A HIGHER ORDER APPROXIMATION TO A PERCENTAGE POINT OF THE DISTRIBUTION OF A NON-CENTRAL T-STATISTIC WITHOUT THE NORMALITY ASSUMPTION

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ABSTRACT

Non-central distributions appear in two sample problems and are often used in several fields, for example, in biostatistics. A higher order approximation for a percentage point of the non-central t-distribution under normality is given by Akahira (1995) and is also shown to be numerically better than others. In this paper, without the normality assumption, we obtain a higher order approximation to the percentage point of the distribution of a non-central t-statistic, in a similar way to Akahira (1995) where the statistic based on a linear combination of a normal random variable and a chi-statistic takes an important role. Its application to the confidence limit and the confidence interval for a non-centrality parameter are also given. Further, a numerical comparison of the higher order approximation with the limiting normal distribution is done and the former one is shown to be more accurate. As a result of the numerical calculation, the higher order approximation seems to be useful in practical situations, when the size of sample is not so small.

1. INTRODUCTION

The non-central t-distribution was derived by Fisher (1931), and tables obtaining its percentage points were given by Johnson and Welch (1940), Resnikoff and Lieberman (1957), Bagui (1993) and others. Comparisons of some approximations for its percentage points were provided by van Eeden (1961), Owen (1963) and others (see also Johnson et al. (1995)).

In the two sample problem, assuming the normality on the sample distributions, we usually use the t-statistic in hypothesis testing and interval estimation. For example, let $(X_1, \ldots, X_{n_1})$ and $(Y_1, \ldots, Y_{n_2})$ be random samples from the normal distributions with means θ_1 and θ_2 and a common variance σ^2, respectively. Denote the sample means and the sample variances by

$$\bar{X} := \frac{1}{n_1}\sum_{i=1}^{n_1} X_i, \quad \bar{Y} := \frac{1}{n_2}\sum_{i=1}^{n_2} Y_i, \quad S_1^2 := \frac{1}{n_1}\sum_{i=1}^{n_1}(X_i - \bar{X})^2 \ \text{ and } \ S_2^2 := \frac{1}{n_2}\sum_{i=1}^{n_2}(Y_i - \bar{Y})^2,$$

respectively. Then the statistic

$$T := \frac{\bar{X} - \bar{Y}}{\sqrt{\dfrac{n_1 + n_2}{n_1 n_2}}\sqrt{\dfrac{n_1 S_1^2 + n_2 S_2^2}{n_1 + n_2 - 2}}} \tag{1.1}$$

follows a non-central t-distribution with $n_1 + n_2 - 2$ degrees of freedom and a non-centrality parameter $\delta := \lambda\sqrt{n_1 n_2/(n_1 + n_2)}$, where $\lambda := (\theta_1 - \theta_2)/\sigma$. So, we can consider the problems of the testing hypothesis on λ and the interval estimation on λ using the statistic T given by (1.1). In the above situation, we assume the normality condition on the underlying distribution, and using the Cornish-Fisher expansion, derive the higher order approximation to a percentage point of the non-central t-distribution. The approximation is shown to be numerically accurate (see Akahira (1995) and Akahira et al. (1995)). However, the normality assumption is too strict to apply the result to practical cases. Hence it seems to be meaningful to derive a higher order approximation to the distribution of a non-central t-statistic without the normality assumption.

Some works in the line of non-central distributions can be seen in Díaz-García et al. (2002) and Díaz-García and Leiva-Sánchez (2003). Recently, the limiting behaviour of the non-central t-statistic under non-normality has been studied by Bentkus et al. (2007), but its speed of covergence does not seem to be high. Here we derive a higher order approximation to the upper 100α percentile, without the normality assumption, in a similar way to Akahira (1995). We also obtain the confidence limit and the confidence interval of a non-centrality parameter.

In Section 2, the non-central t-statistic T_n and a percentile of its distribution are defined. Further, in order to obtain the approximation to a percentage point of the distribution of T_n without the normality assumption we calculate the mean, variance and covariance of the statistic based on a linear combination of a normal random variable and a chi-statistic. In Section 3, using the Cornish-Fisher expansion we derive higher order approximations to a percentage point of the distribution of T_n and the lower confidence limit and the confidence interval of the non-centrality parameter. In Section 4, we compare the higher order approximation with the limiting normal distribution and show it to be numerically more accurate. In Section 5, an application to distribution patterns of plant species is discussed. In Section 6, some conclusions are mentioned.

2. THE CALCULATION OF THE MEAN, VARIANCE AND COVARIANCE OF SOME STATISTICS FROM DERIVED FROM THE NON-CENTRAL T-STATISTIC

In this section, we define the non-central t-statistic, and in order to obtain the approximation to a percentage point of the distribution of the non-central t-statistic, we calculate the mean, variance and covariance of the sample mean $\bar{X}$ and the sample standard deviation S_n.

Suppose that $X_1, \ldots, X_n$ are independent and identically distributed (i.i.d.) non-degenerate continuous random variables with mean μ, variance 1 and finite sixth moment. Let $\mu_j := E[(X_1 - \mu)^j]$ $(j = 2, \ldots, 6)$, $\bar{X} := (1/n)\sum_{i=1}^{n} X_i$, $S_n^2 := \sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)$. Define $T_n := \sqrt{n}\bar{X}/S_n$ as the non-central t-statistic when $\mu \neq 0$ where $S_n = \sqrt{S_n^2}$. In particular, if the underlying distribution is $N(\mu, 1)$, then T_n follows the non-central t-distribution with $n-1$ degrees of freedom and a non-centrality parameter $\mu\sqrt{n}$. Here we put $\sigma_n := E(S_n)$. For any α with $0 < \alpha < 1$, there exists t_α such that $P\{T_n < t_\alpha\} = 1 - \alpha$. The t_α is called the upper 100α percentile of the distribution of the non-central t-statistic T_n.

First we have for any $t \in (-\infty, \infty)$

$$P_\mu\{T_n \le t\} = P_\mu\left\{\frac{\sqrt{n}\bar{X}}{S_n} \le t\right\} = P_\mu\{\sqrt{n}\bar{X} - tS_n \le 0\}$$

$$= P_\mu\left\{\sqrt{n}(\bar{X} - \mu) - t(S_n - \sigma_n) \le -\mu\sqrt{n} + t\sigma_n\right\}. \tag{2.1}$$

Here, for each $i = 1, 2, \dots$, let $X_i' := X_i - \mu$. Note that $\mu_j = E(X_i'^j)$ $(j = 2, \dots .6)$. Then (2.1) is written as

$$P_\mu\{T_n \le t\} = P_0\left\{\sqrt{n}\bar{X}' - t(S_n - \sigma_n) \le a_n(t)\right\} = P_0\left\{Z - t(S_n - \sigma_n) \le a_n(t)\right\}, \tag{2.2}$$

where $a_n(t) := -\mu\sqrt{n} + t\sigma_n$ and $Z := \sqrt{n}\bar{X}'$ with $\bar{X}' := (1/n)\sum_{i=1}^n X_i'$. Since $E(X_i') = 0$ $(i = 1, 2, \dots)$, putting

$$Y_n := Z - t(S_n - \sigma_n),$$

we obtain the mean $E_t(Y_n) = 0$ and the variance

$$V_t(Y_n) = nV(\bar{X}') + t^2V(S_n) - 2\sqrt{n}t\mathrm{Cov}(\bar{X}', S_n), \tag{2.3}$$

where $\mathrm{Cov}(\bar{X}', S_n)$ denotes the covariance between $\bar{X}'$ and S_n. In the right-hand side of (2.3), we have

$$nV(\bar{X}') = 1, \quad V(S_n) = E(S_n^2) - \sigma_n^2 = 1 - \sigma_n^2, \quad \mathrm{Cov}(\bar{X}', S_n) = E(\bar{X}'S_n). \tag{2.4}$$

Here, we consider the case when $t = O(\sqrt{n})$.

