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Degenerate Elliptic Eigenvalue Problems with Indefinite Weights

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Dedicated to the memory of Professor Sigeru Mizohata (1924–2002)

Abstract. The purpose of this paper is to provide a careful and accessible exposition of the Kreĭn and Rutman theory of degenerate elliptic eigenvalue problems with indefinite weights that model population dynamics in environments with spatial heterogeneity. We prove that the first eigenvalue of our problem is algebraically simple and its corresponding eigenfunction may be chosen to be positive everywhere. The approach here is distinguished by the extensive use of the ideas and techniques characteristic of the recent developments in the theory of partial differential equations. The results extend an earlier theorem due to Manes and Micheletti to the degenerate case.

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1. Introduction and Main Results

Let Ω be a bounded domain of Euclidean space $\mathbf{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega$; its closure $\overline{\Omega} = \Omega \cup \partial\Omega$ is an N dimensional, compact smooth manifold with boundary. Let A_0be a second-order, elliptic differential operator with real coefficients such that

$$Au := -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N} a^{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)u. \tag{1.1}$$

Here:

(1) $a^{ij}(x) \in C^\infty(\overline{\Omega})$ and $a^{ij}(x) = a^{ji}(x)$ on $\overline{\Omega}$.

(2) There exists a positive constant a_0 such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2, \quad (x,\xi) \in \overline{\Omega} \times \mathbf{R}^N.$$

(3) The function $c(x)$ is real-valued and may be *discontinuous* in Ω. More precisely, $c(x) \in L^\infty(\Omega)$ and $c(x) \geq 0$ almost everywhere in Ω.

Let B be a first-order, boundary condition with real coefficients such that

$$Bu := a(x')\frac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u \tag{1.2}$$

Here:

(4) $a(x') \in C^\infty(\partial\Omega)$ and $a(x') \geq 0$ on $\partial\Omega$.
(5) $b(x') \in C^\infty(\partial\Omega)$ and $b(x') \geq 0$ on $\partial\Omega$.
(6) $\partial/\partial\boldsymbol{\nu}$ is the conormal derivative associated with the operator A:

$$\frac{\partial}{\partial \boldsymbol{\nu}} = \sum_{i=1}^{N} a^{ij}(x')\, n_j \frac{\partial}{\partial x_i},$$

where $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ is the unit exterior normal to the boundary $\partial\Omega$.

In this paper we study the following elliptic eigenvalue problem with an indefinite weight function:

$$\begin{cases} Au = \lambda m(x)u & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Here:

(7) λ is a real parameter.
(8) The weight function $m(x)$ is real-valued and may be *discontinuous* in Ω.

The main purpose of this paper is to study the existence and uniqueness of solutions of problem (1.3) in the framework of Sobolev spaces of L^p type (Theorem 1.1), substantially improving the previous work [19]. In particular, we prove a theorem of the Kreĭn and Rutman type which asserts that the first eigenvalue of problem (1.3) is algebraically simple and its corresponding eigenfunction is strictly positive in Ω (Theorem 1.2).

We discuss our motivation and some of the modeling process leading to problem (1.3) (see [9]; [18]). The basic interpretation of the various terms in problem (1.3) is that $u(x)$ represents the population density of a species inhabiting the region Ω. The members of the population are assumed to move about Ω via the type of random walks occurring in Brownian motion that is modeled by the diffusive term $(1/\lambda)A$; hence $1/\lambda$ represents the diffusion rate, so small values of λ the population spreads more rapidly than for larger values of λ. The term $m(x)$ describes the rate at which the population would grow or decline at the location x in the absence of crowding or limitations on the availability of resources. The sign of $m(x)$ will be positive on favorable habitats for population growth and negative on unfavorable ones. Specifically $m(x)$ may be considered as a food source or any resource that will be good in some areas and bad in some others.

A solution $u(x) \in W^{2,p}(\Omega)$, $N < p < \infty$, of problem (1.3) is said to be non-trivial if it does not identically equal zero on $\overline{\Omega}$. We call a non-trivial solution u of problem (1.3) a positive solution if $u(x) \geq 0$ on $\overline{\Omega}$. Here it should be noticed that we have, by Sobolev's imbedding theorem (see [1, Theorem 4.12]),

$$W^{2,p}(\Omega) \subset C^1(\overline{\Omega}),$$

if $N < p < \infty$.

In this paper we study problem (1.3) under the following two conditions on the functions $m(x)$, $a(x')$ and $b(x')$:

(H.1) The weight function $m(x)$ is in the space $L^\infty(\Omega)$, and takes a positive value in a subset of positive measure in Ω.

(H.2) $a(x') + b(x') > 0$ on $\partial\Omega$, and $b(x') \not\equiv 0$ on $\partial\Omega$.

Condition (H.1) implies that there exists a region endowed with a nice food source, while condition (H.2) implies that the exterior of the domain is not totally reflective, that is, the boundary condition B is not the pure Neumann condition. It should be emphasized that problem (1.3) is a degenerate elliptic boundary value problem from an analytical point of view. This is due to the fact that the so-called Shapiro and Lopatinskii complementary condition is violated at the points $x' \in \partial\Omega$ where $a(x') = 0$. Amann [3] studied the non-degenerate case; more precisely, he assumes that the boundary $\partial\Omega$ is the disjoint union of the two closed subsets $M = \{x' \in \partial\Omega : a(x') = 0\}$ and $\partial\Omega \setminus M = \{x' \in \partial\Omega : a(x') > 0\}$, each of which is an $N-1$ dimensional compact smooth manifold.

First, in order to study problem (1.3) we consider the following non-homogeneous elliptic boundary value problem: Given functions $f(x)$ and $\varphi(x')$ defined in Ω and on $\partial\Omega$, respectively, find a function $u(x)$ in Ω such that

$$\begin{cases} Au = f & \text{in } \Omega, \\ Bu = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

We prove an existence and uniqueness theorem for problem (1.4) in the framework of Sobolev spaces of L^p type that will play an essential role in the study of problem (1.3).

