2016年2月1日(月)

微積分第27回講義ノート

連立微分方程式

$$\begin{cases} x' = a_{11}x + a_{12}y & x = x(t) \\ y' = a_{21}x + a_{22}y & y = y(t) \end{cases}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\overset{2\times 2}{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

固有多項式　　固有多項式

$|tE - A| = 0$　tに関する2次式

異なる2実解　　(虚数解)複素解

$x = \sin t$　$y = x'$

$x' = \cos t$　$x' = y$

$x'' = -\sin t$　$y' = -x$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\left|tE - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right| = \begin{vmatrix} t & -1 \\ 1 & t \end{vmatrix} = t^2 + 1 = 0$$

$$t = \pm i$$

$x + iy$

実部　虚部

$(a+ib)(x+iy) = (ax - by) + i(ay + bx)$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad |tE - A| = \begin{vmatrix} t-a & b \\ -b & t-a \end{vmatrix}$$

$$= (t-a)^2 + b^2 = t^2 - 2at + a^2 + b^2$$

$D/4 = a^2 - (a^2 + b^2) = -b^2$　$t = a \pm bi$

2×2　異なる2実解を持つ

A　a, b

$$A = P \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} P^{-1}$$

$$e^A = P \underbrace{e^{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}}}_{\parallel} P^{-1}$$

$$\begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix}$$

複素解

A　a, b

$$A = P \underline{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$|tE - A| = \underbrace{\left|tE - \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\right|}_{\parallel}$$

$$0$$

$$t = a \pm ib$$

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underset{2\times 2}{A} \begin{pmatrix} x \\ y \end{pmatrix}$　$e^{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} t} \to a+ib$　$(a+ib)^2 = \underline{a^2-b^2} + 2iab$　$(a+ib)^3 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^3$

$|tE-A|=0$

複素解

$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^2 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} a^2-b^2 & -2ab \\ 2ab & a^2-b^2 \end{pmatrix}$

$E + \begin{pmatrix} a & -b \\ b & a \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^2 t^2 + \cdots$

$e^{a+ib} = e^{at} \underline{e^{ibt}}$

$\cos bt + i\sin bt$

$\boxed{e^{at}\begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix}}$

$A = P\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} P^{-1}$

$At = P\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} t P^{-1}$

e^{At}

$\begin{pmatrix} x \\ y \end{pmatrix} = e^{At}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$

$= P e^{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} t} P^{-1} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$

$\begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix}$

$x = a_{11} e^{at} + a_{12} e^{bt}$

$y = a_{21} e^{at} + a_{22} e^{bt}$

未定定数

複素解

$A = P\begin{pmatrix} a & -b \\ b & a \end{pmatrix} P^{-1}$

$e^{At} = P \overline{e^{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} t}} P^{-1}$

$\begin{cases} x = a_{11} e^{at}\cos bt + a_{12} e^{at}\sin bt \\ y = a_{21} e^{at}\cos bt + a_{22} e^{at}\sin bt \end{cases}$

$tE = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$　$\begin{cases} x' = y \\ y' = x \end{cases}$

$tE = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$= \begin{vmatrix} t & -1 \\ 1 & t \end{vmatrix} = t^2+1$

$t = \pm i$　$e^{at} = 1$

$x = a_{11}\cos t + a_{12}\sin t$　$(-a_{11}-a_{22})\sin t + (a_{12}-a_{21})\cos t = 0$

$y = a_{21}\cos t + a_{22}\sin t$　$(-a_{21}+a_{12})\sin t + (a_{22}+a_{11})\cos t = 0$

$x' = \boxed{-a_{11}\sin t + a_{12}\cos t = a_{21}\cos t + a_{22}\sin t}$

$y' = \boxed{-a_{21}\sin t + a_{22}\cos t = -(a_{11}\cos t + a_{12}\sin t)}$

$a_{11} = -a_{22}$

$a_{12} = a_{21}$

初期条件

$|tE-A|=0$ (重解)
2通り

$A=\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

$P\underbrace{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}}_{aE}P^{-1}=aE$

指数法則
$e^{z_1+z_2}=e^{z_1}e^{z_2}$
$z_1z_2=z_2z_1$
$AB=BA$ ならば $e^{A+B}=e^Ae^B$

行列は $AB\neq BA$

$A=\begin{pmatrix} a & 0 \\ b & a \end{pmatrix}=\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$

$|tE-A|=\begin{vmatrix} t-a & 0 \\ -b & t-a \end{vmatrix}=(t-a)^2$ 重解

$e^{tA}=\underbrace{e^{t\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}}}_{\begin{pmatrix} e^{at} & 0 \\ 0 & e^{at} \end{pmatrix}}\underline{e^{\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}t}}$

$e^{\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}t}=E+\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}t+\frac{1}{2}\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}^2t^2+\frac{1}{3!}\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}^3t^3+\cdots$

$\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}^2=0$ $\quad\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} e^{at} & 0 \\ 0 & e^{at} \end{pmatrix}\left(E+\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}t\right)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$