2015年11月2日(月)

微積分第19回講義ノート

古典的

ベクトル解析　微分作用素

$$\overset{\varphi}{\text{スカラー場}} \xrightarrow{\text{grad(ient)}} \overset{\mathbb{f}}{\text{ベクトル場}} \xrightarrow{\text{rot(ation)}} \overset{\mathbb{f}}{\text{ベクトル場}} \xrightarrow{\text{div(ergence)}} \text{スカラー場}$$

曲線 $\gamma : [a, b] \to \mathbb{R}^3$

$$\boxed{\int_\gamma (\mathrm{grad}\,\varphi) \cdot d\mathbb{r} = \varphi(\gamma(b)) - \varphi(\gamma(a))}$$

閉曲線 γ を境界とする曲面 Σ

$$\underbrace{\boxed{\int_\Sigma (\mathrm{rot}\,\mathbb{f}) \cdot d\mathbb{S}}}_{\text{面積分}} = \underbrace{\boxed{\int_\gamma \mathbb{f} \cdot d\mathbb{r}}}_{\text{線積分}}$$

Stokesの定理

無限小のLevelで成り立てば…

閉曲面 Σ で囲まれる領域 Ω

$$\underbrace{\boxed{\int_\Omega (\mathrm{div}\,\mathbb{f})\, dV}}_{\text{体積分}} = \underbrace{\boxed{\int_\Sigma \mathbb{f} \cdot d\mathbb{S}}}_{\text{面積分}}$$

nabla

ナブラ

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

ベクトルのようなもの

擬似ベクトル

$$\mathbb{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \quad f_i : \mathbb{R}^3 \to \mathbb{R}$$

スカラー

$$\mathrm{grad}\,\varphi = \nabla \varphi = \begin{pmatrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \\ \frac{\partial \varphi}{\partial z} \end{pmatrix}$$

ベクトル積

$$\mathrm{rot}\,\mathbb{f} = \nabla \times \mathbb{f} = \begin{pmatrix} \begin{vmatrix} \frac{\partial}{\partial y} & f_2 \\ \frac{\partial}{\partial z} & f_3 \end{vmatrix} \\ \begin{vmatrix} \frac{\partial}{\partial z} & f_3 \\ \frac{\partial}{\partial x} & f_1 \end{vmatrix} \\ \begin{vmatrix} \frac{\partial}{\partial x} & f_1 \\ \frac{\partial}{\partial y} & f_2 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \\ \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{pmatrix}$$

$$\mathrm{div}\,\mathbb{f} = \nabla \cdot \mathbb{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Gaussの発散定理

$\mathbb{R}^3 \ni \mathbb{r}$（点） $\longrightarrow \mathbb{r} \in \mathbb{R}^3$（ベクトル）　$\mathbb{r}$ ベクトル場

$$\mathrm{div}\,\mathbb{f} = 1 + 1 + 1 = 3$$

$$\int_\Omega \underbrace{(\mathrm{div}\,\mathbb{f})}_{\substack{3 \\ \text{定値}}} dV = 3 \times \frac{4}{3}\pi a^3 = 4\pi a^3$$

積分定理　解析学の天王山

Report　I, IIを証明せよ

I. スカラー場 φ

$\mathrm{rot}(\mathrm{grad}\,\varphi) = 0$

II. ベクトル場 $\mathbb{f}$

$\mathrm{div}(\mathrm{rot}\,\mathbb{f}) = 0$

微分形式

$\mathbb{R}^3 \to \mathbb{R}$ 1次の交代形式

線形写像

3次元

$dx : \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \mapsto a_1$

$dy : \mapsto a_2$

$dz : \mapsto a_3$

2次の交代形式

$dy \wedge dz \quad \left(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right) \mapsto \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

$dz \wedge dx \quad \mapsto \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix}$

$dx \wedge dy \quad \mapsto \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

2つの1次の交代形式

Ⅲ $\omega_1, \omega_2 \quad \mathbb{a}, \mathbb{b} \in \mathbb{R}^3$

$(\omega_1 \overset{\text{くさび型積}}{\wedge} \omega_2)(\mathbb{a}, \mathbb{b}) = \omega_1(\mathbb{a})\,\omega_2(\mathbb{b}) - \omega_1(\mathbb{b})\,\omega_2(\mathbb{a})$ が2次の交代形式になることを示せ.