If k is a positive integer and $1 < p < \infty$, we define the Sobolev space

$$\begin{aligned} W^{k,p}(\Omega) = {} & \text{the space of (equivalence classes of) functions} \\ & u \in L^p(\Omega) \text{ whose derivatives } D^\alpha u,\ |\alpha| \leq k, \text{ in the} \\ & \text{sense of distributions are in } L^p(\Omega), \end{aligned}$$

and the boundary space

$$\begin{aligned} B^{k-1/p,p}(\partial\Omega) = {} & \text{the space of the boundary values } u|_{\partial\Omega} \text{ of functions} \\ & u \in W^{k,p}(\Omega). \end{aligned}$$

In the space $B^{k-1/p,p}(\partial\Omega)$, we define a norm

$$|\varphi|_{B^{k-1/p,p}(\partial\Omega)} = \inf\left\{\|u\|_{W^{k,p}(\Omega)} : u \in W^{k,p}(\Omega),\ u|_{\partial\Omega} = \varphi\right\}.$$

It is easy to verify that the space $B^{k-1/p,p}(\partial\Omega)$ is a Banach space with respect to the norm $|\cdot|_{B^{k-1/p,p}(\partial\Omega)}$; more precisely, it is a Besov space (see [1]; [6]).

We introduce a subspace of $B^{1-1/p,p}(\partial\Omega)$ which is associated with the degenerate boundary condition

$$Bu = a(x')\frac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u$$

in the following way: We let

$$\begin{aligned} &B_B^{1-1/p,p}(\partial\Omega) \\ &= \left\{\varphi = a(x')\varphi_1 + b(x')\varphi_2 : \varphi_1 \in B^{1-1/p,p}(\partial\Omega),\ \varphi_2 \in B^{2-1/p,p}(\partial\Omega)\right\}, \end{aligned}$$

and define a norm

$$\begin{aligned} &|\varphi|_{B_B^{1-1/p,p}(\partial\Omega)} \\ &= \inf\left\{|\varphi_1|_{B^{1-1/p,p}(\partial\Omega)} + |\varphi_2|_{B^{2-1/p,p}(\partial\Omega)} : \varphi = a(x')\varphi_1 + b(x')\varphi_2\right\}. \end{aligned}$$

It is easy to verify (see [16, Lemma 4.7]) that the space $B_B^{1-1/p,p}(\partial\Omega)$ is a Banach space with respect to the norm $|\cdot|_{B_B^{1-1/p,p}(\partial\Omega)}$.

We remark that the space $B_B^{1-1/p,p}(\partial\Omega)$ is an intermediate space between the Besov spaces $B^{2-1/p,p}(\partial\Omega)$ and $B^{1-1/p,p}(\partial\Omega)$. In fact, we have the assertions

$$B_B^{1-1/p,p}(\partial\Omega) = \begin{cases} B^{2-1/p,p}(\partial\Omega) & \text{if } a(x') \equiv 0 \text{ on } \partial\Omega \text{ (the Dirichlet case)}, \\ B^{1-1/p,p}(\partial\Omega) & \text{if } a(x') > 0 \text{ on } \partial\Omega \text{ (the Robin case)}. \end{cases}$$

The first main result of this paper is stated as follows:

Theorem 1.1. *Assume that condition (H.2) is satisfied. Then the mapping*

$$\mathcal{A} := (A, B) : W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \bigoplus B_B^{1-1/p,p}(\partial\Omega)$$

is an algebraic and topological isomorphism for all $N < p < \infty$. *In particular, for any* $f \in L^p(\Omega)$ *and any* $\varphi \in B_B^{1-1/p,p}(\partial\Omega)$, *there exists a unique solution* $u \in W^{2,p}(\Omega)$ *of problem* (1.4).

The essential point in the proof of Theorem 1.1 is to consider the discontinuous term $c(x)$ of the differential operator A as a perturbation of a compact operator in the framework of Sobolev spaces.

The next theorem of the Kreĭn and Rutman type is a generalization of a result due to Manes–Micheletti [15] (see [11, Theorem 1.13]) to the degenerate case:

Theorem 1.2. *Assume that conditions (H.1) and (H.2) are satisfied. Then the first eigenvalue* $\lambda_1(m)$ *of problem* (1.3) *is positive and algebraically simple, and its corresponding eigenfunction* $\phi_1(x) \in W^{2,p}(\Omega)$, $N < p < \infty$, *may be chosen to be strictly positive in* Ω. *Moreover, no other eigenvalues,* $\lambda_j(m)$, $j \geq 2$, *have positive eigenfunctions.*

Remark 1.1. Theorem 1.2 is proved in the previous paper [19] under the condition that $A = -\Delta$ and $m(x) \in C(\overline{\Omega})$ (see [19, Theorem 1.2]).

The crucial point in the proof of Theorem 1.2 is how to controll the discontinuous weight function $m(x)$ in terms of Sobolev spaces.

By the Rayleigh principle, we can prove that the first eigenvalue $\lambda_1(m)$ is characterized by the variational formula

$$\begin{aligned} &\lambda_1(m) \\ &= \inf\left\{ \frac{(A\phi, \phi)_{L^2(\Omega)}}{\int_\Omega m(x)\phi^2\,dx} : \phi \in W^{2,2}(\Omega),\ B\phi = 0,\ \int_\Omega m(x)\phi^2\,dx > 0 \right\}. \end{aligned} \tag{1.5}$$

As an application of Theorem 1.2, we consider the following boundary value problem with an indefinite weight function: For a given non-negative function $h(x)$ defined in Ω, find a function $u(x)$ in Ω such that

$$\begin{cases} (A - \lambda\, m(x))\, u = h & \text{in } \Omega, \\ Bu = a(x')\dfrac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

By making good use of the Kreĭn and Rutman theory ([14]), we can generalize a result due to de Figueiredo (see [11, Theorem 1.14]) to the degenerate case. In fact, we can prove the following:

Theorem 1.3. *Assume that conditions (H.1) and (H.2) are satisfied. Then we have the following two assertions for* $N < p < \infty$:

(i) *If* $0 \leq \lambda < \lambda_1(m)$, *then problem* (1.6) *has a unique positive solution* $u(x) \in W^{2,p}(\Omega)$ *for any given function* $h(x) \in L^p(\Omega)$ *such that* $h(x) \geq 0$ *almost everywhere in* Ω.

