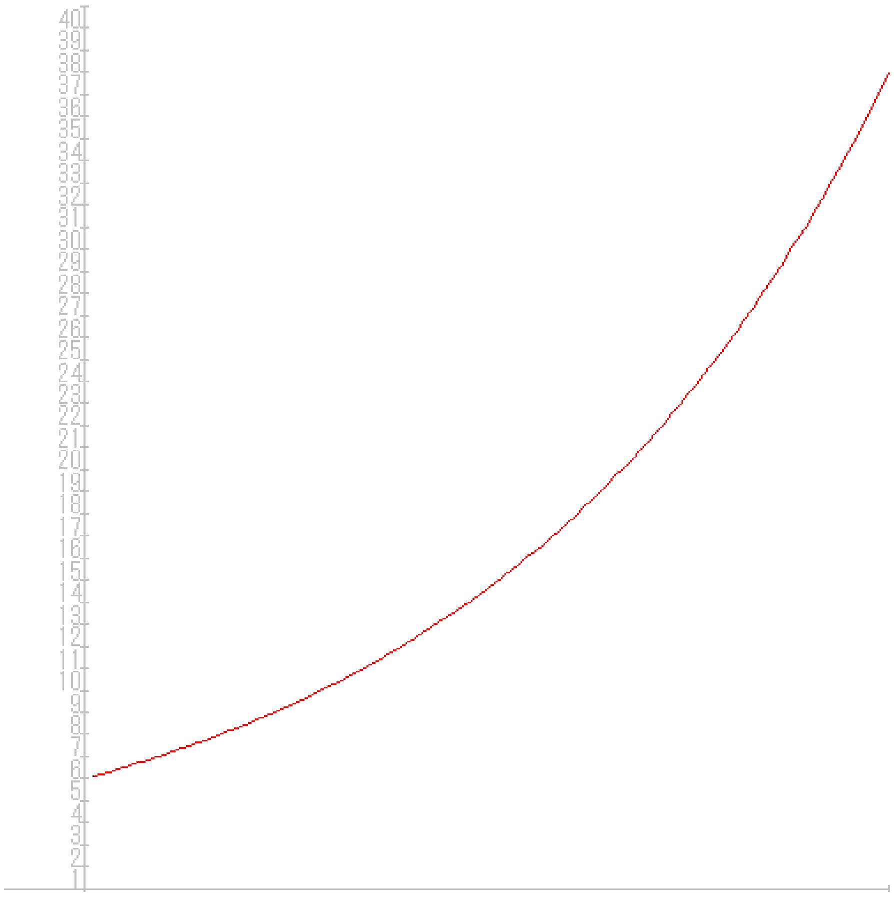
$$\Longrightarrow$$

$$\boxed{x(t) = 5e^{2t}}$$

Runge-Kutta Method

```
DEF F(x, y)=2*y
SET WINDOW -0.1,3,-0.1,60
DRAW axes
LET  x = 0
LET  y = 5
LET  h = 0.01
LET  N = 10

FOR i = 0 TO N STEP 0.01
   LET  k1 = F(x, y)
   LET  k2 = F(x + h / 2, y + h * k1 / 2)
   LET  k3 = F(x + h / 2, y + h * k2 / 2)
   LET  k4 = F(x + h, y + h * k3)

   LET  x = x + h
   LET  y = y + h * (k1 + 2 * k2 + 2 * k3 + k4) / 6
   PLOT LINES: x,y;
   SET LINE COLOR "red"
   WAIT DELAY 0.01
NEXT i
END
```

