微積分 第21回 講義ノート

複素関数論

$\mathbb{R} \to \mathbb{R}$

$\mathbb{C} \to \mathbb{C}$

$\mathbb{R}^3$ (空間) ← 特化

古典的ベクトル解析

微分形式

スカラー場 $\xrightarrow{\text{grad}}$ ベクトル場 $\xrightarrow{\text{rot}}$ ベクトル場 $\xrightarrow{\text{div}}$ スカラー場

↕

0次の微分形式 $\xrightarrow{d}$ 1次の微分形式 $\xrightarrow{d}$ 2次の微分形式 $\xrightarrow{d}$ 3次の微分形式

$\mathbb{R}$のベクトル解析 $\mathbb{R} \to \mathbb{R}$

スカラー関数	1次の交代形式	2次の交代形式 $\begin{cases}\text{二重線形} \\ \text{交代}\end{cases}$
0次の微分形式	1次の微分形式	$\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$\varphi(x, y) = \varphi(x1, y1) = xy\underbrace{\varphi(1,1)}_{0} = 0$$

$f : \mathbb{R} \to \mathbb{R}$ 1次の微分形式

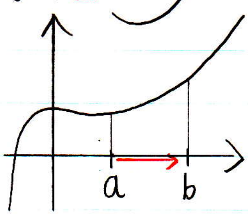
$df = f' \, dx$

$dx : a \in \mathbb{R} \to a \in \mathbb{R}$ (恒等関数)

$(df)(x) = f'(x)\,dx$ 定数で決まるため省いている

$\int_a^b g(x)\,dx$

線積分



積分定理

$f' = g$

$df = g\,dx$

$$\boxed{\int_a^b g\,dx = f(b) - f(a)}$$

aからbまでの道の $1g$ の線積分

rot grad = 0 , div rot = 0

$\varphi : \mathbb{R}^3 \to \mathbb{R}$

$$d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy + \frac{\partial \varphi}{\partial z}dz$$

$$d(d\varphi) = \left(\frac{\partial}{\partial x}\frac{\partial \varphi}{\partial x}dx + \frac{\partial}{\partial y}\frac{\partial}{\partial x}dy + \frac{\partial}{\partial z}\frac{\partial}{\partial x}dz\right)\wedge dx + \left(\frac{\partial}{\partial x}\frac{\partial \varphi}{\partial y}dx + \cdots\right)\wedge dy + \left(\frac{\partial}{\partial x}\frac{\partial \varphi}{\partial z}dx + \cdots\right)\wedge dz$$

$$= \frac{\partial^2\varphi}{\partial x^2}\underset{\overset{\shortparallel}{0}}{dx\wedge dx} + \frac{\partial^2\varphi}{\partial y\partial x}dy\wedge dx + \frac{\partial^2\varphi}{\partial z\partial x}dz\wedge dx$$
$$+ \frac{\partial^2\varphi}{\partial x\partial y}dx\wedge dy + \frac{\partial^2\varphi}{\partial y^2}\underset{\overset{\shortparallel}{0}}{dy\wedge dy} + \frac{\partial^2\varphi}{\partial z\partial y}dz\wedge dy$$
$$+ \frac{\partial^2\varphi}{\partial x\partial z}dx\wedge dz + \frac{\partial^2\varphi}{\partial y\partial z}dy\wedge dz + \frac{\partial^2\varphi}{\partial z^2}\underset{\overset{\shortparallel}{0}}{dz\wedge dz}$$
$$= 0$$

$dx\wedge dy = -dy\wedge dx$

$$\frac{\partial^2\varphi}{\partial y\partial x} = \frac{\partial^2\varphi}{\partial x\partial y}$$

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$\omega = f_1dx + f_2dy + f_3dz$ のとき $d(d\omega)$ を示す

$\mathbb{R}^2$ での ベクトル解析

1次の交代形式

$\mathbb{R}^2 \to \mathbb{R}$ $\quad dx : \begin{pmatrix} a \\ b \end{pmatrix} \mapsto a$

