POSITIVE SOLUTIONS OF SUBLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

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1. Introduction and Results

This paper is a continuation of the previous papers Taira-Umezu [TU1] and [TU2] where we studied global static bifurcation theory for a class of *degenerate* boundary value problems for nonlinear second-order elliptic differential operators. The previous papers treated the asymptotic linear and nonlinear cases, for example, such nonlinear terms as $u + 1/u$ and u^p, $p > 1$, near $u = +\infty$, by using the Leray-Schauder degree theory. The purpose of this paper is to study more general nonlinear terms such as $\sqrt{u}$, $\log(1+u)$ and e^{-u}, and is to prove the existence and uniqueness of positive solutions of nonlinear elliptic boundary value problems, by making good use of the super-subsolution method. We remark that the variational method would break down, since our boundary condition is degenerate.

Let D be a bounded domain of Euclidean space $\mathbf{R}^N$, $N \geq 2$, with C^∞ boundary ∂D; its closure $\overline{D} = D \cup \partial D$ is an N-dimensional, compact C^∞ manifold with boundary. We let

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N} a^{ij}(x)\frac{\partial u}{\partial x_j}(x)\right) + c(x)u(x)$$

be a second-order, *elliptic* differential operator with real C^∞ coefficients on $\overline{D}$ such that:

(1) $a^{ij}(x) = a^{ji}(x)$, $x \in \overline{D}$, $1 \leq i, j \leq N$, and there exists a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2, \ x \in \overline{D}, \ \xi \in \mathbf{R}^N.$$

(2) $c(x) \geq 0$ on $\overline{D}$.

In this paper we consider the following general nonlinear elliptic boundary value problem: Given function $f(x,\xi)$ defined on $\overline{D} \times [0,\infty)$, find a nonnegative function u in D such that

$$(*) \qquad \begin{cases} Au = f(x,u) & \text{in } D, \\ Bu = a\frac{\partial u}{\partial \boldsymbol{\nu}} + bu = 0 & \text{on } \partial D. \end{cases}$$

Here:

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(1) $a \in C^\infty(\partial D)$ and $a \geq 0$ on ∂D.

(2) $b \in C^\infty(\partial D)$ and $b \geq 0$ on ∂D.

(3) $\partial/\partial\boldsymbol{\nu}$ is the conormal derivative associated with the operator A: $\partial/\partial\boldsymbol{\nu} = \sum_{i,j=1}^N a^{ij} n_j \partial/\partial x_i$, where $\mathbf{n} = (n_1, n_2, \cdots, n_N)$ is the unit exterior normal to the boundary ∂D.

First we state our fundamental hypotheses on the functions a, b and c:

(H.1) $b(x') > 0$ on $M = \{x' \in \partial D : a(x') = 0\}$.

(H.2) $c(x) > 0$ in D.

It is worth pointing out here that the boundary condition B is non-degenerate if and only if either $a \neq 0$ on ∂D or $a \equiv 0$ and $b \neq 0$ on ∂D. In particular, if $a \equiv 1$ and $b \equiv 0$ on ∂D (resp. $a \equiv 0$ and $b \equiv 1$ on ∂D), then the boundary condition B is the so-called Neumann (resp. Dirichlet) condition.

A solution $u \in C^2(\overline{D})$ of problem $(*)$ is said to be *nontrivial* if it does not identically equal zero on $\overline{D}$. We call a nontrivial solution u of problem $(*)$ a *positive solution* if $u(x) \geq 0$ on $\overline{D}$.

Let λ_1 be the first eigenvalue of the linearized boundary value problem

$$(\dagger) \qquad \begin{cases} Au = \lambda u & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases}$$

By [Ta2, Theorem 1], we know that the eigenvalue λ_1 is positive and simple with positive eigenfunction in D.