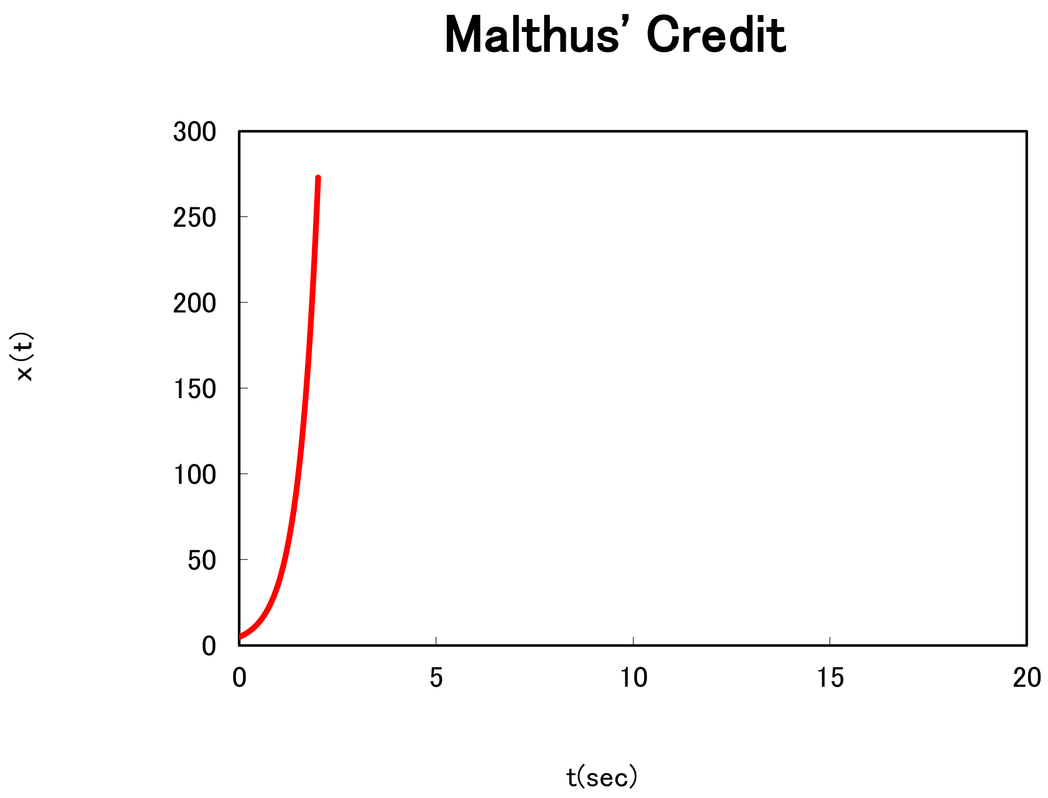
Runge-Kutta Method



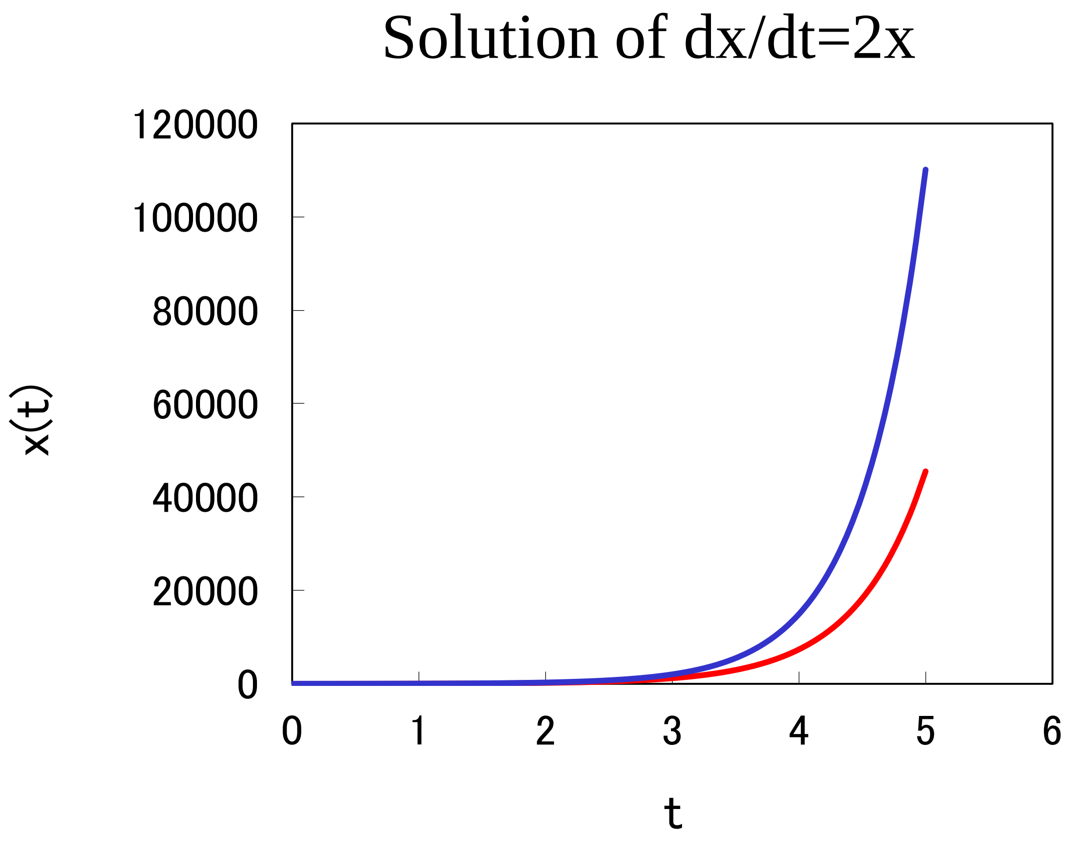
A population will grow exponentially.