2(i) *The calculation of* $\sigma_n := E(S_n)$.

First we have by the Taylor expansion

$$\sigma_n = E(S_n) = E\left[\sqrt{1 + (S_n^2 - 1)}\right]$$

$$= 1 - \frac{1}{8}E\left[(S_n^2 - 1)^2\right] + \frac{1}{16}E\left[(S_n^2 - 1)^3\right] - \frac{15}{128}E\left[(S_n^2 - 1)^4\right] + O\left(\frac{1}{n^3}\right). \tag{2.5}$$

Since

$$E(S_n^4) = 1 + \frac{1}{n}\left(\mu_4 - \frac{n-3}{n-1}\right),$$

$$E(S_n^6) = 1 + \frac{3}{n}(\mu_4 - 1) - \frac{1}{n^2}(6\mu_3^2 - \mu_6 + 3\mu_4 - 8) + \frac{2}{n^3}(6\mu_4 - 7),$$

and

$$E(S_n^8) = 1 + \frac{6}{n}(\mu_4 - 1) + \frac{1}{n^2}(4\mu_6 + 3\mu_4^2 - 18\mu_4 - 24\mu_3^2 + 23) + O\left(\frac{1}{n^3}\right),$$

it follows that

$$E\left[(S_n^2 - 1)^2\right] = \frac{1}{n}(\mu_4 - 1) + \frac{2}{n^2} + \frac{2}{n^3} + O\left(\frac{1}{n^4}\right), \tag{2.6}$$

$$E\left[(S_n^2 - 1)^3\right] = \frac{1}{n^2}\{\mu_6 - 15 - 3(\mu_4 - 3) + 8 - 6\mu_3^2\} + \frac{2}{n^3}\{6(\mu_4 - 3) + 8\} + O\left(\frac{1}{n^4}\right), \tag{2.7}$$

and

$$E\left[(S_n^2 - 1)^4\right] = \frac{3}{n^2}(\mu_4 - 1)^2 + O\left(\frac{1}{n^3}\right). \tag{2.8}$$

From (2.5) to (2.8) we have

$$\sigma_n = 1 - \frac{1}{8n}(\mu_4 - 1) - \frac{1}{128n^2}\left\{8(6\mu_3^2 - \mu_6 + 3\mu_4 + 2) + 15(\mu_4 - 1)^2\right\} + O\left(\frac{1}{n^3}\right). \tag{2.9}$$

In particular, if the underlying distribution is $N(0,1)$, then

$$\sigma_n = 1 - \frac{1}{4n} - \frac{7}{32n^2} + O\left(\frac{1}{n^3}\right), \tag{2.10}$$

which coincides with the value b_{n-1} in (2.6) of Akahira (1995).

2(ii) *The calculation of* $\mathrm{Cov}(\bar{X}', S_n) = E(\bar{X}' S_n)$.

In a similar way to the above (i), we have by the Taylor expansion

$$\begin{aligned} E(\bar{X}' S_n) &= E\left[\bar{X}'\sqrt{1 + (S_n^2 - 1)}\right] \\ &= \frac{1}{2}E(\bar{X}' S_n^2) - \frac{1}{8}E\left[\bar{X}'(S_n^2 - 1)^2\right] + \frac{1}{16}E\left[\bar{X}'(S_n^2 - 1)^3\right] + O\left(\frac{1}{n^3}\right). \end{aligned} \tag{2.11}$$

Since

$$E(\bar{X}' S_n^2) = \frac{\mu_3}{n}, \tag{2.12}$$

$$E(\bar{X}'S_n^4) = \frac{2\mu_3}{n} + \frac{1}{n^2}(\mu_5 - 6\mu_3) + O\left(\frac{1}{n^3}\right),$$

$$E(\bar{X}'S_n^6) = \frac{3\mu_3}{n} + \frac{1}{n^2}(\mu_3\mu_4 + \mu_5 - 7\mu_3) + O\left(\frac{1}{n^3}\right),$$

it follows that

$$E\left[\bar{X}'(S_n^2 - 1)^2\right] = \frac{1}{n^2}(\mu_5 - 6\mu_3) + O\left(\frac{1}{n^3}\right), \tag{2.13}$$

$$E\left[\bar{X}'(S_n^2 - 1)^3\right] = \frac{3}{n^2}\mu_3(\mu_4 - 1) + O\left(\frac{1}{n^3}\right). \tag{2.14}$$

From (2.11), (2.13) and (2.14) we obtain

$$E(\bar{X}'S_n) = \frac{\mu_3}{2n} + \frac{1}{16n^2}(9\mu_3 - 2\mu_5 + 3\mu_3\mu_4) + O\left(\frac{1}{n^3}\right). \tag{2.15}$$

2(iii) *The calculation of the variance and the third cumulant of* Y_n.

From (2.3), (2.4) and (2.15) we have

$$V_t(Y_n) = 1 - \frac{t\mu_3}{\sqrt{n}} + \frac{t^2}{4n}(\mu_4 - 1) - \frac{t}{8n\sqrt{n}}(9\mu_3 - 2\mu_5 + 3\mu_3\mu_4) + O\left(\frac{1}{n\sqrt{n}}\right). \tag{2.16}$$

Note that $t = O(\sqrt{n})$. Let $W_n := Y_n/\sqrt{V_t(Y_n)}$. Then $E_t(W_n) = 0$ and $V_t(W_n) = 1$.

