THE YAMABE PROBLEM AND NONLINEAR BOUNDARY VALUE PROBLEMS

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Dedicated to Professor Sigeru Mizohata on the occasion of his 70th birthday

Abstract. We study the Yamabe problem in the context of manifolds with boundary - a basic problem in Riemannian geometry - from the point of view of nonlinear elliptic boundary value problems. By making good use of bifurcation theory from a simple eigenvalue, we show that nonpositive scalar curvatures and nonpositive mean curvatures are not always conformal to constant negative scalar curvatures and the zero mean curvature.

Introduction

Let $(\overline{M}, g)$ be a smooth compact, connected Riemannian manifold with boundary ∂M of dimension $n \geq 3$, and let $M = \overline{M} \backslash \partial M$ be the interior of $\overline{M}$. A metric g' of $\overline{M}$ is said to be *conformal* to the metric g if there exists a strictly positive function $\varphi \in C^\infty(\overline{M})$ such that

$$g' = \varphi^{\frac{4}{n-2}}\, g.$$

A basic problem in Riemannian geometry is to seek a conformal change of the metric g that makes the scalar curvature of M *constant* and the mean curvature of ∂M *zero*. When the boundary ∂M is empty, this problem is the so-called Yamabe problem. The solution of the Yamabe problem is completely given by H. Yamabe [Ya], N. S. Trudinger [Tr], T. Aubin [Au] and R. Schoen [Sc] (cf. [LP]). Recently, J. Escobar [Es] has studied the problem in the context of manifolds with boundary, and has given an affirmative solution to the problem formulated above in almost every case.

In this paper we consider the case where the given metric g *already* has a constant negative scalar curvature k of M and the zero mean curvature of ∂M as in Ouyang [Ou] (cf. [Ka], [KW]). Our problem is stated as follows:

Problem. *Given a nonpositive function $R' \in C^\infty(M)$ and a nonpositive function $h' \in C^\infty(\partial M)$, find a metric g' of $\overline{M}$, conformal to g, such that R' and h' are the scalar curvature of M and the mean curvature of ∂M with respect to g', respectively.*

Now we let

$$\mathcal{M}_-(R') = \{x \in M; R'(x) < 0\},$$

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and

$$\mathcal{M}_0(R') = M \backslash \overline{\mathcal{M}_-(R')}.$$

Our fundamental hypothesis is the following (cf. Figure 1):

(H) The open set $\mathcal{M}_0(R')$ consists of a finite number of connected components with smooth boundary, say $\mathcal{M}_i(R')$, $1 \leq i \leq \ell$, which are bounded away from ∂M, and consists of a finite number of connected components with smooth boundary, say $\mathcal{M}_j(R')$, $\ell+1 \leq j \leq N$, such that each closure $\overline{\mathcal{M}_j(R')}$ is a neighborhood of some connected component $\mathcal{S}_j$ of ∂M.

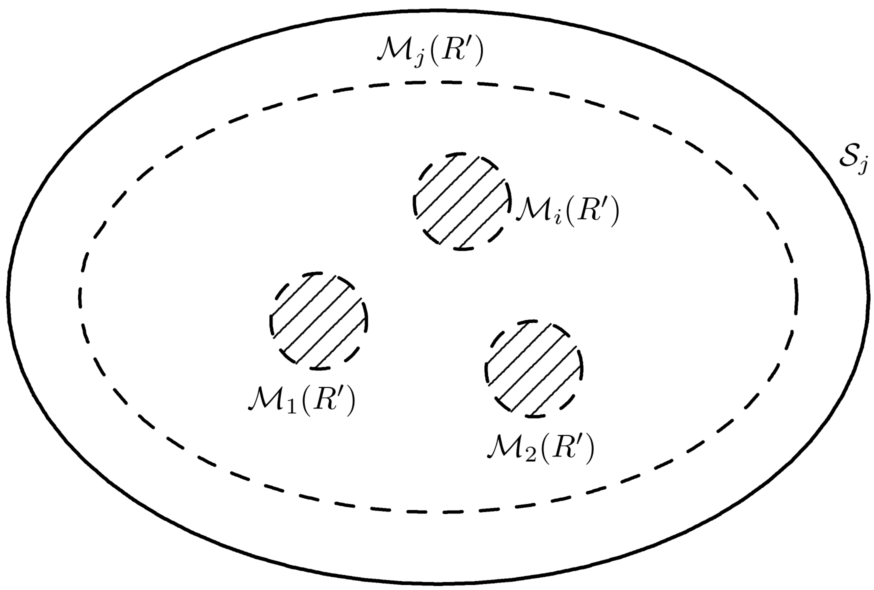

Figure 1

Under hypothesis (H), we shall show that nonpositive scalar curvatures R' and nonpositive mean curvatures h' are not always conformal to the negative scalar curvature k and the zero mean curvature; it depends on the shape of the set $\mathcal{M}_0(R')$ (see Theorem 1 below).

First we consider the Dirichlet eigenvalue problem in each connected component $\mathcal{M}_i(R')$, $1 \leq i \leq \ell$, which is bounded away from ∂M:

$$(D_i) \qquad \begin{cases} \Delta\psi = \lambda\psi & \text{in } \mathcal{M}_i(R'), \\ \psi = 0 & \text{on } \partial\mathcal{M}_i(R'). \end{cases}$$

We remark that the Laplacian Δ has the sign so that $\Delta\psi = -\psi''$ on $\mathbf{R}$. By the celebrated Rayleigh theorem, we know that the *first eigenvalue* $\lambda_1(\mathcal{M}_i(R'))$ of problem (D_i) is given by the formula

$$\lambda_1(\mathcal{M}_i(R')) = \inf\left\{\int_{\mathcal{M}_i(R')} |\nabla\psi|^2\, dV; \psi \in H_0^1(\mathcal{M}_i(R')), \|\psi\|_{L^2(\mathcal{M}_i(R'))} = 1\right\}.$$

Here dV is the Riemannian density of g, and $H_0^1(\mathcal{M}_i(R'))$ is the closure of C^∞ functions with compact support in $\mathcal{M}_i(R')$ in the Sobolev space $H^1(\mathcal{M}_i(R'))$.

Next we consider the Dirichlet-Neumann eigenvalue problem in each connected component $\mathcal{M}_j(R')$, $\ell+1 \leq j \leq N$, whose closure is a neighborhood of some connected component $\mathcal{S}_j$ of ∂M:

$$(M_j) \qquad \begin{cases} \Delta\psi = \mu\psi & \text{in } \mathcal{M}_j(R'), \\ \psi = 0 & \text{on } \partial\mathcal{M}_j(R')\backslash\mathcal{S}_j, \\ \frac{\partial\psi}{\partial\mathbf{n}} = 0 & \text{on } \mathcal{S}_j, \end{cases}$$

where $\mathbf{n}$ is the unit outward normal vector to $\mathcal{S}_j$ with respect to the metric g. Similarly, by Rayleigh's theorem, we know that the *first eigenvalue* $\mu_1(\mathcal{M}_j(R'))$ of problem (M_j) is given by the formula

$$\mu_1(\mathcal{M}_j(R')) = \inf\Big\{\int_{\mathcal{M}_j(R')} |\nabla\psi|^2\, dV; \psi \in H^1(\mathcal{M}_j(R')), \psi = 0 \,\text{on}\, \partial\mathcal{M}_j(R')\backslash\mathcal{S}_j, \\ \|\psi\|_{L^2(\mathcal{M}_j(R'))} = 1\Big\}.$$

We let

$$\widetilde{\lambda}_1(\mathcal{M}_0(R')) = \min\{\lambda_1(\mathcal{M}_1(R')), \cdots, \lambda_1(\mathcal{M}_\ell(R')), \\ \mu_1(\mathcal{M}_{\ell+1}(R')), \cdots, \mu_1(\mathcal{M}_N(R'))\}.$$

We remark (cf. [Ch]) that the minimal eigenvalue $\widetilde{\lambda}_1(\mathcal{M}_0(R'))$ is monotone decreasing with respect to the set $\mathcal{M}_0(R')$; more precisely, it tends to $+\infty$ if $\mathcal{M}_0(R') \to \emptyset$, and it tends to zero if $\mathcal{M}_0(R') \to M$.

Then our main result of this paper is stated as follows.

Theorem 1. *Assume that the given metric g has a constant negative scalar curvature k of M and the zero mean curvature of ∂M, and that:*

(A) $h' \leq 0$ on $\partial M \backslash \cup_{j=\ell+1}^N \mathcal{S}_j$, and $h' = 0$ on $\cup_{j=\ell+1}^N \mathcal{S}_j$.

Then there exists a conformally related metric $g' = \varphi^{4/(n-2)}\, g$, $\varphi > 0$ on $\overline{M}$, such that R' and h' are respectively the scalar curvature of M and the mean curvature of ∂M with respect to g' if and only if we have

$$\widetilde{\lambda}_1(\mathcal{M}_0(R')) > -\frac{n-2}{4(n-1)}\, k.$$

The rest of this paper is organized as follows.

Section 1 is devoted to analytic and geometric preliminaries. In particular, we formulate our problem more precisely, and show that the problem is equivalent to finding a strictly positive solution $\varphi \in C^\infty(\overline{M})$ of the nonlinear boundary value problem:

$$(*) \qquad \begin{cases} 4\,\frac{n-1}{n-2}\,\Delta\varphi + k\varphi - R'\varphi^{\frac{n+2}{n-2}} = 0 & \text{in } M, \\ \frac{2}{n-2}\,\frac{\partial\varphi}{\partial\mathbf{n}} - h'\varphi^{\frac{n}{n-2}} = 0 & \text{on } \partial M. \end{cases}$$

If we let

$$\lambda = -\frac{n-2}{4(n-1)}\,k, \quad h = -\frac{n-2}{4(n-1)}\,R', \quad a = -\frac{n-2}{2}\,h',$$

then the boundary value problem $(*)$ can be written in the following form:

$$(**) \qquad \begin{cases} \Delta u - \lambda u + hu^p = 0 & \text{in } M, \\ \frac{\partial u}{\partial \mathbf{n}} + au^q = 0 & \text{on } \partial M, \end{cases}$$

where

$$p = \frac{n+2}{n-2} > 1, \quad q = \frac{n}{n-2} > 1.$$

Here we remark that

$$\begin{cases} \lambda > 0, \\ h \in C^\infty(M) \text{ and } h \geq 0 \text{ in } M, \\ a \in C^\infty(\partial M) \text{ and } a \geq 0 \text{ on } \partial M. \end{cases}$$

In Section 2, we free our problem from geometry, and study the existence and nonexistence of positive solutions of problem $(**)$ in the framework of Hölder spaces. Our approach to problem $(**)$ is a modification of that of Ouyang [Ou] adapted to the present context. However we do not use the sub-super-solution method as in Ouyang [Ou] (cf. [Ka], [KW]).

First, by using a bifurcation theorem from a simple eigenvalue due to Crandall-Rabinowitz [CR], we prove that there exists a positive solution $u(\lambda)$ of problem $(**)$ starting at the point $(0,0)$ (Lemma 2.1). Next we show that the solution $u(\lambda)$ is strictly positive on $\overline{M}$ (Lemma 2.2), and is monotone increasing with respect to the parameter λ (Lemma 2.5). In the proof we make essential use of the positivity of the resolvent of the linearized problem on the space $C(\overline{M})$ due to Taira [Ta2] (Claim 2.6). Furthermore, by virtue of the implicit function theorem, we can find a constant $0 < \overline{\lambda}(h) < \infty$ such that there occurs no *secondary bifurcation* along the bifurcation solution curve $(\lambda, u(\lambda))$ of problem $(**)$ for all $0 < \lambda < \overline{\lambda}(h)$ (Lemma 2.3). The situation may be represented schematically by the following bifurcation diagram:

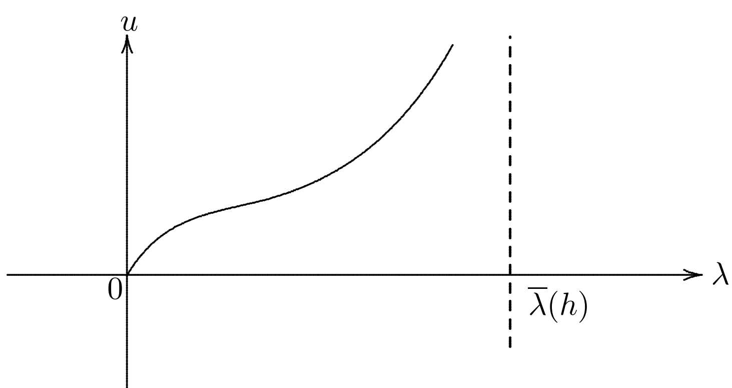

Figure 2

In order to characterize explicitly the critical value $\overline{\lambda}(h)$, we let

$$\mathcal{M}_+(h) = \{x \in M; h(x) > 0\},$$

and

$$\mathcal{M}_0(h) = M \backslash \overline{\mathcal{M}_+(h)}.$$

Our fundamental hypothesis is the following (cf. hypothesis (H)):

(η) The open set $\mathcal{M}_0(h)$ consists of a finite number of connected components with smooth boundary, say $\mathcal{M}_i(h)$, $1 \leq i \leq \ell$, which are bounded away from ∂M, and consists of a finite number of connected components with smooth boundary, say $\mathcal{M}_j(h)$, $\ell+1 \leq j \leq N$, such that each closure $\overline{\mathcal{M}_j(h)}$ is a neighborhood of some connected component $\mathcal{S}_j$ of ∂M.

Then we have the formula

$$\overline{\lambda}(h) = \widetilde{\lambda}_1(\mathcal{M}_0(h)), \tag{0.1}$$

where the quantity $\widetilde{\lambda}_1(\mathcal{M}_0(h))$ is defined similarly, with R' replaced by $-h$.

More precisely, we can prove the following existence and nonexistence theorem of positive solutions of problem $(**)$ (cf. [Cr, Théorème 6], [Ou, Theorem 3]):

Theorem 2. *Assume that:*

(α) $a \geq 0$ on $\partial M \backslash \mathcal{S}_j$, and $a = 0$ on $\mathcal{S}_j$, $\ell+1 \leq j \leq N$.

Then we have the following:

(i) For any $0 < \lambda < \widetilde{\lambda}_1(\mathcal{M}_0(h))$, there exists a strictly positive solution $u(\lambda)$ of problem $(**)$.

(ii) For any $\lambda \geq \widetilde{\lambda}_1(\mathcal{M}_0(h))$, there exists no positive solution of problem $(**)$.

Our main Theorem 1 is an immediate consequence of Theorem 2.

The proof of formula (0.1) and Theorem 2 is carried out in Section 3 through Section 6.

First, in Section 3, by using Green's formula, we prove the inequality (Proposition 3.1):

$$\overline{\lambda}(h) \leq \widetilde{\lambda}_1(\mathcal{M}_0(h)). \tag{0.2}$$

Next, in Section 4 through Section 6, we prove the reverse inequality of inequality (0.2) (Proposition 6.1):

$$\widetilde{\lambda}_1(\mathcal{M}_0(h)) \leq \overline{\lambda}(h). \tag{0.3}$$

In Section 4 we study the behavior of the positive solutions $u(\lambda)$ $(0 < \lambda < \overline{\lambda}(h))$ in the set $\mathcal{M}_+(h)$. Roughly speaking, we prove that, for each $\varepsilon > 0$, there exists a constant $C(\varepsilon, \lambda) > 0$ such that (Lemma 4.2)

$$(u(\lambda)(x))^{p-1-\varepsilon} \leq \frac{C(\varepsilon,\lambda)}{h(x)} \quad \text{for all } x \in \mathcal{M}_+(h).$$

This is an essential step in the proof of inequality (0.3) in Section 6 (cf. estimate (6.9)). On the other hand, in Section 5, we prove that (Lemma 5.1)

$$\lim_{\lambda \to \overline{\lambda}(h)} \|u(\lambda)\|_{L^2(M)} = +\infty,$$

that is, we show that the solution $u(\lambda)$ "blows up" at the critical value $\overline{\lambda}(h)$. In Section 6, we prove that the critical value $\overline{\lambda}(h)$ is an eigenvalue of either the Dirichlet problem (D_i) or the Dirichlet-Neumann problem (M_j). By Rayleigh's theorem, this implies the desired reverse inequality (0.3).

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1. Analytic and geometric preliminaries

In this section we collect some notation and well-known facts from the theory of partial differential equations and Riemannian geometry which will be used in the subsequent sections.

1.1 Function spaces. Let Ω be an open subset of Euclidean space $\mathbf{R}^n$. If m is a nonnegative integer and $1 \leq p \leq \infty$, we let

$$\begin{aligned} W^{m,p}(\Omega) = {} & \text{the space of (equivalence classes of) functions} \\ & u \in L^p(\Omega) \text{ all of whose derivatives } \partial^\alpha u,\ |\alpha| \leq m, \\ & \text{in the sense of distributions are in } L^p(\Omega), \end{aligned}$$

and

$$W_0^{m,p}(\Omega) = \text{the closure of } C_0^\infty(\Omega) \text{ in the space } W^{m,p}(\Omega).$$

In the case $p = 2$, we customarily write

$$H^m(\Omega) = W^{m,2}(\Omega), \quad H_0^m(\Omega) = W_0^{m,2}(\Omega).$$

Furthermore, if m is a nonnegative integer and $0 < \theta < 1$, we let

$$\begin{aligned} C^{m+\theta}(\overline{\Omega}) = {} & \text{the space of functions in } C^m(\overline{\Omega}) \text{ all of whose } m\text{-th order} \\ & \text{derivatives are Hölder continuous with exponent } \theta \text{ on } \Omega. \end{aligned}$$

If M is an n-dimensional compact smooth manifold without boundary, then the spaces $W^{m,p}(M)$ and $C^{m+\theta}(M)$ are defined respectively to be locally the spaces $W^{m,p}(\mathbf{R}^n)$ and $C^{m+\theta}(\mathbf{R}^n)$, upon using local coordinate systems flattening out M, together with a partition of unity.

1.2 Bifurcation theory. Let $F(t, x)$ be a map of a neighborhood of the point $(0, 0)$ in a Banach space $\mathbf{R} \times X$ into a Banach space Y such that

$$F(t, 0) = 0 \quad \text{for } |t| < 1.$$

Of particular interest is the process of bifurcation whereby a given solution of $F(t, x) = 0$ splits into two or more solutions as t passes through some critical value.

The point $(0, 0)$ is called a *bifurcation point* of the equation $F(t, x) = 0$ if every neighborhood of $(0, 0)$ in $\mathbf{R} \times X$ contains a solution of the equation $F(t, x) = 0$ with $x \neq 0$.

