# Group-graded and group-bigraded rings

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**Abstract**

Let $I$ be a non-trivial finite multiplicative group with the unit element $e$ and $A = \oplus_{x \in I} A_x$ an $I$-graded ring. We construct a Frobenius extension $\Lambda$ of $A$ and study when the ring extension $A$ of $A_e$ can be a Frobenius extension. Also, formulating the ring structure of $\Lambda$, we introduce the notion of $I$-bigraded rings and show that every $I$-bigraded ring is isomorphic to the $I$-bigraded ring $\Lambda$ constructed above.

Let $I$ be a non-trivial finite multiplicative group with the unit element $e$ and $A = \oplus_{x \in I} A_x$ an $I$-graded ring. In this note, assuming $A_e$ is a local ring, we study when a ring extension $A$ of $A_e$ can be a Frobenius extension, the notion of which we recall below. Auslander-Gorenstein rings (see Definition 1.2) appear in various fields of current research in mathematics. For instance, regular 3-dimensional algebras of type $A$ in the sense of Artin and Schelter, Weyl algebras over fields of characteristic zero, enveloping algebras of finite dimensional Lie algebras and Sklyanin algebras are Auslander-Gorenstein rings (see [2], [5], [6] and [15], respectively). However, little is known about constructions of Auslander-Gorenstein rings. We have shown in [9, Section 3] that a left and right noetherian ring is an Auslander-Gorenstein ring if it admits an Auslander-Gorenstein resolution over another Auslander-Gorenstein ring. A Frobenius extension $A$ of a left and right noetherian ring $R$ is a typical example such that $A$ admits an Auslander-Gorenstein resolution over $R$.

Now we recall the notion of Frobenius extensions of rings due to Nakayama and Tsuzuku [11, 12] which we modify as follows (cf. [1, Section 1]). We use the notation $A/R$ to denote that a ring $A$ contains a ring $R$ as a subring. We say that $A/R$ is a Frobenius extension if the following conditions are satisfied: (F1) $A$ is finitely generated as a left $R$-module; (F2) $A$ is finitely generated projective as a right $R$-module; (F3) there exists an isomorphism $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ in Mod-$A$. Note that $\phi$ induces a unique ring homomorphism $\theta : R \to A$ such that

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$x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$. A Frobenius extension $A/R$ is said to be of first kind if $A \cong \mathrm{Hom}_R(A, R)$ as $R$-$A$-bimodules, and to be of second kind if there exists an isomorphism $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ in Mod-$A$ such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism of $R$. Note that a Frobenius extension of first kind is a special case of a Frobenius extension of second kind. Let $A/R$ be a Frobenius extension. Then $A$ is an Auslander-Gorenstein ring if so is $R$, and the converse holds true if $A$ is projective as a left $R$-module, and if $A/R$ is split, i.e., the inclusion $R \to A$ is a split monomorphism of $R$-$R$-bimodules. It should be noted that $A$ is projective as a left $R$-module if $A/R$ is of second kind.

To state our main theorem we have to construct a Frobenius extension $\Lambda/A$ of first kind. Namely, we will define an appropriate multiplication on a free right $A$-module $\Lambda$ with a basis $\{v_x\}_{x\in I}$ so that $\Lambda/A$ is a Frobenius extension of first kind. Denote by $\{\gamma_x\}_{x\in I}$ the dual basis of $\{v_x\}_{x\in I}$ for the free left $A$-module $\mathrm{Hom}_A(\Lambda, A)$ and set $\gamma = \Sigma_{x\in I}\gamma_x$. Assume $A_e$ is local, $A_x A_{x^{-1}} \subseteq \mathrm{rad}(A_e)$ for all $x \neq e$ and $A$ is reflexive as a right $A_e$-module. Our main theorem states that the following are equivalent: (1) $A \cong \mathrm{Hom}_{A_e}(A, A_e)$ as right $A$-modules; (2) There exist a unique $s \in I$ and some $\alpha \in \mathrm{Hom}_{A_e}(A, A_e)$ such that $\phi_{sx,x} : v_{sx}\Lambda \xrightarrow{\sim} \mathrm{Hom}_{A_e}(\Lambda v_x, A_e), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu)))$ for all $x \in I$; (3) There exist a unique $s \in I$ and some $\alpha_s \in \mathrm{Hom}_{A_e}(A_s, A_e)$ such that $\psi_x : A_{sx} \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A_{x^{-1}}, A_e), a \mapsto (b \mapsto \alpha_s(ab))$ for all $x \in I$ (Theorem 3.3). Assume $A/A_e$ is a Frobenius extension. We show that it is of second kind (Corollary 3.5), and that $A$ is an Auslander-Gorenstein ring if and only if so is $\Lambda$ (Theorem 3.6).

As we saw above, the ring $\Lambda$ plays an essential role in our argument. Formulating the ring structure of $\Lambda$, we introduce the notion of group-bigraded rings as follows. A ring $\Lambda$ together with a group homomorphism $\eta : I^{\mathrm{op}} \to \mathrm{Aut}(\Lambda), x \mapsto \eta_x$ is said to be an $I$-bigraded ring, denoted by $(\Lambda, \eta)$, if $1 = \sum_{x\in I} v_x$ with the $v_x$ orthogonal idempotents and $\eta_y(v_x) = v_{xy}$ for all $x, y \in I$. A homomorphism $\varphi : (\Lambda, \eta) \to (\Lambda', \eta')$ is defined as a ring homomorphism $\varphi : \Lambda \to \Lambda'$ such that $\varphi(v_x) = v'_x$ and $\varphi\eta_x = \eta'_x\varphi$ for all $x \in I$. We conclude that every $I$-bigraded ring is isomorphic to the $I$-bigraded ring $\Lambda$ constructed above (Proposition 4.3).

This note is organized as follows. In Section 1, we recall basic facts on Auslander-Gorenstein rings and Frobenius extensions. In Section 2, we construct a Frobenius extension $\Lambda/A$ of first kind and study the ring structure of $\Lambda$. In Section 3, we prove the main theorem. In Section 4, we introduce the notion of group-bigraded rings and study the structure of such rings. In Section 5, we provide a systematic construction of $I$-graded rings $A$ such that $A/A_e$ is a Frobenius extension of second kind.

# 1 Preliminaries

For a ring $R$ we denote by $\mathrm{rad}(R)$ the Jacobson radical of $R$, by $R^\times$ the set of units in $R$, by $\mathrm{Z}(R)$ the center of $R$ and by $\mathrm{Aut}(R)$ the group of ring automorphisms of $R$. Usually, the identity element of a ring is simply denoted by 1. Sometimes, we use the notation $1_R$ to stress that it is the identity element of the ring $R$. We denote by Mod-$R$ the category of right $R$-modules. Left $R$-modules are considered as right $R^{\mathrm{op}}$-modules, where $R^{\mathrm{op}}$ denotes the opposite ring of $R$. In particular, we denote by inj dim $R$ (resp., inj dim $R^{\mathrm{op}}$) the injective dimension of $R$ as a right (resp., left) $R$-module and by $\mathrm{Hom}_R(-,-)$ (resp., $\mathrm{Hom}_{R^{\mathrm{op}}}(-,-)$) the set of homomorphisms in Mod-$R$ (resp., Mod-$R^{\mathrm{op}}$). Sometimes, we use the notation $X_R$ (resp., ${}_RX$) to stress that the module $X$ considered is a right (resp., left) $R$-module.

