微積分第11回講義ノート

極値

1変数(高校)

$f:\mathbb{R}\to\mathbb{R}$, $x\in\mathbb{R}$

必要条件: $f(x)=0$

十分条件: $f''(x)>0$ 極小値

$f''(x)<0$ 極大値

多変数

$f:\mathbb{R}^n\to\mathbb{R}$, $x\in\mathbb{R}^n$

必要条件: $f(x)=0$

$f'(x):\mathbb{R}^n\to\mathbb{R}$の線型写像

$1\times n$の行列

$$\left[\frac{\partial f}{\partial x_1}(x)\ \frac{\partial f}{\partial x_2}(x)\ \cdots\ \frac{\partial f}{\partial x_n}(x)\right]$$

$$\frac{\partial f}{\partial x_1}(x)=\frac{\partial f}{\partial x_2}(x)=\cdots=\frac{\partial f}{\partial x_n}(x)=0$$

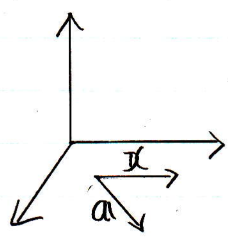
$f:\mathbb{R}^n\to\mathbb{R}$ が $x\in\mathbb{R}^n$ で極大値(極小値)

$\Updownarrow$

$(\forall a\in\mathbb{R}^n$ with $a=\mathbf{0})$

1変数 $\boxed{t\in\mathbb{R}\mapsto f(x+ta)\in\mathbb{R}}$ が

$t=0$ で極大値(極小値)



$F(t)=f(x+ta)$

合成 $t\in\mathbb{R}\to x+ta\in\mathbb{R}^n$, $y\in\mathbb{R}^n\to f(y)\in\mathbb{R}$

$F(t)=f(x+ta)$

$F'(t)=f'(x+ta)(a)$ $\quad a=\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}$

$F'(0)=f'(x)(a)$

$=\frac{\partial f}{\partial x_1}(x)a_1+\frac{\partial f}{\partial x_2}(x)a_2+\cdots+\frac{\partial f}{\partial x_n}(x)a_n$

$f'(x+at):\mathbb{R}\to L(\mathbb{R}^n;\mathbb{R})$

$\mathbb{R}^n\to\mathbb{R}$の線型写像の全体

:線型空間

$1\times n$の行列, $\mathbb{R}^n$

補題
$g: \mathbb{R} \to \mathbb{R}^n \quad g'(t)$
$\varphi: \mathbb{R}^n \to \mathbb{R}$ 線型写像
$(\varphi \circ g)'(t) = \varphi(g'(t))$

$\varphi'(x) = \varphi$
$\underline{\varphi(x + d a) - \varphi(x)} = \varphi'(x)(a)d$
$\cancel{\varphi(x)} + d\varphi(a) - \cancel{\varphi(x)}$
結論 $\varphi'(x)(a) = \varphi(a)$

$\underline{F'(t) = f'(x + at)(a)}$
$t \in \mathbb{R} \mapsto x + at \in \mathbb{R}^n$
$y \in \mathbb{R}^n \mapsto f'(y)(a)$
$y \in \mathbb{R}^n \mapsto f'(y)$

$\underset{G(t)}{\underline{f'(x+at)}} \xmapsto{\text{線型}} f'(x+at)(a)$
$t \longmapsto f(x + at)$
$f'(y) \longmapsto f'(y)(a)$

$G'(t) = f''(x + at)(a)$
$F''(t) = f''(x + at)(a)(a)$
$F''(0) = f''(x)(a)(a)$

$F(t): t \in \mathbb{R} \longmapsto f(x + at)$
$F'(0) = f'(x)(a) = 0$
$F''(0) = f''(x)(a)(a)$
$= f''(x)(a, a) > 0$ 極小値

$\boxed{\dfrac{\partial^2 f}{\partial x^2} > 0}$

判別式 $\left(\dfrac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2 - \dfrac{\partial^2 f}{\partial x_1^2}\dfrac{\partial^2 f}{\partial x_2^2} < 0 \quad \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix} > 0$

$\boxed{n=2 \text{とする}}$ 二重線型 $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \dfrac{\partial^2 f}{\partial x_1 \partial x_2} = \dfrac{\partial^2 f}{\partial x_2 \partial x_1}$
$f''(x)$
$f'(x)(a) = \dfrac{\partial f}{\partial x_1}(x) a_1 + \dfrac{\partial f}{\partial x_2}(x) a_2 = 0 \quad \dfrac{\partial f}{\partial x_1}(x) = \dfrac{\partial f}{\partial x_2}(x) = 0$
$f''(x)(a, a) = \dfrac{\partial^2 f}{\partial x_1^2}(x) a_1^2 + 2\dfrac{\partial^2 f}{\partial x_1 \partial x_2} a_1 a_2 + \dfrac{\partial^2 f}{\partial x_2^2} a_2^2 > 0$
a_2^2で割る $a_2 \neq 0$
$\dfrac{\partial^2 f}{\partial x_1^2}\left(\dfrac{a_1}{a_2}\right)^2 + 2\dfrac{\partial^2 f}{\partial x_1 \partial x_2}\dfrac{a_1}{a_2} + \dfrac{\partial^2 f}{\partial x_2^2} > 0$ 2次方程式
十分条件 $\dfrac{\partial^2 f}{\partial x_1^2} < 0$ 増減表

Report
$f(x, y) = (x^2 + y^2)^2 - 2(x^2 - y^2)$ の極大値・極小値を求めよ。