Our existence theorem for positive solutions of problem $(*)$ is stated as follows (cf. [BO, Theorem 2]):

Theorem 1. *Assume that hypotheses (H.1) and (H.2) are satisfied and that the function $f(x,\xi)$ belongs to $C^\theta(\overline{D} \times [0,\sigma])$, $0 < \theta < 1$, for every $\sigma > 0$, and satisfies the* slope condition*: For every $\sigma > 0$, there exists a constant $\omega = \omega(\sigma) > 0$, independent of $x \in \overline{D}$, such that*

$$(\mathrm{R})_\sigma \qquad f(x,\xi) - f(x,\eta) > -\omega(\xi - \eta), \quad x \in \overline{D},\ 0 \leq \eta < \xi \leq \sigma.$$

If in addition the two limits

$$\ell(x) = \lim_{\xi \downarrow 0} \frac{f(x,\xi)}{\xi}$$

and

$$m(x) = \lim_{\xi \to \infty} \frac{f(x,\xi)}{\xi}$$

exist uniformly for all $x \in \overline{D}$ and if we have

$$(1.1) \qquad m(x) < \lambda_1 < \ell(x), \quad x \in \overline{D},$$

then problem $(*)$ has a positive solution $u \in C^{2+\theta}(\overline{D})$.

If the nonlinear term $f(x,\xi)$ is *independent of* x, then we can prove that condition (1.1) is necessary and sufficient for the existence of positive solutions of problem $(*)$; more precisely, we have the following generalization of [BO, Theorem 1] to the degenerate case:

Theorem 2. *Assume that hypotheses (H.1) and (H.2) are satisfied, and that the function* $f(x,\xi) = f(\xi)$ *is independent of* $x \in \overline{D}$ *and further that the function* $f(\xi)/\xi$ *is strictly decreasing for* $0 < \xi < \infty$*. We let*

$$\ell = \lim_{\xi \downarrow 0} \frac{f(\xi)}{\xi}, \quad m = \lim_{\xi \to \infty} \frac{f(\xi)}{\xi}.$$

Then problem $(*)$ *has a positive solution* $u \in C^{2+\theta}(\overline{D})$ *if and only if*

$$m < \lambda_1 < \ell. \tag{1.2}$$

Furthermore, the solution u *is unique in the space* $C^2(\overline{D})$*.*

Now, as an application of Theorem 2, we consider global static bifurcation problems for the following semilinear elliptic boundary value problem:

$$(**) \qquad \begin{cases} Au - \lambda u + h(u) = 0 & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases}$$

The next corollary, which is an immediate consequence of Theorem 2, allows us to treat more general nonlinear terms h than [TU1, Theorem 1] as is shown in Examples 1–4 below.

Corollary 1. *Assume that* $h(\xi)$ *is a function in* $C^\theta([0,\sigma])$, $0 < \theta < 1$, *for every* $\sigma > 0$, *and that the function* $h(\xi)/\xi$ *is strictly increasing for* $0 < \xi < \infty$*. We let*

$$\alpha = \lim_{\xi \downarrow 0} \frac{h(\xi)}{\xi}, \quad \beta = \lim_{\xi \to \infty} \frac{h(\xi)}{\xi}.$$

Then problem $(**)$ *has a unique positive solution* $u \in C^{2+\theta}(\overline{D})$ *if and only if* $\lambda_1 + \alpha < \lambda < \lambda_1 + \beta$*.*

For Corollary 1, we give four simple examples of the function $h(\xi)$:

Example 1. $h(\xi) = (k/6)\xi^3$ for $0 \le \xi \le 1$ and $h(\xi) = k(\xi + 1/(2\xi) - 4/3)$ for $1 < \xi < \infty$, where k is a positive constant. In this case, $\alpha = 0$, $\beta = k$ and so $\lambda_1 < \lambda < \lambda_1 + k$.

Example 2. $h(\xi) = \xi^p$, $p > 1$. In this case, $\alpha = 0$, $\beta = \infty$ and so $\lambda_1 < \lambda < \infty$.

Example 3. $h(\xi) = -\sqrt{\xi}$. In this case, $\alpha = -\infty$, $\beta = 0$ and so $-\infty < \lambda < \lambda_1$.

Example 4. $h(\xi) = -e^{-\xi}$. In this case, $\alpha = -\infty$, $\beta = 0$ and so $-\infty < \lambda < \lambda_1$.