We start by recalling the notion of Auslander-Gorenstein rings.

**Proposition 1.1** (Auslander)**.** *Let $R$ be a right and left noetherian ring. Then for any $n \geq 0$ the following are equivalent.*

*(1) In a minimal injective resolution $I^\bullet$ of $R$ in* Mod-$R$, flat dim $I^i \leq i$ *for all $0 \leq i \leq n$.*

*(2) In a minimal injective resolution $J^\bullet$ of $R$ in* Mod-$R^{\mathrm{op}}$, flat dim $J^i \leq i$ *for all $0 \leq i \leq n$.*

*(3) For any $1 \leq i \leq n+1$, any $M \in$* mod-$R$ *and any submodule $X$ of $\mathrm{Ext}^i_R(M,R) \in$* mod-$R^{\mathrm{op}}$ *we have $\mathrm{Ext}^j_{R^{\mathrm{op}}}(X,R) = 0$ for all $0 \leq j < i$.*

*(4) For any $1 \leq i \leq n+1$, any $X \in$* mod-$R^{\mathrm{op}}$ *and any submodule $M$ of $\mathrm{Ext}^i_{R^{\mathrm{op}}}(X,R) \in$* mod-$R$ *we have $\mathrm{Ext}^j_R(M,R) = 0$ for all $0 \leq j < i$.*

*Proof.* See e.g. [7, Theorem 3.7]. □

**Definition 1.2** ([6])**.** A right and left noetherian ring $R$ is said to satisfy the Auslander condition if it satisfies the equivalent conditions in Proposition 1.1 for all $n \geq 0$, and to be an Auslander-Gorenstein ring if it satisfies the Auslander condition and inj dim $R =$ inj dim $R^{\mathrm{op}} < \infty$.

It should be noted that for a right and left noetherian ring $R$ we have inj dim $R =$ inj dim $R^{\mathrm{op}}$ whenever inj dim $R < \infty$ and inj dim $R^{\mathrm{op}} < \infty$ (see [16, Lemma A]).

Next, we recall the notion of Frobenius extensions of rings due to Nakayama and Tsuzuku [11, 12], which we modify as follows (cf. [1, Section 1]).

**Definition 1.3.** A ring $A$ is said to be an extension of a ring $R$ if $A$ contains $R$ as a subring, and the notation $A/R$ is used to denote that $A$ is an extension ring of $R$. A ring extension $A/R$ is said to be Frobenius if the following conditions are satisfied:

(F1) $A$ is finitely generated as a left $R$-module;
(F2) $A$ is finitely generated projective as a right $R$-module;
(F3) $A \cong \mathrm{Hom}_R(A,R)$ as right $A$-modules.

In case $R$ is a right and left noetherian ring, for any Frobenius extension $A/R$ the isomorphism $A \xrightarrow{\sim} \mathrm{Hom}_R(A,R)$ in Mod-$A$ yields an Auslander-Gorenstein resolution of $A$ over $R$ in the sense of [9, Definition 3.5].

The next proposition is well-known and easily verified.

**Proposition 1.4.** *Let $A/R$ be a ring extension and $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ an isomorphism in* Mod-$A$*. Then the following hold.*

*(1) There exists a unique ring homomorphism $\theta : R \to A$ such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$.*

*(2) If $\phi' : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ is another isomorphism in* Mod-$A$*, then there exists $u \in A^\times$ such that $\phi'(1) = \phi(1)u$ and $\theta'(x) = u^{-1}\theta(x)u$ for all $x \in R$.*

*(3) $\phi$ is an isomorphism of $R$-$A$-bimodules if and only if $\theta(x) = x$ for all $x \in R$.*

**Definition 1.5** (cf. [11, 12])**.** A Frobenius extension $A/R$ is said to be of first kind if $A \cong \mathrm{Hom}_R(A, R)$ as $R$-$A$-bimodules, and to be of second kind if there exists an isomorphism $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ in Mod-$A$ such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism $\theta : R \xrightarrow{\sim} R$.

**Proposition 1.6.** *If $A/R$ is a Frobenius extension of second kind, then $A$ is projective as a left $R$-module.*

*Proof.* Let $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ be an isomorphism in Mod-$A$ such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism $\theta : R \xrightarrow{\sim} R$. Then $\theta$ induces an equivalence $U_\theta :$ Mod-$R^{\mathrm{op}} \xrightarrow{\sim}$ Mod-$R^{\mathrm{op}}$ such that for any $M \in$ Mod-$R^{\mathrm{op}}$ we have $U_\theta M = M$ as an additive group and the left $R$-module structure of $U_\theta M$ is given by the law of composition $R \times M \to M, (x, m) \mapsto \theta(x)m$. Since $\phi$ yields an isomorphism of $R$-$A$-bimodules $U_\theta A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$, and since $\mathrm{Hom}_R(A, R)$ is projective as a left $R$-module, it follows that $U_\theta A$ and hence $A$ are projective as left $R$-modules. □

**Proposition 1.7.** *For any Frobenius extensions $\Lambda/A$, $A/R$ the following hold.*

*(1) $\Lambda/R$ is a Frobenius extension.*

*(2) Assume $\Lambda/A$ is of first kind. If $A/R$ is of second (resp., first) kind, then so is $\Lambda/R$.*

*Proof.* (1) Obviously, (F1) and (F2) are satisfied. Also, we have

$$\begin{aligned}\Lambda &\cong \mathrm{Hom}_A(\Lambda, A)\\ &\cong \mathrm{Hom}_A(\Lambda, \mathrm{Hom}_R(A, R))\\ &\cong \mathrm{Hom}_R(\Lambda \otimes_A A, R)\\ &\cong \mathrm{Hom}_R(\Lambda, R)\end{aligned}$$

in Mod-$\Lambda$.

(2) Let $\psi : \Lambda \xrightarrow{\sim} \mathrm{Hom}_A(\Lambda, A)$ be an isomorphism of $A$-$\Lambda$-bimodules and $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ an isomorphism in Mod-$A$ such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism $\theta : R \xrightarrow{\sim} R$. Setting $\gamma = \psi(1)$ and $\alpha = \phi(1)$, as in (1), we have an isomorphism in Mod-$\Lambda$

$$\xi : \Lambda \xrightarrow{\sim} \mathrm{Hom}_R(\Lambda, R), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu))).$$

For any $x \in R$, we have

$$\begin{aligned} x\xi(1)(\mu) &= x\alpha(\gamma(\mu)) \\ &= \alpha(\theta(x)\gamma(\mu)) \\ &= \alpha(\gamma(\theta(x)\mu)) \\ &= \xi(1)(\theta(x)\mu) \end{aligned}$$

for all $\mu \in \Lambda$ and $x\xi(1) = \xi(1)\theta(x)$. □

**Definition 1.8** ([1])**.** A ring extension $A/R$ is said to be split if the inclusion $R \to A$ is a split monomorphism of $R$-$R$-bimodules.

**Proposition 1.9** (cf. [1])**.** *For any Frobenius extension $A/R$ the following hold.*

*(1) If $R$ is an Auslander-Gorenstein ring, then so is $A$ with* inj dim $A \leq$ inj dim $R$.

*(2) Assume $A$ is projective as a left $R$-module and $A/R$ is split. If $A$ is an Auslander-Gorenstein ring, then so is $R$ with* inj dim $R =$ inj dim $A$.

*Proof.* (1) See [9, Theorem 3.6].