$\quad dy : \begin{pmatrix} a \\ b \end{pmatrix} \mapsto b$

2次の交代形式

$\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$

$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2$

$\varphi(\mathbf{e}_1, \mathbf{e}_2) = -\varphi(\mathbf{e}_2, \mathbf{e}_1)$

0次の微分形式

$$\varphi(\mathbf{a}, \mathbf{b}) = \varphi(a_1\mathbf{e}_1 + a_2\mathbf{e}_2, b_1\mathbf{e}_1 + b_2\mathbf{e}_2)$$
$$= a_1b_1\underset{\overset{\shortparallel}{0}}{\varphi(\mathbf{e}_1, \mathbf{e}_1)} + a_2b_2\underset{\overset{\shortparallel}{0}}{\varphi(\mathbf{e}_2, \mathbf{e}_2)} + a_1b_2\varphi(\mathbf{e}_1, \mathbf{e}_2) + a_2b_1\varphi(\mathbf{e}_2, \mathbf{e}_1)$$
$$= (a_1b_2 - a_2b_1)\varphi(\mathbf{e}_1, \mathbf{e}_2)$$

$\mathbb{R}^2$でのベクトル解析

0次の微分形式 $\xrightarrow{d}$ 1次 $\xrightarrow{d}$ 2次

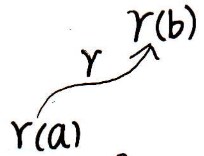
積分定理

φ: 0次の微分形式 $\mathbb{R}^2 \to \mathbb{R}$

$d\omega = \omega$

$$\boxed{\int_\gamma \omega = \omega(\gamma(b)) - \omega(\gamma(a))}$$

$\gamma : [a,b] \to \mathbb{R}^2$ 1次の微分形式



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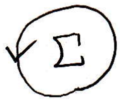
$d\omega$



γ 閉路

$\int_\Sigma d\omega = \int_\gamma \omega$

$\mathbb{C}$: 複素数 $\quad \mathbb{C} = \mathbb{R}^2 \quad dz = dx + idy \quad f : \mathbb{C} \to \mathbb{C} \quad f = u + iv \quad u, v : \mathbb{C} \to \mathbb{R}$

$z = x + iy$ (実部, 虚部) $\quad \uparrow$ 掛け算

$df = du + idv \qquad g : \mathbb{R} \to \mathbb{R} \quad dg = g' dx$

$$= \underbrace{\frac{\partial u}{\partial x}}_{\alpha} dx + \underbrace{\frac{\partial u}{\partial y}}_{-\beta} dy + i\left\{\underbrace{\frac{\partial v}{\partial x}}_{\beta} dx + \underbrace{\frac{\partial v}{\partial y}}_{\alpha} dy\right\}$$

$g = \alpha + i\beta \quad \alpha, \beta : \mathbb{C} \to \mathbb{R}$

$g : \mathbb{C} \to \mathbb{C} \qquad df = g\,dz = (\alpha + i\beta)(dx + idy) = (\alpha dx - \beta dy) + i(\alpha dy + \beta dx)$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned}\right\}$$ Cauchy − Riemannの方程式 正則な関数 解析関数

$f : \mathbb{C} \to \mathbb{C} \qquad f = u + iv$

$$\begin{aligned}
d(f\,dz) &= d((u+iv)(dx+idy)) = d\{(u\,dx - v\,dy) + i(v\,dx + u\,dy)\} \\
&= \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)\wedge dx - \left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)\wedge dy \\
&\quad + i\left\{\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)\wedge dx + \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)\wedge dy\right\} \\
&= \frac{\partial u}{\partial y}dy\wedge dx - \frac{\partial v}{\partial x}dx\wedge dy + i\left(\frac{\partial v}{\partial y}dy\wedge dx + \frac{\partial u}{\partial x}dx\wedge dy\right) = 0 \\
&= \underbrace{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)}_{=0} dy\wedge dx + i\underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{=0} dx\wedge dy
\end{aligned}$$

定理 $f : \mathbb{C} \to \mathbb{C}$

(1) $\exists g : \mathbb{C} \to \mathbb{C}, \ df = g\,dz$

(2) $d(f\,dz) = 0$

(3) $\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned}\right\}$