On the third cumulant of Y_n, we have

$$\begin{aligned}\kappa_{3,t}(Y_n) :&= E\left[\{\sqrt{n}\bar{X}' - t(S_n - \sigma_n)\}^3\right] \\ &= n\sqrt{n}E\left(\bar{X}'^3\right) - 3ntE\left[\bar{X}'^2(S_n - \sigma_n)\right] + 3\sqrt{n}t^2E\left[\bar{X}'(S_n - \sigma_n)^2\right] \\ &\qquad - t^3E\left[(S_n - \sigma_n)^3\right]. \end{aligned} \tag{2.17}$$

Then we obtain by the Taylor expansion

$$\begin{aligned}E(\bar{X}'^2S_n) &= E\left[\bar{X}'^2\sqrt{1 + (S_n^2 - 1)^2}\right] \\ &= E(\bar{X}'^2) + \frac{1}{2}E\left[\bar{X}'^2(S_n^2 - 1)\right] - \frac{1}{8}E\left[\bar{X}'^2(S_n^2 - 1)^2\right] + O\left(\frac{1}{n^3}\right)\end{aligned} \tag{2.18}$$

for large n. Since

$$E(\bar{X}'^2) = \frac{1}{n},$$

$$
\begin{aligned}
E(\bar{X}'^2 S_n^2) &= \frac{1}{n-1} E\left(\bar{X}'^2 \sum_{i=1}^n X_i'^2\right) - \frac{n}{n-1} E(\bar{X}'^4) \\
&= \frac{1}{n-1}\left(\frac{\mu_4}{n} + \frac{n-1}{n}\right) - \frac{n}{n-1}\left\{\frac{\mu_4}{n^3} + \frac{3(n-1)}{n^3}\right\} \\
&= \frac{1}{n} + \frac{\mu_4 - 3}{n^2}, \\
E\left[\bar{X}'^2 (S_n^2 - 1)^2\right] &= \frac{1}{n^2}(\mu_4 + 2\mu_3^2 - 1) + \frac{1}{n^3}(\mu_6 - 11\mu_4 - 6\mu_3^2 + 20) + O\left(\frac{1}{n^4}\right), \\
E\left[\bar{X}'^2 (S_n^2 - 1)^3\right] &= \frac{1}{n^3}(\mu_6 + 6\mu_3\mu_5 + 3\mu_4^2 - 15\mu_4 - 42\mu_3^2 + 11) + O\left(\frac{1}{n^4}\right),
\end{aligned}
$$

it follows from (2.18) that

$$
E(\bar{X}'^2 S_n) = \frac{1}{n} + \frac{1}{8n^2}(3\mu_4 - 2\mu_3^2 - 11) + O\left(\frac{1}{n^3}\right). \tag{2.19}
$$

Since, by (2.6), (2.7) and (2.8),

$$
\begin{aligned}
E(S_n^3) &= E\left[\{1 + (S_n^2 - 1)\}^{3/2}\right] \\
&= 1 + \frac{3}{2}E(S_n^2 - 1) + \frac{3}{8}E\left[(S_n^2 - 1)^2\right] - \frac{1}{16}E\left[(S_n^2 - 1)^3\right] + \frac{3}{128}E\left[(S_n^2 - 1)^4\right] \\
&\qquad + O\left(\frac{1}{n^3}\right) \\
&= 1 + \frac{3}{8n}\left(\mu_4 - \frac{n-3}{n-1}\right) - \frac{1}{16n^2}(2 - 3\mu_4 + \mu_6 - 6\mu_3^2) + \frac{9}{128n^2}(\mu_4 - 1)^2 \\
&\qquad + O\left(\frac{1}{n^3}\right) \\
&= 1 + \frac{3}{8n}(\mu_4 - 1) - \frac{1}{16n^2}(\mu_6 - 3\mu_4 - 6\mu_3^2 - 10) + \frac{9}{128n^2}(\mu_4 - 1)^2 + O\left(\frac{1}{n^3}\right)
\end{aligned}
$$

for large n, substituting $E(\bar{X}^3) = \mu_3/n^2$, (2.12), (2.15) and (2.19) into (2.17), we have

$$
\begin{aligned}
\kappa_{3,t}(Y_n) =& \frac{\mu_3}{\sqrt{n}} - 3nt\left\{\frac{1}{n} + \frac{1}{8n^2}(3\mu_4 - 2\mu_3^2 - 11) - \frac{\sigma_n}{n} + O\left(\frac{1}{n^3}\right)\right\} \\
&+ 3\sqrt{n}t^2\left[\frac{\mu_3}{n} - 2\sigma_n\left\{\frac{\mu_3}{2n} + \frac{1}{16n^2}(9\mu_3 - 2\mu_5 + 3\mu_3\mu_4) + O\left(\frac{1}{n^3}\right)\right\}\right]
\end{aligned}
$$

$$
\begin{aligned}
&- t^3\left\{1+\frac{3}{8n}(\mu_4-1)+\frac{1}{16n^2}(3\mu_4-\mu_6+6\mu_3^2+10)+\frac{9}{128n^2}(\mu_4-1)^2-3\sigma_n\right.\\
&\qquad\left.+2\sigma_n^3+O\left(\frac{1}{n^3}\right)\right\}\\
=&3t(\sigma_n-1)-t^3+\frac{\mu_3}{\sqrt{n}}\left\{3t^2(1-\sigma_n)+1\right\}-\frac{3t}{8n}\left\{3\mu_4-2\mu_3^2-11+t^2(\mu_4-1)\right\}\\
&-\frac{3t^2\sigma_n}{8n\sqrt{n}}(9\mu_3-2\mu_5+3\mu_3\mu_4)-\frac{t^3}{16n^2}(10+3\mu_4-\mu_6+6\mu_3^2)-\frac{9t^3}{128n^2}(\mu_4-1)^2\\
&-t^3\sigma_n(2\sigma_n^2-3)+O\left(\frac{t^3}{n^3}\right). \qquad (2.20)
\end{aligned}
$$

Substituting (2.9) in (2.20) we also obtain

$$
\begin{aligned}
\kappa_{3,t}(Y_n)=&\frac{\mu_3}{\sqrt{n}}-\frac{3t}{4n}\left\{2(\mu_4-3)-\mu_3^2\right\}+\frac{3t^2}{4n\sqrt{n}}\left\{\mu_5-\mu_3(\mu_4+5)\right\}\\
&-\frac{t^3}{16n^2}(1+2\mu_6-12\mu_3^2-3\mu_4^2)+O\left(\frac{t^3}{n^3}\right). \qquad (2.21)
\end{aligned}
$$

In particular, if the underlying distribution is $N(\mu,1)$, then

$$
\kappa_{3,t}(Y_n)=-\frac{t^3}{4n^2}+O\left(\frac{t^3}{n^3}\right),
$$

which coincides with the result in Lemma 1 in Akahira (1995).

Letting $t=c\sqrt{n}+d$ with some constants c and d, from (2.16) and (2.21) we have

$$
\begin{aligned}
V_{c,d}(Y_n):=&1-c\mu_3+\frac{c^2}{4}(\mu_4-1)+\frac{d}{2\sqrt{n}}\left\{c(\mu_4-1)-2\mu_3\right\}+\frac{d^2}{4n}(\mu_4-1)\\
&-\frac{c}{8n}(9\mu_3-2\mu_5+3\mu_3\mu_4)+O\left(\frac{1}{n\sqrt{n}}\right), \qquad (2.22)
\end{aligned}
$$

$$
\begin{aligned}
&\kappa_{3,c,d}(Y_n)\\
&:=\frac{1}{\sqrt{n}}\left[\mu_3-\frac{3c}{4}\left\{2(\mu_4-3)-\mu_3^2\right\}+\frac{3c^2}{4}\left\{\mu_5-\mu_3(\mu_4+5)\right\}-\frac{c^3}{16}(1+2\mu_6-12\mu_3^2-3\mu_4^2)\right]\\
&\quad-\frac{3d}{16n}\left[4\left\{2(\mu_4-3)-\mu_3^2\right\}-8c\left\{\mu_5-\mu_3(\mu_4+5)\right\}+c^2(1+2\mu_6-12\mu_3^2-3\mu_4^2)\right]\\
&\quad+O\left(\frac{1}{n\sqrt{n}}\right). \qquad (2.23)
\end{aligned}
$$

Remark 2.1 In the right-hand side of (2.22), the term of constant order is nonnegative, i.e.

$$1 - c\mu_3 + \frac{c^2}{4}(\mu_4 - 1) \geq 0,$$

since, for a random variable X with $E(X) = 0$ and $E(X^2) = 1$

$$E\left[\left\{X - \frac{c}{2}(X^2 - 1)\right\}^2\right] = 1 - c\mu_3 + \frac{c^2}{4}(\mu_4 - 1)$$

(see Bentkus et al. (2007)).