The next theorem gives sufficient conditions in order that the point $(0, 0)$ be a bifurcation point of the equation $F(t, x) = 0$ (cf. [CR, Theorem 1.7], [Ni, Theorem 3.2.2]; [CH, Chapter 6, Theorem 6.1]):

Theorem 1.1 (The bifurcation theorem). *Let X, Y be Banach spaces, and let V be a neighborhood of 0 in X and let $F : (-1, 1) \times V \to Y$ have the following properties:*

(1) $F(t, 0) = 0$ for $|t| < 1$.

(2) The partial Fréchet derivatives F_t, F_x and F_{tx} of F exist and are continuous.

(3) $N(F_x(0, 0))$ and $Y/R(F_x(0, 0))$ are one dimensional.

(4) $F_{tx}(0, 0)x_0 \notin R(F_x(0, 0))$ where $N(F_x(0, 0)) = \mathrm{span}\,\{x_0\}$.

If Z is a complement of $N(F_x(0, 0))$ in X, that is, if it is a closed subspace of X such that

$$X = N(F_x(0, 0)) \oplus Z,$$

then there exist a neighborhood U of $(0, 0)$ in $\mathbf{R} \times X$ and an open interval $(-a, a)$ such that the set of solutions of $F(t, x) = 0$ in U consists of two continuous curves Γ_1 and Γ_2 which may be parametrized respectively by t and α as follows:

$$\begin{aligned}\Gamma_1 &= \{(t, 0); (t, 0) \in U\},\\ \Gamma_2 &= \{(\varphi(\alpha), \alpha x_0 + \alpha\psi(\alpha)); |\alpha| < a\}.\end{aligned}$$

Here

$$\begin{aligned}&\varphi : (-a, a) \to \mathbf{R}, \quad \varphi(0) = 0,\\ &\psi : (-a, a) \to Z, \quad \psi(0) = 0.\end{aligned}$$

1.3 Formulation of Problem. Let $(\overline{M}, g)$ be a smooth compact, connected Riemannian manifold with boundary ∂M of dimension $n \geq 3$ and $M = \overline{M} \backslash \partial M$ the interior of $\overline{M}$. If g_{jk} are the components of the metric tensor g with respect to a local coordinate system $x^1, x^2, \cdots, x^n$, then g_{jk} and its inverse g^{jk} are used to raise and lower indices. Covariant differentiation is denoted by ∇. If f is a function on M, then its covariant derivative is the one-tensor ∇f with components

$$\nabla_i f = \frac{\partial f}{\partial x^i}.$$

The second covariant derivative of f is the two-tensor $\nabla^2 f$ with components

$$\nabla_{ij} f = \frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_{\ell=1}^{n} \Gamma^\ell_{ij} \frac{\partial f}{\partial x^\ell}.$$

Here the functions

$$\Gamma^\ell_{ij} = \frac{1}{2}\left[\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right] g^{k\ell}$$

are the *Christoffel symbols*. The metric extends to an inner product on tensors of any type; for example, the norm of ∇f is

$$|\nabla f|^2 = \sum_{j=1}^{n} \nabla^j f \nabla_j f = \sum_{i,j=1}^{n} g^{ij} \nabla_i f \nabla_j f.$$

The *divergence operator* is the formal adjoint ∇^* of ∇ given on one-forms $u = \sum_{i=1}^n u_i\, dx^i$ by

$$\nabla^* u = -\sum_{i=1}^n \nabla^i u_i = -\sum_{i,j=1}^n g^{ij}\nabla_j u_i = -\sum_{i,j=1}^n g^{ij}\,\frac{\partial u_i}{\partial x^j} + \sum_{i,j,\ell=1}^n g^{ij}\,\Gamma^{\ell}_{ji}\,u_\ell.$$

The *Laplace-Beltrami operator*, or simply *Laplacian*, is the second-order differential operator Δ given on functions f by

$$\Delta f = \nabla^*\nabla f = -\sum_{i=1}^n \nabla^i\nabla_i f = -\sum_{i,j=1}^n g^{ij}\,\frac{\partial^2 f}{\partial x^i\partial x^j} + \sum_{i,j,\ell=1}^n g^{ij}\,\Gamma^{\ell}_{ji}\,\frac{\partial f}{\partial x^\ell}.$$

The Riemannian *curvature tensor* is the tensor with components $R^{\ell}{}_{kij}$ computed in a local coordinate system $x^1, x^2, \cdots, x^n$ by

$$R^{\ell}{}_{kij} = \frac{\partial}{\partial x^i}(\Gamma^{\ell}{}_{jk}) - \frac{\partial}{\partial x^j}(\Gamma^{\ell}{}_{ik}) + \sum_{m=1}^n \Gamma^{\ell}{}_{im}\,\Gamma^{m}{}_{jk} - \sum_{m=1}^n \Gamma^{\ell}{}_{jm}\,\Gamma^{m}{}_{ik}.$$

The *Ricci tensor* is the contraction of the curvature tensor

$$R_{ij} = \sum_{k=1}^n R^{k}{}_{ikj},$$

and the *scalar curvature* is the trace of the Ricci tensor

$$R = \sum_{i,j=1}^n g^{ij} R_{ij}.$$

Let $(x^1, x^2, \cdots, x^{n-1}, x^n)$ be a local coordinate system on $\overline{M}$ in which ∂M is the plane $x^n = 0$ and for which $\partial/\partial x^n$ is a unit outward normal vector to ∂M. Then the components h_{ij} of the second fundamental form of g are given by

$$h_{ij} = \frac{1}{2}\,\frac{\partial g_{ij}}{\partial x^n},\ 1 \leq i, j \leq n-1.$$

The *mean curvature* of ∂M is the trace

$$h = \frac{1}{n-1}\sum_{i,j=1}^{n-1} g^{ij} h_{ij}.$$

A metric g' of $\overline{M}$ is said to be *conformal* to the metric g if there exists a real-valued function $f \in C^\infty(\overline{M})$ such that

$$g' = e^{2f} g.$$

If $g' = e^{2f} g$ is a metric conformal to g, then we have the following transformation laws for the Ricci curvatures R_{ij}, R'_{ij} and the scalar curvatures R, R', respectively:

$$
\begin{aligned}
R'_{ij} &= R_{ij} - (n-2)\nabla_{ij} f + (n-2)\nabla_i f \nabla_j f + \left(\Delta f - (n-2)|\nabla f|^2\right) g_{ij}, \\
R' &= e^{-2f}\left(R + 2(n-1)\Delta f - (n-1)(n-2)|\nabla f|^2\right).
\end{aligned}
$$

Furthermore, if we make the substitution

$$e^{2f} = \varphi^{\frac{4}{n-2}}, \quad \varphi > 0 \text{ on } \overline{M},$$

then the second formula can be simplified as follows:

$$4\,\frac{n-1}{n-2}\,\Delta\varphi + R\varphi - R'\varphi^{\frac{n+2}{n-2}} = 0 \quad \text{in } M. \tag{1.1}$$

Similarly, one can compute the components h'_{ij} of the second fundamental form of $g' = e^{2f} g$ in terms of the second fundamental form of g. We have the following transformation laws for the components h_{ij}, h'_{ij} and the mean curvatures h, h', respectively:

$$
\begin{aligned}
h'_{ij} &= e^f h_{ij} + \frac{\partial}{\partial \mathbf{n}}(e^f) g_{ij}, \\
h' &= e^{-f}\left(h + \frac{\partial f}{\partial \mathbf{n}}\right),
\end{aligned}
$$

where $\mathbf{n}$ is the unit outward normal vector with respect to the metric g. Furthermore, if we make the substitution $e^{2f} = \varphi^{4/(n-2)}$ as above, then the second formula can be simplified as follows:

$$\frac{2}{n-2}\frac{\partial\varphi}{\partial\mathbf{n}} + h\varphi - h'\varphi^{\frac{n}{n-2}} = 0 \quad \text{on } \partial M. \tag{1.2}$$

Therefore, if we take $R = k$ in equation (1.1) and $h = 0$ in condition (1.2), our problem is equivalent to finding a strictly positive solution $\varphi \in C^\infty(\overline{M})$ of the nonlinear boundary value problem:

$$
(*) \qquad \begin{cases}
4\,\frac{n-1}{n-2}\,\Delta\varphi + k\varphi - R'\varphi^{\frac{n+2}{n-2}} = 0 & \text{in } M, \\
\frac{2}{n-2}\frac{\partial\varphi}{\partial\mathbf{n}} - h'\varphi^{\frac{n}{n-2}} = 0 & \text{on } \partial M.
\end{cases}
$$

1.4 Regularity theorem for nonlinear Neumann problems. The next regularity theorem for *nonlinear Neumann problems*, due to Cherrier [Cr, Théorème 1], will play an important role in the proof of Theorem 2:

Theorem 1.2. *Let $H \in C^\infty(\mathbf{R} \times \overline{M})$ and $L \in C^\infty(\mathbf{R} \times \partial M)$. Assume that there exist constants $C_1 > 0$ and $C_2 > 0$ such that*

$$|H(t,x)| \le C_1\left(1 + |t|^{\frac{n+2}{n-2}}\right) \quad \textit{for all } (t,x) \in \mathbf{R} \times \overline{M},$$

$$|L(t,y)| \le C_2 \left(1 + |t|^{\frac{n}{n-2}}\right) \quad \textit{for all } (t,y) \in \mathbf{R} \times \partial M.$$

If a function $\varphi \in H^1(M)$ is a weak solution *of the problem*

$$(+) \qquad \begin{cases} \Delta\varphi + H(\varphi, x) = 0 & \textit{in } M, \\ \frac{\partial \varphi}{\partial \mathbf{n}} + L(\varphi, y) = 0 & \textit{on } \partial M, \end{cases}$$

that is, if it satisfies, for all $\psi \in H^1(M)$,

$$\int_M \left(\sum_{i=1}^n \nabla^i \varphi \nabla_i \psi + H(\varphi, x)\psi \right) dV(x) + \int_{\partial M} L(\varphi, y)\psi \, d\sigma(y) = 0,$$

then φ belongs to $C^\infty(\overline{M})$, and is a solution of problem $(+)$. Here dV is the Riemannian density of M and $d\sigma$ is the induced Riemannian density of ∂M, respectively.

2. Existence of positive solutions of problem $(**)$

In the subsequent sections, we shall prove Theorem 2, the existence and nonexistence theorem of positive solutions of problem $(**)$.

Now we associate with problem $(**)$ a nonlinear mapping $F : \mathbf{R} \times C^{2+\theta}(\overline{M}) \longmapsto C^\theta(\overline{M}) \times C^{1+\theta}(\partial M)$ $(0 < \theta < 1)$ as follows:

$$\begin{aligned} F : \mathbf{R} \times C^{2+\theta}(\overline{M}) &\longrightarrow C^\theta(\overline{M}) \times C^{1+\theta}(\partial M) \\ (\lambda, u) &\longmapsto \left(\Delta u - \lambda u + h u^p, \frac{\partial u}{\partial \mathbf{n}} + a u^q \right). \end{aligned}$$

We remark that a function $u \in C^{2+\theta}(\overline{M})$ is a solution of problem $(**)$ if and only if $F(\lambda, u) = 0$.

I) First we prove an existence result of positive solutions of problem $(**)$ near the point $(0,0)$.

Lemma 2.1. *There exists a bifurcation solution curve $(\lambda, u(\lambda))$ of the equation $F(\lambda, u) = 0$ starting at $(0,0)$:*

$$(**) \qquad \begin{cases} \Delta u(\lambda) - \lambda u(\lambda) + h u(\lambda)^p = 0 & \textit{in } M, \\ \frac{\partial u(\lambda)}{\partial \mathbf{n}} + a u(\lambda)^q = 0 & \textit{on } \partial M. \end{cases}$$

Proof. The proof of Lemma 2.1 is based on the bifurcation theorem 1.1.

We have for partial Fréchet derivatives of the mapping $F(\lambda, u)$

$$\begin{aligned} F_u(\lambda, u) : C^{2+\theta}(\overline{M}) &\longrightarrow C^\theta(\overline{M}) \times C^{1+\theta}(\partial M) \\ v &\longmapsto \left(\Delta v - \lambda v + p h u^{p-1} v, \frac{\partial v}{\partial \mathbf{n}} + q a u^{q-1} v \right), \end{aligned}$$

and

$$\begin{aligned} F_{\lambda u}(\lambda, u) : C^{2+\theta}(\overline{M}) &\longrightarrow C^\theta(\overline{M}) \times C^{1+\theta}(\partial M) \\ v &\longmapsto (-v, 0). \end{aligned}$$

In particular we have

$$F_u(0,0) : C^{2+\theta}(\overline{M}) \longrightarrow C^{\theta}(\overline{M}) \times C^{1+\theta}(\partial M)$$
$$v \longmapsto \left(\Delta v, \frac{\partial v}{\partial \mathbf{n}} \right).$$

It is easy to see that

(2.1)
$$N(F_u(0,0)) = \{\text{constant functions}\} = \text{span}\,\{1\},$$

(2.2)
$$R(F_u(0,0)) = \{(f,\varphi) \in C^{\theta}(\overline{M}) \times C^{1+\theta}(\partial M); \int_M f\,dV + \int_{\partial M} \varphi\,d\sigma = 0\},$$

and

(2.3)
$$F_{\lambda u}(0,0)1 = (-1,0) \notin R(F_u(0,0)).$$

First we prove assertion (2.1): Assume that a function $v \in C^{2+\theta}(\overline{M})$ is a solution of the homogeneous Neumann problem

$$\begin{cases} \Delta v = 0 & \text{in } M, \\ \frac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial M. \end{cases}$$

Then, by Green's formula, it follows that

$$0 = \int_M \Delta v \cdot v\,dV = \int_M |\nabla v|^2\,dV.$$

Hence we find that the function v is constant. This proves that

$$N(F_u(0,0)) = \{\text{constant functions}\} = \text{span}\,\{1\}.$$

Next we prove assertion (2.2): Assume that a function $v \in C^{2+\theta}(\overline{M})$ is a solution of the nonhomogeneous Neumann problem

$$(N) \qquad \begin{cases} \Delta v = f & \text{in } M, \\ \frac{\partial v}{\partial \mathbf{n}} = \varphi & \text{on } \partial M, \end{cases}$$

with $(f,\varphi) \in C^{\theta}(\overline{M}) \times C^{1+\theta}(\partial M)$.

Then we have by Green's formula

$$\begin{aligned} \int_M f\,dV + \int_{\partial M} \varphi\,d\sigma &= \int_M \Delta v\,dV + \int_{\partial M} \frac{\partial v}{\partial \mathbf{n}}\,d\sigma \\ &= -\int_{\partial M} \frac{\partial v}{\partial \mathbf{n}}\,d\sigma + \int_{\partial M} \frac{\partial v}{\partial \mathbf{n}}\,d\sigma \end{aligned}$$

$$= 0.$$

Conversely, assume that a function $(f, \varphi) \in C^\theta(\overline{M}) \times C^{1+\theta}(\partial M)$ satisfies the condition

$$\int_M f \, dV + \int_{\partial M} \varphi \, d\sigma = 0.$$

If we choose a function $w \in C^{2+\theta}(\overline{M})$ such that

$$\frac{\partial w}{\partial \mathbf{n}} = \varphi \quad \text{on } \partial M,$$

then we have by Green's formula

$$\int_M (f - \Delta w) \, dV = \int_M f \, dV + \int_{\partial M} \varphi \, d\sigma = 0. \tag{2.4}$$

Now we introduce a densely defined, closed linear operator

$$\mathfrak{A}_N : L^2(M) \to L^2(M)$$

as follows.

(a) The domain of definition $D(\mathfrak{A}_N)$ is the space

$$D(\mathfrak{A}_N) = \left\{ u \in H^2(M); \frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial M \right\}.$$

(b) $\mathfrak{A}_N u = \Delta u$, $u \in D(\mathfrak{A}_N)$.

Then it is known (cf. [LM, Chapter 2, Section 8.4, Theorem 8.4]) that the operator $\mathfrak{A}_N$ is self-adjoint and

$$L^2(M) = N(\mathfrak{A}_N) \oplus R(\mathfrak{A}_N),$$

where

$$N(\mathfrak{A}_N) = N(F_u(0,0)) = \{\text{constant functions}\} = \text{span}\,\{1\}.$$

Therefore, one can find a solution $u \in C^{2+\theta}(\overline{M})$ of the problem

$$\begin{cases} \Delta u = f - \Delta w & \text{in } M, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial M, \end{cases}$$

since formula (2.4) tells us that the function $f - \Delta w$ is orthogonal to the constant function 1.

Summing up, we find that the function $v = u + w \in C^{2+\theta}(\overline{M})$ is a solution of problem (N).

Finally we prove assertion (2.3): Since we have

$$F_{\lambda u}(0,0)1 = (-1, 0),$$

it follows that

$$\int_M (-1) \, dV + \int_{\partial M} 0 \, d\sigma = -\text{vol(M)} < 0.$$

By assertion (2.2), this proves that

$$F_{\lambda u}(0,0)1 \notin R(F_u(0,0)).$$

The proof of Lemma 2.1 is complete. $\square$

I-2) Next we show that the solution $u(\lambda)$ is strictly positive on $\overline{M}$:

$$u(\lambda) > 0 \quad \text{on } \overline{M}.$$

To do so, it suffices to prove the following:

Lemma 2.2. *If a function $v \in C^2(\overline{M})$ satisfies the conditions*

$$\begin{cases} \Delta v - \lambda v + h v^p = 0 & \text{in } M, \\ \frac{\partial v}{\partial \mathbf{n}} + a v^q = 0 & \text{on } \partial M, \\ v \geq 0 & \text{on } \overline{M}, \end{cases}$$

then we have

$$v > 0 \quad \text{on } \overline{M}.$$

Proof. a) First assume to the contrary that there exists a point $x_0 \in M$ such that

$$v(x_0) = \min_{\overline{M}} v = 0.$$

Then, since $\lambda > 0$ and $v \geq 0$ on $\overline{M}$, one can find a neighborhood Ω of x_0 such that

$$\Delta v = v(\lambda - h v^{p-1}) \geq 0 \quad \text{in } \Omega.$$

Hence, applying the strong maximum principle (cf. [PW, Chapter 2, Section 3, Theorem 6]; [Ta1, Theorem 7.2.1]), we obtain that

$$v \equiv 0 \quad \text{in } \Omega.$$

This implies that $v \equiv 0$ in M, since M is connected.