Similarly, we consider the following semilinear elliptic eigenvalue problem:

$$(***) \qquad \begin{cases} Au - \lambda g(u) = 0 & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases}$$

Then we have the following generalization of [SC, Theorem 2.1] to the degenerate case:

Corollary 2. *Assume that $g(\xi)$ is a function in $C^\theta([0,\sigma])$, $0<\theta<1$, for every $\sigma>0$, and that the function $g(\xi)/\xi$ is strictly decreasing for $0<\xi<\infty$. We let*

$$\gamma = \lim_{\xi \downarrow 0} \frac{g(\xi)}{\xi}, \quad \delta = \lim_{\xi \to \infty} \frac{g(\xi)}{\xi}.$$

Then problem $(***)$ has a unique positive solution $u \in C^{2+\theta}(\overline{D})$ if and only if $\lambda_1/\gamma < \lambda < \lambda_1/\delta$.

For Corollary 2, we give three simple examples of the function $g(\xi)$:

Example 5. $g(\xi)=\sqrt{\xi}$. In this case, $\gamma=\infty$, $\delta=0$ and so $0<\lambda<\infty$.

Example 6. $g(\xi)=e^{-\xi}$. In this case, $\gamma=\infty$, $\delta=0$ and so $0<\lambda<\infty$.

Example 7. $g(\xi)=\log(1+\xi)$. In this case, $\gamma=1$, $\delta=0$ and so $\lambda_1<\lambda<\infty$.

2. Proof of Theorem 1

Our proof of Theorem 1 is carried out by making use of the super-subsolution method just as in the proof of [De, Theorem 2.2].

A nonnegative function $\psi \in C^2(\overline{D})$ is said to be a *supersolution* of problem $(*)$ if it satisfies the conditions:

$$\begin{cases} A\psi - f(x,\psi) \geq 0 & \text{in } D, \\ B\psi \geq 0 & \text{on } \partial D. \end{cases}$$

Similarly, a nonnegative function $\phi \in C^2(\overline{D})$ is said to be a *subsolution* of problem $(*)$ if it satisfies the conditions:

$$\begin{cases} A\phi - f(x,\phi) \leq 0 & \text{in } D, \\ B\phi \leq 0 & \text{on } \partial D. \end{cases}$$

(I) First we construct a subsolution of problem $(*)$.

By condition (1.1), we can find a constant $c_1>0$ such that

$$f(x,\xi) \geq \lambda_1 \xi, \quad x \in \overline{D},\ 0<\xi<c_1. \tag{2.1}$$

On the other hand, it is known (cf. [Ta2, Theorem 1]) that the linearized boundary value problem

$$\begin{cases} A\varphi = \lambda_1 \varphi & \text{in } D, \\ B\varphi = 0 & \text{on } \partial D \end{cases}$$

has a positive eigenfunction $\varphi \in C^{2+\theta}(\overline{D})$. If we let

$$\phi_\varepsilon = \varepsilon \varphi$$

for $\varepsilon>0$ sufficiently small, we may assume that

$$\max_{\overline{D}} \phi_\varepsilon < c_1.$$

Then we have by condition (2.1)

$$\begin{cases} A\phi_\varepsilon - f(x, \phi_\varepsilon) \leq \lambda_1\phi_\varepsilon - \lambda_1\phi_\varepsilon = 0 & \text{in } D, \\ B\phi_\varepsilon = 0 & \text{on } \partial D. \end{cases}$$

This proves that the function $\phi_\varepsilon \in C^{2+\theta}(\overline{D})$ is a subsolution of problem (∗).

(II) In order to construct a supersolution of problem (∗), we make use of the theory of positive operators in ordered Banach spaces (cf. [Am]).

A Banach space X is called an *ordered Banach space* if it is an ordered set. For an ordered Banach space X having the ordering $\leq$, the set $Q = \{x \in X : x \geq 0\}$ is called the *positive cone* in X.

For functions u and v in $C(\overline{D})$, we write $u \leq v$ if $u(x) \leq v(x)$ for all $x \in \overline{D}$. Then the space $C(\overline{D})$ becomes an ordered Banach space with the ordering $\leq$. Moreover, if we let

$$P = \{u \in C(\overline{D}) : u \geq 0\},$$

then the set P is the positive cone in $C(\overline{D})$.