Numerical Computing
with
Excel (VBA)

Runge-Kutta Method



Euler's Method



Pierre Francois Verhulst

Verhulst

- **Pierre Francois Verhulst (1804-1849)**
  Belgian Mathematical Biologist
  Notice sur la loi que la population poursuit dans son accroissement (1838)

Idea Credited to Verhulst

- **The growth rate of a population will depend on the effect of crowding within the population.**

Logistic Model (1)

$$\begin{cases} \dfrac{dx}{dt} = x(t)(\varepsilon - \lambda x(t)) \\ x(0) = x_0 \quad \textbf{(Initial Condition)} \end{cases}$$

$x(t)$: **Population Density**

ε : **Intrinsic Growth Rate**

λ : **Coefficient of Intraspecific Competition**

Logistic Model (2)

$$\begin{cases} \dfrac{dx}{dt} = \varepsilon x(t)\left(1 - \dfrac{x(t)}{K}\right) \\ x(0) = x_0 \quad \textbf{(Initial Condition)} \end{cases}$$

$x(t)$: **Population Density**

$K = \dfrac{\varepsilon}{\lambda}$: **Carrying Capacity**

Logistic Model (3)

$$\begin{cases} \dfrac{dx}{dt} = ax(t)(A - x(t)) \\ x(0) = x_0 \quad \textbf{(Initial Condition)} \end{cases}$$

$x(t)$: **Population Density**

$a = \dfrac{\varepsilon}{K}$: **Growth Rate**

$A = K$: **Carrying Capacity of the Environment**

Simplified Logistic Model (1)

$$\begin{cases} \dfrac{dx}{dt} = a\left(A - x(t)\right)x(t) \\ x(0) = x_0 \end{cases}$$

Simplified Logistic Model (2)

$$x(t) = \frac{x_0 A}{x_0 + (A - x_0)e^{-aAt}}$$

$$\to \frac{x_0 A}{x_0} = A \quad (t \to +\infty)$$

Biological Interpretation

- For small populations, we get exponential growth with rate aA.
- As $x(t)$ increases, the growth slows down and the population gradually reaches the carrying capacity of the environment.