3. HIGHER ORDER APPROXIMATIONS TO A PERCENTAGE POINT OF THE DISTRIBUTION OF T_n

In this section we derive higher order approximations to the upper percentile using the results of Section 2. From (2.2) we obtain

$$\begin{aligned} P_\mu\{T_n \leq t\} &= P_0\left\{Z - t(S_n - \sigma_n) \leq a_n(t)\right\} \\ &= P_0\{Y_n \leq a_n(t)\} \\ &= P_0\left\{W_n \leq \frac{t\sigma_n - \mu\sqrt{n}}{\sqrt{V_t(Y_n)}}\right\}. \end{aligned} \tag{3.1}$$

Using the Cornish-Fisher expansion, we can obtain higher order approximation formulae of a percentage point of the distribution of T_n.

Theorem 3.1 The upper 100α percentile t_α of the distribution of T_n can be derived from the formula

$$\frac{t_\alpha\sigma_n - \mu\sqrt{n}}{\sqrt{V_{t_\alpha}(Y_n)}} = u_\alpha + \frac{1}{6}\kappa_{3,t_\alpha}(W_n)(u_\alpha^2 - 1) + O\left(\frac{1}{n}\right), \tag{3.2}$$

where u_α is the upper 100α percentile of the standard normal distribution, σ_n and $V_t(Y_n)$ are given by (2.9) and (2.16), respectively, and

$$\kappa_{3,t}(W_n) = \kappa_{3,t}(Y_n)\left\{V_t(Y_n)\right\}^{-3/2}$$

with (2.21).

The proof is straightforward from (3.1) and the Cornish-Fisher expansion. From Theorem 3.1 we have the following.

Corollary 3.1 Let T_n be the non-central t-statistic. Then the lower confidence limit $\hat{\delta}$ of the non-centrality parameter $\delta := \mu\sqrt{n}$ of level $1-\alpha$ and the confidence interval $[\underline{\delta}, \bar{\delta}]$ of δ of level $1-\alpha$ are given by

$$\hat{\delta} = \sigma_n T_n - \sqrt{V_{T_n}(Y_n)} \left\{ u_\alpha + \frac{1}{6}\kappa_{3,T_n}(W_n)(u_\alpha^2 - 1) \right\} + O_p\left(\frac{1}{n}\right),$$

$$\underline{\delta} = \sigma_n T_n - \sqrt{V_{T_n}(Y_n)} \left\{ u_{\alpha/2} + \frac{1}{6}\kappa_{3,T_n}(W_n)(u_{\alpha/2}^2 - 1) \right\} + O_p\left(\frac{1}{n}\right),$$

$$\bar{\delta} = \sigma_n T_n + \sqrt{V_{T_n}(Y_n)} \left\{ u_{\alpha/2} + \frac{1}{6}\kappa_{3,T_n}(W_n)(u_{\alpha/2}^2 - 1) \right\} + O_p\left(\frac{1}{n}\right).$$

Remark 3.1 In particular, if the underlying distribution is $N(\mu, 1)$, then

$$V_t(Y_n) = 1 + \frac{t^2}{2n} + O\left(\frac{1}{n\sqrt{n}}\right), \tag{3.3}$$

$$\kappa_{3,t}(Y_n) = -\frac{t^3}{4n^2} + O\left(\frac{1}{n\sqrt{n}}\right), \tag{3.4}$$

hence, from (3.2)

$$\frac{t_\alpha \sigma_n - \mu\sqrt{n}}{\sqrt{1 + \frac{t_\alpha^2}{2n} + O\left(\frac{1}{n\sqrt{n}}\right)}} = u_\alpha - \frac{t_\alpha^3(u_\alpha^2 - 1)}{24n^2}\left(1 + \frac{t_\alpha^2}{2n}\right)^{-3/2} \left\{1 + O\left(\frac{1}{n}\right)\right\}. \tag{3.5}$$

The approximation formula (3.5) of the non-central t-distribution with $n-1$ degrees of freedom and a non-centrality parameter $\mu\sqrt{n}$ is also derived from (2.8) in the paper by Akahira (1995), i.e.

$$\frac{t_\alpha b_\nu - \mu\sqrt{n}}{\sqrt{1 + t_\alpha^2(1 - b_\nu^2)}} = u_\alpha - \frac{t_\alpha^3(u_\alpha^2 - 1)}{24\{1 + t_\alpha^2(1 - b_\nu^2)\}^{3/2}} \left\{\frac{1}{\nu^2} + \frac{1}{4\nu^3} + O\left(\frac{1}{\nu^4}\right)\right\}, \tag{3.6}$$

where $\nu = n - 1$ and

$$b_\nu = \sqrt{\frac{2}{\nu}}\frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)} = 1 - \frac{1}{4\nu} + \frac{1}{32\nu^2} + \frac{5}{128\nu^3} + O\left(\frac{1}{\nu^4}\right).$$

The existence and uniqueness of a solution of the equation (3.6) on t_α are shown to be guaranteed in Akahira et al. (1995). Under the normality assumption, the variance and the third cumulant of Y_n are exactly given by

$$V_t(Y_n) = E\left[\{Z - t(S_n - b_{n-1})\}^2\right] = 1 + t^2(1 - b_{n-1}^2),$$

$$\kappa_{3,t}(Y_n) = t^3 b_{n-1}\left\{2(1 - b_{n-1}^2) - \frac{1}{n-1}\right\},$$

respectively (see Akahira (1995)) which are used in (3.6), but, instead of them, the approximate values (3.3) and (3.4) derived from (2.16) and (2.21) respectively are done in (3.5). Hence the higher order approximation (3.6) is seen to be much better than (3.5).