(ii) *If* $\lambda \geq \lambda_1(m)$ *and if problem* (1.6) *has a positive solution* $u(x) \in W^{2,p}(\Omega)$ *for a given function* $h(x) \in L^p(\Omega)$ *such that* $h(x) \geq 0$ *almost everywhere in* Ω, *then it follows that* $\lambda = \lambda_1(m)$, $h(x) = 0$ *in* Ω *and that* $u(x) = t\,\phi_1(x)$ *for some positive constant* t.

Remark 1.2. Theorem 1.3 will play an essential role in the study of semilinear degenerate elliptic boundary value problems, by using the variational method. In a forthcoming paper, we shall derive lower bounds on the number of solutions of a class of semilinear degenerate elliptic boundary value problems, extending an earlier theorem due to Ambrosetti–Prodi [4] (see also [5]) to the degenerate case.

The rest of this paper is organized as follows. In Section 2 we summarize the basic definitions and results about ordered Banach spaces and the well-known

Kreĭn and Rutman theorem for strongly positive, compact linear operators (Theorem 2.1) that enter naturally in connection with elliptic eigenvalue problems. In Section 3 we study the non-homogeneous boundary value problem (1.4), and we prove Theorem 1.1 (Theorem 3.10). In Section 4 we introduce an ordered Banach subspace $C_\phi(\overline{\Omega})$ of $C(\overline{\Omega})$ which combines the good properties of the resolvent operator K with the good properties of the natural ordering of $C(\overline{\Omega})$, and we characterize the eigenvalues and positive eigenfunctions of K (Theorem 4.2). Section 5 is devoted to the proof of Theorem 1.2. Our proof is carried out just as in the proof of Brown–Lin [8, Theorem 3.5] by using Theorem 4.2 and a variant of the Kreĭn and Rutman theorem (Theorem 5.8). In the final Section 6 we prove Theorem 1.3. Our proof is based on an abstract version of Theorem 1.3 in the framework of ordered Banach spaces (Theorem 2.2).

2. Theory of Ordered Banach Spaces

In this section we present a brief description of basic definitions and results about the theory of positive mappings in ordered Banach spaces ([3]; [13]). A general class of second-order elliptic boundary value problems satisfies the maximum principle. Roughly speaking, this additional information means that the operators associated with the boundary value problems are compatible with the natural ordering of the underlying function spaces. In this way, we are led to the study of operator equations in the framework of ordered Banach spaces.

2.1. Ordered Banach Spaces and the Kreĭn and Rutman Theorem

Let X be a non-empty set. An ordering $\leq$ in X is a relation in X which is reflexive, transitive and antisymmetric. A non-empty set together with an ordering is called an ordered set.

Let V be a real vector space. An ordering $\leq$ in V is said to be linear if the following two conditions are satisfied:

(i) If x, $y \in V$ and $x \leq y$, then we have $x + z \leq y + z$ for all $z \in V$.
(ii) If x, $y \in V$ and $x \leq y$, then we have $\alpha x \leq \alpha y$ for all $\alpha \geq 0$.

A real vector space together with a linear ordering is called an ordered vector space.

If x, $y \in V$ and $x \leq y$, then the set

$$[x, y] = \{z \in X : x \leq z \leq y\}$$

is called an order interval.

If we let

$$P = \{x \in V : x \geq 0\},$$

then it is easy to verify that the set P has the following two conditions:

(iii) If x, $y \in P$, then $\alpha x + \beta y \in P$ for all α, $\beta \geq 0$.
(iv) If $x \neq 0$, then at least one of x and $-x$ does not belong to P, that is, $P \cap (-P) = \{0\}$.

The set P is called the positive cone of the ordering $\leq$.

Let E be a Banach space E with a linear ordering $\leq$. The Banach space E is called an ordered Banach space if the positive cone P is closed in E. We say that P is generating if, for each $x \in E$ there exist vectors u, $v \in P$ such that $x = u - v$. It is to be expected that the topology and the ordering of an ordered Banach space are closely related if the norm is monotone: If $0 \leq u \leq v$, then we have $\|u\| \leq \|v\|$.

For x, $y \in E$, we write

$$\begin{aligned} x &\geq y \quad \text{if } x - y \in P, \\ x &> y \quad \text{if } x - y \in P \setminus \{0\}. \end{aligned}$$

If the interior $\mathrm{Int}(P)$ is non-empty, then we write

$$x \gg y \quad \text{if } x - y \in \mathrm{Int}(P).$$

A linear operator $L\colon E \to E$ is said to be strongly positive if Lx belongs to $\mathrm{Int}(P)$ for every $x \in P \setminus \{0\}$, that is, if it satisfies the condition

$$x > 0 \implies Lx \gg 0.$$

A linear operator $L\colon E \to E$ is said to be compact (or completely continuous) if it is continuous (bounded) and maps bounded sets into relatively compact sets.

The next sharper version of the famous Kreĭn and Rutman theorem for strongly positive, compact linear operators will play a fundamental role in the sequel (see [14, Theorem 6.3]; [13, Chapter 2]; [10, Theorem 3.6.12]):

Theorem 2.1 (Kreĭn–Rutman). *Let (E, P) be an ordered Banach space with non-empty $\mathrm{Int}(P)$ and $L\colon E \to E$ a linear operator. If L is strongly positive and compact, then we have the following four assertions:*

(1) *The spectral radius*

$$\lambda_0 := \lim_{n \to \infty} \sqrt[n]{\|L^n\|}$$

of L is positive and λ_0 is the unique eigenvalue of L having a positive eigenfunction x_0: $Lx_0 = \lambda_0 x_0$.