Computational Approach

Numerical Computing with BASIC

Example (Large Initial Data)

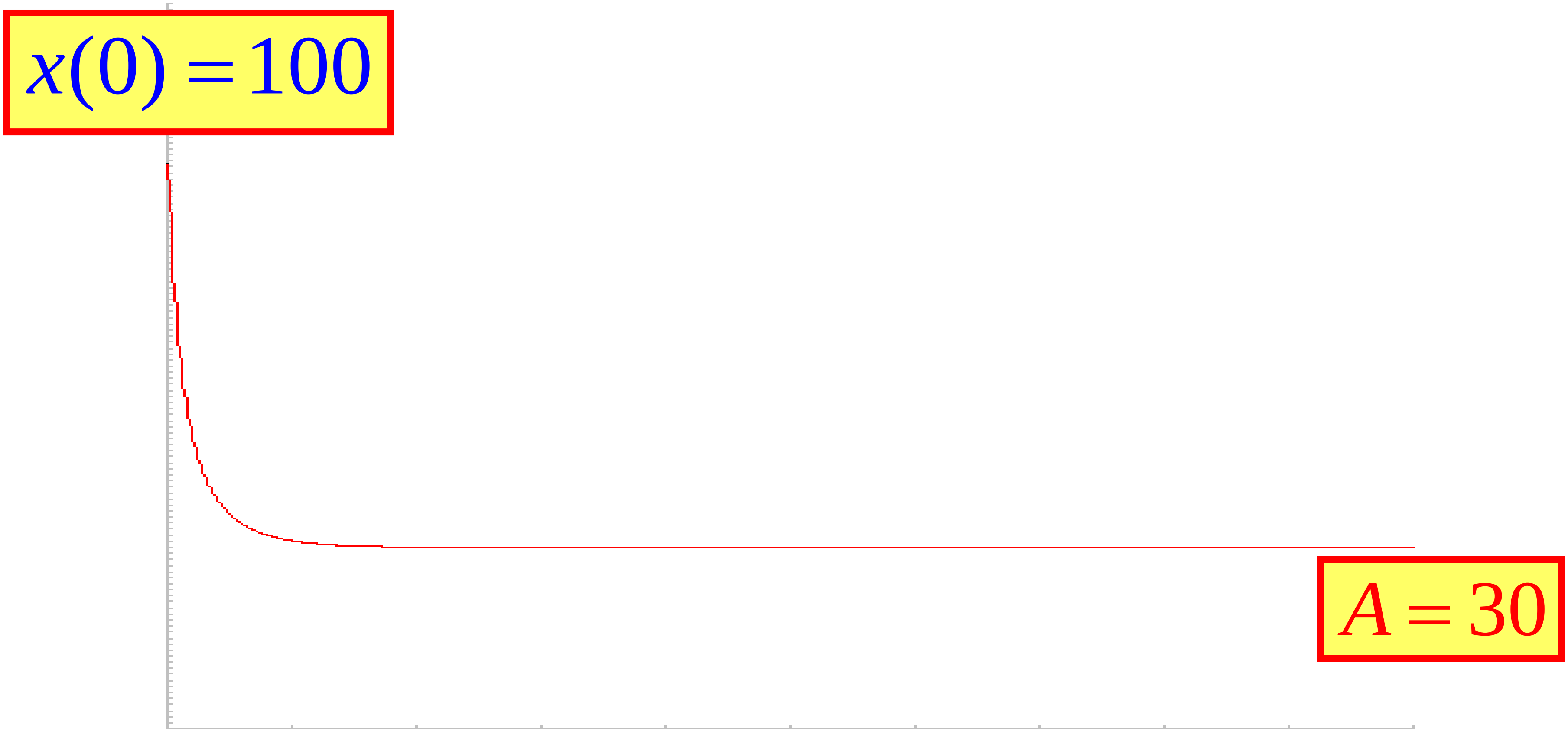
$$\begin{cases} \dfrac{dx}{dt} = \dfrac{1}{10}\big(30 - x(t)\big)\,x(t) \\ x(0) = 100 > 30 \end{cases}$$