Since t_α is the upper 100α percentile of the distribution of the non-central t-statistic T_n, it follows from (3.1), (2.22) and the first order approximation that

$$\frac{t_\alpha \sigma_n - \mu\sqrt{n}}{\sqrt{V_{c,d}(Y_n)}} = u_\alpha + o(1),$$

i.e.

$$t_\alpha = \frac{1}{\sigma_n}\left\{\mu\sqrt{n} + u_\alpha\sqrt{V_{c,d}(Y_n)}\right\} + o(1).$$

From (2.9) and (2.22) we have

$$t_\alpha = \mu\sqrt{n} + u_\alpha\left\{1 - c\mu_3 + \frac{c^2}{4}(\mu_4 - 1)\right\}^{1/2} + o(1). \tag{3.7}$$

Here, letting $c = \mu$, we obtain $d = \sigma_0 u_\alpha$, where

$$\sigma_0 := \left\{1 - \mu\mu_3 + \frac{\mu^2}{4}(\mu_4 - 1)\right\}^{1/2}.$$

From (2.22) and (2.23) we have

$$\begin{aligned} V_{\mu,\sigma_0 u_\alpha}(Y_n) =& \sigma_0^2 + \frac{\sigma_0 u_\alpha}{2\sqrt{n}}\{\mu(\mu_4 - 1) - 2\mu_3\} \\ &+ \frac{\sigma_0^2 u_\alpha^2}{4n}(\mu_4 - 1) - \frac{\mu}{8n}(9\mu_3 - 2\mu_5 + 3\mu_3\mu_4) + O\left(\frac{1}{n\sqrt{n}}\right), \end{aligned} \tag{3.8}$$

$$\kappa_{3,\mu,\sigma_0 u_\alpha}(Y_n) = \frac{1}{\sqrt{n}}\Big[\mu_3 - \frac{\mu}{16}\{12(2(\mu_4 - 3) - \mu_3^2) - 12\mu(\mu_5 - \mu_3(\mu_4 + 5))$$

$$+\mu^2(1+2\mu_6-12\mu_3^2-3\mu_4^2)\}\Big]$$

$$-\frac{3\sigma_0 u_\alpha}{16n}\left[4\left\{2(\mu_4-3)-\mu_3^2\right\}-8\mu\left\{\mu_5-\mu_3(\mu_4+5)\right\}\right.$$

$$\left.+\mu^2(1+2\mu_6-12\mu_3^2-3\mu_4^2)\right]+O\left(\frac{1}{n\sqrt{n}}\right)$$

$$=:\frac{A}{\sqrt{n}}+\frac{B}{n}+O\left(\frac{1}{n\sqrt{n}}\right)\qquad\text{(say)}.\tag{3.9}$$

Then we have the following.

Theorem 3.2 The upper 100α percentile t_α of the distribution of the non-central t-statistic T_n is given by

$$t_\alpha=\frac{1}{\sigma_n}\left[\mu\sqrt{n}+\sqrt{V_{\mu,\sigma_0 u_\alpha}(Y_n)}\left\{u_\alpha+\frac{1}{6}\kappa_{3,\mu,\sigma_0 u_\alpha}(W_n)(u_\alpha^2-1)+O\left(\frac{1}{n}\right)\right\}\right],\tag{3.10}$$

where σ_n and $V_{\mu,\sigma_0 u_\alpha}(Y_n)$ are given by (2.9) and (3.8), respectively and

$$\kappa_{3,\mu,\sigma_0 u_\alpha}(W_n)=\sigma_0^{-3}\left[\frac{A}{\sqrt{n}}-\frac{3Au_\alpha}{4n}\left\{\mu(\mu_4-1)-2\mu_3\right\}+\frac{B}{n}+O\left(\frac{1}{n\sqrt{n}}\right)\right]\tag{3.11}$$

with

$$A=\mu_3-\frac{\mu}{16}\left[12\left\{2(\mu_4-3)-\mu_3^2\right\}-12\mu\left\{\mu_5-\mu_3(\mu_4+5)\right\}\right.$$

$$\left.+\mu^2(1+2\mu_6-12\mu_3^2-3\mu_4^2)\right],$$

$$B=-\frac{3\sigma_0 u_\alpha}{16}\left[4\left\{2(\mu_4-3)-\mu_3^2\right\}-8\mu\left\{\mu_5-\mu_3(\mu_4+5)\right\}\right.$$

$$\left.+\mu^2(1+2\mu_6-12\mu_3^2-3\mu_4^2)\right].$$

Proof Since $1-\alpha=P_\mu\{T_n\le t_\alpha\}$ for $0<\alpha<1$, by the Cornish-Fisher expansion, we obtain from (3.1)

$$\frac{t_\alpha\sigma_n-\mu\sqrt{n}}{\sqrt{V_{\mu,\sigma_0 u_\alpha}(Y_n)}}=u_\alpha+\frac{1}{6}\kappa_{3,\mu,\sigma_0 u_\alpha}(W_n)(u_\alpha^2-1)+O\left(\frac{1}{n}\right),\tag{3.12}$$

where $V_{\mu,\sigma_0 u_\alpha}(Y_n)$ is given by (3.8) and

$$\kappa_{3,\mu,\sigma_0 u_\alpha}(W_n)=\frac{1}{\{V_{\mu,\sigma_0 u_\alpha}(Y_n)\}^{3/2}}\kappa_{3,\mu,\sigma_0 u_\alpha}(Y_n)$$

with $\kappa_{3,\mu,\sigma_0 u_\alpha}(Y_n)$ given by (3.9). A straightforward calculation derives (3.11) from (3.8) and (3.9). From (3.12) we obtain (3.10). This completes the proof.

Remark 3.2 The approximate value (3.10) of t_α can be easily obtained by a pocket calculator, which is a merit.

Remark 3.3 If the underlying distribution has a symmetric density $f(x)$ around $x = k$, then

$$\mu := E(X_1) = k, \qquad \mu_3 = \mu_5 = 0,$$

hence, in Theorem 3.2,

$$A = -\frac{3k}{2}(\mu_4 - 3),$$
$$B = -\frac{3\sigma_0 u_\alpha}{16}\left\{8(\mu_4 - 3) + k^2(1 + 2\mu_6 - 3\mu_4^2)\right\}.$$

4. NUMERICAL COMPARISON OF THE HIGHER ORDER APPROXIMATION WITH THE LIMITING NORMAL DISTRIBUTION

The limiting distribution of the non-central t-statistic T_n is given by Bentkus et al. (2007), i.e. the statistic $\sigma_0^{-1}(T_n - \mu\sqrt{n})$ converges in law to $N(0,1)$ as $n \to \infty$. Then the upper 100α percentile t_α of the distribution of T_n is asymptotically given by

$$t_\alpha = \mu\sqrt{n} + \sigma_0 u_\alpha + o(1) \tag{4.1}$$

as $n \to \infty$, since

$$\alpha = P\{T_n > t_\alpha\} = P\left\{\sigma_0^{-1}(T_n - \mu\sqrt{n}) > \sigma_0^{-1}(t_\alpha - \mu\sqrt{n})\right\}.$$

On the other hand, since $\sigma_n = 1 + o(1)$ as $n \to \infty$ from (2.9), it follows from the first order approximation (3.7) with $c = \mu$ that $t_\alpha = \mu\sqrt{n} + \sigma_0 u_\alpha + o(1)$ as $n \to \infty$, which concides with (4.1) derived from the limiting normal distribution. Note that the approximation (3.10) is a higher order one than (4.1). In order to compare the higher order approximation (3.10) with the first order one (4.1), in the case when $\alpha = 0.05$, we give various examples including asymmetric distributions. In the below tables except for Table 4.6 in Section 4, the true value of the upper 5 percentile of the distribution of the non-central t-statistic means the 9,500th one from the smallest one among the ones of the statistic calculated from the total repeated number 10,000 of size n of sample. In Table 5.2 of Section 5, the total repeated number is 100,000, hence 95,000th one from the smallest one is used as the true value of the upper 5 percentile.