By [TU2, Theorem 1.1], we can introduce a continuous linear operator

$$K : C^\theta(\overline{D}) \longrightarrow C^{2+\theta}(\overline{D})$$

as follows: For any $v \in C^\theta(\overline{D})$, the function $u = Kv \in C^{2+\theta}(\overline{D})$ is the unique solution of the boundary value problem

$$\begin{cases} Au = v & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases}$$

Now we introduce an ordered Banach subspace of $C(\overline{D})$ which is associated with the resolvent K.

We let

$$e = K1 \in C^{2+\theta}(\overline{D}),$$

and

$$C_e(\overline{D}) = \{u \in C(\overline{D}) : \text{ there is a constant } c > 0 \text{ such that } -ce \leq u \leq ce\}.$$

Then the space $C_e(\overline{D})$ is given a norm by the formula

$$\|u\|_e = \inf\{c > 0 : -ce \leq u \leq ce\}.$$

If we let

$$P_e = \{u \in C_e(\overline{D}) : u \geq 0\},$$

it is easy to verify that the space $C_e(\overline{D})$ is an ordered Banach space having the positive cone P_e with nonempty interior. Moreover, by [TU1, Proposition 2.2], we can extend uniquely the resolvent K to a strongly positive, compact linear operator

$$K : C_e(\overline{D}) \longrightarrow C_e(\overline{D}).$$

Here we recall that K is said to be *strongly positive* if $v \in P_e$ and $v \not\equiv 0$ on $\overline{D}$, then the function Kv is an interior point of P_e.

The next lemma plays an important role in the construction of a supersolution of problem (∗) (cf. [Kr, Theorem 2.16]):

Lemma (The positivity lemma). *Let $T : C_e(\overline{D}) \to C_e(\overline{D})$ be a strongly positive, compact linear operator and λ_0 the largest eigenvalue of T. Then, for any given positive function $g \in C_e(\overline{D})$, the equation*

$$\lambda v - Tv = g$$

has a unique positive solution $v \in C_e(\overline{D})$ for each $\lambda > \lambda_0$.

(III) By condition (1.1), we can find constants $c_2 > 0$ and $0 < d < \lambda_1$ such that

$$f(x, \xi) \leq (\lambda_1 - d)\xi, \quad x \in \overline{D},\ \xi > c_2.$$

Hence, if we let

$$k = \max \left\{ |f(x, \xi)| : x \in \overline{D},\ 0 \leq \xi \leq c_2 \right\},$$

then we have

$$f(x, \xi) \leq (\lambda_1 - d)\xi + k, \quad x \in \overline{D},\ \xi \geq 0. \tag{2.2}$$

We show that the boundary value problem

$$\begin{cases} Au = (\lambda_1 - d)u + k & \text{in } D, \\ Bu = 0 & \text{on } \partial D \end{cases} \tag{2.3}$$

has a positive solution $u \in C^{2+\theta}(\overline{D})$.

First it is easy to see that $u \in C^{2+\theta}(\overline{D})$ is a solution of problem (2.3) if and only if it satisfies the following operator equation:

$$u = (\lambda_1 - d)Ku + Kk \quad \text{in } C_e(\overline{D}). \tag{2.4}$$

But we remark that the largest eigenvalue $(\lambda_1 - d)/\lambda_1$ of the operator $(\lambda_1 - d)K$ is less than 1, and that the function Kk is positive on $\overline{D}$. Thus, applying the positivity lemma to our situation, we can find a solution $\psi \in C^{2+\theta}(\overline{D})$ of equation (2.4), or equivalently, a solution of problem (2.3).

Then we have by condition (2.2)

$$\begin{cases} A\psi - f(x, \psi) \geq (\lambda_1 - d)\psi + k - ((\lambda_1 - d)\psi + k) = 0 & \text{in } D, \\ B\psi = 0 & \text{on } \partial D. \end{cases}$$

This proves that the function $\psi \in C^{2+\theta}(\overline{D})$ is a supersolution of problem $(*)$.