(2) *The eigenvalue λ_0 is algebraically simple and $x_0 \gg 0$.*

(3) *The eigenvalue λ_0 is greater than all the remaining eigenvalues λ of L: $\lambda_0 > |\lambda|$.*

(4) *The adjoint operator $L^*\colon E^* \to E^*$ has λ_0 as an algebraically simple eigenvalue with a strictly positive eigenfunction x_0^*: $L^* x_0^* = \lambda_0 x_0^*$.*

The eigenvalue λ_0 is called the principal eigenvalue of L.

We recall that the algebraic multiplicity $\widetilde{k}_\lambda$ of an eigenvalue λ of L is the dimension of the generalized eigenspace of L

$$\widetilde{k}_\lambda := \dim \left(\bigcup_{\ell=1}^{\infty} N\left((L - \lambda I)^\ell\right) \right),$$

while the geometric multiplicity k_λ of the eigenvalue λ of L is the dimension of the eigenspace of L

$$k_\lambda := \dim N\left(L - \lambda I\right).$$

The eigenvalue λ of L is said to be algebraically (resp. geometrically) simple if $\widetilde{k}_\lambda = 1$ (resp. $k_\lambda = 1$).

2.2. Application of the Kreĭn and Rutman Theorem

Let E be an ordered Banach space and let $K\colon E \to E$ be a strongly positive, compact linear operator. As an application of the Kreĭn and Rutman theorem (Theorem 2.1), we consider the following non-homogeneous operator equation: For a given $h > 0$ in E, find an element $u \in E$ such that

$$\lambda u - Ku = h, \tag{2.1}$$

where λ is a real parameter.

The next theorem will play an important role in the proof of Theorem 1.3 in Section 6 (see [3]; [12]):

Theorem 2.2. *Let $K\colon E \to E$ be a strongly positive, compact linear operator and let $r(K) := \lim_{n\to\infty} \sqrt[n]{\|K^n\|}$ be its principal eigenvalue. Then we have the following three assertions:*

(i) *If $\lambda > r(K)$, then equation (2.1) has a unique positive solution u and $u \gg 0$ for any given $h > 0$ in E.*
(ii) *If $\lambda < r(K)$, then equation (2.1) has no positive solution for any given $h > 0$ in E.*
(iii) *If $\lambda = r(K)$, then equation (2.1) has no solution for any given $h > 0$ in E.*

Proof. The proof of Theorem 2.2 is divided into three steps.

Step 1: First, we assume that equation (2.1) has a positive solution $u \in P$:

$$\lambda u - Ku = h, \quad h > 0.$$

Let $x^* \in E^*$ be the strictly positive eigenfunction of K^* as in assertion (4) of Theorem 2.1:

$$K^* x^* = r(K) x^*, \quad x^* \gg 0. \tag{2.2}$$

Then it follows that

$$\begin{aligned}
(\lambda - r(K)) \langle x^*, u\rangle &= \lambda \langle x^*, u\rangle - \langle r(K) x^*, u\rangle \\
&= \langle x^*, \lambda u\rangle - \langle K^* x^*, u\rangle \\
&= \langle x^*, \lambda u - Ku\rangle \\
&= \langle x^*, h\rangle.
\end{aligned}$$

This proves that

$$\lambda - r(K) = \frac{\langle x^*, h\rangle}{\langle x^*, u\rangle} > 0,$$

so that

$$\lambda > r(K).$$

Step 2: Conversely, we assume that

$$\lambda > r(K).$$

In order to prove assertion (i), we need the following the positivity lemma (see [13, Theorem 2.16]):

Lemma 2.3 (the positivity lemma). *Let (E, P) be an ordered Banach space with non-empty interior $\mathrm{Int}(P)$. Assume that a linear operator $L\colon E \to E$ is strongly positive and compact. If λ_0 is the largest eigenvalue of L, then, for any given $h \in P$, the equation*

$$\lambda v - Lv = h \tag{2.3}$$

has a unique solution $v \in \mathrm{Int}(P)$ for each $\lambda > \lambda_0$.

Then, by applying Lemma 2.3 we obtain that equation (2.1) has a unique positive solution u. More precisely, we can prove that the unique solution u is given by the formula

$$u = R_\lambda h := \frac{h}{\lambda} + \frac{Kh}{\lambda^2} + \ldots = \sum_{k=0}^{\infty} \frac{K^k h}{\lambda^{k+1}}, \quad \lambda > r(K).$$

Furthermore, the strong positivity of K implies that

$$h > 0 \implies u = R_\lambda h \gg 0.$$

Therefore, we have proved that, for any given $h > 0$ equation (2.1) has a unique positive solution $u = R_\lambda h$ if and only if $\lambda > r(K)$, and further that $u \gg 0$.

Step 3: Finally, we assume, to the contrary, that there exists a solution $u \in E$ of equation (2.1) with $\lambda := r(K)$

$$r(K)u - Ku = h, \quad h > 0.$$

Then it follows from formula (2.2) that

$$0 < \langle x^*, h\rangle = \langle x^*, r(K)u - Ku\rangle = \langle r(K)x^* - K^*x^*, u\rangle = 0.$$

This is the desired contradiction.

Now the proof of Theorem 2.2 is complete. □

3. Elliptic Boundary Value Problems

In this section we study the non-homogeneous elliptic boundary value problem (1.4)

$$\begin{cases} Au = (A_0 + c(x))\, u = f & \text{in } \Omega, \\ Bu = a(x') \dfrac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Here it should be emphasized that the differential operator

$$A_0 u := -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{N} a^{ij}(x) \frac{\partial u}{\partial x_j} \right)$$

has smooth coefficients and that the zero-th order term $c(x)$ is a discontinuous function in $L^\infty(\Omega)$ satisfying the condition

$$c(x) \geq 0 \quad \text{almost everywhere in } \Omega.$$

In Subsection 3.1 we prove an existence and uniqueness theorem for the non-homogeneous boundary value problem (with $c(x) = 0$)

$$\begin{cases} A_0 u = g & \text{in } \Omega, \\ Bu = \phi & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

in the framework of Sobolev spaces of L^p type (Theorem 3.1). By using this theorem, we prove Theorem 1.1 in Subsection 3.2 (Theorem 3.10), while we prove the selfadjointness of the operator $\mathfrak{A}$ associated with problem (1.4) in Subsection 3.3 (Theorem 3.11) and the positivity of the resolvent K associated with problem (1.4) in Subsection 3.4 (Proposition 3.12), respectively.