(2) It follows by [1, Proposition 1.7] that $R$ is a right and left noetherian ring with inj dim $R =$ inj dim $R^{\mathrm{op}} =$ inj dim $A$. Let $A \to E^\bullet$ be a minimal injective resolution in Mod-$A$. For any $i \geq 0$, $\mathrm{Hom}_R(-, E^i) \cong \mathrm{Hom}_A(- \otimes_R A, E^i)$ as functors on Mod-$R$ and $E^i_R$ is injective, and $E^i \otimes_R - \cong E^i \otimes_A A \otimes_R -$ as functors on Mod-$R^{\mathrm{op}}$ and flat dim $E^i_R \leq$ flat dim $E^i_A \leq i$. Now, since $R_R$ appears in $A_R$ as a direct summand, it follows that $R$ satisfies the Auslander condition. □

## 2 Graded rings

Throughout the rest of this note, $I$ stands for a non-trivial finite multiplicative group with the unit element $e$.

Throughout this and the next sections, we fix a ring $A$ together with a family $\{\delta_x\}_{x\in I}$ in $\mathrm{End}_{\mathbb{Z}}(A)$ satisfying the following conditions:

(D1) $\delta_x\delta_y = 0$ unless $x = y$ and $\sum_{x\in I} \delta_x = \mathrm{id}_A$;

(D2) $\delta_x(a)\delta_y(b) = \delta_{xy}(\delta_x(a)b)$ for all $a, b \in A$ and $x, y \in I$.

Namely, setting $A_x = \mathrm{Im}\ \delta_x$ for $x \in I$, $A = \oplus_{x\in I} A_x$ is an $I$-graded ring. In particular, $A/A_e$ is a split ring extension.

To prove our main theorem (Theorem 3.3), we need an extension ring $\Lambda$ of $A$ such that $\Lambda/A$ is a Frobenius extension of first kind. Let $\Lambda$ be a free right $A$-module with a basis $\{v_x\}_{x\in I}$ and define a multiplication on $\Lambda$ subject to the following axioms:

(M1) $v_xv_y = 0$ unless $x = y$ and $v_xv_x = v_x$ for all $x \in I$;

(M2) $av_x = \sum_{y\in I} v_y\delta_{yx^{-1}}(a)$ for all $a \in A$ and $x \in I$.

We denote by $\{\gamma_x\}_{x\in I}$ the dual basis of $\{v_x\}_{x\in I}$ for the free left $A$-module $\mathrm{Hom}_A(\Lambda, A)$, i.e., $\lambda = \sum_{x\in I} v_x\gamma_x(\lambda)$ for all $\lambda \in \Lambda$. It is not difficult to see that

$$\lambda\mu = \sum_{x,y\in I} v_x\delta_{xy^{-1}}(\gamma_x(\lambda))\gamma_y(\mu)$$

for all $\lambda, \mu \in \Lambda$. Also, setting $\gamma = \sum_{x\in I}\gamma_x$, we define a mapping

$$\phi : \Lambda \to \mathrm{Hom}_A(\Lambda, A), \lambda \mapsto \gamma\lambda.$$

**Proposition 2.1.** *The following hold.*

*(1) $\Lambda$ is an associative ring with $1 = \sum_{x\in I} v_x$ and contains $A$ as a subring via the injective ring homomorphism $A \to \Lambda, a \mapsto \sum_{x\in I} v_x a$.*

*(2) $\phi$ is an isomorphism of $A$-$\Lambda$-bimodules, i.e., $\Lambda/A$ is a Frobenius extension of first kind.*

*Proof.* (1) Let $\lambda \in \Lambda$. Obviously, $\sum_{x\in I} v_x \cdot \lambda = \lambda$. Also, by (D1) we have

$$\begin{aligned}\lambda \cdot \sum_{y\in I} v_y &= \sum_{x,y\in I} v_x\delta_{xy^{-1}}(\gamma_x(\lambda))\\ &= \sum_{x\in I} v_x\gamma_x(\lambda)\\ &= \lambda.\end{aligned}$$

Next, for any $\lambda, \mu, \nu \in \Lambda$ by (D2) we have

$$\begin{aligned}(\lambda\mu)\nu &= \sum_{x,y,z\in I} v_x\delta_{xz^{-1}}(\delta_{xy^{-1}}(\gamma_x(\lambda))\gamma_y(\mu))\gamma_z(\nu)\\ &= \sum_{x,y,z\in I} v_x\delta_{xy^{-1}}(\gamma_x(\lambda))\delta_{yz^{-1}}(\gamma_y(\mu))\gamma_z(\nu)\\ &= \lambda(\mu\nu).\end{aligned}$$

The remaining assertions are obvious.

(2) Let $\lambda \in \mathrm{Ker}\ \phi$. For any $y \in I$ we have $0 = \gamma(\lambda v_y) = \sum_{x\in I}\delta_{xy^{-1}}(\gamma_x(\lambda))$ and $\delta_{xy^{-1}}(\gamma_x(\lambda)) = 0$ for all $x \in I$. Thus for any $x \in I$ we have $\delta_{xy^{-1}}(\gamma_x(\lambda)) = 0$ for all $y \in I$ and by (D1) $\gamma_x(\lambda) = 0$, so that $\lambda = 0$. Next, for any $f = \sum_{x\in I} a_x\gamma_x \in \mathrm{Hom}_A(\Lambda, A)$, setting $\lambda = \sum_{x,z\in I} v_x\delta_{xz^{-1}}(a_z)$, by (D1) we have

$$\begin{aligned}(\gamma\lambda)(v_y) &= \gamma(\lambda v_y)\\ &= \sum_{x\in I}\delta_{xy^{-1}}(\gamma_x(\lambda))\\ &= \sum_{x,z\in I}\delta_{xy^{-1}}(\delta_{xz^{-1}}(a_z))\\ &= a_y\\ &= f(v_y)\end{aligned}$$

for all $y \in I$ and $f = \gamma\lambda$. Finally, for any $a \in A$ by (D1) we have

$$\begin{aligned}(\gamma a)(\lambda) &= \gamma(a\lambda) \\ &= \sum_{x,y\in I} \delta_{yx^{-1}}(a)\gamma_x(\lambda) \\ &= a\gamma(\lambda)\end{aligned}$$

for all $\lambda \in \Lambda$ and $\gamma a = a\gamma$. □

*Remark* 2.2. Denote by $|I|$ the order of $I$. If $|I| \cdot 1_A \in A^\times$, then $\Lambda/A$ is a split ring extension.

**Lemma 2.3.** *The following hold.*

(1) $v_x\lambda v_y = v_x\delta_{xy^{-1}}(\gamma_x(\lambda))$ *for all* $\lambda \in \Lambda$ *and* $x, y \in I$.

(2) $v_x\Lambda v_y = v_x A_{xy^{-1}}$ *for all* $x, y \in I$.

(3) $v_x a \cdot v_y b = v_x ab$ *for all* $x, y, z \in I$ *and* $a \in A_{xy^{-1}}, b \in A_{yz^{-1}}$.