(IV) One may assume that the super- and subsolutions ψ, ϕ_ε satisfy the condition

$$\phi_\varepsilon \leq \psi \quad \text{on } \overline{D}.$$

Furthermore, if we take a constant $\sigma > 0$ such that

$$\max_{\overline{D}} \phi_\varepsilon,\ \max_{\overline{D}} \psi \leq \sigma,$$

then it follows that the functions ψ and ϕ_ε are respectively super- and subsolutions of problem $(*)$ taking values in the interval $[0, \sigma]$.

Therefore our theorem follows from an application of [TU2, Theorem 1].

The proof of Theorem 1 is complete. □

3. Proof of Theorem 2

By Theorem 1, it suffices to prove that condition (1.2) is necessary for the existence of positive solutions of problem $(*)$.

We associate with problem $(\dagger)$ an unbounded linear operator $\mathfrak{A}$ from the Hilbert space $L^2(D)$ into itself as follows:

(a) The domain of definition $D(\mathfrak{A})$ is the space

$$D(\mathfrak{A}) = \left\{u \in H^2(D) : Bu = 0\right\}.$$

(b) $\mathfrak{A}u = Au$, $u \in D(\mathfrak{A})$.

Then it is known (cf. [Ta1, Theorems 7.3 and 7.4]) that the operator $\mathfrak{A}$ is a non-negative, self-adjoint operator in $L^2(D)$, and has a compact resolvent. Hence we find that the first eigenvalue λ_1 of $\mathfrak{A}$ is characterized by the following formula:

$$\lambda_1 = \min\left\{(\mathfrak{A}u, u) : u \in D(\mathfrak{A}),\ \|u\| = 1\right\}, \tag{3.1}$$

where $\|\cdot\|$ is the norm on $L^2(D)$.

First we show that

$$\lambda_1 < \ell. \tag{3.2}$$

Since the function $f(\xi)/\xi$ is strictly decreasing, it follows that

$$m < \frac{f(\xi)}{\xi} < \ell, \quad 0 < \xi < \infty. \tag{3.3}$$

Now let $u \in C^2(\overline{D})$ be a positive solution of problem $(*)$:

$$\begin{cases} Au = f(u) & \text{in } D, \\ u > 0 & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases}$$

Then, since $u \in D(\mathfrak{A})$, we have by inequality (3.3) with $\xi = u(x)$

$$(\mathfrak{A}u, u) = (Au, u) = \int_D f(u)u\,dx < \ell \int_D u^2\,dx.$$

Hence inequality (3.2) follows by using formula (3.1).

Next we show that

$$\lambda_1 > m. \tag{3.4}$$

If $u \in C^2(\overline{D})$ is a positive solution of problem $(*)$, we let

$$d = \frac{f(\|u\|_\infty + 1)}{\|u\|_\infty + 1}, \tag{3.5}$$

where $\|u\|_\infty = \max_{\overline{D}} u$. We remark that $d > m$.

Now we consider the eigenvalue problem

$$\begin{cases} Au - du = \lambda u & \text{in } D, \\ Bu = 0 & \text{on } \partial D, \end{cases}$$

and let $\lambda_1(d)$ be its first eigenvalue. Then, by formula (3.1), we find that

$$\lambda_1(d) = \min\{((\mathfrak{A} - dI)u, u) : u \in D(\mathfrak{A}),\ \|u\| = 1\} = \lambda_1 - d.$$

Furthermore, by [Ta2, Theorem 1], one may assume that the first eigenvalue $\lambda_1(d)$ has a positive eigenfunction $\varphi \in C^{2+\theta}(\overline{D})$:

$$\begin{cases} A\varphi - d\varphi = \lambda_1(d)\varphi & \text{in } D, \\ \varphi > 0 & \text{in } D, \\ B\varphi = 0 & \text{on } \partial D. \end{cases}$$

Then we have the following:

Claim 3.1. $\lambda_1(d) = \lambda_1 - d > 0$.