This contradiction proves that $v > 0$ in M.

b) Next assume to the contrary that there exists a point $x' \in \partial M$ such that

$$v(x') = \min_{\overline{M}} v = 0.$$

Then one can find a neighborhood Ω' of x' such that

$$\Delta v = v(\lambda - h v^{p-1}) \geq 0 \quad \text{in } \Omega'.$$

But we have by step a)

$$\begin{cases} v(x) > 0, \;\; x \in M, \\ v(x') = \min_{\overline{M}} v = 0. \end{cases}$$

Thus, applying the boundary point lemma (cf. [PW, Chapter 2, Section 3, Theorem 8]; [Ta1, Lemma 7.1.7]), we obtain that

$$\frac{\partial v}{\partial \mathbf{n}}(x') < 0.$$

Hence it follows that

$$0 = \frac{\partial v}{\partial \mathbf{n}}(x') + a(x') v(x')^q = \frac{\partial v}{\partial \mathbf{n}}(x') < 0.$$

This contradiction proves that $v > 0$ on ∂M.

Summing up, we have proved that $v > 0$ on $\overline{M}$. □

I-3) By applying the regularity theorem 1.2 for problem (**), we find that

$$u(\lambda) \in C^\infty(\overline{M}).$$

II) Secondly, we prove that there occurs no *secondary bifurcation* along the bifurcation solution curve $(\lambda, u(\lambda))$ of equation $F(\lambda, u) = 0$ starting at $(0, 0)$:

Lemma 2.3. *There exists a constant* $0 < \overline{\lambda}(h) < \infty$ *such that* $F(\lambda, u(\lambda)) = 0$ *and the Fréchet derivative* $F_u(\lambda, u(\lambda))$ *is an algebraic and topological isomorphism for all* $0 < \lambda < \overline{\lambda}(h)$.

Proof. It is known (for example, [Ta1, Theorem 8.4.1]) that the Fréchet derivative $F_u(\lambda, u(\lambda))$ is a Fredholm operator with index *zero*. Hence, in order to prove the bijectivity of $F_u(\lambda, u(\lambda))$, it suffices to show that $F_u(\lambda, u(\lambda))$ is *injective*:

$$\begin{aligned} &\Delta v - \lambda v + phu(\lambda)^{p-1} v = 0 \text{ in } M, \quad \frac{\partial v}{\partial \mathbf{n}} + qau(\lambda)^{q-1} v = 0 \text{ on } \partial M \\ &\Longrightarrow v = 0 \text{ on } \overline{M}. \end{aligned}$$

Indeed, by using the implicit function theorem (cf. [Di, Theorem 10.2.1]), one can find a constant $0 < \overline{\lambda}(h) < \infty$ such that $F(\lambda, u(\lambda)) = 0$ and $F_u(\lambda, u(\lambda))$ is an algebraic and topological isomorphism for all $0 < \lambda < \overline{\lambda}(h)$.

1) In ordr to prove the injectivity of $F_u(\lambda, u(\lambda))$, we need the following:

Claim 2.4. *We define a densely defined, closed linear operator* $\mathfrak{A}(\lambda) : L^2(M) \to L^2(M)$ *as follows.*

(a) *The domain of definition* $D(\mathfrak{A}(\lambda))$ *is the space*

$$D(\mathfrak{A}(\lambda)) = \left\{ v \in H^2(M); \frac{\partial v}{\partial \mathbf{n}} + qau(\lambda)^{q-1} v = 0 \text{ on } \partial M \right\}.$$

(b) $\mathfrak{A}(\lambda) v = \Delta v + phu(\lambda)^{p-1} v$, $v \in D(\mathfrak{A}(\lambda))$.

Then the operator $\mathfrak{A}(\lambda) - \lambda I$ *is positive in* $L^2(M)$ *for* $\lambda > 0$. *More precisely, if* $\mu_1(\lambda)$ *is the first eigenvalue of* $\mathfrak{A}(\lambda) - \lambda I$, *then we have* $\mu_1(\lambda) > 0$ *and*

$$\int_M (\mathfrak{A}(\lambda) - \lambda I) v \cdot v \, dV \geq \mu_1(\lambda) \int_M v^2 \, dV, \quad v \in D(\mathfrak{A}(\lambda)). \tag{2.5}$$

Proof. Let $v_1(\lambda)$ be the eigenfunction of $\mathfrak{A}(\lambda) - \lambda I$ associated with $\mu_1(\lambda)$:

$$(\mathfrak{A}(\lambda) - \lambda I) v_1(\lambda) = \mu_1(\lambda) v_1(\lambda).$$

We remark that $v_1(\lambda) > 0$ on $\overline{M}$. Then we have by Green's formula

$$\begin{aligned} \mu_1(\lambda) \int_M u(\lambda) v_1(\lambda) \, dV &= \int_M (\Delta v_1(\lambda) - \lambda v_1(\lambda) + phu(\lambda)^{p-1} v_1(\lambda)) u(\lambda) \, dV \\ &= \int_M \nabla v_1(\lambda) \cdot \nabla u(\lambda) \, dV - \lambda \int_M u(\lambda) v_1(\lambda) \, dV \\ &\quad + p \int_M hu(\lambda)^p v_1(\lambda) \, dV - \int_{\partial M} \frac{\partial v_1(\lambda)}{\partial \mathbf{n}} u(\lambda) \, d\sigma, \end{aligned}$$

and also

$$-\int_M hu(\lambda)^p v_1(\lambda) \, dV = \int_M (\Delta u(\lambda) - \lambda u(\lambda)) v_1(\lambda) \, dV$$

$$
\begin{aligned}
&= \int_M \nabla u(\lambda) \cdot \nabla v_1(\lambda)\, dV - \lambda \int_M u(\lambda) v_1(\lambda)\, dV \\
&\qquad - \int_{\partial M} \frac{\partial u(\lambda)}{\partial \mathbf{n}} v_1(\lambda)\, d\sigma.
\end{aligned}
$$

But recall that the functions $v_1(\lambda)$ and $u(\lambda)$ satisfy respectively the following boundary conditions:

$$
\begin{aligned}
&\frac{\partial v_1(\lambda)}{\partial \mathbf{n}} + qau(\lambda)^{q-1} v_1(\lambda) = 0 \quad \text{on } \partial M. \\
&\frac{\partial u(\lambda)}{\partial \mathbf{n}} + au(\lambda)^q = 0 \quad \text{on } \partial M.
\end{aligned}
$$

Hence it follows that

$$
\begin{aligned}
&\int_M \nabla v_1(\lambda) \cdot \nabla u(\lambda)\, dV - \lambda \int_M u(\lambda) v_1(\lambda)\, dV \\
&= \mu_1(\lambda) \int_M u(\lambda) v_1(\lambda)\, dV - p \int_M hu(\lambda)^p v_1(\lambda)\, dV - q \int_{\partial M} au(\lambda)^q v_1(\lambda)\, d\sigma,
\end{aligned}
$$

and also

$$
\begin{aligned}
&\int_M \nabla v_1(\lambda) \cdot \nabla u(\lambda)\, dV - \lambda \int_M u(\lambda) v_1(\lambda)\, dV \\
&= - \int_M hu(\lambda)^p v_1(\lambda)\, dV - \int_{\partial M} au(\lambda)^q v_1(\lambda)\, d\sigma.
\end{aligned}
$$

Therefore, we obtain that

$$
\begin{aligned}
&\mu_1(\lambda) \int_M u(\lambda) v_1(\lambda)\, dV \\
&= (p-1) \int_M hu(\lambda)^p v_1(\lambda)\, dV + (q-1) \int_{\partial M} au(\lambda)^q v_1(\lambda)\, d\sigma.
\end{aligned}
$$

This proves that

$$
\begin{aligned}
\mu_1(\lambda) &= \frac{(p-1) \int_M hu(\lambda)^p v_1(\lambda)\, dV + (q-1) \int_{\partial M} au(\lambda)^q v_1(\lambda)\, d\sigma}{\int_M u(\lambda) v_1(\lambda)\, dV} \\
&> 0,
\end{aligned}
$$

since $p > 1$, $q > 1$, and $h \geq 0$ in M and $a \geq 0$ on ∂M.

2) Now let v be an arbitrary function in $C^{2+\theta}(\overline{M})$ such that $F_u(\lambda, u(\lambda))v = 0$. Then we have $v \in D(\mathfrak{A}(\lambda))$ and $(\mathfrak{A}(\lambda) - \lambda I)v = 0$. By estimate (2.5), this implies that $v = 0$ on $\overline{M}$.

The proof of Lemma 2.3 is complete. $\square$

By virtue of Lemma 2.3, one can extend the above bifurcation curve $(\lambda, u(\lambda))$ to all $0 < \lambda < \overline{\lambda}(h)$. Then we have the following:

Lemma 2.5. *The solution $u(\lambda)$ is differentiable with respect to λ for all $0 < \lambda < \overline{\lambda}(h)$, and is monotone increasing; more precisely, we have for all $0 < \lambda < \overline{\lambda}(h)$*

$$u'(\lambda) > 0 \quad \text{on } \overline{M}.$$

Our situation may be represented schematically as in Figure 2.

Proof. 1) First, since the Fréchet derivative $F_u(\lambda, u(\lambda))$ is an algebraic and topological isomorphism for all $0 < \lambda < \overline{\lambda}(h)$, it follows from an application of the implicit function theorem that the solution $u(\lambda)$ is differentiable with respect to λ.

2) Next we show that, for all $0 < \lambda < \overline{\lambda}(h)$, the derivative $u'(\lambda)$ is nonnegative on $\overline{M}$:

$$u'(\lambda) \geq 0 \quad \text{on } \overline{M}.$$

By differentiating problem $(**)$ with respect to λ, we obtain that

$$\begin{cases} \Delta u'(\lambda) - \lambda u'(\lambda) + phu(\lambda)^{p-1}u'(\lambda) - u(\lambda) = 0 & \text{in } M, \\ \frac{\partial u'(\lambda)}{\partial \mathbf{n}} + qau(\lambda)^{q-1}u'(\lambda) = 0 & \text{on } \partial M. \end{cases}$$

This implies that

$$\begin{cases} u'(\lambda) \in D(\mathfrak{A}(\lambda)), \\ (\mathfrak{A}(\lambda) - \lambda I)u'(\lambda) = u(\lambda). \end{cases} \tag{2.6}$$

Now we introduce a linear operator

$$\mathcal{A}(\lambda) : C(\overline{M}) \to C(\overline{M})$$

as follows.

(a) The domain of definition $D(\mathcal{A}(\lambda))$ is the space

$$D(\mathcal{A}(\lambda)) = \left\{ v \in C(\overline{M}); \Delta v \in C(\overline{M}), \frac{\partial v}{\partial \mathbf{n}} + qau(\lambda)^{q-1}v = 0 \text{ on } \partial M \right\}.$$

(b) $\mathcal{A}(\lambda)v = (-\Delta - phu(\lambda)^{p-1})v$, $v \in D(\mathcal{A}(\lambda))$.

Then it follows from an application of the existence theorem of Feller semigroups due to Taira [Ta2, Theorem 3.16 and Theorem 1.3] that:

(α) The resolvent $(\alpha I - \mathcal{A}(\lambda))^{-1}$ is nonnegative on the space $C(\overline{M})$ for all $\alpha > 0$.

But we remark that:

(β) The operator $\mathfrak{A}(\lambda)$ is an extension of the operator $-\mathcal{A}(\lambda)$.

(γ) The point λ belongs to the resolvent set of $\mathfrak{A}(\lambda)$.

Therefore, we obtain the following:

Claim 2.6. *The resolvent $(\mathfrak{A}(\lambda) - \lambda I)^{-1}$ is nonnegative on the space $C(\overline{M})$, for all $0 < \lambda < \overline{\lambda}(h)$.*

By formula (2.6), this claim proves that

$$u'(\lambda) = (\mathfrak{A}(\lambda) - \lambda I)^{-1}u(\lambda) \geq 0 \quad \text{on } \overline{M},$$

since $u(\lambda) \in C^\infty(\overline{M})$ and $u(\lambda) > 0$ on $\overline{M}$.

3) Finally we show that, for all $0 < \lambda < \overline{\lambda}(h)$, the function $u'(\lambda)$ is strictly positive on $\overline{M}$:

$$u'(\lambda) > 0 \quad \text{on } \overline{M}.$$

3-a) First assume to the contrary that there exists a point $x_0 \in M$ such that

$$u'(\lambda)(x_0) = \min_{\overline{M}} u'(\lambda) = 0.$$

Then we have

$$\begin{aligned} 0 < u(\lambda)(x_0) &= (\Delta - \lambda)u'(\lambda)(x_0) + ph(x_0)\left(u(\lambda)(x_0)\right)^{p-1} u'(\lambda)(x_0) \\ &= \Delta u'(\lambda)(x_0) \leq 0. \end{aligned}$$

This contradiction proves that $u'(\lambda) > 0$ in M.

3-b) Next assume to the contrary that there exists a point $x' \in \partial M$ such that

$$u'(\lambda)(x') = \min_{\overline{M}} u'(\lambda) = 0.$$

Thus, applying the boundary point lemma to the function $u'(\lambda)$, we obtain that

$$\frac{\partial u'(\lambda)}{\partial \mathbf{n}}(x') < 0.$$

Hence it follows that

$$0 = \frac{\partial u'(\lambda)}{\partial \mathbf{n}}(x') + qa(x')\left(u(\lambda)(x')\right)^{q-1} u'(\lambda)(x') = \frac{\partial u'(\lambda)}{\partial \mathbf{n}}(x') < 0.$$

This contradiction proves that $u'(\lambda) > 0$ on ∂M.

Summing up, we have proved that $u'(\lambda) > 0$ on $\overline{M}$. $\square$

3. Proof of Theorem 2 -(1)-

Sections 3-6 are devoted to the characterization of the critical value $\overline{\lambda}(h)$, that is, the proof of formula (0.1):

$$\begin{aligned} \overline{\lambda}(h) &= \widetilde{\lambda}_1(\mathcal{M}_0(h)) \\ &= \min\{\lambda_1(\mathcal{M}_1(h)), \cdots, \lambda_1(\mathcal{M}_\ell(h)), \\ &\qquad \mu_1(\mathcal{M}_{\ell+1}(h)), \cdots, \mu_1(\mathcal{M}_N(h))\}. \end{aligned} \tag{0.1}$$

We begin with the following:

Proposition 3.1. *For all $0 < \lambda < \overline{\lambda}(h)$, we have*

$$\lambda < \lambda_1(\mathcal{M}_i(h)), \quad 1 \leq i \leq \ell,$$

and

$$\lambda < \mu_1(\mathcal{M}_j(h)), \quad \ell + 1 \le j \le N.$$

In particular, we have the inequality

$$\overline{\lambda}(h) \le \widetilde{\lambda}_1(\mathcal{M}_0(h)). \tag{0.2}$$

Proof. 1) First we consider the Dirichlet eigenvalue problem in each connected component $\mathcal{M}_i(h)$, $1 \le i \le \ell$:

$$\begin{cases} \Delta\varphi = \lambda\,\varphi & \text{in } \mathcal{M}_i(h), \\ \varphi = 0 & \text{on } \partial\mathcal{M}_i(h). \end{cases} \tag{D_i}$$

Let $\lambda_1(\mathcal{M}_i(h))$ be the first eigenvalue of problem (D_i) with eigenfunction φ:

$$\begin{cases} \Delta\varphi = \lambda_1(\mathcal{M}_i(h))\,\varphi & \text{in } \mathcal{M}_i(h), \\ \varphi = 0 & \text{on } \partial\mathcal{M}_i(h). \end{cases}$$

Here we remark that $\varphi > 0$ in $\mathcal{M}_i(h)$. If we let

$$\varphi^*(x) = \begin{cases} \varphi(x) & x \in \mathcal{M}_i(h), \\ 0 & x \in M \setminus \mathcal{M}_i(h), \end{cases}$$

then it follows that

$$\varphi^* \in H^1(M),$$

and

$$\varphi^* \ge 0 \quad \text{in } M.$$

Now let $u(\lambda) \in C^\infty(\overline{M})$ be a solution of problem $(**)$. Then it follows from an application of Green's formula that

$$\begin{aligned} 0 &= \int_M (\Delta u(\lambda) - \lambda u(\lambda) + h u(\lambda)^p)\,\varphi^*\,dV \\ &= \int_{\mathcal{M}_i(h)} \nabla u(\lambda) \cdot \nabla\varphi\,dV - \lambda \int_{\mathcal{M}_i(h)} u(\lambda)\,\varphi\,dV \\ &\quad + \int_{\mathcal{M}_i(h)} h u(\lambda)^p \varphi\,dV - \int_{\partial\mathcal{M}_i(h)} \frac{\partial u(\lambda)}{\partial \mathbf{n}}\,\varphi\,d\sigma \\ &= \int_{\mathcal{M}_i(h)} \nabla u(\lambda) \cdot \nabla\varphi\,dV - \lambda \int_{\mathcal{M}_i(h)} u(\lambda)\,\varphi\,dV, \end{aligned}$$

since $h = 0$ in $\mathcal{M}_i(h)$ and $\varphi = 0$ on $\partial\mathcal{M}_i(h)$. Hence we obtain that

$$\int_{\mathcal{M}_i(h)} \nabla u(\lambda) \cdot \nabla\varphi\,dV = \lambda \int_{\mathcal{M}_i(h)} u(\lambda)\,\varphi\,dV. \tag{3.1}$$

Similarly, it follows that

$$
\begin{aligned}
0 &= \int_{\mathcal{M}_i(h)} (\Delta\varphi - \lambda_1(\mathcal{M}_i(h))\,\varphi)\, u(\lambda)\, dV \\
&= \int_{\mathcal{M}_i(h)} \nabla u(\lambda)\cdot\nabla\varphi\, dV - \int_{\partial\mathcal{M}_i(h)} \frac{\partial\varphi}{\partial\mathbf{n}}\, u(\lambda)\, d\sigma \\
&\quad - \lambda_1(\mathcal{M}_i(h)) \int_{\mathcal{M}_i(h)} \varphi u(\lambda)\, dV.
\end{aligned} \tag{3.2}
$$

Thus, combining formulas (3.1) and (3.2), we obtain that

$$
\begin{aligned}
&\lambda \int_{\mathcal{M}_i(h)} u(\lambda)\,\varphi\, dV \\
&= \lambda_1(\mathcal{M}_i(h)) \int_{\mathcal{M}_i(h)} u(\lambda)\,\varphi\, dV + \int_{\partial\mathcal{M}_i(h)} \frac{\partial\varphi}{\partial\mathbf{n}}\, u(\lambda)\, d\sigma.
\end{aligned}
$$

But, it follows from an application of the boundary point lemma that

$$
\frac{\partial\varphi}{\partial\mathbf{n}} < 0 \quad \text{on } \partial\mathcal{M}_i(h).
$$

Indeed, it suffices to note that

$$
\begin{cases}
\Delta\varphi = \lambda_1(\mathcal{M}_i(h))\,\varphi > 0 & \text{in } \mathcal{M}_i(h), \\
\varphi > 0 & \text{in } \mathcal{M}_i(h), \\
\varphi = 0 & \text{on } \partial\mathcal{M}_i(h).
\end{cases}
$$