*Proof.* Immediate by the definition. □

Setting $\Lambda_{x,y} = v_x\Lambda v_y$ for $x, y \in I$, we have $\Lambda = \oplus_{x,y\in I}\Lambda_{x,y}$ with $\Lambda_{x,y}\Lambda_{z,w} = 0$ unless $y = z$ and $\Lambda_{x,y}\Lambda_{y,z} \subseteq \Lambda_{x,z}$ for all $x, y, z \in I$. Also, setting $\lambda_{x,y} = \delta_{xy^{-1}}(\gamma_x(\lambda)) \in A_{xy^{-1}}$ for $\lambda \in \Lambda$ and $x, y \in I$, we have a group homomorphism

$$\eta : I^{\mathrm{op}} \to \mathrm{Aut}(\Lambda), x \mapsto \eta_x$$

such that $\eta_x(\lambda)_{y,z} = \lambda_{yx^{-1},zx^{-1}}$ for all $\lambda \in \Lambda$ and $x, y, z \in I$. We denote by $\Lambda^I$ the subring of $\Lambda$ consisting of all $\lambda$ such that $\eta_x(\lambda) = \lambda$ for all $x \in I$.

**Proposition 2.4.** *The following hold.*

(1) $\eta_y(v_x) = v_{xy}$ *for all* $x, y \in I$.

(2) $\Lambda^I = A$.

(3) $(\lambda\mu)_{x,z} = \sum_{y\in I} \lambda_{x,y}\mu_{y,z}$ *for all* $\lambda, \mu \in \Lambda$ *and* $x, z \in I$.

*Proof.* (1) Since $\eta_y(v_x)_{z,w} = \delta_{zw^{-1}}(\gamma_{zy^{-1}}(v_x))$ for all $z, w \in I$, we have

$$\eta_y(v_x)_{z,w} = \begin{cases} 1 & \text{if } z = w \text{ and } x = zy^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

(2) For any $a \in A$, since $\eta_x(a)_{y,z} = a_{yx^{-1},zx^{-1}} = \delta_{(yx^{-1})(zx^{-1})^{-1}}(a) = \delta_{yz^{-1}}(a) = a_{y,z}$ for all $x, y, z \in I$, we have $a \in \Lambda^I$. Conversely, for any $\lambda \in \Lambda^I$ we have $\delta_{y^{-1}}(\gamma_x(\lambda)) = \lambda_{x,yx} = \eta_{x^{-1}}(\lambda)_{e,y} = \lambda_{e,y} = \delta_{y^{-1}}(\gamma_e(\lambda))$ for all $x, y \in I$, so that $\gamma_x(\lambda) = \gamma_e(\lambda)$ for all $x \in I$.

(3) For any $\lambda, \mu \in \Lambda$ and $x, z \in I$ by (D2) we have

$$\begin{aligned}(\lambda\mu)_{x,z} &= \sum_{y \in I} \delta_{xz^{-1}}(\delta_{xy^{-1}}(\gamma_x(\lambda))\gamma_y(\mu)) \\ &= \sum_{y \in I} \delta_{xy^{-1}}(\gamma_x(\lambda))\delta_{yz^{-1}}(\gamma_y(\mu)) \\ &= \sum_{y \in I} \lambda_{x,y}\mu_{y,z}.\end{aligned}$$

□

*Remark* 2.5. We have $\eta_y(v_x a_x)v_y b_y = v_{xy}a_x b_y$ for all $a_x \in A_x$ and $b_y \in A_y$.

**Proposition 2.6.** *The following hold.*

*(1)* $\mathrm{End}_\Lambda(v_x\Lambda) \cong A_e$ *as rings for all* $x \in I$.

*(2)* $v_x\Lambda \ncong v_y\Lambda$ *in* Mod-$\Lambda$ *for all* $x, y \in I$ *with* $A_{xy^{-1}}A_{yx^{-1}} \subseteq \mathrm{rad}(A_e)$.

*Proof.* (1) We have $\mathrm{End}_\Lambda(v_x\Lambda) \cong v_x\Lambda v_x \cong A_e$ as rings.

(2) For any $f : v_x\Lambda \to v_y\Lambda$ and $g : v_y\Lambda \to v_x\Lambda$ in Mod-$\Lambda$, since $f(v_x) = v_y a$ with $a \in A_{yx^{-1}}$ and $g(v_y) = v_x b$ with $b \in A_{xy^{-1}}$, we have $g(f(v_x)) = v_x ba$ with $ba \in \mathrm{rad}(A_e)$. □

The proposition above asserts that if $A_e$ is local and $A_x A_{x^{-1}} \subseteq \mathrm{rad}(A_e)$ for all $x \neq e$ then $\Lambda$ is semiperfect and basic. We refer to [3] for semiperfect rings.

# 3 Auslander-Gorenstein rings

In this section, we will ask when $A/A_e$ is a Frobenius extension.

**Lemma 3.1.** *For any* $x \in I$ *the following hold.*

*(1)* $av_x = v_x a$ *for all* $a \in A_e$ *and* $\Lambda v_x$ *is a* $\Lambda$-$A_e$*-bimodule.*

*(2)* $\Lambda v_x = \sum_{y \in I} v_y A_{yx^{-1}}$.

*(3)* $A \xrightarrow{\sim} \Lambda v_x, a \mapsto \sum_{y \in I} v_y \delta_{yx^{-1}}(a)$ *as* $A$-$A_e$*-bimodules.*

*(4) If* $\Lambda v_x$ *is reflexive as a right* $A_e$*-module, then* $\mathrm{End}_\Lambda(\mathrm{Hom}_{A_e}(\Lambda v_x, A_e)) \cong A_e$ *as rings.*

*Proof.* (1) and (2) Immediate by the definition.

(3) By (2) we have a bijection $f_x : A \xrightarrow{\sim} \Lambda v_x, a \mapsto \sum_{y \in I} v_y \delta_{yx^{-1}}(a)$. Since every $\delta_{yx^{-1}}$ is a homomorphism in Mod-$A_e$, so is $f_x$. Finally, for any $a, b \in A$

we have

$$
\begin{aligned}
a\cdot(\sum_{y\in I} v_y\delta_{yx^{-1}}(b)) &= \sum_{y,z\in I} v_z\delta_{zy^{-1}}(a)\delta_{yx^{-1}}(b)\\
&= \sum_{z\in I} v_z(\sum_{y\in I}\delta_{zy^{-1}}(a)\delta_{yx^{-1}}(b))\\
&= \sum_{z\in I} v_z\delta_{zx^{-1}}(\sum_{y\in I}\delta_{zy^{-1}}(a)b)\\
&= \sum_{z\in I} v_z\delta_{zx^{-1}}(ab)
\end{aligned}
$$

and $f_x$ is a homomorphism in Mod-$A^{\mathrm{op}}$.

(4) Since the canonical homomorphism

$$\Lambda v_x \to \mathrm{Hom}_{A_e^{\mathrm{op}}}(\mathrm{Hom}_{A_e}(\Lambda v_x, A_e), A_e), \lambda \mapsto (f \mapsto f(\lambda))$$

is an isomorphism, $\mathrm{End}_\Lambda(\mathrm{Hom}_{A_e}(\Lambda v_x, A_e)) \cong \mathrm{End}_{\Lambda^{\mathrm{op}}}(\Lambda v_x)^{\mathrm{op}} \cong v_x\Lambda v_x \cong A_e$ as rings. □

It follows by Lemma 3.1(1) that $\delta_e\gamma_e : \Lambda \to A_e$ is a homomorphism of $A_e$-$A_e$-bimodules and $\Lambda/A_e$ is a split ring extension.