Proof. Since the function $f(\xi)/\xi$ is strictly decreasing, it follows from formula (3.5) that

$$f(u(x)) > du(x), \quad x \in D.$$

Hence we have

$$(\mathfrak{A}u, \varphi) = (Au, \varphi) = \int_D f(u)\varphi\, dx > d \int_D u\varphi\, dx. \tag{3.6}$$

On the other hand, by the self-adjointness $\mathfrak{A}$, it follows that

$$(\mathfrak{A}u, \varphi) = (u, \mathfrak{A}\varphi) = (u, A\varphi) = \int_D u(\lambda_1(d) + d)\varphi\, dx. \tag{3.7}$$

Thus, combining formulas (3.6) and (3.7), we obtain that

$$\lambda_1(d) \int_D u\varphi\, dx > 0.$$

This proves Claim 3.1, since we have $u > 0$, $\varphi > 0$ in D. $\square$

Summing up, we have proved that

$$\lambda_1 > d > m.$$

The desired inequality (1.2) follows from inequalities (3.2) and (3.4).

(II) Finally we prove the uniqueness of positive solutions of problem $(*)$ (cf. [BO, Section 2]).

Let $u_i \in C^2(\overline{D})$, $i = 1, 2$, be two positive solutions of problem $(*)$:

$$\begin{cases} Au_i = f(u_i) & \text{in } D, \\ u_i > 0 & \text{in } D, \\ Bu_i = 0 & \text{on } \partial D. \end{cases}$$

The next claim is an essential step in the proof of uniqueness of positive solutions (cf. [BO, Lemma 1]):

Claim 3.2. $u_1/u_2,\ u_2/u_1 \in C(\overline{D})$.

Proof. Since the function $f(\xi)/\xi$ is strictly decreasing, one can find two nonnegative constants ω_i, $i = 1, 2$, such that

$$f(u_i) + \omega_i\, u_i \geq 0 \quad \text{in } D.$$

Indeed, it suffices to take

$$\omega_i = \max\left\{0, -\frac{f(\|u_i\|_\infty)}{\|u_i\|_\infty}\right\}, \qquad i = 1, 2.$$

Then the solutions u_i, $i = 1, 2$, are expressed as follows:

$$\begin{aligned} &u_i = K_{\omega_i}\left(f(u_i) + \omega_i\, u_i\right), \\ &f(u_i) + \omega_i\, u_i \geq 0 \quad \text{in } D. \end{aligned}$$

Here K_{ω_i} is the resolvent of the boundary value problem

$$\begin{cases} (A + \omega_i)\, u = \varphi & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases}$$

Hence Claim 3.2 follows from the strong positivity of the resolvents K_{ω_i}, $i = 1, 2$ (see [TU1, inequality (2.4)]). □

By Claim 3.2, we can apply Green's formula to obtain that

(3.8)

$$\begin{aligned} &\int_D \left(\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2}\right)\left(u_1^2 - u_2^2\right) dx \\ &= \int_D \left(\frac{Au_1}{u_1} - \frac{Au_2}{u_2}\right)\left(u_1^2 - u_2^2\right) dx \\ &= -\int_D \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\sum_{j=1}^N a^{ij}\frac{\partial u_1}{\partial x_j}\right) u_1\, dx + \int_D \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\sum_{j=1}^N a^{ij}\frac{\partial u_1}{\partial x_j}\right)\left(\frac{u_2^2}{u_1}\right) dx \\ &\quad - \int_D \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\sum_{j=1}^N a^{ij}\frac{\partial u_2}{\partial x_j}\right) u_2\, dx + \int_D \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\sum_{j=1}^N a^{ij}\frac{\partial u_2}{\partial x_j}\right)\left(\frac{u_1^2}{u_2}\right) dx \\ &= \int_D \sum_{i,j=1}^N a^{ij}\frac{\partial u_1}{\partial x_i}\frac{\partial u_1}{\partial x_j}\, dx - \int_D \sum_{i,j=1}^N a^{ij}\frac{\partial u_1}{\partial x_i}\frac{\partial}{\partial x_j}\left(\frac{u_2^2}{u_1}\right) dx \\ &\quad - \int_{\partial D} \frac{\partial u_1}{\partial \boldsymbol{\nu}} u_1\, d\sigma + \int_{\partial D} \frac{\partial u_1}{\partial \boldsymbol{\nu}}\left(\frac{u_2^2}{u_1}\right) d\sigma \\ &\quad + \int_D \sum_{i,j=1}^N a^{ij}\frac{\partial u_2}{\partial x_i}\frac{\partial u_2}{\partial x_j}\, dx - \int_D \sum_{i,j=1}^N a^{ij}\frac{\partial u_2}{\partial x_i}\frac{\partial}{\partial x_j}\left(\frac{u_1^2}{u_2}\right) dx \\ &\quad - \int_{\partial D} \frac{\partial u_2}{\partial \boldsymbol{\nu}} u_2\, d\sigma + \int_{\partial D} \frac{\partial u_2}{\partial \boldsymbol{\nu}}\left(\frac{u_1^2}{u_2}\right) d\sigma. \end{aligned}$$

