2016年1月25日(月)

微積分第26回講義ノート

微分方程式

$x' = ax$　　$x = x(t)$

$x = Ce^{at}$

$x = x(t)$

$y = y(t)$　　a_{ij}:定数

$x' = a_{11}x + a_{12}y$

$y' = a_{21}x + a_{22}y$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

連立方程式

鶴亀算(小学校)

⇓

連立方程式(中学校)

鶴　x

亀　y

頭　$m = x + y$

足　$n = 2x + 4y$

前回 $\begin{pmatrix} x \\ y \end{pmatrix} = e^{At}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$　一般解

$\left.\begin{array}{l} C_1 = x(0) \\ C_2 = y(0) \end{array}\right\}$ 初期条件

$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$　αとβは定数

$e^{At} = e^{\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}t} = \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{\beta t} \end{pmatrix}$ 対角化可能

$x = e^{\alpha t}C_1$

$y = e^{\beta t}C_2$

$x' = \alpha x$

$y' = \beta y$

$x' = \alpha x$

$y' = \beta y$

$A = P\underset{\text{対角型}}{B}P^{-1}$　　$PP^{-1} = P^{-1}\underset{\text{正則行列}}{P} = E$

$e^{At} = e^{(PBP^{-1})t} = e^{(PBt)P^{-1}} = Pe^{Bt}P^{-1}$

線型代数 determinant (行列式)

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad |A| \text{ とも書く}$$

<u>ReportI</u> $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ のとき $|AB| = |A||B|$ を示せ.

$$\begin{aligned} A &= PBP^{-1} \\ |A| &= |PBP^{-1}| \\ &= |P||B||P^{-1}| \\ &= |B||PP^{-1}| \quad E \quad |E| = 1 \\ &= |B| \end{aligned}$$

行列Aの固有多項式

$\det(tE - A)$

$A = PBP^{-1}$

$|tE - PBP^{-1}| = |PtEP^{-1} - PBP^{-1}| = |tE - B|$

$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ Collorary (レポート問題)

$$|tE - B| = \left|\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} - \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\right| = \begin{vmatrix} t-\alpha & 0 \\ 0 & t-\beta \end{vmatrix} = (t-\alpha)(t-\beta)$$

定理

$|tE - A| = 0$ が異なる2つの実解をもつ場合、Aは対角化可能

$$\boxed{\begin{aligned} x' &= -x \\ y' &= x + 2y \end{aligned}}$$ 元の方程式

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} |tE - A| &= \begin{vmatrix} t+1 & 0 \\ -1 & t-2 \end{vmatrix} \\ &= (t+1)(t-2) \\ t &= -1, 2 \end{aligned}$$

$$e^{P\begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix} tP^{-1}} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

$$P e^{\begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix} t} P^{-1} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

$$\begin{pmatrix} \; \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} \; \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

$$\begin{cases} x = a_{11}e^{-t} + a_{12}e^{2t} \\ y = a_{21}e^{-t} + a_{22}e^{2t} \end{cases}$$

実戦的

$$\boxed{\begin{aligned} x' &= -a_{11}e^{-t} + 2a_{12}e^{2t} = -(\underline{a_{11}e^{-t}} + a_{12}e^{2t}) \\ y' &= -\underline{a_{21}e^{-t}} + 2a_{22}e^{2t} = (a_{11}e^{-t} + a_{12}e^{2t}) + 2(a_{21}e^{-t} + a_{22}e^{2t}) \end{aligned}}$$

a_{ij}：定数　$2a_{12}e^{2t} = -a_{12}e^{2t}$

$3a_{12}e^{2t} = 0 \Rightarrow a_{12} = 0$ ◎

$(-a_{21} - a_{11} - 2a_{21})e^{-t} + (2a_{22} - a_{12} - 2a_{22})e^{2t} = 0$

$\Rightarrow \boxed{-a_{21} - a_{11} - 2a_{21} = 0}$

$2a_{22} - a_{12} - 2a_{22} = 0$

$-a_{11} - 3a_{21} = 0$

$\boxed{a_{11} = -3a_{21}}$ ◎

初期条件を求めよ

Report II

$x(0) = 0$

$y(0) = 3$

III $x' = 2x + y$　$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$　$|tE - A| = \begin{vmatrix} t-2 & -1 \\ -1 & t-1 \end{vmatrix}$

$y' = x + y$

$= (t-2)(t-1) - 1$

$= t^2 - 3t + 2 - 1 = t^2 - 3t + 1$

IV $x' = 3y$

$y' = x - 2y$

$x(0) = 3$

$y(0) = 0$

$\underline{x' = ax}$　$10万 \overset{1年後}{\Rightarrow} 10.5万$

人口 x　$20万 \Rightarrow 21万$

V 放射性元素の崩壊

半減期

$x(T) = \frac{1}{2}x(0)$

$T = -\frac{\log 2}{a}$ になることを示せ