$$a = \frac{1}{10}, \quad A = 30, \quad x_0 = 100$$

Runge-Kutta Method

```
DEF F(t,x) = (3 - 0.1 * x) * x
SET WINDOW  0,10,0,40
DRAW axes

LET  t = 0
LET  x = 100

LET  h = 0.01
LET  N = 10

FOR i = 0 TO N STEP 0.01
   LET  k1 = F(t, x)
   LET  k2 = F(t + h / 2, x + h * k1 / 2)
   LET  k3 = F(t + h / 2, x + h * k2 / 2)
   LET  k4 = F(t + h, x + h * k3)

   LET  t = t + h
   LET  x = x + h * (k1 + 2 * k2 + 2 * k3 + k4) / 6
   PLOT LINES: t,x;
   SET LINE COLOR 4
   WAIT DELAY 0.01
NEXT i
END
```

Runge-Kutta Method (Large Initial Data)

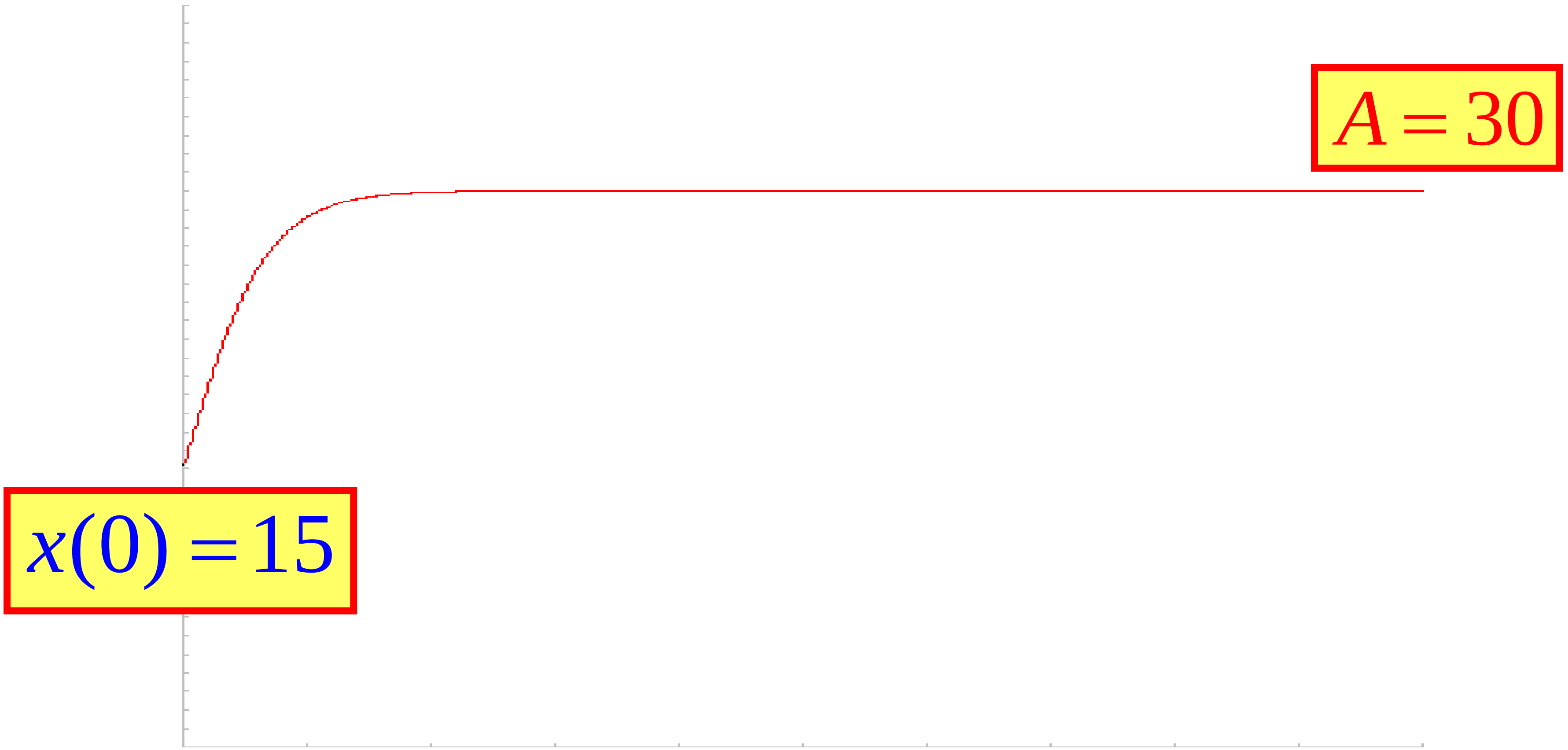


Example (Small Initial Data)