3.1. Existence and Uniqueness Theorem for Problem (3.1)

First, we consider problem (3.1) in the framework of Sobolev spaces of L^p type. If we associate with problem (3.1) a continuous linear operator

$$\mathcal{A}_0 = (A_0, B) \colon W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \bigoplus B_B^{1-1/p,p}(\partial\Omega),$$

then we have the following existence and uniqueness theorem for problem (3.1):

Theorem 3.1. *If condition (H.2) is satisfied, then the mapping $\mathcal{A}_0$ is an algebraic and topological isomorphism for all $1 < p < \infty$. In particular, for any $g \in L^p(\Omega)$ and any $\phi \in B_B^{1-1/p,p}(\partial\Omega)$, there exists a unique solution $u \in W^{2,p}(\Omega)$ of problem (3.1).*

Proof. In order to prove Theorem 3.1, it suffices to show that the operator $\mathcal{A}_0$ is bijective. Indeed, the continuity of the inverse $\mathcal{A}_0^{-1}$ follows from an application of Banach's open mapping theorem (see [23, Chapter II, Section 5, Corollary]). The proof of Theorem 3.1 is divided into three steps.

Step 1: First, the next theorem proves the injectivity of the mapping $\mathcal{A}_0$:

Theorem 3.2. *If condition (H.2) is satisfied, then the mapping $\mathcal{A}_0$ is injective for $1 < p < \infty$. In particular, for any $g \in L^p(\Omega)$ and any $\phi \in B_B^{1-1/p,p}(\partial\Omega)$, there exists at most one solution $u \in W^{2,p}(\Omega)$ of problem (3.1).*

Proof. (1) The next regularity theorem for problem (3.1) due to Taira ([20, Theorem 8.2]) plays an important role in the proof of Theorem 3.2:

Theorem 3.3. *If condition (H.2) is satisfied, then we have, for all $s \in \mathbf{R}$ and all $p > 1$,*

$$u \in L^p(\Omega), \;\; A_0 u \in W^{s-2,p}(\Omega), \;\; Bu \in B_B^{s-1-1/p,p}(\partial\Omega) \implies u \in W^{s,p}(\Omega).$$

In particular, we have the assertion for the null space $N(\mathcal{A}_0)$ of $\mathcal{A}_0$

$$u \in L^p(\Omega), \;\; A_0 u = 0, \;\; Bu = 0 \implies u \in C^\infty(\overline{\Omega}).$$

(2) We make good use of a variant of the Bakel'man and Aleksandrov maximum principle in the framework of Sobolev spaces (see [7, Théorème 2]; [21, Lemmas 3.25 and 3.26 and Theorem 3.27]) in order to prove the uniqueness result in Theorem 1.1:

Theorem 3.4 (the weak maximum principle). *Assume that condition (H.2) is satisfied. If a function $v \in W^{2,p}(\Omega)$, $N < p < \infty$, satisfies the condition*

$$(A_0 + c(x))\, v(x) \leq 0 \quad \textit{almost everywhere in } \Omega,$$

then we have the inequality

$$\max_{\overline{\Omega}} v \leq \max_{\partial\Omega} v^+,$$

where

$$v^+(x) = \max\{v(x), 0\}.$$

Theorem 3.5 (the Hopf boundary point lemma). *Assume that condition (H.2) is satisfied. If a function $v \in W^{2,p}(\Omega)$, $N < p < \infty$, satisfies the condition*

$$(A_0 + c(x))\, v(x) \leq 0 \quad \textit{almost everywhere in } \Omega,$$

and attains a strict local non-negative maximum at a point x_0' of $\partial\Omega$, then we have the inequality

$$\frac{\partial u}{\partial \boldsymbol{\nu}}(x_0') > 0.$$

Theorem 3.6 (the strong maximum principle). *Assume that condition (H.2) is satisfied. If a function $v \in W^{2,p}(\Omega)$, $N < p < \infty$, satisfies the condition*

$$(A_0 + c(x))\, v(x) \leq 0 \quad \textit{almost everywhere in } \Omega$$

and attains a non-negative maximum at a point x_0 of Ω, then it is a constant.

(3) By combining Theorem 3.5 and Theorem 3.6, we can obtain the following:

Theorem 3.7. *Assume that condition (H.2) is satisfied. If a function $u \in W^{2,p}(\Omega)$, $N < p < \infty$, satisfies the conditions*

$$\begin{cases} Au = (A_0 + c(x))\, u \geq 0 & \textit{almost everywhere in } \Omega, \\ Bu = a(x')\dfrac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u \geq 0 & \textit{on } \partial\Omega, \end{cases}$$

then it follows that

$$u(x) \geq 0 \quad \textit{in } \Omega.$$

Proof. Assume, to the contrary, that there exists a point $x_0 \in \overline{\Omega} = \Omega \cup \partial\Omega$ such that

$$u(x_0) = \min_{x \in \overline{\Omega}} u(x) < 0. \tag{3.2}$$

(a) If $x_0 \in \Omega$, then it follows from an application of the strong maximum principle (Theorem 3.6) with $v := -u$ that

$$u(x) \equiv u(x_0) < 0, \quad x \in \Omega.$$

Hence we have, for any point $x' \in \partial\Omega$,

$$0 \leq Bu(x') = a(x')\frac{\partial u}{\partial \boldsymbol{\nu}}(x') + b(x')u(x') = b(x')u(x_0).$$

However, since $b(x') \geq 0$ and $b(x') \not\equiv 0$ on $\partial\Omega$, we obtain that

$$u(x_0) \geq 0.$$

This contradicts condition (3.2).

