2015年 12月7日 (月)

微積分第22回講義ノート

3次元 微分形式

2次元 / 1次元 積分定理

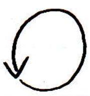
3次元 1次の微分形式 ω

閉曲線 γ を境界とする

曲面 Σ

$\int_\gamma \omega = \int_\Sigma d\omega$

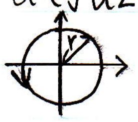
閉曲線 γ, Σ f: 解析 $\frac{1}{z-a}$



$(\cos\theta + i\sin\theta)i$

1次の微分形式 ω $d(fdz)=0$

2次 $\int_\gamma \omega = \int_\Sigma d\omega = 0$



$f(z) = \frac{1}{z}$

円周

$$\int_\gamma f(z)dz = \int_0^{2\pi} f(r(\cos\theta + i\sin\theta))\frac{dz}{d\theta}d\theta$$

$\gamma : \theta \in [0, 2\pi] \mapsto r(\cos\theta + i\sin\theta)$

$\frac{dz}{d\theta} = \frac{dr(\cos\theta + i\sin\theta)}{d\theta} = r(-\sin\theta + i\cos\theta)$

$$\int_0^{2\pi} \frac{r(-\sin\theta + i\cos\theta)}{r(\cos\theta + i\sin\theta)}d\theta = 2\pi i$$

f: 解析

中心 a

半径 R

$\int_\gamma f(z)dz = 0$

$g(z) = \frac{f(z)}{z-w}$ $\int_\gamma \frac{f(z)}{z-w}dz$ $\int_{\gamma\cup\gamma_r} g(z)dz = 0$

$\gamma \cup \gamma_r$

$\int_\gamma g(z)dz = \int_{\gamma_r} g(z)dz$

$r \to 0$

$\int_{\gamma_r} \frac{dz}{z-w} = 2\pi i$

$f(z) \to f(w)$ $\int_{\gamma_r} \frac{f(z)}{z-w}dz = f(w)\int_{\gamma_r}\frac{dz}{z-w} = 2\pi i f(w)$

Cauchyの積分公式

$$\boxed{f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz}$$

(1) $\exists g, df = g\,dz$ $dz = dx + idy$

(2) Cauchy-Rieman gをfの微分

(3) $d(fdz)=0$

$f(z) = C$ (定数)

$df = 0 = 0dz$

$f(z) = z$ $f' = 1$

$dz = dx + idy = 1dz$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

1次の微分形式　　$f: \mathbb{C} \to \mathbb{C}$

$\omega = f\,dx + g\,dy$

$d\omega = (df)\wedge dx + (dg)\wedge dy$　　$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

$$= \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)\wedge dx + \left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy\right)\wedge dy$$

$$= \frac{\partial f}{\partial x}\underline{dx\wedge dx} + \frac{\partial f}{\partial y}dy\wedge dx + \frac{\partial g}{\partial x}dx\wedge dy + \frac{\partial g}{\partial y}\underline{dy\wedge dy}$$

$$= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dx\wedge dy$$

$$d(df) = d\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) = \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right)dx\wedge dy$$

$$= \left(\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial y\partial x}\right)dx\wedge dy = 0$$

$\mathrm{rot}\circ\mathrm{grad} = 0$

$\mathrm{div}\circ\mathrm{rot} = 0$　　$df = f'dz$　　命題

$0 = d(df) = d(f'dz)$　　f: 解析 $\to$ f': 解析

$$d(f+g) = df + dg$$

$$\frac{\partial(f+g)}{\partial x}dx + \frac{\partial(f+g)}{\partial y}dy = \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}\right)dy$$

$$= \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) + \left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy\right)$$

Report I　(1),(2)を証明せよ

(1) $d(fg)=(df)g+f(dg)$

(2) $d\left(\frac{f}{g}\right)=\frac{(df)g-f(dg)}{g^2}$　$(g\neq 0)$

$f,g:\mathbb{C}\to\mathbb{C}$　, $df=f'dz$　, $dg=g'dz$

$d(f+g)=df+dg=f'dz+g'dz=(f'+g')dz$

$d(fg)=(df)g+f(dg)=(f'dz)g+f(g'dz)=(f'g+fg')dz$

Report II　$d\left(\frac{f}{g}\right)$を確認せよ

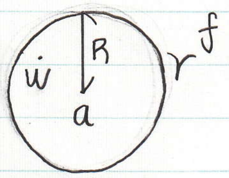
$f(z)=z^2$　e^z

$f(z)=z$　$\sin z$

$g(z)=z$　$\cos z$

$f(z)=a_n z^n+a_{n-1}z^{n-1}+\cdots+a_0$

$f'(z)=a_n n z^{n-1}+a_{n-1}(n-1)z^{n-2}+\cdots+a_1$



$$f(w)=\frac{1}{2\pi i}\int_\gamma\frac{f(z)dz}{z-w}=\int_\gamma\frac{f(z)dz}{(z-a)-(w-a)}$$　$|z-a|=R$
$|w-a|=R$

$$=\int_\gamma f(z)\frac{1}{z-a}\frac{1}{1-\underbrace{\left|\frac{w-a}{z-a}\right|}_{<1}}dz$$　$r<1$　$\frac{1}{1-r}=1+r+r^2+r^3+\cdots$

$$=\int_\gamma\frac{f(z)}{z-a}\left\{1+\frac{w-a}{z-a}+\left(\frac{w-a}{z-a}\right)^2+\left(\frac{w-a}{z-a}\right)^3+\cdots\right\}dz$$

$$=\int_\gamma\frac{f(z)}{z-a}dz+(w-a)\int_\gamma\frac{f(z)}{(z-a)^2}dz+(w-a)^2\int_\gamma\frac{f(z)}{(z-a)^3}dz+\cdots$$