$$(dy \wedge dz)(\mathbb{a}, \mathbb{b}) = \underbrace{dy(\mathbb{a})}_{a_2}\ \underbrace{dz(\mathbb{b})}_{b_3} - \underbrace{dy(\mathbb{b})}_{b_2}\ \underbrace{dz(\mathbb{a})}_{a_3} = a_2 b_3 - b_2 a_3 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$(\omega_1 \wedge \omega_2 \wedge \omega_3)(\mathbb{a}, \mathbb{b}, \mathbb{c}) = \omega_1(\mathbb{a})\,\omega_2(\mathbb{b})\,\omega_3(\mathbb{c})$$

$\begin{matrix} 1, & 2, & 3 \\ \downarrow & \downarrow & \downarrow \\ 2, & 1, & 3 \end{matrix}$ $\ominus\,(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$

$= (x_2 - x_1)(x_2 - x_3)(x_1 - x_3)$

符号が変わる：奇置換

$\begin{matrix} 1, & 2, & 3 \\ \downarrow & \downarrow & \downarrow \\ 2, & 3, & 1 \end{matrix}$ $\bigcirc\,(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$

$= (x_2 - x_3)(x_2 - x_1)(x_3 - x_1)$

変わらない：偶置換

Ⅳ $(\omega_1 \wedge \omega_2 \wedge \omega_3)(\mathbb{a}, \mathbb{b}, \mathbb{c})$ が3次の交代形式になることを示せ

$$(dx \wedge dy \wedge dz)(\mathbb{a}, \mathbb{b}, \mathbb{c}) = |\mathbb{a}, \mathbb{b}, \mathbb{c}|$$

$\omega_1 \wedge \omega_2 = -\omega_2 \wedge \omega_1$

$\omega_1 \wedge \omega_1 = 0$

$\omega_1 = \omega_{11} + \omega_{12}$

$(\omega_{11} + \omega_{12}) \wedge \omega_2 = \omega_{11} \wedge \omega_2 + \omega_{12} \wedge \omega_2$

$(\alpha \omega_1) \wedge \omega_2 = \alpha (\omega_1 \wedge \omega_2)$

古典的ベクトル解析　$f_i : \mathbb{R}^3 \to \mathbb{R}$

$f_1 \mathbb{e}_1 + f_2 \mathbb{e}_2 + f_3 \mathbb{e}_3$

$$
\begin{array}{ccccccc}
\text{スカラー場} & \longrightarrow & \text{ベクトル場} & \longrightarrow & \text{ベクトル場} & \longrightarrow & \text{スカラー場} \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
\text{0次の微分形式} & \xrightarrow{d} & \text{1次の} & \xrightarrow{d} & \text{2次の} & \xrightarrow{d} & \text{3次の} \\
 & & f_1 dx + f_2 dy + f_3 dz & & & & f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy
\end{array}
$$

$\varphi = dx \wedge dy \wedge dz$

$(\varphi)' = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy + \frac{\partial \varphi}{\partial z} dz$

rot

$d(f_1 dx + f_2 dy + f_3 dz) = (df_1) \wedge dx + (df_2) \wedge dy + (df_3) \wedge dz$

$$
\left(
\begin{aligned}
df_1 &= \frac{\partial f_1}{\partial x} dx + \frac{\partial f_1}{\partial y} dy + \frac{\partial f_1}{\partial z} dz \\
df_2 &= \frac{\partial f_2}{\partial x} dx + \frac{\partial f_2}{\partial y} dy + \frac{\partial f_2}{\partial z} dz \\
df_3 &= \frac{\partial f_3}{\partial x} dx + \frac{\partial f_3}{\partial y} dy + \frac{\partial f_3}{\partial z} dz
\end{aligned}
\right)
$$

$$
\begin{aligned}
&= \frac{\partial f_1}{\partial x} \overset{0}{dx \wedge dx} + \frac{\partial f_1}{\partial y} \overset{-dx \wedge dy}{dy \wedge dx} + \frac{\partial f_1}{\partial z} dz \wedge dx \\
&\quad + \frac{\partial f_2}{\partial x} dx \wedge dy + \frac{\partial f_2}{\partial y} \overset{0}{dy \wedge dy} + \frac{\partial f_2}{\partial z} \overset{-dy \wedge dz}{dz \wedge dy} \\
&\quad + \frac{\partial f_3}{\partial x} \overset{-dz \wedge dx}{dx \wedge dz} + \frac{\partial f_3}{\partial y} dy \wedge dz + \frac{\partial f_3}{\partial z} \overset{0}{dz \wedge dz} \\
&= \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx \wedge dy + \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) dy \wedge dz + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) dz \wedge dx
\end{aligned}
$$

2次の交代形式の場

$\nabla \quad d(f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy)$