We also recall that

$$
u(\lambda) > 0 \quad \text{on } \overline{M}.
$$

Therefore, we find that

$$
(\lambda_1(\mathcal{M}_i(h)) - \lambda) \int_{\mathcal{M}_i(h)} u(\lambda)\,\varphi\, dV = -\int_{\partial\mathcal{M}_i(h)} \frac{\partial\varphi}{\partial\mathbf{n}}\, u(\lambda)\, d\sigma > 0,
$$

so that

$$
\lambda < \lambda_1(\mathcal{M}_i(h)), \quad 1 \le i \le \ell.
$$

This proves that

$$
\overline{\lambda}(h) \le \lambda_1(\mathcal{M}_i(h)) \quad \text{for } 1 \le i \le \ell. \tag{3.3}
$$

2) Next we consider the Dirichlet-Neumann eigenvalue problem in each connected component $\mathcal{M}_j(h)$, $\ell + 1 \le j \le N$:

$$
\begin{cases}
\Delta\varphi = \mu\,\varphi & \text{in } \mathcal{M}_j(h), \\
\varphi = 0 & \text{on } \partial\mathcal{M}_j(h)\setminus\mathcal{S}_j, \\
\frac{\partial\varphi}{\partial\mathbf{n}} = 0 & \text{on } \mathcal{S}_j.
\end{cases} \tag{M_j}
$$

Let $\mu_1(\mathcal{M}_j(h))$ be the first eigenvalue of problem (M_j) with eigenfunction ψ:

$$\begin{cases} \Delta\psi = \mu_1(\mathcal{M}_j(h))\,\psi & \text{in } \mathcal{M}_j(h), \\ \psi = 0 & \text{on } \partial\mathcal{M}_j(h)\backslash\mathcal{S}_j, \\ \frac{\partial\psi}{\partial\mathbf{n}} = 0 & \text{on } \mathcal{S}_j. \end{cases}$$

Here we remark that $\psi > 0$ in $\mathcal{M}_j(h)$. If we let

$$\psi^*(x) = \begin{cases} \psi(x) & x \in \mathcal{M}_j(h), \\ 0 & x \in M\backslash\mathcal{M}_j(h), \end{cases}$$

then it follows that

$$\psi^* \in H^1(M),$$

and

$$\psi^* \geq 0 \quad \text{in } M.$$

Now let $u(\lambda) \in C^\infty(\overline{M})$ be a solution of problem $(**)$. We remark that

$$\frac{\partial u(\lambda)}{\partial\mathbf{n}} = 0 \quad \text{on } \mathcal{S}_j,$$

since we have, by condition (α), $a = 0$ on $\mathcal{S}_j$. Then it follows from an application of Green's formula that

$$\begin{aligned} 0 &= \int_M (\Delta u(\lambda) - \lambda u(\lambda) + hu(\lambda)^p)\,\psi^*\,dV \\ &= \int_{\mathcal{M}_j(h)} \nabla u(\lambda)\cdot\nabla\psi\,dV - \lambda\int_{\mathcal{M}_j(h)} u(\lambda)\,\psi\,dV \\ &\quad + \int_{\mathcal{M}_j(h)} hu(\lambda)^p\psi\,dV - \int_{\partial\mathcal{M}_j(h)} \frac{\partial u(\lambda)}{\partial\mathbf{n}}\,\psi\,d\sigma \\ &= \int_{\mathcal{M}_j(h)} \nabla u(\lambda)\cdot\nabla\psi\,dV - \lambda\int_{\mathcal{M}_j(h)} u(\lambda)\,\psi\,dV. \end{aligned}$$

Indeed, since we have

$$\begin{aligned} &\psi = 0 \quad \text{on } \partial\mathcal{M}_j(h)\backslash\mathcal{S}_j, \\ &\frac{\partial u(\lambda)}{\partial\mathbf{n}} = 0 \quad \text{on } \mathcal{S}_j, \end{aligned}$$

it follows that

$$\int_{\partial\mathcal{M}_j(h)} \frac{\partial u(\lambda)}{\partial\mathbf{n}}\,\psi\,d\sigma = \int_{\mathcal{S}_j} \frac{\partial u(\lambda)}{\partial\mathbf{n}}\,\psi\,d\sigma + \int_{\partial\mathcal{M}_j(h)\backslash\mathcal{S}_j} \frac{\partial u(\lambda)}{\partial\mathbf{n}}\,\psi\,d\sigma = 0.$$

Hence we obtain that

$$\int_{\mathcal{M}_j(h)} \nabla u(\lambda)\cdot\nabla\psi\,dV = \lambda\int_{\mathcal{M}_j(h)} u(\lambda)\,\psi\,dV. \tag{3.4}$$

Similarly, it follows that

$$
\begin{aligned}
0 &= \int_{\mathcal{M}_j(h)} (\Delta\psi - \mu_1(\mathcal{M}_j(h))\,\psi)\,u(\lambda)\,dV \\
&= \int_{\mathcal{M}_j(h)} \nabla u(\lambda)\cdot\nabla\psi\,dV - \int_{\partial\mathcal{M}_j(h)} \frac{\partial\psi}{\partial\mathbf{n}}\,u(\lambda)\,d\sigma \\
&\quad - \mu_1(\mathcal{M}_j(h)) \int_{\mathcal{M}_j(h)} \psi u(\lambda)\,dV \\
&= \int_{\mathcal{M}_j(h)} \nabla u(\lambda)\cdot\nabla\psi\,dV - \int_{\partial\mathcal{M}_j(h)\setminus\mathcal{S}_j} \frac{\partial\psi}{\partial\mathbf{n}}\,u(\lambda)\,d\sigma \\
&\quad - \mu_1(\mathcal{M}_j(h)) \int_{\mathcal{M}_j(h)} \psi u(\lambda)\,dV,
\end{aligned}
\tag{3.5}
$$

since $\partial\psi/\partial\mathbf{n} = 0$ on $\mathcal{S}_j$.

Thus, combining formulas (3.4) and (3.5), we obtain that

$$
\begin{aligned}
&\lambda \int_{\mathcal{M}_j(h)} u(\lambda)\,\psi\,dV \\
&= \mu_1(\mathcal{M}_j(h)) \int_{\mathcal{M}_j(h)} u(\lambda)\,\psi\,dV + \int_{\partial\mathcal{M}_j(h)\setminus\mathcal{S}_j} \frac{\partial\psi}{\partial\mathbf{n}}\,u(\lambda)\,d\sigma.
\end{aligned}
$$

But, it follows from an application of the boundary point lemma that

$$
\frac{\partial\psi}{\partial\mathbf{n}} < 0 \quad \text{on } \partial\mathcal{M}_j(h)\setminus\mathcal{S}_j.
$$

Indeed, it suffices to note that

$$
\begin{cases}
\Delta\psi = \mu_1(\mathcal{M}j_(h))\,\psi > 0 & \text{in } \mathcal{M}_j(h), \\
\psi > 0 & \text{in } \mathcal{M}_j(h), \\
\psi = 0 & \text{on } \partial\mathcal{M}_j(h)\setminus\mathcal{S}_j.
\end{cases}
$$

Therefore, we find that

$$
(\mu_1(\mathcal{M}_j(h)) - \lambda) \int_{\mathcal{M}_j(h)} u(\lambda)\,\psi\,dV = -\int_{\partial\mathcal{M}_j(h)\setminus\mathcal{S}_j} \frac{\partial\psi}{\partial\mathbf{n}}\,u(\lambda)\,d\sigma > 0,
$$

so that

$$
\lambda < \mu_1(\mathcal{M}_j(h)), \quad \ell+1 \le j \le N.
$$

This proves that

$$
\overline{\lambda}(h) \le \mu_1(\mathcal{M}_j(h)) \quad \text{for } \ell+1 \le j \le N. \tag{3.6}
$$

The desired inequality (0.2) follows by combining inequalities (3.3) and (3.6).

The proof of Proposition 3.1 is complete. □

4. Properties of positive solutions -(1)-

In Section 4 through Section 6, we shall prove the reverse inequality (0.3) of inequality (0.2).

First we begin with the following:

Lemma 4.1. *One can construct a function $h^* \in C^1(M)$ having the following properties:*

(a) $0 < h^*(x) \leq h(x)$ *for* $x \in \mathcal{M}_+(h)$.

(b) For each $\varepsilon > 0$, *there exists a constant* $C > 0$, *depending on* $\sup_M |\nabla h|$, *such that*

$$\sup_{\mathcal{M}_+(h)} \left| \frac{\nabla h^*}{(h^*)^{1-\varepsilon}} \right| \leq \frac{C}{\varepsilon^2}. \tag{4.1}$$

Proof. We let

$$d(x) = \mathrm{dist}\,(x, \partial\mathcal{M}_+(h)), \quad x \in \mathcal{M}_+(h),$$

and define a function $h^*(x)$ by the following:

$$h^*(x) = \begin{cases} e^{-\frac{1}{h(x)}} & \text{for } x \in \mathcal{M}_+(h),\, d(x) < \delta, \\ h(x) & \text{for } x \in \mathcal{M}_+(h),\, d(x) > 2\delta, \\ 0 & \text{for } x \in M \backslash \mathcal{M}_+(h), \end{cases}$$

and

$$0 < h^*(x) \leq h(x) \quad \text{for } x \in \mathcal{M}_+(h),\, \delta \leq d(x) \leq 2\delta.$$

Then it is easy to verify that the function h^* enjoys properties (a) and (b).

Indeed, property (b) may be verified as follows. Since we have

$$\nabla h^*(x) = e^{-\frac{1}{h(x)}} \frac{\nabla h(x)}{h(x)^2} \quad \text{for } x \in \mathcal{M}_+(h),\, 0 < d(x) < \delta,$$

it follows that

$$\begin{aligned} \left| \frac{\nabla h^*(x)}{h^*(x)^{1-\varepsilon}} \right| &= |\nabla h(x)|\, e^{-\frac{\varepsilon}{h(x)}} \frac{1}{h(x)^2} \\ &\leq \sup_M |\nabla h| \cdot \sup_M \left(\frac{e^{-\frac{\varepsilon}{h}}}{h^2} \right) \\ &\leq \sup_M |\nabla h| \left(\frac{4e^{-2}}{\varepsilon^2} \right). \quad \square \end{aligned}$$

The next lemma will play an essential role in the proof of inequality (0.3) (cf. the proof of Proposition 6.1).

Lemma 4.2. *If $u(\lambda)$ is a positive solution of problem $(**)$ with $0 < \lambda < \overline{\lambda}(h)$, then, we have for any $\varepsilon > 0$*

$$h^*\, u(\lambda)^{p-1-\varepsilon} \in L^\infty(\mathcal{M}_+(h)),$$

and

$$\sup_{\mathcal{M}_+(h)} \left(h^*\, u(\lambda)^{p-1-\varepsilon}\right) \leq C(\varepsilon, \lambda), \tag{4.2}$$

with a constant $C(\varepsilon, \lambda) > 0$. Moreover, if λ is finite, then so is the constant $C(\varepsilon, \lambda)$.

Proof. 1) Let $\mu_1(\lambda)$ and $v_1(\lambda)$ be the first eigenvalue and associated eigenfunction of the Fréchet derivative $F_u(\lambda, u(\lambda))$, that is,

$$\begin{cases} (\Delta - \lambda + phu(\lambda)^{p-1})v_1(\lambda) = \mu_1(\lambda)v_1(\lambda) & \text{in } M, \\ \frac{\partial v_1(\lambda)}{\partial \mathbf{n}} + qau(\lambda)^{q-1}v_1(\lambda) = 0 & \text{on } \partial M. \end{cases}$$

We recall that $\mu_1(\lambda) > 0$ and $v_1(\lambda) > 0$ on $\overline{M}$. Furthermore, by *Rayleigh's theorem* (cf. [Ag, Chapter 10], [Ch, Chapter I]), we know that the first eigenvalue $\mu_1(\lambda)$ can be characterized by the following formula:

$$\begin{aligned} \mu_1(\lambda) \int_M \varphi^2\, dV \leq \int_M |\nabla\varphi|^2\, dV - \lambda \int_M \varphi^2\, dV \\ + p \int_M hu(\lambda)^{p-1}\, \varphi^2\, dV + q \int_{\partial M} au(\lambda)^{q-1}\, \varphi^2\, d\sigma. \end{aligned} \tag{4.3}$$

Now, we take

$$\varphi = (h^*)^s\, u(\lambda)^k, \quad s > 0,\ k > p,\ k > q,$$

where the constants s, k will be chosen later on. Then we have

$$\nabla\varphi = s(h^*)^{s-1}\, u(\lambda)^k\, \nabla h^* + k(h^*)^s\, u(\lambda)^{k-1}\, \nabla u,$$

and so

$$\begin{aligned} |\nabla\varphi|^2 = s^2(h^*)^{2s-2}\, u(\lambda)^{2k}\, |\nabla h^*|^2 + k^2(h^*)^{2s}\, u(\lambda)^{2k-2}\, |\nabla u|^2 \\ + 2sk(h^*)^{2s-1}\, u(\lambda)^{2k-1}\, \nabla h^* \cdot \nabla u. \end{aligned}$$

Hence, we can write inequality (4.3) in the following form:

$$\begin{aligned} &\mu_1(\lambda) \int_M (h^*)^{2s}\, u(\lambda)^{2k}\, dV \\ &\leq s^2 \int_M (h^*)^{2s-2}\, u(\lambda)^{2k}\, |\nabla h^*|^2\, dV \\ &\quad + 2sk \int_M (h^*)^{2s-1}\, u(\lambda)^{2k-1}\, \nabla h^* \cdot \nabla u\, dV \\ &\quad + k^2 \int_M (h^*)^{2s}\, u(\lambda)^{2k-2}\, |\nabla u|^2\, dV - \lambda \int_M (h^*)^{2s}\, u(\lambda)^{2k}\, dV \\ &\quad + p \int_M h\, (h^*)^{2s}\, u(\lambda)^{p-1+2k}\, dV + q \int_{\partial M} a\, (h^*)^{2s}\, u(\lambda)^{q-1+2k}\, d\sigma. \end{aligned} \tag{4.4}$$

2) Next we show that the second term on the right-hand side of inequality (4.4) can be written as

$$
\begin{aligned}
(4.5)\qquad & 2sk \int_M (h^*)^{2s-1}\, u(\lambda)^{2k-1}\, \nabla h^* \cdot \nabla u \, dV \\
&= \lambda k \int_M (h^*)^{2s}\, u(\lambda)^{2k}\, dV - k(2k-1) \int_M (h^*)^{2s}\, u(\lambda)^{2k-2}\, |\nabla u|^2 \, dV \\
&\quad - k \int_M h\, (h^*)^{2s}\, u(\lambda)^{p-1+2k}\, dV - k \int_{\partial M} a\, (h^*)^{2s}\, u(\lambda)^{q-1+2k}\, d\sigma.
\end{aligned}
$$

If we let

$$\psi(\lambda) = k u(\lambda)^{2k-1}\, (h^*)^{2s},$$

then we obtain that

$$
\begin{aligned}
\nabla \psi(\lambda) &= 2sk u(\lambda)^{2k-1}\, (h^*)^{2s-1}\, \nabla h^* \\
&\quad + k(2k-1) u(\lambda)^{2k-2}\, (h^*)^{2s}\, \nabla u(\lambda).
\end{aligned}
$$

Recall that the function $u(\lambda)$ is a solution of problem $(**)$. Hence we have by Green's formula

$$
\begin{aligned}
0 &= \int_M \left(\Delta u(\lambda) - \lambda u(\lambda) + h u(\lambda)^p \right) \psi(\lambda)\, dV \\
&= \int_M \nabla u(\lambda) \cdot \nabla \psi(\lambda)\, dV - \lambda \int_M u(\lambda)\, \psi(\lambda)\, dV \\
&\quad + \int_M h u(\lambda)^p\, \psi(\lambda)\, dV - \int_{\partial M} \frac{\partial u(\lambda)}{\partial \mathbf{n}}\, \psi(\lambda)\, d\sigma \\
&= \int_M \nabla u(\lambda) \left(2sk\, u(\lambda)^{2k-1}\, (h^*)^{2s-1}\, \nabla h^* \right. \\
&\quad \left. + k(2k-1)\, u(\lambda)^{2k-2}\, (h^*)^{2s}\, \nabla u(\lambda) \right) dV \\
&\quad + k \int_{\partial M} a\, u(\lambda)^{2k-1+q}\, (h^*)^{2s}\, d\sigma.
\end{aligned}
$$

This proves formula (4.5).

Thus, carrying formula (4.5) into inequality (4.4), we find that

$$
\begin{aligned}
& \mu_1(\lambda) \int_M (h^*)^{2s}\, u(\lambda)^{2k}\, dV \\
&\leq s^2 \int_M (h^*)^{2s-2}\, u(\lambda)^{2k}\, |\nabla h^*|^2 \, dV + \lambda(k-1) \int_M (h^*)^{2s}\, u(\lambda)^{2k}\, dV \\
&\quad - k(k-1) \int_M (h^*)^{2s}\, u(\lambda)^{2k-2}\, |\nabla u|^2 \, dV \\
&\quad + (p-k) \int_M h\, (h^*)^{2s}\, u(\lambda)^{p-1+2k}\, dV \\
&\quad + (q-k) \int_{\partial M} a\, (h^*)^{2s}\, u(\lambda)^{q-1+2k}\, d\sigma.
\end{aligned}
$$

Furthermore, it follows that

$$
\begin{aligned}
& s^2 \int_M (h^*)^{2s-2}\, u(\lambda)^{2k}\, |\nabla h^*|^2\, dV \\
& \quad + \lambda(k-1) \int_M (h^*)^{2s}\, u(\lambda)^{2k}\, dV \\
& = s^2 \int_{\mathcal{M}_+(h)} (h^*)^{2s-2}\, u(\lambda)^{2k}\, |\nabla h^*|^2\, dV \\
& \quad + \lambda(k-1) \int_{\mathcal{M}_+(h)} (h^*)^{2s}\, u(\lambda)^{2k}\, dV \\
& \geq k(k-1) \int_{\mathcal{M}_+(h)} (h^*)^{2s}\, u(\lambda)^{2k-2}\, |\nabla u|^2\, dV \\
& \quad + (k-p) \int_{\mathcal{M}_+(h)} h\, (h^*)^{2s}\, u(\lambda)^{p-1+2k}\, dV + \mu_1(\lambda) \int_{\mathcal{M}_+(h)} (h^*)^{2s}\, u(\lambda)^{2k}\, dV \\
& \quad + (k-q) \int_{\partial M} a\, (h^*)^{2s}\, u(\lambda)^{q-1+2k}\, d\sigma \\
& \geq k(k-1) \int_{\mathcal{M}_+(h)} (h^*)^{2s}\, u(\lambda)^{2k-2}\, |\nabla u|^2\, dV \\
& \quad + (k-p) \int_{\mathcal{M}_+(h)} (h^*)^{2s+1}\, u(\lambda)^{p+2k-1}\, dV,
\end{aligned}
\tag{4.6}
$$

since $k > p$, $k > q$, $\mu_1(\lambda) > 0$ and also we have, by part (a) of Lemma 4.1, $0 < h^*(x) \leq h(x)$ for $x \in \mathcal{M}_+(h)$.