(b) If $x_0 \in \partial\Omega$, then we may assume that $u(x)$ attains a strict negative minimum at a point x_0, that is,

$$\begin{cases} u(x_0) = \min_{x \in \overline{\Omega}} u(x) < 0, \\ u(x) > u(x_0), \quad x \in \Omega. \end{cases}$$

Thus it follows from an application of the boundary point lemma (Theorem 3.5) with $v := -u$ that

$$\frac{\partial u}{\partial \boldsymbol{\nu}}(x_0) < 0.$$

However, we have, by condition (H.2),

$$0 \leq Bu(x_0) = a(x_0)\frac{\partial u}{\partial \boldsymbol{\nu}}(x_0) + b(x_0)u(x_0) < 0.$$

This is also a contradiction.

The proof of Theorem 3.7 is complete. □

Therefore, by applying Theorem 3.7 to the functions $\pm u(x)$ we can prove the following uniqueness theorem for problem (1.4) (and hence problem (3.1)) in the framework of Sobolev spaces of L^p type:

Theorem 3.8. *Assume that condition (H.2) is satisfied. If a function $u \in W^{2,p}(\Omega)$, $N < p < \infty$, satisfies the conditions*

$$\begin{cases} (A_0 + c(x))\, u = 0 & \text{almost everywhere in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases}$$

then it follows that

$$u(x) \equiv 0 \quad \text{in } \Omega.$$

(4) By combining Theorem 3.3 and Theorem 3.8 with $c(x) \equiv 0$, we obtain that the mapping

$$\mathcal{A}_0 \colon W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \bigoplus B_B^{1-1/p,p}(\partial\Omega)$$

is injective for $1 < p < \infty$.

The proof of Theorem 3.2 is complete. □

Step 2: Secondly, we prove the surjectivity of the mapping $\mathcal{A}_0$. The next theorem due to Taira ([20, Proposition 8.9]) plays an essential role in the proof:

Theorem 3.9. *Assume that condition (H.2) is satisfied. Then the mapping*

$$\mathcal{A}_0 = (A_0, B) \colon W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \bigoplus B_B^{1-1/p,p}(\partial\Omega)$$

is a Fredholm operator with index zero for $1 < p < \infty$*, that is, we have the formula*

$$\operatorname{ind} \mathcal{A}_0 := \dim N(\mathcal{A}_0) - \operatorname{codim} R(\mathcal{A}_0) = 0.$$

By Theorem 3.2, it follows that the mapping $\mathcal{A}_0$ is injective for $1 < p < \infty$, that is, $\dim N(\mathcal{A}_0) = 0$. Hence it is also surjective for $1 < p < \infty$, since we have the formula

$$\operatorname{codim} R(\mathcal{A}_0) = \dim N(\mathcal{A}_0) = 0.$$

Step 3: Summing up, we have proved that the mapping

$$\mathcal{A}_0 = (A_0, B) \colon W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \bigoplus B_B^{1-1/p,p}(\partial\Omega)$$

is an algebraic and topological isomorphism for $1 < p < \infty$.

Now the proof of Theorem 3.1 is complete. □

3.2. Proof of Theorem 1.1

This subsection is devoted to the proof of Theorem 1.1. In fact, we prove the following existence and uniqueness theorem for problem (1.4) (cf. [19, Theorem 3.1]):

Theorem 3.10. *Assume that condition (H.2) is satisfied. Then the mapping*

$$\mathcal{A} := (A, B) \colon W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \bigoplus B_B^{1-1/p,p}(\partial\Omega)$$

is an algebraic and topological isomorphism for all $N < p < \infty$*. In particular, for any* $f \in L^p(\Omega)$ *and any* $\varphi \in B_B^{1-1/p,p}(\partial\Omega)$*, there exists a unique solution* $u \in W^{2,p}(\Omega)$ *of problem* (1.4)*.*

Proof. We have only to show that the operator $\mathcal{A}$ is bijective, since the continuity of the inverse $\mathcal{A}^{-1}$ follows from an application of Banach's open mapping theorem (see [23, Chapter II, Section 5, Corollary]).

The essential point in the proof is to consider the discontinuous term $c(x)$ of the operator $A = A_0 + c(x)$ as a perturbation of a compact operator in the framework of Sobolev spaces. The proof of Theorem 3.10 is divided into three steps.

Step 1: First, if C is the multiplication operator by the function $c(x) \in L^\infty(\Omega)$, then it follows from an application of the Rellich and Kondrachov theorem (see [1, Theorem 6.3]) that the mapping

$$\mathcal{C} \colon W^{2,p}(\Omega) \longrightarrow L^p(\Omega)$$

is compact.

Therefore, we obtain that the mapping

$$\mathcal{A} = \mathcal{A}_0 + (\mathcal{C}, 0) \colon W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \bigoplus B_B^{1-1/p,p}(\partial\Omega)$$

is a Fredholm operator with index zero, since we have, by Theorem 3.9,

$$\operatorname{ind} \mathcal{A} = \operatorname{ind} \mathcal{A}_0 = 0.$$

Step 2: On the other hand, the uniqueness result in Theorem 3.10 follows from a variant of the Bakel'man and Aleksandrov maximum principle in the framework of Sobolev spaces of L^p type, $N < p < \infty$ (Theorem 3.8):

$$\begin{cases} Au = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases} \Longrightarrow u(x) \equiv 0 \quad \text{in } \Omega.$$

Step 3: By Step 2, it follows that the mapping

$$\mathcal{A} = (A, B) \colon W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \bigoplus B_B^{1-1/p,p}(\partial\Omega)$$

is injective for $N < p < \infty$. Hence it is also surjective for $N < p < \infty$, since we have the formula

$$\operatorname{codim} R(\mathcal{A}) = \dim N(\mathcal{A}) = 0.$$

Summing up, we have proved that the mapping $\mathcal{A}$ is an algebraic and topological isomorphism for $N < p < \infty$.

