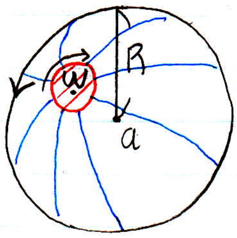
2015年 12月14日(月)

微積分第23回講義ノート

Cauchyの積分定理 (内部を左に見るように回る)

少しインチキ
ただす
$r \to 0 \quad f(z) \to f(w)$



γ:円

f:解析

$$g(z) = \frac{f(z)}{z-w}$$

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)dz}{z-w}$$

$\gamma_r \begin{cases} w \text{ 中心} \\ \text{半径 } r \end{cases}$

$\gamma \cup \gamma_r$

$$\int_{\gamma \cup \gamma_r} \frac{f(z)dz}{z-w} = 0$$

$$= \int_\gamma \frac{f(z)dz}{z-w} - \int_{\gamma_r} \frac{f(z)dz}{z-w} = 0$$

$$\int_\gamma \frac{f(z)dz}{z-w} = \int_{\gamma_r} \frac{f(z)dz}{z-w} \qquad \int_{\gamma_r} \frac{dz}{z-w} = 2\pi i$$

$r \geq 0$ $\quad r > 0$

$$\varphi(r) = \int_{\gamma_r} \frac{f(z)dz}{z-w} = \int_0^{2\pi} \frac{f(w+r(\cos\theta + i\sin\theta))}{w + r(\cos\theta + i\sin\theta) - w} \frac{dz}{d\theta} d\theta$$

$$\left(\begin{aligned} &\gamma_r \ \theta \in [0, 2\pi] \mapsto w + r(\cos\theta + i\sin\theta) \\ &\frac{dz}{d\theta} = \frac{d(w + r(\cos\theta + i\sin\theta))}{d\theta} \\ &\quad = r(-\sin\theta + i\cos\theta) \end{aligned} \right)$$

$$= \int_0^{2\pi} f(w + r(\cos\theta + i\sin\theta)) \frac{-\sin\theta + i\cos\theta}{\cos\theta + i\sin\theta} d\theta$$

$$= i \int_0^{2\pi} f(w + r(\cos\theta + i\sin\theta)) d\theta$$

$$= i \int_0^{2\pi} f(w) d\theta = i f(w) \int_0^{2\pi} d\theta = 2\pi i f(w)$$

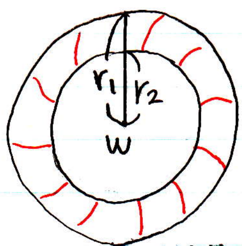
線積分

$r_1 > r_2 > 0$

$\gamma_{r_1} \cup \gamma_{r_2}$

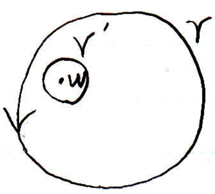
$$\int_{\gamma_{r_1} \cup \gamma_{r_2}} \frac{f(z)dz}{z-w} = 0$$

$$\underbrace{\int_{\gamma_{r_1}} \frac{f(z)dz}{z-w}}_{\varphi(r_1)} = \underbrace{\int_{\gamma_{r_2}} \frac{f(z)dz}{z-w}}_{\varphi(r_2)}$$



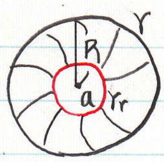
$\varphi(r)$: 連続

$r > 0, \ \varphi(0)$



$$\int_{\gamma \cup \gamma'} \frac{f(z)dz}{z-w} = 0$$

$$\int_\gamma \frac{f(z)dz}{z-w} = \int_{\gamma'} \frac{f(z)dz}{z-w} = 2\pi i f(w)$$

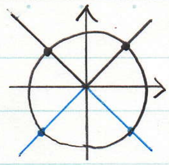


f:解析的

$$f(z)=\frac{z^2}{z^4+1} \qquad z^4+1=0$$

$$z^2=\pm i$$

$$\gamma_R \cup \gamma_r$$



$\pi+\frac{\pi}{4}$

fがaで定義されてなくて

中級数

$$f(w)=\frac{1}{2\pi i}\left(\int_{\gamma_R}\frac{f(z)dz}{z-w}-\int_{\gamma_r}\frac{f(z)dz}{z-w}\right)$$

$$=\frac{1}{2\pi i}\left(\int_{\gamma_R}\frac{f(z)dz}{(z-a)-(w-a)}-\int_{\gamma_r}\frac{f(z)dz}{(z-a)-(w-a)}\right)$$