3) First we show that the first term on the left-hand side of inequality (4.6) can be estimated as follows:

$$
\begin{aligned}
& s^2 \int_M (h^*)^{2s-2}\, u(\lambda)^{2k}\, |\nabla h^*|^2\, dV \\
& = s^2 \int_{\mathcal{M}_+(h)} (h^*)^{2s-2}\, u(\lambda)^{2k}\, |\nabla h^*|^2\, dV \\
& \leq \int_{\mathcal{M}_+(h)} (h^*)^{1+2s}\, u(\lambda)^{2k+p-1}\, dV + |\mathcal{M}_+(h)|\,.
\end{aligned}
\tag{4.7}
$$

By inequality (4.1), it follows that

$$
\begin{aligned}
& \int_M (h^*)^{2s-2}\, u(\lambda)^{2k}\, |\nabla h^*|^2\, dV \\
& = \int_{\mathcal{M}_+(h)} (h^*)^{2(s-\varepsilon)}\, u(\lambda)^{2k}\, \frac{|\nabla h^*|^2}{|(h^*)^{1-\varepsilon}|^2}\, dV \\
& \leq \frac{C^2}{\varepsilon^4} \int_{\mathcal{M}_+(h)} (h^*)^{2(s-\varepsilon)}\, u(\lambda)^{2k}\, dV.
\end{aligned}
$$

If we choose the constant s as

$$
s = \frac{1+2\varepsilon}{p-1}\, k + \varepsilon,
\tag{4.8}
$$

then we obtain from Hölder's inequality that

$$\begin{aligned}
&\int_M (h^*)^{2s-2}\, u(\lambda)^{2k}\, |\nabla h^*|^2\, dV \\
&\le \frac{C^2}{\varepsilon^4} \left(\int_{\mathcal{M}_+(h)} \left((h^*)^{2(s-\varepsilon)}\, u(\lambda)^{2k} \right)^{\frac{2k+p-1}{2k}} dV \right)^{\frac{2k}{2k+p-1}} \\
&\quad \times \left(\int_{\mathcal{M}_+(h)} dV \right)^{\frac{p-1}{2k+p-1}} \\
&= \frac{C^2}{\varepsilon^4} \left(\int_{\mathcal{M}_+(h)} (h^*)^{1+2s}\, u(\lambda)^{2k+p-1}\, dV \right)^{\frac{2k}{2k+p-1}} |\mathcal{M}_+(h)|^{\frac{p-1}{2k+p-1}} .
\end{aligned}$$

Hence, it follows from an application of Young's inequality that

$$\begin{aligned}
&s^2 \int_{\mathcal{M}_+(h)} (h^*)^{2s-2}\, u(\lambda)^{2k}\, |\nabla h^*|^2\, dV \\
&\le \left(\int_{\mathcal{M}_+(h)} (h^*)^{1+2s}\, u(\lambda)^{2k+p-1}\, dV \right)^{\frac{2k}{2k+p-1}} \\
&\quad \times \left(|\mathcal{M}_+(h)| \left(\frac{C^2 s^2}{\varepsilon^4} \right)^{\frac{2k+p-1}{p-1}} \right)^{\frac{p-1}{2k+p-1}} \\
&\le \left(\frac{2k}{2k+p-1} \right) \int_{\mathcal{M}_+(h)} (h^*)^{1+2s}\, u(\lambda)^{2k+p-1}\, dV \\
&\qquad + \left(\frac{p-1}{2k+p-1} \right) |\mathcal{M}_+(h)| \left(\frac{C^2 s^2}{\varepsilon^4} \right)^{\frac{2k+p-1}{p-1}} \\
&\le \int_{\mathcal{M}_+(h)} (h^*)^{1+2s}\, u(\lambda)^{2k+p-1}\, dV + |\mathcal{M}_+(h)| \left(\frac{C^2 s^2}{\varepsilon^4} \right)^{\frac{2k+p-1}{p-1}} .
\end{aligned}$$

This proves inequality (4.7).

Next we show that the second term on the left-hand side of inequality (4.6) can be estimated as follows:

$$\begin{aligned}
&\lambda(k-1) \int_M (h^*)^{2s}\, u(\lambda)^{2k}\, dV \\
&= \lambda(k-1) \int_{\mathcal{M}_+(h)} (h^*)^{2s}\, u(\lambda)^{2k}\, dV \\
&\le \int_{\mathcal{M}_+(h)} (h^*)^{1+2s}\, u(\lambda)^{2k+p-1}\, dV \\
&\qquad + \left(\lambda(k-1) \sup_{\mathcal{M}_+(h)} \left| (h^*)^{2\varepsilon} \right| \right)^{\frac{2k+p-1}{p-1}} |\mathcal{M}_+(h)| .
\end{aligned} \tag{4.9}$$

By using Hölder's inequality and Young's inequality as above, we have the following:

$$
\begin{aligned}
&\lambda(k-1) \int_{\mathcal{M}_+(h)} (h^*)^{2s}\, u(\lambda)^{2k}\, dV \\
&= \lambda(k-1) \int_{\mathcal{M}_+(h)} (h^*)^{2(s-\varepsilon)}\, (h^*)^{2\varepsilon}\, u(\lambda)^{2k}\, dV \\
&\leq \lambda(k-1) \sup_{\mathcal{M}_+(h)} \left|(h^*)^{2\varepsilon}\right| \int_{\mathcal{M}_+(h)} (h^*)^{2(s-\varepsilon)}\, u(\lambda)^{2k}\, dV \\
&\leq \lambda(k-1) \sup_{\mathcal{M}_+(h)} \left|(h^*)^{2\varepsilon}\right| \left(\int_{\mathcal{M}_+(h)} (h^*)^{1+2s}\, u(\lambda)^{2k+p-1}\, dV \right)^{\frac{2k}{2k+p-1}} \\
&\qquad \times |\mathcal{M}_+(h)|^{\frac{p-1}{2k+p-1}} \\
&= \left(\left(\lambda(k-1) \sup_{\mathcal{M}_+(h)} \left|(h^*)^{2\varepsilon}\right| \right)^{\frac{2k+p-1}{p-1}} |\mathcal{M}_+(h)| \right)^{\frac{p-1}{2k+p-1}} \\
&\qquad \times \left(\int_{\mathcal{M}_+(h)} (h^*)^{1+2s}\, u(\lambda)^{2k+p-1}\, dV \right)^{\frac{2k}{2k+p-1}} \\
&\leq \left(\frac{2k}{2k+p-1} \right) \int_{\mathcal{M}_+(h)} (h^*)^{1+2s}\, u(\lambda)^{2k+p-1}\, dV \\
&\qquad + \left(\frac{p-1}{2k+p-1} \right) \left(\lambda(k-1) \sup_{\mathcal{M}_+(h)} \left|(h^*)^{2\varepsilon}\right| \right)^{\frac{2k+p-1}{p-1}} |\mathcal{M}_+(h)| \\
&\leq \int_{\mathcal{M}_+(h)} (h^*)^{1+2s}\, u(\lambda)^{2k+p-1}\, dV \\
&\qquad + \left(\lambda(k-1) \sup_{\mathcal{M}_+(h)} \left|(h^*)^{2\varepsilon}\right| \right)^{\frac{2k+p-1}{p-1}} |\mathcal{M}_+(h)|\,.
\end{aligned}
$$

This proves inequality (4.9).

Therefore, combining inequalities (4.6), (4.7) and (4.9), we obtain that

$$
\begin{aligned}
&k(k-1) \int_{\mathcal{M}_+(h)} (h^*)^{2s}\, u(\lambda)^{2k-2}\, |\nabla u|^2\, dV \\
&\qquad + (k-p) \int_{\mathcal{M}_+(h)} (h^*)^{1+2s}\, u(\lambda)^{p-1+2k}\, dV \\
&\leq 2 \int_{\mathcal{M}_+(h)} (h^*)^{1+2s}\, u(\lambda)^{2k+p-1}\, dV + \left(\frac{C^2 s^2}{\varepsilon^4} \right)^{\frac{2k+p-1}{p-1}} |\mathcal{M}_+(h)| \\
&\qquad + \left(\lambda(k-1) \sup_{\mathcal{M}_+(h)} \left|(h^*)^{2\varepsilon}\right| \right)^{\frac{2k+p-1}{p-1}} |\mathcal{M}_+(h)|\,.
\end{aligned}
$$

In particular, this proves that

$$
\begin{aligned}
(4.10) \qquad & (k-p) \int_{\mathcal{M}_+(h)} (h^*)^{1+2s}\, u(\lambda)^{p-1+2k}\, dV \\
& \leq 2 \int_{\mathcal{M}_+(h)} (h^*)^{1+2s}\, u(\lambda)^{2k+p-1}\, dV + \left(\frac{C^2 s^2}{\varepsilon^4}\right)^{\frac{2k+p-1}{p-1}} |\mathcal{M}_+(h)| \\
& \quad + \left(\lambda(k-1) \sup_{\mathcal{M}_+(h)} \left|(h^*)^{2\varepsilon}\right|\right)^{\frac{2k+p-1}{p-1}} |\mathcal{M}_+(h)|\,.
\end{aligned}
$$

If we take the constant k so large that the first term on the right-hand side of inequality (4.10) may be absorbed into the left-hand side, for example, if we take k so that

$$k - p > 3,$$

then it follows that

$$
\begin{aligned}
& \int_{\mathcal{M}_+(h)} (h^*)^{1+2s}\, u(\lambda)^{p-1+2k}\, dV \\
& \leq \left(\left(\frac{C^2 s^2}{\varepsilon^4}\right)^{\frac{2k+p-1}{p-1}} + \left(\lambda(k-1) \sup_{\mathcal{M}_+(h)} \left|(h^*)^{2\varepsilon}\right|\right)^{\frac{2k+p-1}{p-1}}\right) |\mathcal{M}_+(h)|\,.
\end{aligned}
$$

But, by formula (4.8), we find that the constant s is of order k. Thus one can find a constant $C' > 0$ such that

$$
(4.11) \qquad \int_{\mathcal{M}_+(h)} (h^*)^{1+2s}\, u(\lambda)^{p-1+2k}\, dV \leq \left(C'(1+\lambda)\, \frac{k^2}{\varepsilon^4}\right)^{\frac{2k+p-1}{p-1}} |\mathcal{M}_+(h)|\,.
$$

Here we remark that the constant $C' > 0$ depends on the quantities $\sup_{\mathcal{M}_+(h)} |h^*|$ and $\sup_{\mathcal{M}_+(h)} |\nabla h|$.

On the other hand, since we have by formula (4.8)

$$1 + 2s = \frac{1+2\varepsilon}{p-1}\,(p + 2k - 1),$$

we can write the left-hand side of inequality (4.11) as

$$
\int_{\mathcal{M}_+(h)} (h^*)^{1+2s}\, u(\lambda)^{p-1+2k}\, dV = \int_{\mathcal{M}_+(h)} \left((h^*)^{\frac{1+2\varepsilon}{p-1}}\, u(\lambda)\right)^{p+2k-1} dV.
$$

Therefore we obtain from inequality (4.11) that

$$
(4.12) \qquad \int_{\mathcal{M}_+(h)} \left((h^*)^{\frac{1+2\varepsilon}{p-1}}\, u(\lambda)\right)^{p+2k-1} dV \leq \left(C'(1+\lambda)\, \frac{k^2}{\varepsilon^4}\right)^{\frac{2k+p-1}{p-1}} |\mathcal{M}_+(h)|\,.
$$

4) We let

$$\omega(\lambda) = (h^*)^{\frac{s}{k}}\, u(\lambda),$$

where (cf. formula (4.8))

$$\frac{s}{k} = \frac{1+2\varepsilon}{p-1} + \frac{\varepsilon}{k}.$$

Then we have

$$\nabla\omega(\lambda)^k = s(h^*)^{s-1}\, u(\lambda)^k\, \nabla h^* + k(h^*)^s\, u(\lambda)^{k-1}\, \nabla u,$$

and so

$$\begin{aligned}|\nabla\omega(\lambda)^k|^2 &= s^2(h^*)^{2s-2}\, u(\lambda)^{2k}\, |\nabla h^*|^2 + k^2(h^*)^{2s}\, u(\lambda)^{2k-2}\, |\nabla u|^2 \\ &\quad + 2sk(h^*)^{2s-1}\, u(\lambda)^{2k-1}\, \nabla h^* \cdot \nabla u \\ &\le 2\left(s^2(h^*)^{2s-2}\, u(\lambda)^{2k}\, |\nabla h^*|^2 + k^2(h^*)^{2s}\, u(\lambda)^{2k-2}\, |\nabla u|^2\right).\end{aligned}$$

Hence it follows that

$$\begin{aligned}\int_{\mathcal{M}_+(h)} |\nabla\omega(\lambda)^k|^2\, dV &\le 2s^2 \int_{\mathcal{M}_+(h)} (h^*)^{2s-2}\, u(\lambda)^{2k}\, |\nabla h^*|^2\, dV \\ &\quad + 2k^2 \int_{\mathcal{M}_+(h)} (h^*)^{2s}\, u(\lambda)^{2k-2}\, |\nabla u|^2\, dV.\end{aligned} \tag{4.13}$$

On the other hand, we find from inequality (4.6) that

$$\begin{aligned}&\int_{\mathcal{M}_+(h)} (h^*)^{2s}\, u(\lambda)^{2k-2}\, |\nabla u|^2\, dV \\ &\le \frac{s^2}{k(k-1)} \int_{\mathcal{M}_+(h)} (h^*)^{2s-2}\, u(\lambda)^{2k}\, |\nabla h^*|^2\, dV \\ &\quad + \frac{\lambda}{k} \int_{\mathcal{M}_+(h)} (h^*)^{2s}\, u(\lambda)^{2k}\, dV.\end{aligned} \tag{4.14}$$

Thus, combining inequalities (4.13) and (4.14), we obtain that

$$\begin{aligned}&\int_{\mathcal{M}_+(h)} |\nabla\omega(\lambda)^k|^2\, dV \\ &\le 2s^2 \left(\frac{2k-1}{k-1}\right) \int_{\mathcal{M}_+(h)} (h^*)^{2s-2}\, u(\lambda)^{2k}\, |\nabla h^*|^2\, dV \\ &\quad + 2k\lambda \int_{\mathcal{M}_+(h)} (h^*)^{2s}\, u(\lambda)^{2k}\, dV.\end{aligned} \tag{4.15}$$

But we recall that the two terms on the right-hand side of inequality (4.15) can be estimated respectively as follows:

$$\int_{\mathcal{M}_+(h)} (h^*)^{2s-2}\, u(\lambda)^{2k}\, |\nabla h^*|^2\, dV \le \frac{C^2}{\varepsilon^4} \int_{\mathcal{M}_+(h)} (h^*)^{2(s-\varepsilon)}\, u(\lambda)^{2k}\, dV.$$

$$\int_{\mathcal{M}_+(h)} (h^*)^{2s}\, u(\lambda)^{2k}\, dV \leq \sup_{\mathcal{M}_+(h)} \left|(h^*)^{2\varepsilon}\right| \int_{\mathcal{M}_+(h)} (h^*)^{2(s-\varepsilon)}\, u(\lambda)^{2k}\, dV.$$

Therefore, carrying these inequalities into the right-hand side of inequality (4.15), we obtain that

$$\begin{aligned}
&\int_{\mathcal{M}_+(h)} |\nabla\omega(\lambda)^k|^2\, dV \\
&\leq \left(2s^2 \left(\frac{2k-1}{k-1}\right) \frac{C^2}{\varepsilon^4} + 2\lambda k \sup_{\mathcal{M}_+(h)} \left|(h^*)^{2\varepsilon}\right|\right) \int_{\mathcal{M}_+(h)} (h^*)^{2(s-\varepsilon)}\, u(\lambda)^{2k}\, dV.
\end{aligned}$$

But, by formula (4.8), we find that the constant s is of order k. Thus, if we take the constant k so large that

$$\frac{2k-1}{k-1} < 3,$$

then one can find a constant $C'' > 0$ such that

$$\begin{aligned}
&\int_{\mathcal{M}_+(h)} |\nabla\omega(\lambda)^k|^2\, dV \\
&\leq \left(6s^2 \frac{C^2}{\varepsilon^4} + 2\lambda k \sup_{\mathcal{M}_+(h)} \left|(h^*)^{2\varepsilon}\right|\right) \int_{\mathcal{M}_+(h)} (h^*)^{2(s-\varepsilon)}\, u(\lambda)^{2k}\, dV \\
&\leq C''(1+\lambda)\, \frac{k^2}{\varepsilon^4} \int_{\mathcal{M}_+(h)} (h^*)^{2(s-\varepsilon)}\, u(\lambda)^{2k}\, dV.
\end{aligned} \tag{4.16}$$

Here we remark that the constant $C'' > 0$ depends on the quantities $\sup_{\mathcal{M}_+(h)} |h^*|$ and $\sup_{\mathcal{M}_+(h)} |\nabla h|$.

5) We make use of the Sobolev imbedding theorem (cf. [Ad, Theorem 5.4]):

$$\left(\int_{\mathcal{M}_+(h)} \varphi^{\frac{2n}{n-2}}\, dV\right)^{\frac{n-2}{2n}} \leq C(n) \left(\int_{\mathcal{M}_+(h)} |\nabla\varphi|^2\, dV\right)^{\frac{1}{2}}, \quad \varphi \in H^1(\mathcal{M}_+(h)). \tag{4.17}$$

Here the constant $C(n) > 0$ depends on the dimension $n \geq 3$.