The proof of Theorem 3.10 (and hence Theorem 1.1) is now complete. □

3.3. Selfadjointness of the Operator $\mathfrak{A}$

This subsection is devoted to the study of the eigenvalue problem (1.3) with an indefinite weight function $m(x) \in L^\infty(\Omega)$. First, we introduce a densely defined, closed linear operator $\mathfrak{A}$ from the Hilbert space $L^2(\Omega)$ into itself as follows.

(a) The domain of definition $D(\mathfrak{A})$ is the space

$$D(\mathfrak{A}) := \left\{ v \in W^{2,2}(\Omega) : Bv = a(x')\frac{\partial v}{\partial \boldsymbol{\nu}} + b(x')v = 0 \quad \text{on } \partial\Omega \right\}.$$

(b) $\mathfrak{A}v := Av = (A_0 + c(x))\, v$, $v \in D(\mathfrak{A})$.

First, we show that the operator $\mathfrak{A}$ is non-negative and selfadjoint in $L^2(\Omega)$ (cf. [17, Theorem 0]):

Theorem 3.11. *If condition (H.2) is satisfied, then the operator $\mathfrak{A}$ is non-negative and selfadjoint in $L^2(\Omega)$.*

Proof. The proof is divided into two steps.

Step 1: Let $\mathfrak{A}^*$ be the adjoint operator of $\mathfrak{A}$. First, we show that $\mathfrak{A}$ is selfadjoint:

$$\mathfrak{A}^* = \mathfrak{A}. \tag{3.3}$$

To do this, we introduce an auxiliary closed linear operator $\mathfrak{A}_0 \colon L^2(\Omega) \to L^2(\Omega)$ defined by the formula

$$\begin{cases} \mathfrak{A}_0 := \mathfrak{A} - c(x) I, \\ D\left(\mathfrak{A}_0\right) := D\left(\mathfrak{A}\right). \end{cases}$$

Since the operator $c(x)I$ is non-negative and bounded in $L^2(\Omega)$, we have only to show that

$$\mathfrak{A}_0^* = \mathfrak{A}_0. \tag{3.4}$$

(a) We prove that the adjoint operator $\mathfrak{A}_0^*$ is an extension of the operator $\mathfrak{A}_0$:

$$\mathfrak{A}_0 \subset \mathfrak{A}_0^*. \tag{3.5}$$

By the first Green formula (see [22, Theorem 14.2]), we have, for all functions u and v in $W^{2,2}(\Omega)$,

$$\int_\Omega (A_0 u \cdot \overline{v} - u \cdot \overline{A_0 v})\, dx = \int_{\partial\Omega} \left(\frac{\partial u}{\partial \boldsymbol{\nu}} \cdot \overline{v} - u \cdot \overline{\frac{\partial v}{\partial \boldsymbol{\nu}}} \right) d\sigma, \tag{3.6}$$

where the right-hand side is the inner product of the Hilbert space $L^2(\partial\Omega)$. However, if, in addition, the functions u and v satisfy the boundary conditions

$$a(x')\frac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = 0 \quad \text{on } \partial\Omega,$$
$$a(x')\frac{\partial v}{\partial \boldsymbol{\nu}} + b(x')v = 0 \quad \text{on } \partial\Omega,$$

then it follows that

$$\begin{pmatrix} \dfrac{\partial u}{\partial \boldsymbol{\nu}} & u \\ \overline{\dfrac{\partial v}{\partial \boldsymbol{\nu}}} & \overline{v} \end{pmatrix} \begin{pmatrix} a(x') \\ b(x') \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on } \partial\Omega.$$

Thus we obtain that

$$\begin{vmatrix} \dfrac{\partial u}{\partial \boldsymbol{\nu}} & u \\ \overline{\dfrac{\partial v}{\partial \boldsymbol{\nu}}} & \overline{v} \end{vmatrix} = 0 \quad \text{on } \partial\Omega, \tag{3.7}$$

since we have the condition

$$(a(x'), b(x')) \neq (0,0) \quad \text{on } \partial\Omega.$$

Therefore, combining formulas (3.6) and (3.7) we find that, for all functions u, $v \in D(\mathfrak{A}_0)$,

$$\int_\Omega (A_0 u \cdot \overline{v} - u \cdot \overline{A_0 v})\, dx = 0,$$

or equivalently

$$(\mathfrak{A}_0 u, v) = (u, \mathfrak{A}_0 v), \quad u, v \in D(\mathfrak{A}_0).$$

This proves the desired assertion (3.5).

(b) Secondly, we prove that

$$D(\mathfrak{A}_0^*) \subset D(\mathfrak{A}_0).$$

Let $v(x)$ be an arbitrary element of the domain $D(\mathfrak{A}_0^*)$. It follows from an application of Theorem 3.1 with $p := 2$ that the operator

$$\mathfrak{A}_0 \colon D(\mathfrak{A}_0) \longrightarrow L^2(\Omega)$$

is bijective. Thus there exists an element $v_0 \in D\left(\mathfrak{A}_0\right)$ such that

$$\mathfrak{A}_0 v_0 = \mathfrak{A}_0^* v.$$

Then, by assertion (3.5) it follows that, for all $u \in D\left(\mathfrak{A}_0\right)$,

$$\left(\mathfrak{A}_0 u, v - v_0\right) = \left(u, \mathfrak{A}_0^* v - \mathfrak{A}_0 v_0\right) = 0.$$

This proves that

$$v = v_0 \in D\left(\mathfrak{A}_0\right),$$

since the operator $\mathfrak{A}_0 \colon D\left(\mathfrak{A}_0\right) \to L^2(\Omega)$ is surjective.

Therefore, we have proved assertion (3.4) and hence the desired assertion (3.3).