Example 4.1 (Gamma distribution). Suppose that $X_1, \ldots, X_n$ are i.i.d. random variables according to the gamma distribution $G(2, 1/\sqrt{2})$ with a density

$$f(x) = \begin{cases} 2xe^{-\sqrt{2}x} & \text{for } x > 0, \\ \\ 0 & \text{otherwise.} \end{cases}$$

Then the moments of the distribution up to the 6th order are given by

$$\mu = E(X_1) = \sqrt{2},\ \mu_2 = 1,\ \mu_3 = \sqrt{2},\ \mu_4 = 6,\ \mu_5 = 16\sqrt{2},\ \mu_6 = 110.$$

In order to compare the higher order approximation (3.10) with the first order one (4.1), the numerical calculation is done. As is seen in Table 4.1, the relative errors of (3.10) are much smaller than those of (4.1) when $\alpha = 0.05$ and $n = 5(5)30(10)50,\ 100$.

Table 4.1 The relative errors of the higher order approximation (3.10) and the first order one (4.1).

n	true value	(4.1)	(3.10)
5	7.426089	−0.3028901	−0.2321060
10	7.451146	−0.1294413	−0.0499075
15	8.181845	−0.0843342	−0.0192098
20	8.903006	−0.0633411	−0.0094226
25	9.536338	−0.0472664	−0.0013070
30	10.15138	−0.0385061	0.0013949
40	11.32113	−0.0320048	−0.0005986
50	12.31424	−0.0243409	0.0016650
100	16.38862	−0.0141513	−0.0002046

Example 4.2 (Exponential distribution). Suppose that $X_1, \dots, X_n$ are i.i.d. random variables according to the exponential distribution with a density

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then the moments of X_1 up to the 6th order are given by

$$\mu = \mu_2 = 1,\ \mu_3 = 2,\ \mu_4 = 9,\ \mu_5 = 44,\ \mu_6 = 265.$$

In order to compare the higher order approximation (3.10) with the first order one (4.1), the numerical calculation is done. As is seen in Table 4.2, the relative errors of (3.10) are much smaller than those of (4.1) for $\alpha = 0.05$ and $n = 20(5)30(10)50,\ 100$.

Table 4.2 The relative errors of the higher order approximation (3.10) and the first order one (4.1).

n	true value	(4.1)	(3.10)
5	5.126078	-0.2429066	-0.5034625
10	5.377911	-0.1061343	-0.1478104
15	5.9033	-0.0652974	-0.0604899
20	6.438533	-0.0499404	-0.0320808
25	6.957756	-0.0449723	-0.0233087
30	7.388235	-0.0360242	-0.0134978
40	8.183828	-0.0262002	-0.0048068
50	8.908046	-0.0215677	-0.0021358
100	11.77633	-0.0111605	0.0012118

Example 4.3 (Weibull distribution). Suppose that $Y_1, \dots, Y_n$ are i.i.d. random variables according to the Weibull distribution with a density

$$f(x) = \begin{cases} \frac{\gamma x^{\gamma-1}}{\lambda^\gamma} e^{-(x/\lambda)^\gamma} & \text{for } x > 0, \\ 0 & \text{for } x \le 0, \end{cases}$$

where $\gamma > 0$ and $\lambda > 0$. Then, for each $k = 1, 2, \dots,$ k-th order moment of X_1 around 0 is given by

$$E\left(X_1^k\right) = \lambda^k \Gamma\left(\frac{k}{\gamma} + 1\right).$$

Let $\gamma = 2$, and $X_i = 2Y_i/(\lambda\sqrt{4-\pi})$ $(i = 1, 2, \dots)$. Then the moments of X_i up to the 6th order are given by

$$\mu = \sqrt{\frac{\pi}{4-\pi}}, \quad \mu_2 = 1, \quad \mu_3 = \frac{2\sqrt{\pi}(\pi-3)}{(4-\pi)^{3/2}}, \quad \mu_4 = \frac{32-3\pi^2}{(4-\pi)^2},$$
$$\mu_5 = \frac{4\sqrt{\pi}(\pi^2+5\pi-25)}{(4-\pi)^{5/2}}, \quad \mu_6 = \frac{384+120\pi-60\pi^2-5\pi^3}{(4-\pi)^3}.$$

In order to compare the higher order approximation (3.10) with the first order one (4.1), the numerical calculation is done. As is seen in Table 4.3, the relative errors of (3.10) are much smaller than those of (4.1) for $\alpha = 0.05$ and $n = 5(5)30(10)50,\ 100$.

Table 4.3 The relative errors of the higher order approximation (3.10) and the first order one (4.1) in the case $\lambda = 1$ and $\gamma = 2$.

n	true value	(4.1)	(3.10)
5	9.454586	−0.3111258	−0.1829024
10	9.630487	−0.1397195	−0.0535203
15	10.52915	−0.0840153	−0.0204432
20	11.64788	−0.0735825	−0.0241829
25	12.33538	−0.0433533	−0.0018386
30	13.28742	−0.0431927	−0.0081295
40	14.78891	−0.0307264	−0.0035844
50	16.20312	−0.0271812	−0.0051052
100	21.68477	−0.0147048	−0.0031483

Example 4.4 (Birnbaum-Saunders distribution). When Z is a normal random variable with mean 0 and variance 1, the distribution of a random variable

$$Y = \beta \left\{ \frac{1}{2}\gamma Z + \sqrt{\left(\frac{1}{2}\gamma Z\right)^2 + 1} \right\}^2,$$

where β and γ are positive parameters, which is called the Birnbaum-Saunders (B-S) distribution (see Johnson et al. (1995) and also, e.g. Balakrishnan et al. (2009), (2011), Leiva et al. (2007)). Suppose that $Y_1, \dots, Y_n$ are i.i.d. random variables according to the B-S distribution. Put $X_i = 2Y_i/(\beta\gamma\sqrt{5\gamma^2+4})$ $(i = 1, 2, \dots)$. Then the moments of X_i up to the 6th order are given by

$$\mu = \frac{\gamma^2+2}{\gamma\sqrt{5\gamma^2+4}}, \quad \mu_2 = 1, \quad \mu_3 = \frac{4\gamma(11\gamma^2+6)}{(5\gamma^2+4)^{3/2}}, \quad \mu_4 = \frac{3(211\gamma^4+120\gamma^2+16)}{(5\gamma^2+4)^2},$$
$$\mu_5 = \frac{8\gamma(1433\gamma^4+790\gamma^2+120)}{(5\gamma^2+4)^{5/2}}, \quad \mu_6 = \frac{5(50681\gamma^6+27516\gamma^4+4752\gamma^2+192)}{(5\gamma^2+4)^3}.$$

In order to compare the higher order approximation (3.10) with the first order one (4.1), the numerical calculation is done. As is seen in Table 4.4, the relative errors of (3.10) are much smaller than those of (4.1) for $\alpha = 0.05$, $\gamma = 1$ and $n = 10(5)30(10)50,\ 100$.