Now, applying inequality (4.17) to the function

$$\omega(\lambda)^k = (h^*)^s\, u(\lambda)^k$$

and then using inequality (4.16), we obtain that

$$\begin{aligned}
&\left(\int_{\mathcal{M}_+(h)} \left((h^*)^{\frac{s}{k}}\, u(\lambda)\right)^{\frac{2kn}{n-2}}\, dV\right)^{\frac{n-2}{n}} \\
&= \left(\int_{\mathcal{M}_+(h)} \left((h^*)^{s}\, u(\lambda)^k\right)^{\frac{2n}{n-2}}\, dV\right)^{\frac{n-2}{n}}
\end{aligned}$$

$$
\begin{aligned}
&\leq C(n)^2 \int_{\mathcal{M}_+(h)} |\nabla \omega(\lambda)^k|^2 \, dV \\
&\leq C(n)^2 C''(1+\lambda) \, \frac{k^2}{\varepsilon^4} \int_{\mathcal{M}_+(h)} (h^*)^{2(s-\varepsilon)} \, u(\lambda)^{2k} \, dV \\
&= C(n)^2 \, C''(1+\lambda) \, \frac{k^2}{\varepsilon^4} \int_{\mathcal{M}_+(h)} \left((h^*)^{\frac{s-\varepsilon}{k}} \, u(\lambda) \right)^{2k} \, dV,
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
(4.18) \qquad & \left\| (h^*)^{s/k} \, u(\lambda) \right\|_{L^{2kn/(n-2)}(\mathcal{M}_+(h))} \\
&\leq C(\lambda)^{1/2k} \left(\frac{k^2}{\varepsilon^4} \right)^{1/2k} \left\| (h^*)^{(s-\varepsilon)/k} \, u(\lambda) \right\|_{L^{2k}(\mathcal{M}_+(h))},
\end{aligned}
$$

where

$$
C(\lambda) = C(n)^2 \, C''(1+\lambda).
$$

We let

$$
\chi = \frac{n}{n-2} > 1,
$$

and for a sufficiently large positive integer m

$$
k = \chi^m.
$$

Then we have

$$
\begin{aligned}
\frac{s}{k} &= \frac{1+2\varepsilon}{p-1} + \frac{\varepsilon}{k} = \frac{1+2\varepsilon}{p-1} + \frac{\varepsilon}{\chi^m}, \\
\frac{2kn}{n-2} &= 2\chi^m \, \frac{n}{n-2} = 2\chi^{m+1}, \\
\frac{s-\varepsilon}{k} &= \frac{1+2\varepsilon}{p-1}.
\end{aligned}
$$

Thus we can write inequality (4.18) as

$$
\begin{aligned}
(4.19) \qquad & \left\| (h^*)^{\frac{1+2\varepsilon}{p-1} + \frac{\varepsilon}{\chi^m}} \, u(\lambda) \right\|_{L^{2\chi^{m+1}}(\mathcal{M}_+(h))} \\
&\leq C(\lambda)^{\frac{1}{2\chi^m}} \left(\frac{\chi^{2m}}{\varepsilon^4} \right)^{\frac{1}{2\chi^m}} \left\| (h^*)^{\frac{1+2\varepsilon}{p-1}} \, u(\lambda) \right\|_{L^{2\chi^m}(\mathcal{M}_+(h))}.
\end{aligned}
$$

Furthermore, if we let

$$
\varepsilon_0 = \varepsilon, \quad \varepsilon_1 = \left(1 + \frac{p-1}{2\chi^m} \right) \varepsilon_0,
$$

then it follows that

$$\frac{1+2\varepsilon_0}{p-1} + \frac{\varepsilon_0}{\chi^m} = \frac{1+2\left(1+\frac{p-1}{2\chi^m}\right)\varepsilon_0}{p-1} = \frac{1+2\varepsilon_1}{p-1}.$$

Thus we can rewrite inequality (4.19) in the following form:

$$\begin{aligned}(4.20)_1 \qquad & \left\| (h^*)^{\frac{1+2\varepsilon_1}{p-1}}\, u(\lambda) \right\|_{L^{2\chi^{m+1}}(\mathcal{M}_+(h))} \\ & \leq C(\lambda)^{\frac{1}{2}\chi^{-m}} \left(\frac{\chi^{2m}}{\varepsilon_0^4}\right)^{\frac{1}{2}\chi^{-m}} \left\| (h^*)^{\frac{1+2\varepsilon_0}{p-1}}\, u(\lambda) \right\|_{L^{2\chi^{m}}(\mathcal{M}_+(h))}.\end{aligned}$$

6) By the same procedure as above (replacing χ^m by χ^{m+1}), we have the inequality:

$$\begin{aligned}(4.21) \qquad & \left\| (h^*)^{\frac{1+2\varepsilon_1}{p-1}+\frac{\varepsilon_1}{\chi^{m+1}}}\, u(\lambda) \right\|_{L^{2\chi^{m+2}}(\mathcal{M}_+(h))} \\ & \leq C(\lambda)^{\frac{1}{2}\chi^{-(m+1)}} \left(\frac{\chi^{2(m+1)}}{\varepsilon_1^4}\right)^{\frac{1}{2}\chi^{-(m+1)}} \left\| (h^*)^{\frac{1+2\varepsilon_1}{p-1}}\, u(\lambda) \right\|_{L^{2\chi^{m+1}}(\mathcal{M}_+(h))}.\end{aligned}$$

But we remark that

$$\varepsilon_1 = \left(1+\frac{p-1}{2\chi^m}\right)\varepsilon_0 > \varepsilon_0.$$

Thus, combining inequality (4.21) with inequality $(4.20)_1$, we obtain that

$$\begin{aligned}(4.22) \qquad & \left\| (h^*)^{\frac{1+2\varepsilon_1}{p-1}+\frac{\varepsilon_1}{\chi^{m+1}}}\, u(\lambda) \right\|_{L^{2\chi^{m+2}}(\mathcal{M}_+(h))} \\ & \leq C(\lambda)^{\frac{1}{2}\chi^{-(m+1)}} \left(\frac{\chi^{2(m+1)}}{\varepsilon_0^4}\right)^{\frac{1}{2}\chi^{-(m+1)}} \left\| (h^*)^{\frac{1+2\varepsilon_1}{p-1}}\, u(\lambda) \right\|_{L^{2\chi^{m+1}}(\mathcal{M}_+(h))} \\ & \leq C(\lambda)^{\frac{1}{2}\chi^{-(m+1)}} C(\lambda)^{\frac{1}{2}\chi^{-m}} \left(\frac{\chi^{2(m+1)}}{\varepsilon_0^4}\right)^{\frac{1}{2}\chi^{-(m+1)}} \left(\frac{\chi^{2m}}{\varepsilon_0^4}\right)^{\frac{1}{2}\chi^{-m}} \\ & \quad \times \left\| (h^*)^{\frac{1+2\varepsilon_0}{p-1}}\, u(\lambda) \right\|_{L^{2\chi^{m}}(\mathcal{M}_+(h))} \\ & \leq C(\lambda)^{\frac{1}{2}\left(\chi^{-(m+1)}+\chi^{-m}\right)} \chi^{\left((m+1)\chi^{-(m+1)}+m\chi^{-m}\right)} \left(\frac{1}{\varepsilon_0}\right)^{2\left(\chi^{-(m+1)}+\chi^{-m}\right)} \\ & \quad \times \left\| (h^*)^{\frac{1+2\varepsilon_0}{p-1}}\, u(\lambda) \right\|_{L^{2\chi^{m}}(\mathcal{M}_+(h))}.\end{aligned}$$

If we let

$$\varepsilon_2 = \left(1+\frac{p-1}{2\chi^{m+1}}\right)\varepsilon_1,$$

then we can write inequality (4.22) as

$$(4.20)_2 \qquad \left\| (h^*)^{\frac{1+2\varepsilon_2}{p-1}} u(\lambda) \right\|_{L^{2\chi^{m+2}}(\mathcal{M}_+(h))}$$
$$\leq C(\lambda)^{\frac{1}{2}\left(\chi^{-m}+\chi^{-(m+1)}\right)} \chi^{\left(m\chi^{-m}+(m+1)\chi^{-(m+1)}\right)} \left(\frac{1}{\varepsilon_0}\right)^{2\left(\chi^{-m}+\chi^{-(m+1)}\right)}$$
$$\times \left\| (h^*)^{\frac{1+2\varepsilon_0}{p-1}} u(\lambda) \right\|_{L^{2\chi^{m}}(\mathcal{M}_+(h))}.$$

Continuing this procedure, we have after N steps

$$(4.20)_{N+1} \qquad \left\| (h^*)^{\frac{1+2\varepsilon_{N+1}}{p-1}} u(\lambda) \right\|_{L^{2\chi^{m+N+1}}(\mathcal{M}_+(h))}$$
$$\leq C(\lambda)^{\frac{1}{2}\left(\sum_{i=0}^N \chi^{-(m+i)}\right)} \chi^{\left(\sum_{i=0}^N (m+i)\chi^{-(m+i)}\right)} \left(\frac{1}{\varepsilon_0}\right)^{2\left(\sum_{i=0}^N \chi^{-(m+i)}\right)}$$
$$\times \left\| (h^*)^{\frac{1+2\varepsilon_0}{p-1}} u(\lambda) \right\|_{L^{2\chi^{m}}(\mathcal{M}_+(h))}.$$

But we remark that:

$$\sum_{i=0}^{\infty} \frac{1}{\chi^{m+i}} = \frac{1}{\chi^m} \sum_{i=0}^{\infty} \frac{1}{\chi^i} = \frac{1}{\chi^m} \left(\frac{\chi}{\chi-1}\right) = \frac{n}{2} \frac{1}{\chi^m},$$
$$\sum_{i=0}^{\infty} (m+i) \frac{1}{\chi^{m+i}} = \frac{1}{\chi} \sum_{i=0}^{\infty} (m+i) \frac{1}{\chi^{m+i-1}} = \frac{n(2m+n-2)}{4} \frac{1}{\chi^m}.$$

Thus it follows from inequality $(4.20)_{N+1}$ that

$$(4.23) \qquad \left\| (h^*)^{\frac{1+2\varepsilon_{N+1}}{p-1}} u(\lambda) \right\|_{L^{2\chi^{m+N+1}}(\mathcal{M}_+(h))}$$
$$\leq C(\lambda)^{\frac{n}{4\chi^m}} \chi^{\frac{n(2m+n-2)}{4\chi^m}} \left(\frac{1}{\varepsilon_0}\right)^{\frac{n}{\chi^m}} \left\| (h^*)^{\frac{1+2\varepsilon_0}{p-1}} u(\lambda) \right\|_{L^{2\chi^{m}}(\mathcal{M}_+(h))}.$$

Furthermore we find that

$$\begin{aligned}
\varepsilon_{N+1} &= \left(1 + \frac{p-1}{2\chi^{m+N}}\right) \varepsilon_N \\
&= \left(1 + \frac{p-1}{2\chi^{m+N}}\right) \left(1 + \frac{p-1}{2\chi^{m+N-1}}\right) \varepsilon_{N-1} \\
&\vdots \\
&= \left(\prod_{i=0}^{N} \left(1 + \frac{p-1}{2\chi^{m+i}}\right)\right) \varepsilon_0,
\end{aligned}$$

and that the limit (infinite product)

$$\lim_{N\to\infty} \varepsilon_{N+1} = \left(\prod_{i=0}^{\infty} \left(1 + \frac{p-1}{2\chi^{m+i}} \right) \right) \varepsilon_0$$

exists, since $\chi > 1$.

Therefore, letting $N \to \infty$ in inequality (4.23), we obtain that

$$\begin{aligned} \sup_{\mathcal{M}_+(h)} \left| (h^*)^{\frac{1+2\sigma\varepsilon_0}{p-1}} u(\lambda) \right| \\ \leq C(\lambda)^{\frac{n}{4\chi^m}} \chi^{\frac{n(2m+n-2)}{4\chi^m}} \left(\frac{1}{\varepsilon_0} \right)^{\frac{n}{\chi^m}} \left\| (h^*)^{\frac{1+2\varepsilon_0}{p-1}} u(\lambda) \right\|_{L^{2\chi^m}(\mathcal{M}_+(h))}, \end{aligned} \tag{4.24}$$

where

$$\sigma = \prod_{i=0}^{\infty} \left(1 + \frac{p-1}{2\chi^{m+i}} \right) > 1.$$

7) On the other hand, by Hölder's inequality, it follows that

$$\begin{aligned} \left\| (h^*)^{\frac{1+2\varepsilon_0}{p-1}} u(\lambda) \right\|_{L^{2\chi^m}(\mathcal{M}_+(h))} \leq \left\| (h^*)^{\frac{1+2\varepsilon_0}{p-1}} u(\lambda) \right\|_{L^{p+2k-1}(\mathcal{M}_+(h))} \\ \times |\mathcal{M}_+(h)|^{\frac{1}{2\chi^m} - \frac{1}{p+2k-1}}. \end{aligned} \tag{4.25}$$

Furthermore, we have by inequality (4.12) with $\varepsilon = \varepsilon_0$

$$\left\| (h^*)^{\frac{1+2\varepsilon_0}{p-1}} u(\lambda) \right\|_{L^{p+2k-1}(\mathcal{M}_+(h))} \leq C(\lambda)'^{\frac{1}{p-1}} \left(\frac{k^2}{\varepsilon_0^4} \right)^{\frac{1}{p-1}} |\mathcal{M}_+(h)|^{\frac{1}{p+2k-1}}, \tag{4.26}$$

where

$$C(\lambda)' = C' (1 + \lambda).$$

Here we recall that the constant C' depends on the quantities $\sup_{\mathcal{M}_+(h)} |h^*|$ and $\sup_{\mathcal{M}_+(h)} |\nabla h|$.

Therefore, combining inequalities (4.25) and (4.26), we get the following inequality:

$$\left\| (h^*)^{\frac{1+2\varepsilon_0}{p-1}} u(\lambda) \right\|_{L^{2\chi^m}(\mathcal{M}_+(h))} \leq C(\lambda)'^{\frac{1}{p-1}} \chi^{\frac{2m}{p-1}} \left(\frac{1}{\varepsilon_0} \right)^{\frac{4}{p-1}} |\mathcal{M}_+(h)|^{\frac{1}{2\chi^m}}.$$

Carrying this inequality into the right-hand side of inequality (4.24), we obtain that

$$\begin{aligned} \sup_{\mathcal{M}_+(h)} \left| (h^*)^{\frac{1+2\sigma\varepsilon_0}{p-1}} u(\lambda) \right| \leq C(\lambda)^{\frac{n}{4\chi^m}} \chi^{\frac{n(2m+n-2)}{4\chi^m}} C(\lambda)'^{\frac{1}{p-1}} \chi^{\frac{2m}{p-1}} \left(\frac{1}{\varepsilon_0} \right)^{\frac{n}{\chi^m} + \frac{4}{p-1}} \\ \times |\mathcal{M}_+(h)|^{\frac{1}{2\chi^m}}. \end{aligned}$$

Summing up, we have proved that there exists a constant $C(\lambda)'' > 0$ such that for each $\varepsilon_0 > 0$

$$\sup_{\mathcal{M}_+(h)} \left| (h^*)^{\frac{1+2\sigma\varepsilon_0}{p-1}} u(\lambda) \right| \leq C(\lambda)'' \, \varepsilon_0{}^{-\mu}, \tag{4.27}$$

where

$$\mu = \frac{n}{\chi^m} + \frac{4}{p-1}.$$

It is easy to see that inequality (4.27) is equivalent to inequality (4.2). Moreover, we find that if λ is finite, then so is the constant $C(\lambda)''$.

The proof of Lemma 4.2 is now complete. □

5. Properties of positive solutions -(2)-

The next lemma asserts that the solution $u(\lambda)$ “blows up” at the critical value $\overline{\lambda}(h)$:

Lemma 5.1. *If $u(\lambda) \in C^\infty(\overline{M})$, $0 < \lambda < \overline{\lambda}(h)$, is a solution of problem $(**)$, then we have:*

$$\lim_{\lambda \to \overline{\lambda}(h)} \| u(\lambda) \|_{L^2(M)} = +\infty. \tag{5.1}$$

Proof. Assume to the contrary that there exists a constant $C > 0$ such that

$$\int_M u(\lambda)^2 \, dV \leq C \quad \text{for all } 0 < \lambda < \overline{\lambda}(h). \tag{5.2}$$

Then, using Green's formula, we obtain that

$$\begin{aligned}
0 &= -\int_M \left(\Delta u(\lambda) - \lambda u(\lambda) + h u(\lambda)^p \right) u(\lambda) \, dV \\
&= -\int_M |\nabla u(\lambda)|^2 \, dV + \int_{\partial M} \frac{\partial u(\lambda)}{\partial \mathbf{n}} \cdot u(\lambda) \, d\sigma + \lambda \int_M u(\lambda)^2 \, dV \\
&\quad - \int_M h \, u(\lambda)^{p+1} \, dV \\
&= -\int_M |\nabla u(\lambda)|^2 \, dV - \int_{\partial M} a \, u(\lambda)^{q+1} \, d\sigma + \lambda \int_M u(\lambda)^2 \, dV \\
&\quad - \int_M h u(\lambda)^{p+1} \, dV.
\end{aligned}$$

Thus it follows that

$$\int_M |\nabla u(\lambda)|^2 \, dV + \int_M h \, u(\lambda)^{p+1} \, dV + \int_{\partial M} a \, u(\lambda)^{q+1} \, d\sigma = \lambda \int_M u(\lambda)^2 \, dV.$$

In particular, this proves that

$$\int_M |\nabla u(\lambda)|^2 \, dV \leq \lambda \int_M u(\lambda)^2 \, dV. \tag{5.3}$$

On the other hand, applying Sobolev's inequality (4.17) to the function $u(\lambda)$, we obtain that

$$\left(\int_M u(\lambda)^{\frac{2n}{n-2}}\,dV\right)^{\frac{n-2}{n}} \leq C(n)^2 \int_M |\nabla u(\lambda)|^2\,dV. \tag{5.4}$$

Thus, combining inequalities (5.4) and (5.3), we obtain that

$$\left(\int_M u(\lambda)^{\frac{2n}{n-2}}\,dV\right)^{\frac{n-2}{n}} \leq \lambda\, C(n)^2 \int_M u(\lambda)^2\,dV,$$

or equivalently

$$\|u(\lambda)\|_{L^{\frac{2n}{n-2}}(M)} \leq C(\lambda)^{\frac{1}{2}}\,\|u(\lambda)\|_{L^2(M)}\,, \tag{5.5}$$

where

$$C(\lambda) = \lambda\, C(n)^2.$$

Furthermore, if we let

$$\chi = \frac{n}{n-2} > 1,$$

then we can write inequality (5.5) in the following form:

$$\|u(\lambda)\|_{L^{2\chi}(M)} \leq C(\lambda)^{\frac{1}{2}}\,\|u(\lambda)\|_{L^2(M)}\,. \tag*{(5.6)$_1$}$$

Continuing this procedure as in the proof of Lemma 4.2, we have after N steps

$$\begin{aligned}\|u(\lambda)\|_{L^{2\chi^{N+1}}(M)} &\leq C(\lambda)^{\frac{1}{2}(\Sigma_{i=0}^N \chi^{-i})}\,\|u(\lambda)\|_{L^2(M)} \\ &\leq C(\lambda)^{\frac{n}{4}}\,\|u(\lambda)\|_{L^2(M)}\,.\end{aligned} \tag*{(5.6)$_{N+1}$}$$

Therefore, letting $N \to \infty$ in inequality $(5.6)_{N+1}$, we obtain that

$$\sup_M |u(\lambda)| \leq C(\lambda)^{\frac{n}{4}}\,\chi^{\frac{n(n-2)}{4}}\,\|u(\lambda)\|_{L^2(M)} \quad \text{for all } 0 < \lambda < \overline{\lambda}(h). \tag{5.7}$$

By inequalities (5.2) and (5.3), it follows that for all $0 < \lambda < \overline{\lambda}(h)$

$$\int_M u(\lambda)^2\,dV \leq C, \tag{5.8a}$$

$$\int_M |\nabla u(\lambda)|^2\,dV \leq \overline{\lambda}(h)\,C. \tag{5.8b}$$

This proves that the functions $u(\lambda)$ are bounded in the Sobolev space $H^1(M)$, for all $0 < \lambda < \overline{\lambda}(h)$.