**Lemma 3.2.** *For any $x, y \in I$ and $a, b \in A$ we have*

$$v_x a\cdot(\sum_{z\in I} v_z\delta_{zy^{-1}}(b)) = v_x(\sum_{z\in I}\delta_{xz^{-1}}(a)\delta_{zy^{-1}}(b))$$

*Proof.* Immediate by the definition. □

**Theorem 3.3.** *Assume $A_e$ is local, $A_xA_{x^{-1}} \subseteq \mathrm{rad}(A_e)$ for all $x \neq e$ and $A$ is reflexive as a right $A_e$-module. Then the following are equivalent.*

*(1) $A \cong \mathrm{Hom}_{A_e}(A, A_e)$ as right $A$-modules.*

*(2) There exist a unique $s \in I$ and some $\alpha \in \mathrm{Hom}_{A_e}(A, A_e)$ such that*

$$\phi_{sx,x} : v_{sx}\Lambda \xrightarrow{\sim} \mathrm{Hom}_{A_e}(\Lambda v_x, A_e), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu)))$$

*for all $x \in I$.*

*(3) There exist a unique $s \in I$ and some $\alpha_s \in \mathrm{Hom}_{A_e}(A_s, A_e)$ such that*

$$\psi_x : A_{sx} \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A_{x^{-1}}, A_e), a \mapsto (b \mapsto \alpha_s(ab))$$

*for all $x \in I$.*

*Proof.* (1) ⇒ (2). Let $A \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A, A_e), 1 \mapsto \alpha$ in Mod-$A$. Then, since by Proposition 2.1(2) $\Lambda \xrightarrow{\sim} \mathrm{Hom}_A(\Lambda, A), \lambda \mapsto \gamma\lambda$ in Mod-$\Lambda$, by adjointness we have an isomorphism in Mod-$\Lambda$

$$\Lambda \xrightarrow{\sim} \mathrm{Hom}_{A_e}(\Lambda, A_e), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu))).$$

By Proposition 2.6(1) $\Lambda = \oplus_{x\in I} v_x\Lambda$ with the $\mathrm{End}_\Lambda(v_x\Lambda)$ local. Also, by (1) and (4) of Lemma 3.1

$$\mathrm{Hom}_{A_e}(\Lambda, A_e) \cong \oplus_{x\in I}\mathrm{Hom}_{A_e}(\Lambda v_x, A_e)$$

with the $\mathrm{End}_\Lambda(\mathrm{Hom}_{A_e}(\Lambda v_x, A_e))$ local. Now, according to Proposition 2.6(2), it follows by the Krull-Schmidt theorem that there exists a unique $s \in I$ such that

$$\phi_{s,e} : v_s\Lambda \xrightarrow{\sim} \mathrm{Hom}_{A_e}(\Lambda v_e, A_e), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu))).$$

Thus, setting $\alpha_s = \alpha|_{A_s}$, by Lemmas 3.1(2) and 3.2 we have

$$\psi : A \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A, A_e), a \mapsto (b \mapsto \alpha_s(\delta_s(ab))).$$

It then follows again by Lemmas 3.1(2) and 3.2 that

$$\phi_{sx,x} : v_{sx}\Lambda \xrightarrow{\sim} \mathrm{Hom}_{A_e}(\Lambda v_x, A_e), \lambda \mapsto (\mu \mapsto \alpha(\gamma(\lambda\mu)))$$

for all $x \in I$.

(2) $\Rightarrow$ (3). Since $A = \oplus_{x\in I} A_{sx} = \oplus_{x\in I} A_{x^{-1}}$, and since $A_{sx}A_{x^{-1}} \subseteq A_s$ for all $x \in I$, $\psi$ induces $\psi_x : A_{sx} \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A_{x^{-1}}, A_e), a \mapsto (b \mapsto \alpha_s(ab))$ for all $x \in I$.

(3) $\Rightarrow$ (1). Setting $\psi_x : A_{sx} \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A_{x^{-1}}, A_e), a \mapsto (b \mapsto \alpha_s(ab))$ for each $x \in I$, the $\psi_x$ yields $\psi : A \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A, A_e), a \mapsto (b \mapsto \alpha_s(\delta_s(ab)))$. □

*Remark* 3.4. In the theorem above, $\alpha_s$ is an isomorphism and $A_e \xrightarrow{\sim} \mathrm{End}_{A_e}(A_s)$ canonically.

*Proof.* For any $b \in A_e$, setting $f : A_e \to A_e, 1 \mapsto b$, we have $f = \psi_e(a)$ and hence $b = \alpha_s(a)$ for some $a \in A_s$. Also, $\mathrm{Ker}\ \alpha_s = \mathrm{Ker}\ \psi_s = 0$. Then, since the composite $A_e \to \mathrm{End}_{A_e}(A_s) \to \mathrm{Hom}_{A_e}(A_s, A_e)$ is an isomorphism, the last assertion follows. □

**Corollary 3.5.** *Assume $A_e$ is local and $A_xA_{x^{-1}} \subseteq \mathrm{rad}(A_e)$ for all $x \neq e$. If $A/A_e$ is a Frobenius extension, then it is of second kind.*

*Proof.* Set $t = \alpha_s^{-1}(1) \in A_s$. Then for any $u \in A_s$ there exists $f \in \mathrm{End}_{A_e}(A_s)$ such that $u = f(t)$ and hence $u = at$ for some $a \in A_e$. Thus $A_et = A_s$ and there exists $\theta \in \mathrm{Aut}(A_e)$ such that $\theta(a)t = ta$ for all $a \in A_e$. Then $(\alpha_s\theta(a))(t) = \alpha_s(\theta(a)t) = \alpha_s(ta) = \alpha_s(t)a = a = (a\alpha_s)(t)$ and $\alpha_s\theta(a) = a\alpha_s$ for all $a \in A_e$. Now, setting $\psi : A \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A, A_e), a \mapsto (b \mapsto \alpha_s(\delta_s(ab)))$, we have $(a\psi(1))(b) = a\alpha_s(\delta_s(b)) = (a\alpha_s)(\delta_s(b)) = (\alpha_s\theta(a))(\delta_s(b)) = \alpha_s(\theta(a)\delta_s(b)) = \alpha_s(\delta_s(\theta(a)b)) = (\psi(1)\theta(a))(b)$ for all $a, b \in A$, so that $a\psi(1) = \psi(1)\theta(a)$ for all $a \in A$. □

**Theorem 3.6.** *Assume $A_e$ is local, $A_xA_{x^{-1}} \subseteq \mathrm{rad}(A_e)$ for all $x \neq e$, and $A/A_e$ is a Frobenius extension. Then $A$ is an Auslander-Gorenstein ring if and only if so is $\Lambda$.*

*Proof.* The "only if" part follows by Propositions 1.9(1) and 2.1(2). Assume $\Lambda$ is an Auslander-Gorenstein ring. By Proposition 2.1(2) $\Lambda/A$ is a Frobenius extension of first kind, and by Corollary 3.5 $A/A_e$ is a Frobenius extension of second kind. Thus by Proposition 1.7 $\Lambda/A_e$ is a Frobenius extension of second kind. Also, by Lemma 3.1(1) $\Lambda/A_e$ is split. Hence by Propositions 1.6 and 1.9(2) $A_e$ is an Auslander-Gorenstein ring and by Proposition 1.9(1) so is $A$. □

*Remark* 3.7. Assume $A_e$ is local, $A_x A_{x^{-1}} \subseteq \mathrm{rad}(A_e)$ for all $x \neq e$ and $A/A_e$ is a Frobenius extension. Let $s \in I$ be as in Theorem 3.3. Then the following hold.