Here we remark that the four integrals over ∂D on the last line of formula (3.8) vanish. Indeed, it suffices to note that

$$\begin{vmatrix} \frac{\partial u_1}{\partial \boldsymbol{\nu}} & u_1 \\ \frac{\partial u_2}{\partial \boldsymbol{\nu}} & u_2 \end{vmatrix} = 0 \quad \text{on } \partial D,$$

since the solutions u_1 and u_2 satisfy the boundary conditions

$$\begin{pmatrix} \frac{\partial u_1}{\partial \boldsymbol{\nu}} & u_1 \\ \frac{\partial u_2}{\partial \boldsymbol{\nu}} & u_2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on } \partial D,$$

and since $(a, b) \neq (0, 0)$ on ∂D.

Therefore we find that

$$\begin{aligned} &\int_D \left(\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2} \right) \left(u_1^2 - u_2^2 \right) dx \\ &= \int_D \sum_{i,j=1}^{N} a^{ij} \left(\frac{\partial u_1}{\partial x_i} - \frac{u_2}{u_1} \frac{\partial u_1}{\partial x_i} \right) \left(\frac{\partial u_1}{\partial x_j} - \frac{u_2}{u_1} \frac{\partial u_1}{\partial x_j} \right) dx \\ &\quad + \int_D \sum_{i,j=1}^{N} a^{ij} \left(\frac{\partial u_2}{\partial x_i} - \frac{u_1}{u_2} \frac{\partial u_2}{\partial x_i} \right) \left(\frac{\partial u_2}{\partial x_j} - \frac{u_1}{u_2} \frac{\partial u_2}{\partial x_j} \right) dx \\ &\geq 0. \end{aligned}$$

This implies that $u_1 \equiv u_2$ in D, since the function $f(\xi)/\xi$ is strictly decreasing.

The proof of Theorem 2 is now complete. □

References

[Am] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. **18** (1976), 620–709.

[BO] H. Brezis and L. Oswald, *Remarks on sublinear elliptic equations*, Nonlinear Analysis TMA **10** (1986), 55–64.

[De] D. G. de Figueiredo, *Positive solutions of semilinear elliptic problems*, Lecture Notes in Mathematics, No. 957, Springer-Verlag, Berlin Heidelberg New York, 1982, pp. 34–87.

[Kr] M. A. Krasnosel'skii, *Positive solutions of operator equations*, P. Noordhoff, Groningen, 1964.

[SC] R. B. Simpson and D. S. Cohen, *Positive solutions of nonlinear elliptic eigenvalue problems*, J. Math. Mech. **19** (1970), 895–910.

[Ta1] K. Taira, *On some degenerate oblique derivative problems*, J. Fac. Sci. Univ. Tokyo Sect. IA **23** (1976), 259–287.

[Ta2] K. Taira, *Bifurcation for nonlinear elliptic boundary value problems I*, Collectanea Mathematica **47** (1996), 207–229.

[TU1] K. Taira and K. Umezu, *Bifurcation for nonlinear elliptic boundary value problems II*, Tokyo J. Math. **19** (1996), 387–396.

[TU2] K. Taira and K. Umezu, *Bifurcation for nonlinear elliptic boundary value problems III*, Adv. Differential Equations **1** (1996), 709–727.

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