Table 4.4 The relative errors of the higher order approximation (3.10) and the first order one (4.1).

n	true value	(4.1)	(3.10)
10	5.881246	−0.1261818	−0.0495977
15	6.382393	−0.0834394	−0.0015297
20	6.88221	−0.0629464	0.0109349
25	7.383125	−0.0550262	0.0099599
30	7.826788	−0.0476183	0.0099903
40	8.63897	−0.0390729	0.0074187
50	9.285513	−0.0255864	0.0135670
100	12.1612	−0.0151548	0.0061507

Example 4.5 (Two-sided exponential distribution). Suppose that $Y_1, \dots, Y_n$ are i.i.d. random variables according to the two-sided exponential distribution with a density

$$f(x) = \frac{1}{2} e^{-|x-\mu|} \qquad \text{for } -\infty < x < \infty, \tag{4.2}$$

where $-\infty < \mu < \infty$. Put $X_i = Y_i/\sqrt{2}$ $(i = 1, 2, \dots)$. Then the moments of X_1 up to the 6th order are given by

$$\mu_2 = 1, \ \mu_4 = 6, \ \mu_6 = 90, \ \mu_3 = \mu_5 = 0.$$

It is clear that $E(X_1) = \mu$. Since the density (4.2) is symmetric around $x = \mu$, it is seen that the situation is a typical case in Remark 3.3. Comparing the higher order approximation (3.10) with the first order one (4.1), from Table 4.5 we see that (3.10) is much better than (4.1) for $\alpha = 0.05$ and $n = 5(5)30(10)50, \ 100$.

Table 4.5 The relative errors of the higher order approximation (3.10) and the first order one (4.1) in the case $\mu = 1/\sqrt{2}$.

n	true value	(4.1)	(3.10)
5	5.73092	−0.3582322	−0.2062444
10	5.35649	−0.1911028	−0.0601495
15	5.57938	−0.1333464	−0.0290928
20	5.86239	−0.1029154	−0.0169521
25	6.14907	−0.0840387	−0.0109561
30	6.42849	−0.0713589	−0.0077872
40	6.95202	−0.0551063	−0.0045828
50	7.42971	−0.0448106	−0.0028036
100	9.39077	−0.0237382	−0.0007294

Example 4.6 (Normal distribution). Suppose that $X_1, \ldots, X_n$ are i.i.d. random variables according to the normal distribution $N(\mu, 1)$. Then the moments of X_1 up to the 6th order are given by

$$\mu_3 = \mu_5 = 0,\ \ \mu_2 = 1,\ \ \mu_4 = 3,\ \ \mu_6 = 15.$$

Comparing the higher order approximation (3.5) with (3.6) for $\alpha = 0.05$, we see that (3.6) is good although it is worse than (3.5).

Table 4.6 The errors of the higher order approximation formula of the upper 5 percentile for $\eta = \mu\sqrt{n}/\sqrt{2\nu + n\mu^2}$ which is a transformation from $\mu\sqrt{n}$ in the domain $(-\infty, \infty)$ to η in the range $(-1, 1)$ where $\nu = n - 1$. The true values are referred from Yamauti et al. (1972), and the errors of the higher order approximation formula (3.6) are taken from Table 1 in Akahira (1995).

ν	η	true value	(3.5)	(3.6)
5	0.9	14.0781	−0.59430	0.044
	0.5	4.9462	−0.16158	0.009
	0.1	2.4764	−0.05493	0.001
	−0.1	1.5774	−0.02473	0.000
	−0.5	−0.1872	−0.000222	0.000
	−0.9	−4.0292	−0.085520	0.001
10	0.9	15.1240	−0.22070	0.009
	0.5	5.1564	−0.05465	0.001
	0.1	2.3534	−0.01457	0.000
	−0.1	1.2912	−0.00466	0.000
	−0.5	−0.9254	−0.00219	0.000
	−0.9	−6.4634	−0.05788	0.000
20	0.9	18.1294	−0.09030	0.003
	0.5	5.9580	−0.02040	0.000
	0.1	2.4235	−0.00413	0.000
	−0.1	1.0439	−0.00076	0.000
	−0.5	−1.9539	−0.00257	0.000
	−0.9	−10.0559	−0.03550	0.000

Example 4.7 (t-distribution). Suppose that $Y_1, \ldots, Y_n$ are i.i.d. random variables according to the t-distribution with ν degrees of freedom with a density

$$f_\nu(x) = \frac{\Gamma\left((\nu+1)/2\right)}{\sqrt{\pi\nu}\,\Gamma(\nu/2)} \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2} \tag{4.3}$$

for $-\infty < x < \infty$ and $\nu > 0$. Put $X_i = \sqrt{(\nu-2)/\nu}Y_i$ $(i = 1, 2, \dots)$. Then the moments of X_i up to the 6th order are given by

$$\mu = 0 \ (\nu > 1), \quad \mu_2 = 1 \ (\nu > 2), \quad \mu_3 = 0 \ (\nu > 3),$$
$$\mu_4 = \frac{3(\nu-2)}{\sqrt{2}(\nu-4)} \ (\nu > 4), \quad \mu_5 = 0 \ (\nu > 5), \quad \mu_6 = \frac{15(\nu-2)^2}{\sqrt{2}(\nu-4)(\nu-6)} \ (\nu > 6).$$

Since the density (4.3) is symmetric around $x = 0$, it is seen that the situation is in Remark 4.3 for $\nu > 6$. Comparing the higher order approximation (3.10) with the first order one (4.1), from Table 4.7 we see that (3.10) is much better than (4.1) for $\alpha = 0.05$, $\nu = 7$ and $n = 5(5)30(10)50,\ 100$.

Table 4.7 The relative errors of the higher order approximation (3.10) and the first order one (4.1).

n	true value	(4.1)	(3.10)
5	2.054912	−0.1995521	−0.1922330
10	1.779899	−0.0758745	−0.0304899
15	1.717872	−0.0425072	−0.0012178
20	1.706298	−0.0360125	−0.0011182
25	1.731649	−0.0501251	−0.0208813
30	1.702207	−0.0336957	−0.0079291
40	1.648981	−0.0025052	0.0183744
50	1.678272	−0.0199145	−0.0030698
100	1.662832	−0.0108141	−0.0018775

Example 4.8 (Logistic distribution). Suppose that $Y_1, \dots, Y_n$ are i.i.d. random variables according to the logistic distribution with a density

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2} \quad \text{for } -\infty < x < \infty. \tag{4.4}$$

Put $X_i = \sqrt{3}Y_i/\pi$ $(i = 1, 2, \dots)$. Then the moments of X_i up to the 6th order are given by

$$\mu = \mu_3 = \mu_5 = 0,\ \mu_2 = 1,\ \mu_4 = \frac{21}{5},\ \mu_6 = \frac{279}{7}.$$

Since the density (4.4) is symmetric around $x = 0$, it is seen that the situation is in Remark 3.3. Comparing the higher order approximation (3.10) with the first order one (4.1), from Table 4.8 we see that (3.10) is much better than (4.1) for $\alpha = 0.05$ and $n = 5(5)30(10)50,\ 100$.