(1) $s \neq e$ unless $A = A_e$.

(2) Let $J$ be a subgroup of $I$ containing $s$ and $A_J = \oplus_{x\in J} A_x$. Then $A_J/A_e$ is a Frobenius extension and, unless $s = e$, the mapping cone of the multiplication map

$$\bigoplus_{x\in J} \Lambda v_x \otimes_{A_e} v_x \Lambda \to \Lambda$$

is a tilting complex for right $\Lambda$-modules (see [14] for tilting complexes).

*Proof.* (1) Suppose to the contrary that $s = e$. Let $x \in I$ with $x \neq e$ and $A_x \neq 0$. Then by Remark 3.4 there exists $u \in {A_e}^{\times}$ such that $A_x \xrightarrow{\sim} \mathrm{Hom}(A_{x^{-1}}, A_e), a \mapsto (b \mapsto uab)$. Note that $uab \in \mathrm{rad}(A_e)$ for all $a \in A_x$ and $b \in A_{x^{-1}}$. On the other hand, since $A_{x^{-1}}$ is nonzero projective, and since $A_e$ is local, there exists an epimorphism $f : A_{x^{-1}} \to A_e$ in Mod-$A_e$, a contradiction.

(2) Since $\psi_x : A_{sx} \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A_{x^{-1}}, A_e), a \mapsto (b \mapsto \alpha_s(ab))$ for all $x \in J$, the $\psi_x$ yields $\psi_J : A_J \xrightarrow{\sim} \mathrm{Hom}_{A_e}(A_J, A_e), a \mapsto (b \mapsto \alpha_s(\delta_s(ab)))$. The first assertion follows by Theorem 3.3.

Next, let $v_J = \sum_{x\in J} v_x$. Then by Lemma 3.1(1) $av_J = v_J a$ for all $a \in A_e$. Since $\Lambda/A_e$ is a Frobenius extension, $\Lambda v_J$ is finitely generated projective as a right $A_e$-module and by Theorem 3.3 $v_J\Lambda \cong \mathrm{Hom}_{A_e}(\Lambda v_J, A_e)$ as right $\Lambda$-modules. Note that $v_x \Lambda v_x \neq 0$ and $v_{sx}\Lambda v_x \neq 0$ for all $x \in J$. Thus the last assertion follows by the same argument as in [1, Example 4.3]. □

We will see in the final section that the element $s \in I$ in Theorem 3.3 does not necessarily depend on the structure of the group $I$ (Example 5.3).

# 4 Bigraded rings

Formulating the ring structure of $\Lambda$ constructed in Section 2, we make the following.

**Definition 4.1.** A ring $\Lambda$ together with a group homomorphism

$$\eta : I^{\mathrm{op}} \to \mathrm{Aut}(\Lambda), x \mapsto \eta_x$$

is said to be an $I$-bigraded ring, denoted by $(\Lambda, \eta)$, if $1 = \sum_{x\in I} v_x$ with the $v_x$ orthogonal idempotents and $\eta_y(v_x) = v_{xy}$ for all $x, y \in I$. A homomorphism $\varphi : (\Lambda, \eta) \to (\Lambda', \eta')$ is defined as a ring homomorphism $\varphi : \Lambda \to \Lambda'$ such that $\varphi(v_x) = v'_x$ and $\varphi\eta_x = \eta'_x\varphi$ for all $x \in I$.

Throughout this section, we fix an $I$-bigraded ring $(\Lambda, \eta)$. Set $A_x = v_x \Lambda v_e$ for $x \in I$ and $A = \oplus_{x \in I} A_x$. Note that $\eta_y(A_x) = v_{xy} \Lambda v_y$ for all $x, y \in I$. For any $a_x \in A_x$ and $b_y \in A_y$ we define the multiplication $a_x \cdot b_y$ in $A$ as the multiplication $\eta_y(a_x) b_y$ in $\Lambda$ (cf. Remark 2.5).

**Proposition 4.2.** *The following hold.*

*(1) $A$ is an associative ring with $1 = v_e$.*

*(2) $A$ is an $I$-graded ring.*

*Proof.* (1) For any $a_x \in A_x$, $b_y \in A_y$ and $c_z \in A_z$ we have

$$\begin{aligned}(a_x \cdot b_y) \cdot c_z &= \eta_y(a_x) b_y \cdot c_z \\ &= \eta_z(\eta_y(a_x) b_y) c_z \\ &= \eta_{yz}(a_x) \eta_z(b_y) c_z \\ &= a_x \cdot (b_y \cdot c_z).\end{aligned}$$

Also, for any $a_x \in A_x$ we have $v_e \cdot a_x = \eta_x(v_e) a_x = v_x a_x = a_x$ and $a_x \cdot v_e = \eta_e(a_x) v_e = a_x v_e = a_x$.

(2) Obviously, $A_x A_y \subseteq A_{xy}$ for all $x, y \in I$. □

In the following, for each $x \in I$ we denote by $\delta_x : A \to A_x$ the projection. Then, setting $\lambda_{x,y} = v_x \lambda v_y$ for $\lambda \in \Lambda$ and $x, y \in I$, we have a mapping $\varphi : A \to \Lambda$ such that $\varphi(a)_{x,y} = \eta_y(\delta_{xy^{-1}}(a))$ for all $a \in A$ and $x, y \in I$.

**Proposition 4.3.** *The following hold.*

*(1) $\varphi : A \to \Lambda$ is an injective ring homomorphism with $\operatorname{Im} \varphi = \Lambda^I$.*

*(2) $v_x \Lambda v_y = v_x \varphi(A_{xy^{-1}})$ for all $x, y \in I$.*

*(3) $\{v_x\}_{x \in I}$ is a basis for the right $A$-module $\Lambda$.*

*(4) $\varphi(a) v_x = \sum_{y \in I} v_y \varphi(\delta_{yx^{-1}}(a))$ for all $a \in A$ and $x \in I$.*

*(5) $v_x \varphi(a) v_y \varphi(b) = v_x \varphi(ab)$ for all $x, y, z \in I$ and $a \in A_{xy^{-1}}, b \in A_{yz^{-1}}$.*

*Proof.* (1) Obviously, $\varphi$ is a monomorphism of additive groups. Also, we have

$$\varphi(v_e)_{x,y} = \begin{cases} v_x & \text{if } x = y, \\ 0 & \text{otherwise} \end{cases}$$

and $\varphi(1_A) = 1_\Lambda$. Let $a_x \in A_x, b_y \in A_y$ and $z, w \in I$. Since $\varphi(a_x \cdot b_y)_{z,w} = \varphi(\eta_y(a_x) b_y)_{z,w} = \eta_w(\delta_{zw^{-1}}(\eta_y(a_x) b_y))$, $\varphi(a_x \cdot b_y)_{z,w} = 0$ unless $xy = zw^{-1}$. If $xy = zw^{-1}$, then $\eta_w(\delta_{zw^{-1}}(\eta_y(a_x) b_y)) = \eta_{yw}(a_x) \eta_w(b_y)$. On the other hand,

$$\begin{aligned}(\varphi(a_x)\varphi(b_y))_{z,w} &= \sum_{u \in I} \varphi(a_x)_{z,u} \varphi(b_y)_{u,w} \\ &= \sum_{u \in I} \eta_u(\delta_{zu^{-1}}(a_x)) \eta_w(\delta_{uw^{-1}}(b_y)).\end{aligned}$$

Thus $(\varphi(a_x)\varphi(b_y))_{z,w} = 0$ unless $zu^{-1} = x$ and $uw^{-1} = y$, i.e., $zw^{-1} = xy$. If $zw^{-1} = xy$, then $\sum_{u\in I} \eta_u(\delta_{zu^{-1}}(a_x))\eta_w(\delta_{uw^{-1}}(b_y)) = \eta_{yw}(a_x)\eta_w(b_y)$. As a consequence, $\varphi(a_x \cdot b_y)_{z,w} = (\varphi(a_x)\varphi(b_y))_{z,w}$. The first assertion follows.