$|z-a|=R$

$|w-a|<R$

$$\int_{\gamma_R}\frac{f(z)dz}{(z-a)-(w-a)}=\int_{\gamma_R}\frac{f(z)}{z-a}\,\frac{dz}{1-\frac{w-a}{z-a}} \qquad \left|\frac{w-a}{z-a}\right|<1 \ (\text{等比級数})(※)$$

$$\left((※)\ \frac{1}{1-\frac{w-a}{z-a}}=1+\frac{w-a}{z-a}+\left(\frac{w-a}{z-a}\right)^2+\left(\frac{w-a}{z-a}\right)^3+\cdots\right)$$

$$=\int_{\gamma_R}\frac{f(z)dz}{z-a}+(w-a)\int_{\gamma_R}\frac{f(z)dz}{(z-a)^2}+(w-a)^2\int_{\gamma_R}\frac{f(z)dz}{(z-a)^3}+\cdots$$

$$\boxed{\sum_{n=0}^{\infty}a_n(w-a)^n}$$

$$\int_{\gamma_r}\frac{f(z)dz}{(z-a)-(w-a)}=\int_{\gamma_r}\frac{f(z)}{w-a}\,\frac{dz}{\frac{z-a}{w-a}-1}=-\int_{\gamma_r}\frac{f(z)}{w-a}\,\frac{dz}{1-\frac{z-a}{w-a}}$$

$$=-\int_{\gamma_r}\frac{f(z)}{w-a}\left\{1+\frac{f(z)}{w-a}+\left(\frac{f(z)}{w-a}\right)^2+\cdots\right\}dz$$

$$=-\left\{\frac{1}{w-a}\int_{\gamma_r}f(z)dz+\frac{1}{(w-a)^2}\int_{\gamma_r}(z-a)f(z)dz+\frac{1}{(w-a)^3}\int_{\gamma_r}(z-a)^2f(z)dz+\cdots\right\}$$

$$\boxed{\sum_{-\infty}^{n=-1}a_n(w-a)^n}$$

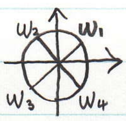
$f(w) = \sum_{-\infty}^{\infty} a_n (w-a)^n$ Laurent 展開 n ℓr
ローラン (仏)

a_{-1} = fのaにおける留数 $n \neq -1$
$\mathrm{Res}\,(f;a)$ $(w-a)^n$ $\frac{(w-a)^{n+1}}{n+1}$
セミコロン

$F' = f$ $\int_\gamma f(z)\,dz = \underbrace{F(\gamma(b)) - F(\gamma(a))}_{=0}$

$\int_\gamma f(z)\,dz = \sum_{-\infty}^{\infty} a_n \int_\gamma (z-a)^n dz$
$= a_{-1} \underbrace{\int_\gamma \frac{dz}{z-a}}_{2\pi i} = 2\pi i\, a_{-1}$

$f(z) = \frac{z^2}{z^4+1}$



$\mathrm{Res}\,(f;w_i)$ w_1

$= \frac{z^2}{(z-w_1)(z-w_2)(z-w_3)(z-w_4)} = \frac{1}{z-w_1} \overbrace{\frac{z^2}{(z-w_2)(z-w_3)(z-w_4)}}^{g(z)}$

$a_0 = g(w_1)$
$g(z) = a_0 + a_1(z-w_1) + a_2(z-w_1)^2 + \cdots$
$\frac{a_0}{z-w_1} + a_1 + a_2(z-w_1) + \cdots$

$\frac{w_1^2}{(w_1-w_2)(w_1-w_3)(w_1-w_4)}$

Report

$\mathrm{Res}\,(f;w_1)$, $\mathrm{Res}\,(f;w_2)$, $\mathrm{Res}\,(f;w_3)$, $\mathrm{Res}\,(f;w_4)$ の留数を求めよ.