But we remark the following:

(a) Rellich's theorem tells us that the injection of $H^1(M)$ into $L^2(M)$ is compact (or completely continuous) if the dimension n is greater than 3.

(b) It is well known that the unit ball in the Hilbert space is *sequentially weakly compact*. Therefore, by inequalities (5.8a) and (5.8b), we can find a sequence $\{\lambda_n\}$ and a function $u(\overline{\lambda}(h)) \in H^1(M)$ such that

$$\lambda_n \longrightarrow \overline{\lambda}(h), \tag{5.9a}$$

and

$$u(\lambda_n) \longrightarrow u(\overline{\lambda}(h)) \quad \text{strongly in } L^2(M), \tag{5.9b}$$

$$\nabla u(\lambda_n) \longrightarrow \nabla u(\overline{\lambda}(h)) \quad \text{weakly in } L^2(M). \tag{5.9c}$$

On the other hand, by combining inequalities (5.2) and (5.7), we obtain that

$$\sup_M |u(\lambda)| \leq C^{\frac{1}{2}} \left(\overline{\lambda}(h)\, C(n)^2\right)^{\frac{n}{4}} \chi^{\frac{n(n-2)}{4}} \quad \text{for all } 0 < \lambda < \overline{\lambda}(h).$$

But, Lemma 2.5 tells us that the solution $u(\lambda)$ is monotone increasing for all $0 < \lambda < \overline{\lambda}(h)$. Thus, we find that the finite limit

$$u(\overline{\lambda}(h))(x) = \lim_{\lambda_n \to \overline{\lambda}(h)} u(\lambda_n)(x) \tag{5.10}$$

exists for each point x of $\overline{M}$.

Now, since $u(\lambda_n)$ is a solution of problem $(**)$, it follows that for all $\psi \in H^1(M)$

$$\begin{aligned} &\int_M \nabla u(\lambda_n) \cdot \nabla \psi \, dV + \int_M h\, u(\lambda_n)^p \, \psi \, dV - \lambda \int_M h\, u(\lambda_n)\, \psi \, dV \\ &\quad + \int_{\partial M} a\, u(\lambda_n)^q \, \psi \, d\sigma = 0. \end{aligned} \tag{5.11}$$

But we have the following:

(1) By assertion (5.9), it follows that

$$\int_M u(\lambda_n)\, \psi \, dV \longrightarrow \int_M u(\overline{\lambda}(h))\, \psi \, dV,$$

and

$$\int_M \nabla u(\lambda_n) \cdot \nabla \psi \, dV \longrightarrow \int_M \nabla u(\overline{\lambda}(h)) \cdot \nabla \psi \, dV.$$

(2) By assertion (5.10), it follows from an application of the Lebesgue monotone convergence theorem that

$$\int_M h\, u(\lambda_n)^p \, \psi \, dV \longrightarrow \int_M h\, u(\overline{\lambda}(h))^p \, \psi \, dV,$$

$$\int_{\partial M} a\, u(\lambda_n)^q \, \psi \, d\sigma \longrightarrow \int_{\partial M} a\, u(\overline{\lambda}(h))^q \, \psi \, d\sigma.$$

By passing to the limit in formula (5.11), we obtain that the function $u(\overline{\lambda}(h))$ satisfies, for all $\psi \in H^1(M)$,

$$\int_M \nabla u(\overline{\lambda}(h)) \cdot \nabla \psi \, dV + \int_M h\, u(\overline{\lambda}(h))^p \, \psi \, dV - \overline{\lambda}(h) \int_M h\, u(\overline{\lambda}(h)) \, \psi \, dV + \int_{\partial M} a\, u(\overline{\lambda}(h))^q \, \psi \, d\sigma = 0.$$

This proves that the function $u(\overline{\lambda}(h)) \in H^1(M)$ is a weak solution of problem (∗∗).

Thus, we have by the regularity theorem 1.2

$$u(\overline{\lambda}(h)) \in C^\infty(\overline{M}).$$

Furthermore, we recall that the solution $u(\lambda)$ is strictly positive on $\overline{M}$ and is monotone increasing for all $0 < \lambda < \overline{\lambda}(h)$. Thus it follows that

$$u(\overline{\lambda}(h)) > 0 \quad \text{on } \overline{M}.$$

Finally, it is easy to see that the Fréchet derivative $F_u(\overline{\lambda}(h), u(\overline{\lambda}(h)))$ is an algebraic and topological isomorphism. Indeed, if $\mu_1(\overline{\lambda}(h))$ is the first eigenvalue of the Fréchet derivative $F_u(\overline{\lambda}(h), u(\overline{\lambda}(h)))$ with eigenfunction $v_1(\overline{\lambda}(h))$, then, arguing as in the proof of Lemma 2.3, we obtain that

$$\begin{aligned} &\mu_1(\overline{\lambda}(h)) \\ &= \frac{(p-1)\int_M hu(\overline{\lambda}(h))^p v_1(\overline{\lambda}(h))\, dV + (q-1)\int_{\partial M} au(\overline{\lambda}(h))^q v_1(\overline{\lambda}(h))\, d\sigma}{\int_M u(\overline{\lambda}(h)) v_1(\overline{\lambda}(h))\, dV} \\ &> 0. \end{aligned}$$

Therefore, by virtue of the implicit function theorem, one can extend the bifurcation curve $(\lambda, u(\lambda))$ beyond the point $(\overline{\lambda}(h), u(\overline{\lambda}(h)))$. This contradicts the definition of $\overline{\lambda}(h)$.

The proof of Lemma 5.1 is complete. □

6. Proof of Theorem 2 -(2)-

The next proposition proves the inequality

$$\widetilde{\lambda}_1(\mathcal{M}_0(h)) \leq \overline{\lambda}(h), \tag{0.3}$$

which completes the proof of Theorem 2.

Proposition 6.1. *The critical value $\overline{\lambda}(h)$ is an eigenvalue of either the Dirichlet problem (D_i) or the Dirichlet-Neumann problem (M_j).*

Proof. 1) Let $u(\lambda) \in C^\infty(\overline{M})$, $0 < \lambda < \overline{\lambda}(h)$, be a solution of the problem (∗∗), and let

$$\omega(\lambda) = \frac{u(\lambda)}{\|u(\lambda)\|_{L^2(M)}}.$$

Then it follows that

$$\begin{aligned}&\Delta\omega(\lambda) - \lambda\omega(\lambda) + hu(\lambda)^{p-1}\omega(\lambda)\\ &= \frac{1}{\|u(\lambda)\|_{L^2(M)}}\left(\Delta u(\lambda) - \lambda u(\lambda) + hu(\lambda)^p\right) = 0 \quad \text{in } M,\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial\omega(\lambda)}{\partial\mathbf{n}} + au(\lambda)^{q-1}\omega(\lambda)\\ &= \frac{1}{\|u(\lambda)\|_{L^2(M)}}\left(\frac{\partial u(\lambda)}{\partial\mathbf{n}} + au(\lambda)^q\right) = 0 \quad \text{on } \partial M.\end{aligned}$$

Hence we have by Green's formula

$$\begin{aligned}&\int_M hu(\lambda)^{p-1}\omega(\lambda)\cdot\omega(\lambda)\,dV\\ &= \int_M (-\Delta\omega(\lambda) + \lambda\omega(\lambda))\,\omega(\lambda)\,dV\\ &= -\int_M |\nabla\omega(\lambda)|^2\,dV + \int_{\partial M}\frac{\partial\omega(\lambda)}{\partial\mathbf{n}}\cdot\omega(\lambda)\,d\sigma + \lambda\int_M \omega(\lambda)^2\,dV\\ &= -\int_M |\nabla\omega(\lambda)|^2\,dV - \int_{\partial M} a\,u(\lambda)^{q-1}\omega(\lambda)^2\,d\sigma + \lambda\int_M \omega(\lambda)^2\,dV.\end{aligned}$$

This proves that

$$\begin{aligned}&\int_M |\nabla\omega(\lambda)|^2\,dV\\ &\leq \int_M |\nabla\omega(\lambda)|^2\,dV + \int_{\partial M} a\,u(\lambda)^{q-1}\omega(\lambda)^2\,d\sigma + \int_M hu(\lambda)^{p-1}\omega(\lambda)^2\,dV\\ &= \lambda\int_M \omega(\lambda)^2\,dV.\end{aligned}\tag{6.1}$$

By inequality (6.1), it follows that for all $0 < \lambda < \overline{\lambda}(h)$

$$\int_M \omega(\lambda)^2\,dV = 1,\tag{6.2a}$$

$$\int_M |\nabla\omega(\lambda)|^2\,dV \leq \lambda\int_M \omega(\lambda)^2\,dV = \lambda \leq \overline{\lambda}(h).\tag{6.2b}$$

Thus, just as in the proof of Lemma 5.1, we can find a sequence $\{\lambda_n\}$ and a function $\omega(\overline{\lambda}(h)) \in H^1(M)$ such that

$$\lambda_n \longrightarrow \overline{\lambda}(h),\tag{6.3a}$$

and

$$\omega(\lambda_n) \longrightarrow \omega(\overline{\lambda}(h)) \quad \text{strongly in } L^2(M),\tag{6.3b}$$

$$\nabla\omega(\lambda_n) \longrightarrow \nabla\omega(\overline{\lambda}(h)) \quad \text{weakly in } L^2(M). \tag{6.3c}$$

Furthermore, arguing as in the proof of Lemma 5.1 (cf. inequality (5.7)), we can find a constant $C(\overline{\lambda}(h)) > 0$ such that

$$\sup_M |\omega(\lambda)| \leq C(\overline{\lambda}(h)) \, \|\omega(\lambda)\|_{L^2(M)} = C(\overline{\lambda}(h)) \quad \text{for all } 0 < \lambda < \overline{\lambda}(h). \tag{6.4}$$

Therefore, we obtain from assertions (6.2), (6.3) and (6.4) that the limit function $\omega(\overline{\lambda}(h)) \in H^1(M)$ satisfies the following conditions:

$$\omega(\overline{\lambda}(h)) \geq 0 \quad \text{in } M. \tag{6.5a}$$

$$\int_M \omega(\overline{\lambda}(h))^2 \, dV = 1. \tag{6.5b}$$

$$\int_M |\nabla\omega(\overline{\lambda}(h))|^2 \, dV \leq \overline{\lambda}(h). \tag{6.5c}$$

$$\sup_M \left|\omega(\overline{\lambda}(h))\right| \leq C(\overline{\lambda}(h)). \tag{6.5d}$$

On the other hand, we remark that the functions $\omega(\lambda_n)$ satisfy the equation

$$\Delta\omega(\lambda_n) - \lambda_n \omega(\lambda_n) = 0 \quad \text{in } \mathcal{M}_0(h),$$

since $h \equiv 0$ in $\mathcal{M}_0(h)$. By passing to the limit, we find that the function $\omega(\overline{\lambda}(h))$ is a weak solution of the equation

$$\Delta\omega(\overline{\lambda}(h)) - \overline{\lambda}(h)\omega(\overline{\lambda}(h)) = 0 \quad \text{in } \mathcal{M}_0(h).$$

Hence it follows from an application of the *interior regularity theorem* in linear elliptic theory (cf. [GT, Corollary 8.11]) that

$$\omega(\overline{\lambda}(h)) \in C^\infty(\mathcal{M}_0(h)).$$

Summing up, we have proved that

$$\omega(\overline{\lambda}(h)) \in C^\infty(\mathcal{M}_0(h)) \cap H^1(M) \cap L^\infty(M). \tag{6.6a}$$

$$\Delta\omega(\overline{\lambda}(h)) - \overline{\lambda}(h)\omega(\overline{\lambda}(h)) = 0 \quad \text{in } \mathcal{M}_0(h). \tag{6.6b}$$

2) Next, we shall prove that the function $\omega(\overline{\lambda}(h))$ is an eigenfunction of problem (D_i) or problem (M_j), more precisely, we shall show that one of the following two cases (i) and (ii) holds:

(i) In some connected component $\mathcal{M}_i(h)$, $1 \leq i \leq \ell$, the function $\omega(\overline{\lambda}(h))$ satisfies the conditions:

$$\begin{cases} \Delta\omega(\overline{\lambda}(h)) = \overline{\lambda}(h)\omega(\overline{\lambda}(h)) & \text{in } \mathcal{M}_i(h), \\ \omega(\overline{\lambda}(h)) = 0 & \text{on } \partial\mathcal{M}_i(h), \\ \omega(\overline{\lambda}(h)) > 0 & \text{in } \mathcal{M}_i(h). \end{cases} \tag{D_i}$$

(ii) In some connected component $\mathcal{M}_j(h)$, $\ell+1 \leq j \leq N$, the function $\omega(\overline{\lambda}(h))$ satisfies the conditions:

$$(M_j) \qquad \begin{cases} \Delta\omega(\overline{\lambda}(h)) = \overline{\lambda}(h)\omega(\overline{\lambda}(h)) & \text{in } \mathcal{M}_j(h), \\ \omega(\overline{\lambda}(h)) = 0 & \text{on } \partial\mathcal{M}_j(h)\backslash\mathcal{S}_j, \\ \frac{\partial\omega(\overline{\lambda}(h))}{\partial\mathbf{n}} = 0 & \text{on } \mathcal{S}_j, \\ \omega(\overline{\lambda}(h)) > 0 & \text{in } \mathcal{M}_j(h). \end{cases}$$

2-a) First, by assertion (5.1), we remark that

$$\lim_{\lambda_n \to \overline{\lambda}(h)} \|u(\lambda_n)\|_{L^2(M)} = +\infty. \tag{6.7}$$

But, Lemma 4.2 tells us that, for each $\varepsilon > 0$, there exists a constant $C(\varepsilon, \overline{\lambda}(h)) > 0$ such that

$$(u(\lambda_n)(x))^{p-1-\varepsilon} \leq \frac{C(\varepsilon, \overline{\lambda}(h))}{h^*(x)} \quad \text{for all } x \in \mathcal{M}_+(h). \tag{6.8}$$

Hence it follows from assertion (6.7) and inequality (6.8) that

$$\omega(\overline{\lambda}(h))(x) = \lim_{\lambda_n \to \overline{\lambda}(h)} \omega(\lambda_n)(x) = \lim_{\lambda_n \to \overline{\lambda}(h)} \frac{u(\lambda_n)(x)}{\|u(\lambda_n)\|_{L^2(M)}} = 0 \tag{6.9}$$

for almost every $x \in \mathcal{M}_+(h)$.

Therefore, combining assertions (6.5b) and (6.9), we find that

$$\int_{\mathcal{M}_0(h)} \omega(\overline{\lambda}(h))^2 \, dV = \int_M \omega(\overline{\lambda}(h))^2 \, dV = 1.$$

This proves that, in some connected component $\mathcal{M}_k(h)$, $1 \leq k \leq N$, the function $\omega(\overline{\lambda}(h))$ is strictly positive:

$$\omega(\overline{\lambda}(h)) > 0 \quad \text{in } \mathcal{M}_k(h),$$

since we have

$$\mathcal{M}_0(h) = \left(\cup_{i=1}^{\ell} \mathcal{M}_i(h)\right) \cup \left(\cup_{j=\ell+1}^{N} \mathcal{M}_j(h)\right).$$

2-b) Furthermore, we can prove the following:

Lemma 6.2. *The function $\omega(\overline{\lambda}(h))$ satisfies the boundary conditions*

$$\omega(\overline{\lambda}(h)) = 0 \quad \text{on } \partial\mathcal{M}_0(h) \backslash \cup_{j=\ell+1}^{N} \mathcal{S}_j.$$

$$\frac{\partial\omega(\overline{\lambda}(h))}{\partial\mathbf{n}} = 0 \quad \text{on } \cup_{j=\ell+1}^{N} \mathcal{S}_j.$$

2-c) Assuming Lemma 6.2 for the moment, we shall prove Proposition 6.1.

By Lemma 6.2, we find that the function $\omega(\overline{\lambda}(h))$ is an eigenfunction of problem (D_i) or problem (M_j). This implies that $\overline{\lambda}(h)$ is an eigenvalue of problem (D_i) or problem (M_j). Hence we have, by Rayleigh's theorem,

$$\overline{\lambda}(h) \geq \lambda_1(\mathcal{M}_i(h)) \geq \widetilde{\lambda}_1(\mathcal{M}_0(h))$$

or

$$\overline{\lambda}(h) \geq \mu_1(\mathcal{M}_j(h)) \geq \widetilde{\lambda}_1(\mathcal{M}_0(h)).$$

The proof of Proposition 6.1 (and hence that of Theorem 2) is complete, apart from the proof of Lemma 6.2. $\square$

Proof of Lemma 6.2.