Table 4.8 The relative errors of the higher order approximation (3.10) and the first order one (4.1).

n	true value	(4.1)	(3.10)
5	2.169503	−0.2418310	−0.1288189
10	1.813658	−0.0930760	−0.0152498
15	1.764209	−0.0676558	−0.0119935
20	1.713897	−0.0402866	0.0035842
25	1.721701	−0.0446367	−0.0092705
30	1.71441	−0.0405737	−0.0107384
40	1.671952	−0.0162098	0.0069667
50	1.674737	−0.0187458	0.0007780
100	1.656844	−0.0072391	0.0022851

As is seen from the above examples, the higher order approximation (3.10) is useful in practical situations, when the size n of sample is not smaller than 15.

5. An application to the practical case

In Andrews and Herzberg (1985), distribution patterns of plant species are stated as follows. Cain and Evans (1952) mapped in detail an old-field grasslands community in southeastern Michigan, plotting the occurrence of three plant species: *Lespedeza capitata, Liatris aspera* and *Solidago rigida*. From these, Evans (1952) prepared quadrat converages of 16, 8, 4, 2, 1, 1/2, 1/4, 1/8 and 1/16 square metres, recording the frequencies with which each of the species appeared in the quadrats. For *Solidago rigida*, golden rod, the frequency distributions for the three largest quadrat sizes are given in Table 5.1.

Table 5.1 Frequency Distribution of *Solidago rigida*

Quadrat coverage	Frequency												
	0	1	2	3	4	5	6	7	8	9	10	14	15+
$16m^2$	245	94	36	31	8	10	0	2	0	1	1	1	0
$8m^2$	615	162	48	20	5	4	1	1	1	1	0	0	0
$4m^2$	1425	222	51	13	2	2	0	1	0	0	0	0	0

In an application to a practical case, we often have a frequency distribution which gives us information on the true distribution of data. In order to extract the information from the frequency distribution of Table 5.1, we consider two different approximate distributions instead of the true distribution which is unknown and obtain the moments up to the 6th order. One depends on the gamma approximation with parameters estimated by the moment method, and another follows from the direct approximation with sample moments up to the 6th order calculated from the frequency distribution of Table 5.1 based on the data from the true distribution. Here, it is also remarked that the underlying distribution is assumed to be continuous but not discrete in this paper.

(i) *Gamma approximation.* We consider the approximation of the frequency distribution of Table 5.1 by the gamma distribution $G(\alpha, \beta)$ with a density

$$f(x) = \begin{cases} \dfrac{1}{\beta\Gamma(\alpha)} \left(\dfrac{x}{\beta}\right)^{\alpha-1} e^{-x/\beta} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0, \end{cases}$$

where $\alpha > 0$ and $\beta > 0$. Since the mean and the variance are given by $\mu = \alpha\beta$ and $\mu_2 = \alpha\beta^2$, respectively, we take the sample mean $\hat{\mu}$ and the sample variance $\hat{\mu_2}$ calculated from the frequency distribution in Table 5.1 as approximate values of μ and μ_2. Letting $\hat{\mu} = \alpha\beta$ and $\hat{\mu_2} = \alpha\beta^2$, we have $\hat{\alpha}$ and $\hat{\beta}$ as the solutions α and β of the equations. Next, suppose that $Y_1, \dots, Y_n$ are i.i.d. random variables according to the gamma distribution $G(\hat{\alpha}, \hat{\beta})$. Put $X_i := Y_i/\sqrt{\hat{\alpha}\hat{\beta}}$ $(i = 1, 2, \dots)$. Then the moments of X_1 up to the 6th order are given by

$$\mu_1 = \sqrt{\hat{\alpha}}, \qquad \mu_2 = 1, \qquad \mu_3 = \frac{2}{\sqrt{\hat{\alpha}}}, \qquad \mu_4 = 3\left(1 + \frac{2}{\hat{\alpha}}\right),$$

$$\mu_5 = 4\left(\frac{5}{\sqrt{\hat{\alpha}}} + \frac{6}{\hat{\alpha}^{3/2}}\right), \qquad \mu_6 = 5\left(3 + \frac{26}{\hat{\alpha}} + \frac{24}{\hat{\alpha}^2}\right).$$

In a similar way to Example 4.1, we have the following table to compare the higher order approximation (3.10) with the first order one (4.1).

Table 5.2 The relative errors of the higher order approximation (3.10) and the first order one (4.1) to a percentage point.

Quadrat coverage	n	$\hat{\alpha}$	$\hat{\beta}$	true value	(4.1)	(3.10)
$16m^2$	429	0.3443477	2.6264989	13.53494	−0.0023746	0.0006546
$8m^2$	858	0.2283017	1.9807752	15.30205	−0.0011273	0.0006176
$4m^2$	1716	0.1496402	1.5110062	17.27925	−0.0004485	0.0006106

As is seen from Table 5.2, the tendency seems to be similar to Table 4.1 where the approximation (3.10) and (4.1) are numerically shown to be accurate.

(ii) *A relative comparison of the gamma approximation with the direct one under the (higher order) approximations.* First we consider the direct approximation by Table 5.1. From the frequency distribution in Table 5.1 we obtain the sample moments $\hat{\mu}$ and $\hat{\mu}_j$ $(j = 2, \dots, 6)$. Substituting $\hat{\mu}/\sqrt{\hat{\mu}_2}$ and $\hat{\mu}_j/\hat{\mu}_2^{j/2}$ $(j = 2, \dots, 6)$ for $\hat{\mu}$, $\hat{\mu}_j$ $(j = 2, \dots, 6)$, respectively, in (3.10) and (4.1), we have the following table to compare the gamma approximation with the direct one.

Table 5.3 The relative differences of the gamma approximation to the direct approximation under the approximations (3.10) and (4.1) to a percentage point.

Quadrat coverage	Relative difference under (4.1)	Relative difference under (3.10)
$16m^2$	−0.0034540	−0.0135256
$8m^2$	0.0089510	0.0041513
$4m^2$	0.0099052	0.0089399

Since, as is seen from Table 5.3, the relative differences are small, the gamma approximation to the true distribution seems to be numerically accurate through the approximations (3.10) and (4.1) to a percentage point.

6. Conclusions

In this paper, without the normality assumption, we derive the higher order approximation to the percentage point of the distribution of a non-central t-statistic, using the Cornish-Fisher expansion. It is also seen that the approximations give extensions of the results under the normality assumption. In particular, the value of the higher order approximation to a percentage point is easily obtained by a pocket calculator, provided that the moments of the underlying distribution up to the sixth order are known. From numerical results in the cases of Birnbaum-Saunders, exponential, gamma, logistic, normal, t-, two-sided exponential and Weibull distributions, the higher order approximation is seen to be numerically better than the first order one. Indeed, the relative errors of the higher order approximation are smaller than the first order one, when the size of sample is not so small in the above cases. Hence it seems to be useful for symmetric and asymmetric distributions. In the applications to distribution patterns of plant species we consider two different approximate distributions instead of the true distribution which is unknown and obtain the moments up to the 6th order. One depends on the gamma approximation with parameters estimated by the moment method, and another follows from the direct approximation with sample moments up to the 6th order calculated from the frequency distribution based on the data from the true distribution. Since the relative differences of the gamma approximation to the direct one under the (higher order) approximation to a percentage point are seen to be small, the gamma approximation to the true distribution seems to be numerically accurate. Hence the approach is seen to deserve a practical application.

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