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{1}{10}\big(30 - x(t)\big)\,x(t) \\ x(0) = 15 < 30 \end{cases}$$

$$a = \frac{1}{10}, \quad A = 30, \quad x_0 = 15$$

Runge-Kutta Method (Small Initial Data)



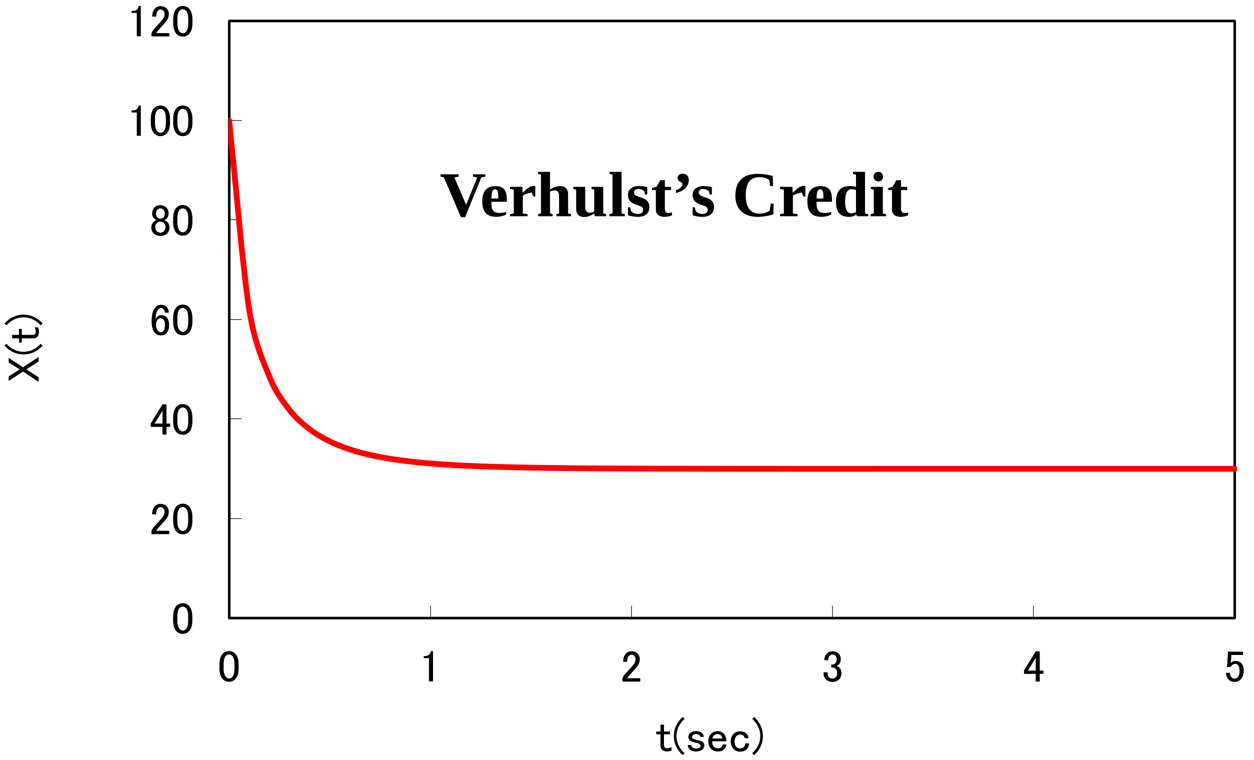
Numerical Computing
with
Excel (VBA)