1) First we show that the function $\omega(\overline{\lambda}(h))$ satisfies the Neumann boundary conditions:

$$\frac{\partial \omega(\overline{\lambda}(h))}{\partial \mathbf{n}} = 0 \quad \text{on } \cup_{j=\ell+1}^{N} \mathcal{S}_j.$$

We recall that

$$\omega(\lambda_n) \longrightarrow \omega(\overline{\lambda}(h)) \quad \text{in } L^2(M),$$

and

$$\varDelta \omega(\lambda_n) = \lambda_n \omega(\lambda_n) \longrightarrow \overline{\lambda}(h) \omega(\overline{\lambda}(h)) = \varDelta \omega(\overline{\lambda}(h)) \quad \text{in } L^2(\mathcal{M}_0(h)).$$

Hence it follows from an application of the *trace theorem* (cf. [Ta1, Proposition 8.3.1] with $\sigma = 0$ and $\tau = 0$) that

$$\frac{\partial \omega(\lambda_n)}{\partial \mathbf{n}} \longrightarrow \frac{\partial \omega(\overline{\lambda}(h))}{\partial \mathbf{n}} \quad \text{in } H^{-3/2}(\partial \mathcal{M}_0(h)).$$

But we have by condition (α)

$$\frac{\partial \omega(\lambda_n)}{\partial \mathbf{n}} = \frac{\partial \omega(\lambda_n)}{\partial \mathbf{n}} + a u(\lambda_n)^{q-1} \omega(\lambda_n) = 0 \quad \text{on } \cup_{j=\ell+1}^{N} \mathcal{S}_j.$$

Thus we find that

$$\frac{\partial \omega(\overline{\lambda}(h))}{\partial \mathbf{n}} = 0 \quad \text{on } \cup_{j=\ell+1}^{N} \mathcal{S}_j.$$

2) Next we show that the function $\omega(\overline{\lambda}(h))$ satisfies the Dirichlet boundary conditions:

$$\omega(\overline{\lambda}(h)) = 0 \quad \text{on } \partial \mathcal{M}_0(h) \setminus \cup_{j=\ell+1}^{N} \mathcal{S}_j.$$

2-a) We show that

$$\omega(\overline{\lambda}(h)) = 0 \quad \text{on } \cup_{i=1}^{\ell} \partial \mathcal{M}_i(h),$$

or equivalently

$$\omega(\overline{\lambda}(h)) \in H_0^1 \left(\cup_{i=1}^{\ell} \mathcal{M}_i(h) \right).$$

In this case, without loss of generality, one may assume that

$$\mathcal{M}_0(h) = \cup_{i=1}^{\ell} \mathcal{M}_i(h).$$

2-a-i) First we recall that

$$\omega(\overline{\lambda}(h)) \in C^\infty(\mathcal{M}_0(h)) \cap H^1(M) \cap L^\infty(M), \tag{6.6a}$$

and

$$\omega(\overline{\lambda}(h)) = 0 \quad \text{almost everywhere in } \mathcal{M}_+(h). \tag{6.9}$$

For $r > 0$ sufficiently small, we let

$$\mathcal{M}_r = \{x \in \mathcal{M}_0(h); \operatorname{dist}(x, \partial\mathcal{M}_0(h)) > r\},$$

and let $\mathbf{n}$ be the unit exterior normal vector to the boundary $\partial\mathcal{M}_r$. One can construct a C^∞ vector function $\mathbf{\Psi}$ on M such that

$$\mathbf{\Psi} \cdot \mathbf{n} \geq \frac{1}{2} \quad \text{on } \partial\mathcal{M}_r, \tag{6.10}$$

and

$$\|\mathbf{\Psi}\|_{C^1(M)} \leq C, \tag{6.11}$$

with a constant $C > 0$. Here and in the following the letter C denotes a generic positive constant *independent* of r.

Since $\omega(\overline{\lambda}(h))$ is in the Sobolev space $H_0^1(M)$, we have by Green's formula

(6.12)
$$\begin{aligned}\int_M \nabla\omega(\overline{\lambda}(h)) \cdot \mathbf{\Psi}\, dV &= -\int_M \omega(\overline{\lambda}(h)) \operatorname{div} \mathbf{\Psi}\, dV \\ &= -\int_{\mathcal{M}_0(h)} \omega(\overline{\lambda}(h)) \operatorname{div} \mathbf{\Psi}\, dV,\end{aligned}$$

and also

(6.13)
$$\begin{aligned}\int_M \nabla\omega(\overline{\lambda}(h)) \cdot \mathbf{\Psi}\, dV &= \int_{\mathcal{M}_r} \nabla\omega(\overline{\lambda}(h)) \cdot \mathbf{\Psi}\, dV + \int_{M\setminus\mathcal{M}_r} \nabla\omega(\overline{\lambda}(h)) \cdot \mathbf{\Psi}\, dV \\ &= \int_{\mathcal{M}_r} \nabla\omega(\overline{\lambda}(h)) \cdot \mathbf{\Psi}\, dV + \int_{\mathcal{M}_0(h)\setminus\mathcal{M}_r} \nabla\omega(\overline{\lambda}(h)) \cdot \mathbf{\Psi}\, dV \\ &= \int_{\partial\mathcal{M}_r} \omega(\overline{\lambda}(h))(\mathbf{\Psi} \cdot \mathbf{n})\, d\sigma - \int_{\mathcal{M}_r} \omega(\overline{\lambda}(h)) \operatorname{div} \mathbf{\Psi}\, dV \\ &\quad + \int_{\mathcal{M}_0(h)\setminus\mathcal{M}_r} \nabla\omega(\overline{\lambda}(h)) \cdot \mathbf{\Psi}\, dV.\end{aligned}$$

Thus, combining formulas (6.12) and (6.13), we obtain that

$$\begin{aligned}&\int_{\partial\mathcal{M}_r} \omega(\overline{\lambda}(h))(\mathbf{\Psi} \cdot \mathbf{n})\, d\sigma \\ &= -\int_{\mathcal{M}_0(h)\setminus\mathcal{M}_r} \omega(\overline{\lambda}(h)) \operatorname{div} \mathbf{\Psi}\, dV - \int_{\mathcal{M}_0(h)\setminus\mathcal{M}_r} \nabla\omega(\overline{\lambda}(h)) \cdot \mathbf{\Psi}\, dV.\end{aligned} \tag{6.14}$$

But, by using inequalities (6.11) and (6.4), we can estimate the first term on the right-hand side of formula (6.14) as follows:

$$\left| \int_{\mathcal{M}_0(h)\setminus\mathcal{M}_r} \omega(\overline{\lambda}(h)) \operatorname{div} \mathbf{\Psi}\, dV \right| \leq C \int_{\mathcal{M}_0(h)\setminus\mathcal{M}_r} dV \leq Cr.$$

Further, by using the Schwarz inequality and inequality (6.5c), we can estimate the second term on the right-hand side of formula (6.14) as follows:

$$\begin{aligned}
&\left| \int_{\mathcal{M}_0(h)\setminus \mathcal{M}_r} \nabla\omega(\overline{\lambda}(h)) \cdot \mathbf{\Psi}\, dV \right| \\
&\leq C \int_{\mathcal{M}_0(h)\setminus \mathcal{M}_r} |\nabla\omega(\overline{\lambda}(h))|\, dV \\
&\leq C \left(\int_{\mathcal{M}_0(h)\setminus \mathcal{M}_r} |\nabla\omega(\overline{\lambda}(h))|^2\, dV \right)^{1/2} \left(\int_{\mathcal{M}_0(h)\setminus \mathcal{M}_r} dV \right)^{1/2} \\
&\leq C r^{\frac{1}{2}}.
\end{aligned}$$

Hence, by formula (6.14), we have for all $r > 0$ sufficiently small

$$\int_{\partial\mathcal{M}_r} \omega(\overline{\lambda}(h))(\mathbf{\Psi}\cdot\mathbf{n})\, d\sigma \leq Cr + Cr^{\frac{1}{2}} \leq Cr^{\frac{1}{2}}.$$

By inequality (6.10), this proves that for all $r > 0$ sufficiently small

$$\int_{\partial\mathcal{M}_r} \omega(\overline{\lambda}(h))\, d\sigma \leq 2 \int_{\partial\mathcal{M}_r} \omega(\overline{\lambda}(h))(\mathbf{\Psi}\cdot\mathbf{n})\, d\sigma \leq Cr^{\frac{1}{2}}. \tag{6.15}$$

If we let

$$\mathcal{M}_r^* = \mathcal{M}_0(h)\setminus\mathcal{M}_r,$$

then it follows from inequality (6.15) that

$$\begin{aligned}
\int_{\mathcal{M}_r^*} \omega(\overline{\lambda}(h))\, dV &= \int_0^r \left(\int_{\partial\mathcal{M}_r} \omega(\overline{\lambda}(h))\, d\sigma \right) dt \\
&\leq Cr^{\frac{3}{2}}.
\end{aligned}$$

Therefore we have for all $r > 0$ sufficiently small

$$\begin{aligned}
\left(\int_{\mathcal{M}_r^*} \omega(\overline{\lambda}(h))^2\, dV \right)^{\frac{1}{2}} &= \left(\int_{\mathcal{M}_r^*} \omega(\overline{\lambda}(h)) \cdot \omega(\overline{\lambda}(h))\, dV \right)^{\frac{1}{2}} \\
&\leq C \left(\int_{\mathcal{M}_r^*} \omega(\overline{\lambda}(h))\, dV \right)^{\frac{1}{2}} \\
&\leq Cr^{\frac{3}{4}}.
\end{aligned} \tag{6.16}$$

2-a-ii) Now we construct a sequence $\{\omega_r\}$ in the space $W^{1,2n/(n+1)}(\mathcal{M}_0(h))$ such that

$$\omega_r \longrightarrow \omega(\overline{\lambda}(h)) \quad \text{in } W^{1,\frac{2n}{n+1}}(\mathcal{M}_0(h)) \text{ as } r \downarrow 0.$$

We let

$$S_r = \{x \in M; \mathrm{dist}\,(x, \partial\mathcal{M}_0(h)) < r\},$$

$$S_r^- = \mathcal{M}_0(h) \backslash \overline{\mathcal{M}_r},$$

and

$$\partial S_r^1 = \{x \in \mathcal{M}_+(h); \operatorname{dist}(x, \partial\mathcal{M}_0(h)) = r\},$$
$$\partial S_r^2 = \{x \in \mathcal{M}_0(h); \operatorname{dist}(x, \partial\mathcal{M}_0(h)) = r\}.$$

Then it is easy to see that, for $r > 0$ sufficiently small, there exists a "shrinking" diffeomorphism

$$\Psi : S_r \longrightarrow S_r^-$$

with the following properties:

(a) $$\Psi(\partial S_r^1) = \partial\mathcal{M}_0(h), \quad \Psi(\partial S_r^2) = \partial S_r^2.$$

(b) $$\sup_{S_r} |\nabla\Psi| \leq C, \quad \sup_{S_r^-} |\nabla\Psi^{-1}| \leq C.$$

Indeed, in terms of local coordinates $(x_1, x_2, \cdots, x_{n-1}, x_n)$ such that

$$\partial S_r^1 = \{x_n = -r\}, \quad \partial S_r^2 = \{x_n = +r\},$$

the diffeomorphism Ψ is given by the formula

$$\Psi(x_1, x_2, \cdots, x_{n-1}, x_n) = \left(x_1, x_2, \cdots, x_{n-1}, \frac{x_n + r}{2}\right).$$

We let

$$\omega_r(x) = \begin{cases} \omega(\overline{\lambda}(h))(\Psi^{-1}(x)) & \text{if } x \in S_r^-, \\ \omega(\overline{\lambda}(h))(x) & \text{if } x \in \mathcal{M}_r. \end{cases}$$

Then, in view of assertion (6.9), it follows that the functions $\{\omega_r\}$ are in the space $H_0^1(\mathcal{M}_0(h))$ for all $r > 0$ sufficiently small. Next, by inequality (6.16), we have for all $r > 0$ sufficiently small

(6.17) $$\begin{aligned} &\|\omega_r - \omega(\overline{\lambda}(h))\|_{L^2(\mathcal{M}_0(h))} \\ &= \|\omega_r - \omega(\overline{\lambda}(h))\|_{L^2(\mathcal{M}_0(h)\backslash\mathcal{M}_r)} \\ &\leq \|\omega_r\|_{L^2(\mathcal{M}_0(h)\backslash\mathcal{M}_r)} + \|\omega(\overline{\lambda}(h))\|_{L^2(\mathcal{M}_0(h)\backslash\mathcal{M}_r)} \\ &\leq 2\|\omega(\overline{\lambda}(h))\|_{L^2(\mathcal{M}_r^*)} \\ &\leq C r^{\frac{3}{4}}. \end{aligned}$$

Furthermore, using Hölder's inequality, we obtain that for all $r > 0$ sufficiently small

(6.18) $$\begin{aligned} &\|\omega_r - \omega(\overline{\lambda}(h))\|_{L^{2n/(n+1)}(\mathcal{M}_0(h))} \\ &\leq |\mathcal{M}_0(h)|^{\frac{n+1}{2n} - \frac{1}{2}} \|\omega_r - \omega(\overline{\lambda}(h))\|_{L^2(\mathcal{M}_r^*)} \\ &\leq C r^{\frac{3}{4}}. \end{aligned}$$

Similarly, it follows that for all $r > 0$ sufficiently small

$$\begin{aligned}
&\|\nabla(\omega_r - \omega(\overline{\lambda}(h)))\|_{L^{2n/(n+1)}(\mathcal{M}_0(h))} \\
&= \|\nabla(\omega_r - \omega(\overline{\lambda}(h))\|_{L^{2n/(n+1)}(\mathcal{M}_0(h)\setminus\mathcal{M}_r)} \\
&\leq \|\nabla\omega_r\|_{L^{2n/(n+1)}(\mathcal{M}_0(h)\setminus\mathcal{M}_r)} + \|\nabla\omega(\overline{\lambda}(h))\|_{L^{2n/(n+1)}(\mathcal{M}_0(h)\setminus\mathcal{M}_r)} \\
&\leq \sup_{S_r^{-1}} |\nabla\Psi^{-1}| \, \|\nabla\omega(\overline{\lambda}(h))\|_{L^{2n/(n+1)}(\mathcal{M}_0(h)\setminus\mathcal{M}_r)} \\
&\quad + \|\nabla\omega(\overline{\lambda}(h))\|_{L^{2n/(n+1)}(\mathcal{M}_0(h)\setminus\mathcal{M}_r)} \\
&\leq C\|\nabla\omega(\overline{\lambda}(h))\|_{L^{2n/(n+1)}(\mathcal{M}_0(h)\setminus\mathcal{M}_r)}.
\end{aligned}$$

But we find that the last term can be estimated as follows:

$$\begin{aligned}
&\|\nabla\omega(\overline{\lambda}(h))\|_{L^{2n/(n+1)}(\mathcal{M}_0(h)\setminus\mathcal{M}_r)} \\
&= \left(\int_{\mathcal{M}_0(h)\setminus\mathcal{M}_r} |\nabla\omega(\overline{\lambda}(h))|^{\frac{2n}{n+1}} \, dV \right)^{\frac{n+1}{2n}} \\
&\leq \left(|\mathcal{M}_0(h)\setminus\mathcal{M}_r|^{1-\frac{n}{n+1}} \left(\int_{\mathcal{M}_0(h)\setminus\mathcal{M}_r} |\nabla\omega(\overline{\lambda}(h))|^2 \, dV \right)^{\frac{n+1}{n}} \right)^{\frac{n+1}{2n}} \\
&\leq |\mathcal{M}_0(h)\setminus\mathcal{M}_r|^{\frac{1}{2n}} \|\nabla\omega(\overline{\lambda}(h))\|_{L^2(M)} \\
&\leq C r^{\frac{1}{2n}}.
\end{aligned}$$

Hence we obtain that for all $r > 0$ sufficiently small

$$\|\nabla(\omega_r - \omega(\overline{\lambda}(h)))\|_{L^{2n/(n+1)}(\mathcal{M}_0(h))} \leq C r^{\frac{1}{2n}}. \tag{6.19}$$

Therefore, combining inequalities (6.18) and (6.19), we have proved that

$$\omega(\overline{\lambda}(h)) \in W_0^{1,\frac{2n}{n+1}}(\mathcal{M}_0(h)),$$

and

$$\omega_r \longrightarrow \omega(\overline{\lambda}(h)) \quad \text{in } W_0^{1,\frac{2n}{n+1}}(\mathcal{M}_0(h)) \text{ as } r \downarrow 0.$$

2-a-iii) Finally we show that:

$$\omega(\overline{\lambda}(h)) \in H_0^1(\mathcal{M}_0(h)), \tag{6.20a}$$

and

$$\omega_r \longrightarrow \omega(\overline{\lambda}(h)) \quad \text{in } H_0^1(\mathcal{M}_0(h)) \text{ as } r \downarrow 0. \tag{6.20b}$$

We recall that the function

$$\omega(\overline{\lambda}(h)) \in W_0^{1,\frac{2n}{n+1}}(\mathcal{M}_0(h)) \cap L^\infty(\mathcal{M}_0(h))$$

satisfies the equation

$$\Delta\omega(\overline{\lambda}(h)) - \overline{\lambda}(h))\omega(\overline{\lambda}(h)) = 0 \quad \text{in } \mathcal{M}_0(h).$$

Thus, by using L^p *estimates* for elliptic equations (cf. [GT, Theorem 9.14]), we obtain that

$$\omega(\overline{\lambda}(h)) \in W^{2,\frac{2n}{n+1}}(\mathcal{M}_0(h)) \cap W_0^{1,\frac{2n}{n+1}}(\mathcal{M}_0(h)). \tag{6.21}$$

On the other hand, by applying the Sobolev imbedding theorem (cf. [Ad, Theorem 5.4]), we find that the injection

$$W^{2,\frac{2n}{n+1}}(\mathcal{M}_0(h)) \subset W^{1,\frac{2n}{n-1}}(\mathcal{M}_0(h)) \tag{6.22}$$

is continuous.

Hence it follows from assertions (6.21) and (6.22) that

$$\|\nabla\omega(\overline{\lambda}(h))\|_{L^{2n/(n-1)}(\mathcal{M}_0(h))} \leq C.$$

By virtue of Hölder's inequality, this proves that for all $r > 0$ sufficiently small

$$\begin{aligned}\left(\int_{\mathcal{M}_r^*} |\nabla\omega(\overline{\lambda}(h))|^2\, dV\right)^{\frac{1}{2}} &\leq \left(\int_{\mathcal{M}_r^*} |\nabla\omega(\overline{\lambda}(h))|^{\frac{2n}{n-1}}\, dV\right)^{\frac{n-1}{2n}} \left(\int_{\mathcal{M}_r^*} dV\right)^{\frac{1}{2n}} \\ &= \|\nabla\omega(\overline{\lambda}(h))\|_{L^{2n/(n-1)}(\mathcal{M}_0(h))} |\mathcal{M}_r^*|^{\frac{1}{2n}} \\ &\leq Cr^{\frac{1}{2n}}.\end{aligned}$$

Thus we have for all $r > 0$ sufficiently small

$$\begin{aligned}\|\nabla(\omega_r - \omega(\overline{\lambda}(h)))\|_{L^2(\mathcal{M}_0(h))} &= \|\nabla(\omega_r - \omega(\overline{\lambda}(h))\|_{L^2(\mathcal{M}_r^*)} \\ &\leq C\|\nabla\omega(\overline{\lambda}(h))\|_{L^2(\mathcal{M}_r^*)} \\ &\leq Cr^{\frac{1}{2n}}.\end{aligned} \tag{6.23}$$

Therefore, assertion (6.20) follows by combining inequalities (6.17) and (6.23).

2-b) Similarly, one can prove that

$$\omega(\overline{\lambda}(h)) = 0 \quad \text{on } \cup_{j=\ell+1}^N \partial\mathcal{M}_j(h) \backslash \mathcal{S}_j.$$

The proof of Lemma 6.2 and hence that of Proposition 6.1 is complete. $\square$

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