Next, for any $a \in A$ and $x, y, z \in I$ we have

$$\begin{aligned}
\eta_x(\varphi(a))_{y,z} &= v_y\eta_x(\varphi(a))v_z \\
&= \eta_x(v_{yx^{-1}}\varphi(a)v_{zx^{-1}}) \\
&= \eta_x(\varphi(a)_{yx^{-1},zx^{-1}}) \\
&= \eta_x(\eta_{zx^{-1}}(\delta_{yz^{-1}}(a))) \\
&= \eta_z(\delta_{yz^{-1}}(a)) \\
&= \varphi(a)_{y,z},
\end{aligned}$$

so that $\mathrm{Im}\ \varphi \subseteq \Lambda^I$. Conversely, let $\lambda \in \Lambda^I$. Then $\lambda_{x,y} = \eta_y(\lambda_{xy^{-1},e}) = \lambda_{xy^{-1},e}$ for all $x, y \in I$. Thus, setting $a = \sum_{x\in I} \lambda_{x,e}$, we have $\varphi(a)_{x,y} = \eta_y(\delta_{xy^{-1}}(a)) = \eta_y(\lambda_{xy^{-1},e}) = \lambda_{xy^{-1},e} = \lambda_{x,y}$ for all $x, y \in I$ and $\varphi(a) = \lambda$.

(2) Let $x, y \in I$ and $a \in A_{xy^{-1}}$. For any $z \neq y$ we have $\delta_{xz^{-1}}(a) = 0$ and hence $v_x\varphi(a)v_z = \varphi(a)_{x,z} = \eta_z(\delta_{xz^{-1}}(a)) = 0$. Thus $v_x\varphi(a) = \varphi(a)_{x,y} = \eta_y(a)$. It follows that $v_x\Lambda v_y = \eta_y(v_{xy^{-1}}\Lambda v_e) = \eta_y(A_{xy^{-1}}) = v_x\varphi(A_{xy^{-1}})$.

(3) This follows by (2).

(4) Note that $\eta_x(\delta_{yx^{-1}}(a)) = v_y\eta_x(\delta_{yx^{-1}}(a))$ for all $y \in I$. Thus $\varphi(a)v_x = \sum_{y\in I} v_y\varphi(a)v_x = \sum_{y\in I} \eta_x(\delta_{yx^{-1}}(a)) = \sum_{y\in I} v_y\eta_x(\delta_{yx^{-1}}(a))$. Also,

$$\begin{aligned}
v_y\varphi(\delta_{yx^{-1}}(a)) &= \sum_{z\in I} v_y\varphi(\delta_{yx^{-1}}(a))v_z \\
&= \sum_{z\in I} v_y\eta_z(\delta_{yz^{-1}}(\delta_{yx^{-1}}(a))) \\
&= v_y\eta_x(\delta_{yx^{-1}}(a))
\end{aligned}$$

for all $y \in I$.

(5) This follows by (2) and (4). □

Let us call the $I$-bigraded ring constructed in Section 2 standard. Then the proposition above asserts that every $I$-bigraded ring is isomorphic to a standard one. Namely, according to Lemma 2.3, $\varphi : A \to \Lambda$ can be extended to an isomorphism of $I$-bigraded rings.

# 5 Examples

In this section, we will provide a systematic construction of $I$-graded rings $A$ such that $A/A_e$ is a Frobenius extension of second kind.

Let $(s, \chi)$ be a pair of an element $s \in I$ and a mapping $\chi : I \to \mathbb{Z}$ satisfying the following conditions:

(X1) $\chi(x) + \chi(y) \geq \chi(xy)$ for all $x, y \in I$;

(X2) $\chi(x) + \chi(x^{-1}s) = \chi(s)$ for all $x \in I$.

These are obviously satisfied if $s$ is arbitrary and $\chi(x) = 0$ for all $x \in I$. We set

$$\omega(x, y) = \chi(x) + \chi(y) - \chi(xy)$$

for $x, y \in I$.

**Lemma 5.1.** *The following hold.*

(1) $\omega(x, y) \geq 0$ *for all* $x, y \in I$.

(2) $\omega(e, x) = \omega(x, e) = \chi(e) = 0$ *for all* $x \in I$.

(3) $\chi(x) + \chi(y) = \omega(x, y) + \chi(xy)$ *for all* $x, y \in I$.

(4) $\omega(xy, z) + \omega(x, y) = \omega(x, yz) + \omega(y, z)$ *for all* $x, y, z \in I$.

(5) $\omega(x, x^{-1}s) = 0$ *for all* $x \in I$.

*Proof.* It follows by (X2) that $\chi(e) = 0$. The other assertions are obvious. $\square$

In the following, we fix a ring $R$ together with a pair $(\sigma, c)$ of $\sigma \in \mathrm{Aut}(R)$ and $c \in R$ satisfying the following condition:

$$(*) \quad \sigma(c) = c \quad \text{and} \quad ac = c\sigma(a) \text{ for all } a \in R.$$

This is obviously satisfied if either $\sigma = \mathrm{id}_R$ and $c \in \mathrm{Z}(R)$, or $\sigma$ is arbitrary and $c = 0$. As usual, we require $c^0 = 1$ even if $c = 0$.

Let $A$ be a free right $R$-module with a basis $\{u_x\}_{x \in I}$. By abuse of notation we denote by $\{\delta_x\}_{x \in I}$ the dual basis of $\{u_x\}_{x \in I}$ for the free left $R$-module $\mathrm{Hom}_R(A, R)$, i.e., $a = \sum_{x \in I} u_x \delta_x(a)$ for all $a \in A$. According to Lemma 5.1(1), we can define a multiplication on $A$ subject to the following axioms:

(M1) $u_x u_y = u_{xy} c^{\omega(x,y)}$ for all $x, y \in I$;

(M2) $a u_x = u_x \sigma^{\chi(x)}(a)$ for all $a \in R$ and $x \in I$.