Example (Large Initial Data)

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{1}{10}\big(30 - x(t)\big)\,x(t) \\ x(0) = 100 > 30 \end{cases}$$

$$a = \frac{1}{10}, \quad A = 30, \quad x_0 = 100$$

Runge-Kutta Method (Large Initial Data)

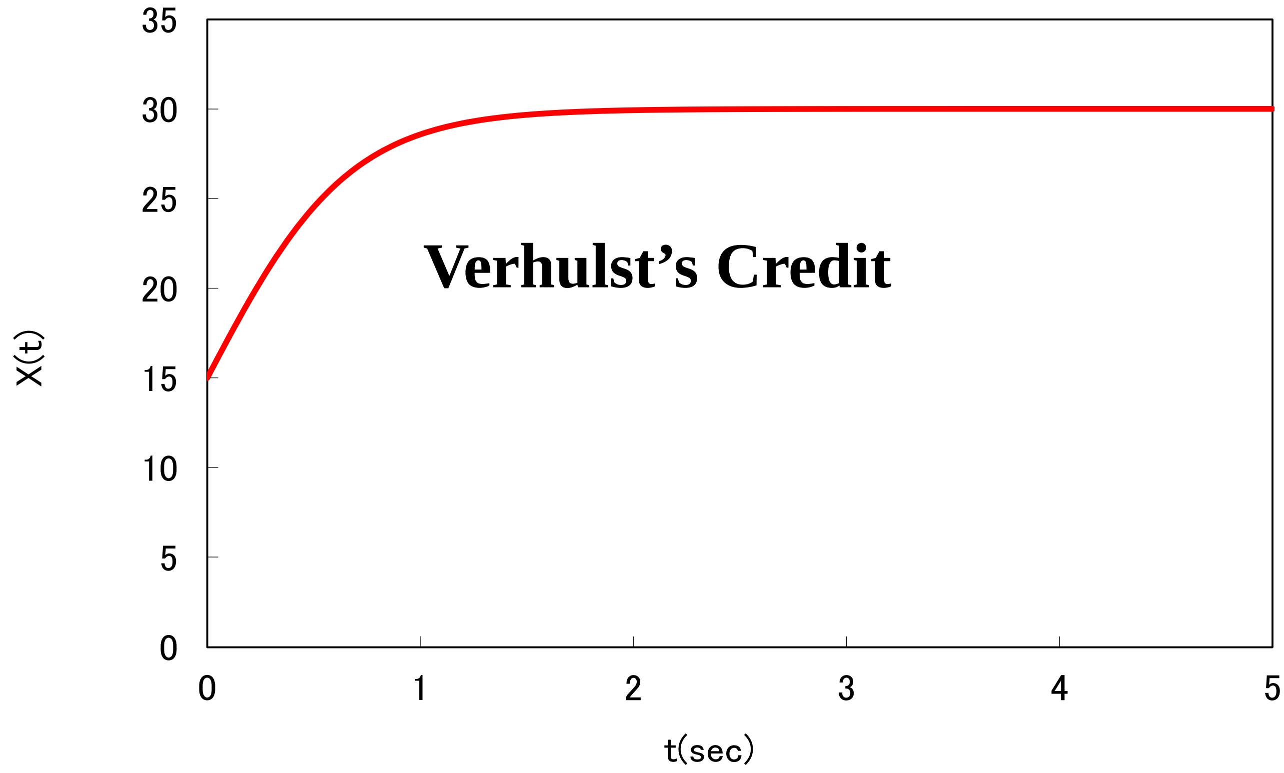


Example (Small Initial Data)

$$\begin{cases} \frac{dx}{dt} = \frac{1}{10}\left(30 - x(t)\right)x(t) \\ x(0) = 15 < 30 \end{cases}$$

$$a = \frac{1}{10}, \quad A = 30, \quad x_0 = 15$$

Runge-Kutta Method (Small Initial Data)



END