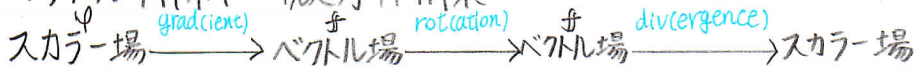


微積分第19回講義ノート

古典的

ベクトル解析 微分作用素



曲線 $\gamma: [a, b] \rightarrow \mathbb{R}^3$

閉曲線 γ を境界とする曲面 Σ

$$\int_{\gamma} (\text{grad } \varphi) \cdot d\mathbf{r} = \varphi(\gamma(b)) - \varphi(\gamma(a))$$

$$\int_{\Sigma} (\text{rot } \mathbf{f}) \cdot d\mathbf{S} = \int_{\gamma} \mathbf{f} \cdot d\mathbf{r}$$

面積分 線積分

Stokesの定理

無限小のLevelで成り立てば...

閉曲面 Σ で囲まれる領域 Ω

$$\int_{\Omega} (\text{div } \mathbf{f}) dV = \int_{\Sigma} \mathbf{f} \cdot d\mathbf{S}$$

体積分 面積分

nabla
ナブラ

$$\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \quad f_i: \mathbb{R}^3 \rightarrow \mathbb{R}$$

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

$$\text{grad } \varphi = \nabla \varphi = \begin{pmatrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \\ \frac{\partial \varphi}{\partial z} \end{pmatrix}$$

rot $\mathbf{f} = \nabla \times \mathbf{f}$

$$\nabla \times \mathbf{f} = \begin{pmatrix} \frac{\partial f_2}{\partial y} - \frac{\partial f_3}{\partial z} \\ \frac{\partial f_3}{\partial z} - \frac{\partial f_1}{\partial x} \\ \frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y} \end{pmatrix}$$

$$\text{div } \mathbf{f} = \nabla \cdot \mathbf{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

ベクトル場の
擬似ベクトル

Gaussの発散定理

\mathbb{R}^3 上の \mathbb{R} ベクトル場
 $\mathbb{R} \rightarrow \mathbb{R} \in \mathbb{R}^3$
点 ベクトル

$$\text{div } \mathbf{f} = 1+1+1=3$$

$$\int_{\Omega} (\text{div } \mathbf{f}) dV = 3 \times \frac{4}{3} \pi a^3 = 4\pi a^3$$

3
定値

積分定理 解析学の天王山

Report I, IIを証明せよ

I. スカラー場 φ
 $\text{rot}(\text{grad } \varphi) = 0$

II. ベクトル場 \mathbf{f}
 $\text{div}(\text{rot } \mathbf{f}) = 0$

微分形式

 $\mathbb{R}^3 \rightarrow \mathbb{R}$ 1次の交代形式
線形写像

3次元

$$dx: \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \mapsto a_1$$

$$dy: \mapsto a_2$$

$$dz: \mapsto a_3$$

2次の交代形式

$$dy \wedge dz \left(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right) \mapsto \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$dz \wedge dx \mapsto \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix}$$

$$dx \wedge dy \mapsto \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

2つの1次の交代形式

III $w_1, w_2 \quad a, b \in \mathbb{R}^3$ $(w_1 \wedge w_2)(a, b) = w_1(a)w_2(b) - w_1(b)w_2(a)$ が2次の交代形式になることを示せ。

$$(dy \wedge dz)(a, b) = \frac{dy(a)}{a_2} \frac{dz(b)}{b_3} - \frac{dy(b)}{b_2} \frac{dz(a)}{a_3} = a_2 b_3 - b_2 a_3 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$(w_1 \wedge w_2 \wedge w_3)(a, b, c) = w_1(a)w_2(b)w_3(c)$$

$$\begin{array}{l} 1, 2, 3 \\ \downarrow \downarrow \downarrow \\ 2, 1, 3 \end{array} \ominus (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$$

符号が変わる: 奇置換

$$\begin{array}{l} 1, 2, 3 \\ \downarrow \downarrow \downarrow \\ 2, 3, 1 \end{array} \circ (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$$

変わらない: 偶置換

IV $(w_1 \wedge w_2 \wedge w_3)(a, b, c)$ が3次の交代形式になることを示せ

$$(dx \wedge dy \wedge dz)(a, b, c) = |a, b, c|$$

$$W_1 \wedge W_2 = -W_2 \wedge W_1$$

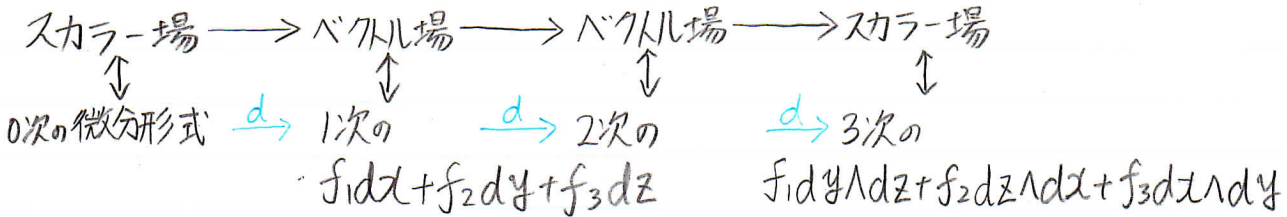
$$W_1 \wedge W_1 = 0$$

$$W_1 = W_{11} + W_{12}$$

$$(W_{11} + W_{12}) \wedge W_2 = W_{11} \wedge W_2 + W_{12} \wedge W_2$$

$$(\alpha W_1) \wedge W_2 = \alpha (W_1 \wedge W_2)$$

古典的ベクトル解析 $f_i: \mathbb{R}^3 \rightarrow \mathbb{R}$
 $f_1 e_1 + f_2 e_2 + f_3 e_3$



$$\varphi = dx \wedge dy \wedge dz$$

$$(\varphi)' = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy + \frac{\partial \varphi}{\partial z} dz$$

rot

$$\begin{aligned}
 d(f_1 dx + f_2 dy + f_3 dz) &= (df_1) \wedge dx + (df_2) \wedge dy + (df_3) \wedge dz \\
 \left(\begin{aligned} df_1 &= \frac{\partial f_1}{\partial x} dx + \frac{\partial f_1}{\partial y} dy + \frac{\partial f_1}{\partial z} dz \\ df_2 &= \frac{\partial f_2}{\partial x} dx + \frac{\partial f_2}{\partial y} dy + \frac{\partial f_2}{\partial z} dz \\ df_3 &= \frac{\partial f_3}{\partial x} dx + \frac{\partial f_3}{\partial y} dy + \frac{\partial f_3}{\partial z} dz \end{aligned} \right) &= \frac{\partial f_1}{\partial x} dx \wedge dx + \frac{\partial f_1}{\partial y} dy \wedge dx + \frac{\partial f_1}{\partial z} dz \wedge dx \\
 &+ \frac{\partial f_2}{\partial x} dx \wedge dy + \frac{\partial f_2}{\partial y} dy \wedge dy + \frac{\partial f_2}{\partial z} dz \wedge dy \\
 &+ \frac{\partial f_3}{\partial x} dx \wedge dz + \frac{\partial f_3}{\partial y} dy \wedge dz + \frac{\partial f_3}{\partial z} dz \wedge dz \\
 &= (\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}) dx \wedge dy + (\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}) dy \wedge dz + (\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}) dz \wedge dx
 \end{aligned}$$

2次の交代形式の場

$$\nabla d(f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy)$$