**Proposition 5.2.** *The following hold.*

(1) *$A$ is an $I$-graded ring with $A_e \cong R$.*

(2) *$A/A_e$ is a Frobenius extension of second kind.*

(3) *If $c \in \mathrm{rad}(R)$, then $A_x A_{x^{-1}} \subseteq \mathrm{rad}(A_e)$ for all $x \neq e$ with $\omega(x, x^{-1}) > 0$.*

*Proof.* (1) It follows by Lemma 5.1(2) that $u_e \cdot u_x a = u_x a = u_x a \cdot u_e$ for all $x \in I$ and $a \in R$. For any $x, y, z \in I$ and $a_x, a_y, a_z \in R$ we have

$$\begin{aligned}
(u_x a_x \cdot u_y a_y) \cdot u_z a_z &= u_{xy} c^{\omega(x,y)} \sigma^{\chi(y)}(a_x) a_y \cdot u_z a_z \\
&= u_{xyz} c^{\omega(xy,z)} \sigma^{\chi(z)}(c^{\omega(x,y)} \sigma^{\chi(y)}(a_x) a_y) a_z \\
&= u_{xyz} c^{\omega(xy,z)} c^{\omega(x,y)} \sigma^{\chi(z)+\chi(y)}(a_x) \sigma^{\chi(z)}(a_y) a_z \\
&= u_{xyz} c^{\omega(xy,z)+\omega(x,y)} \sigma^{\chi(z)+\chi(y)}(a_x) \sigma^{\chi(z)}(a_y) a_z, \\
u_x a_x \cdot (u_y a_y \cdot u_z a_z) &= u_x a_x \cdot u_{yz} c^{\omega(y,z)} \sigma^{\chi(z)}(a_y) a_z \\
&= u_{xyz} c^{\omega(x,yz)} \sigma^{\chi(yz)}(a_x) c^{\omega(y,z)} \sigma^{\chi(z)}(a_y) a_z \\
&= u_{xyz} c^{\omega(x,yz)} c^{\omega(y,z)} \sigma^{\omega(y,z)}(\sigma^{\chi(yz)}(a_x)) \sigma^{\chi(z)}(a_y) a_z \\
&= u_{xyz} c^{\omega(x,yz)+\omega(y,z)} \sigma^{\omega(y,z)+\chi(yz)}(a_x) \sigma^{\chi(z)}(a_y) a_z
\end{aligned}$$

and by (3), (4) of Lemma 5.1 $(u_x a_x \cdot u_y a_y) \cdot u_z a_z = u_x a_x \cdot (u_y a_y \cdot u_z a_z)$. Thus $A$ is an associative ring with $1 = u_e$. Obviously, $A$ contains $R$ as a subring via the injective ring homomorphism $R \to A, a \mapsto u_e a$, i.e., setting $A_x = u_x R$ for $x \in I$, $A = \oplus_{x \in I} A_x$ is an $I$-graded ring with $A_e = R$.

(2) It follows by (M2) that $\delta_x a = \sigma^{\chi(x)}(a)\delta_x$ for all $a \in R$ and $x \in I$. In particular, $\{\delta_x\}_{x \in I}$ is a basis for the right $R$-module $\mathrm{Hom}_R(A, R)$. Also, for any $x \in I$ by Lemma 5.1(5) $u_x u_{x^{-1}s} = u_s$ and hence $\delta_s u_x = \delta_{x^{-1}s}$. It follows that $A \xrightarrow{\sim} \mathrm{Hom}_R(A, R), a \mapsto \delta_s a$ in Mod-$A$. Obviously, $A$ is a free left $R$-module with a basis $\{u_x\}_{x \in I}$. Thus, since $\delta_s a = \sigma^{\chi(s)}(a)\delta_s$ for all $a \in R$, $A/R$ is a Frobenius extension of second kind.

(3) Immediate by (M1). □

**Example 5.3.** For any $s \in I \setminus \{e\}$, setting

$$\chi(x) = \begin{cases} 0 & \text{if } x = e, \\ 2 & \text{if } x = s, \\ 1 & \text{otherwise}, \end{cases}$$

we have a pair $(s, \chi)$ satisfying the conditions (X1), (X2).

**Example 5.4.** Consider the case where $I = I_1 \times \cdots \times I_n$ with the $I_k$ cyclic. For each $1 \le k \le n$, fix a generator $x_k \in I_k$ and set $m_k = |I_k|$. Set $s = (x_1^{m_1 - 1}, \ldots, x_n^{m_n - 1})$ and $\chi((x_1^{i_1}, \ldots, x_n^{i_n})) = i_1 + \cdots + i_n$, where $0 \le i_k \le m_k - 1$ for all $1 \le k \le n$. Then the pair $(s, \chi)$ satisfies the conditions (X1), (X2).

*Remark* 5.5. The following hold.

(1) $0 \le \chi(x) \le \chi(s)$ for all $x \in I$.

(2) $I_0 = \chi^{-1}(0)$ is a subgroup of $I$ with $sI_0 = I_0 s$.

(3) $\chi$ takes the constant value $\chi(x)$ on $I_0 x I_0$ for all $x \in I$.

(4) $\omega(x, x^{-1}) > 0$ for all $x \ne e$ if and only if $I_0 = \{e\}$.

*Proof.* (1) For any $x \in I$, since $x^m = e$ for some $m > 0$, it follows by (X1) that $m\chi(x) \ge \chi(x^m) = \chi(e) = 0$ and $\chi(x) \ge 0$. It then follows by (X2) that $\chi(x) \le \chi(s)$ for all $x \in I$.

(2) We have $e \in I_0$ and by (X1) $xy \in I_0$ for all $x, y \in I_0$. Also, by (X2) we have $sI_0 = \chi^{-1}(\chi(s)) = I_0 s$.

(3) It follows by (X1) that $\chi(x) \ge \chi(xy)$ for all $x \in I$ and $y \in I_0$. It then follows that $\chi(xy) \ge \chi(xyy^{-1}) = \chi(x)$ for all $x \in I$ and $y \in I_0$. Similarly, $\chi(x) = \chi(yx)$ for all $x \in I$ and $y \in I_0$.

(4) By the fact that $I_0$ is a subgroup of $I$. □

*Remark* 5.6. Set $A_0 = \oplus_{x \in I_0} A_x$, which is the group ring of $I_0$ over $R$. It follows by Remark 5.5(3) that $A$ is free as a right (resp., left) $A_0$-module. Next, define mappings $\delta_0 : A \to A_0$ and $\theta : A_0 \to A_0$ as follows:

$$\delta_0(a) = \sum_{x \in I_0} u_x \delta_{sx}(a) \quad \text{and} \quad \theta(b) = \sum_{x \in I_0} u_x \sigma^{\chi(s)}(\delta_{sxs^{-1}}(b))$$

for $a \in A$ and $b \in A_0$, respectively. Then $\delta_0 \in \mathrm{Hom}_{A_0}(A, A_0)$ and $\theta \in \mathrm{Aut}(A_0)$. Furthermore, $A \xrightarrow{\sim} \mathrm{Hom}_{A_0}(A, A_0), a \mapsto \delta_0 a$ in Mod-$A$ and $\delta_0 b = \theta(b)\delta_0$ for all $b \in A_0$. Consequently, $A/A_0$ is a Frobenius extension of second kind.

*Remark* 5.7. Consider the case where $R$ is commutative, $\sigma = \mathrm{id}_R$ and $s$ lies in the center of $I$. Then $A \xrightarrow{\sim} \mathrm{Hom}_R(A, R), a \mapsto \delta_s a$ as $A$-$A$-bimodules.

*Proof.* Note first that $A \xrightarrow{\sim} \mathrm{Hom}_R(A, R), a \mapsto \delta_s a$ in Mod-$A$, which we have shown in the proof of Proposition 5.2(2). Next, for any $a, b \in A$ we have

$$\begin{aligned}
\delta_s(ab) &= \sum_{x \in I} \delta_x(a)\delta_{x^{-1}s}(b) \\
&= \sum_{x \in I} \delta_{sx^{-1}}(b)\delta_x(a) \\
&= \sum_{y \in I} \delta_y(b)\delta_{y^{-1}s}(a) \\
&= \delta_s(ba),
\end{aligned}$$

so that $\delta_s a = a\delta_s$ for all $a \in I$. □

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