Step 2: Finally, it remains to show that the operator $\mathfrak{A} = \mathfrak{A}_0 + c(x)I$ is non-negative:

$$(\mathfrak{A}u, u) \geq 0, \quad u \in D(\mathfrak{A}). \tag{3.8}$$

By condition (H.2), it follows that

$$u(x') = 0 \quad \text{on } M = \{x' \in \partial\Omega : a(x') = 0\},$$

and further that

$$\frac{\partial u}{\partial \boldsymbol{\nu}}(x') = -\frac{b(x')}{a(x')}\, u(x') \quad \text{on } \partial\Omega \setminus M.$$

Hence we have, by the second Green formula (see [22, Theorem 14.8]),

$$\begin{aligned}
\int_\Omega Au \cdot \overline{u}\, dx &= \sum_{i,j=1}^N \int_\Omega a^{ij}(x) \frac{\partial u}{\partial x_i}(x) \cdot \overline{\frac{\partial u}{\partial x_j}(x)}\, dx + \int_\Omega c(x)\, |u(x)|^2\, dx \\
&\quad - \int_{\partial\Omega} \frac{\partial u}{\partial \boldsymbol{\nu}}(x') \cdot \overline{u(x')}\, d\sigma \\
&= \sum_{i,j=1}^N \int_\Omega a^{ij}(x) \frac{\partial u}{\partial x_i}(x) \cdot \overline{\frac{\partial u}{\partial x_j}(x)}\, dx + \int_\Omega c(x)\, |u(x)|^2\, dx \\
&\quad + \int_{\partial\Omega \setminus M} \frac{b(x')}{a(x')}\, |u(x')|^2\, d\sigma \\
&\geq \int_{\partial\Omega \setminus M} \frac{b(x')}{a(x')}\, |u(x')|^2\, d\sigma \\
&\geq 0.
\end{aligned}$$

This proves the desired inequality (3.8).

The proof of Theorem 3.11 is complete. □

3.4. Positivity of the Resolvent $\boldsymbol{K}$

In this subsection we study the following homogeneous boundary value problem:

$$\begin{cases} Au = (A_0 + c(x))\,u = g & \text{in } \Omega, \\ Bu = a(x')\dfrac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.9}$$

First, we let

$$W^{2,p}_B(\Omega) := \left\{u \in W^{2,p}(\Omega) : Bu = 0 \quad \text{on } \partial\Omega\right\}, \qquad N < p < \infty.$$

By applying Theorem 3.10, we find that problem (3.9) has a unique solution $u \in W^{2,p}_B(\Omega)$ for any $g \in L^p(\Omega)$. Therefore, we can introduce a continuous linear operator (resolvent)

$$K\colon L^p(\Omega) \longrightarrow W^{2,p}_B(\Omega)$$

by the formula $u = Kg$. Moreover, by the Ascoli and Arzelà theorem it follows that the resolvent K, considered as an operator

$$K\colon C(\overline{\Omega}) \longrightarrow C^1(\overline{\Omega}),$$

is compact if $N < p < \infty$. Indeed, it suffices to note that, by Sobolev's imbedding theorem, the Sobolev space $W^{2,p}(\Omega)$ is continuously imbedded into the Hölder space $C^{2-N/p}(\overline{\Omega})$ with $2 - N/p > 1$, for all $N < p < \infty$.

Then, by using Theorem 3.10 we can easily prove the following:

Claim 3.1. A function $u(x) \in L^p(\Omega)$, $N < p < \infty$, is a solution of the problem

$$\begin{cases} Au = \lambda u & \text{in } \Omega, \\ Bu = a(x')\dfrac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = 0 & \text{on } \partial\Omega \end{cases}$$

if and only if it satisfies the operator equation

$$u = \lambda K u \quad \text{in } C(\overline{\Omega}). \tag{3.10}$$

For two functions u and v in $C(\overline{\Omega})$, we write $u \leq v$ if $u(x) \leq v(x)$ for all $x \in \overline{\Omega}$. Then it is easy to verify that the space $C(\overline{\Omega})$ is an ordered Banach space with the linear ordering $\leq$ and the positive cone

$$P = \{u \in C(\overline{\Omega}) : u \geq 0 \text{ on } \overline{\Omega}\}.$$

However, we shall introduce another ordered Banach subspace of $C(\overline{\Omega})$ for the fixed point equation (3.10) which combines the good properties of the resolvent K with the good properties of the natural ordering of $C(\overline{\Omega})$.

To do this, we need the following (see [19, Lemma 3.7]):

Proposition 3.12. *Assume that condition (H.2) is satisfied. If $v(x) \in C(\overline{\Omega})$ and if $v(x) \geq 0$ but $v(x) \not\equiv 0$ on $\overline{\Omega}$, then the function $u = Kv \in W^{2,p}(\Omega)$, $N < p < \infty$, satisfies the following three conditions:*

(a) $u(x') = 0$ *on* $M = \{x' \in \partial\Omega : a(x') = 0\}$.

(b) $u(x) > 0$ *on* $\overline{\Omega} \setminus M$.

(c) *For the conormal derivative $\partial u/\partial \boldsymbol{\nu}$ of u, we have the inequality*

$$\frac{\partial u}{\partial \boldsymbol{\nu}}(x') < 0 \quad \text{on } M.$$

In particular, the resolvent $K\colon C(\overline{\Omega}) \to C(\overline{\Omega})$ is positive, that is, K maps the positive cone P into itself.

4. The Ordered Banach Space $C_\phi(\overline{\Omega})$

Now we can introduce an ordered Banach subspace $C_\phi(\overline{\Omega})$ of $C(\overline{\Omega})$ which combines the good properties of the resolvent operator K with the good properties of the natural ordering of $C(\overline{\Omega})$.

If we let

$$\phi(x) := K1(x),$$

then it follows from an application of Theorem 3.10 that the function $\phi(x) \in W^{2,p}(\Omega)$, $N < p < \infty$, is the unique solution of the problem

$$\begin{cases} A\phi = 1 & \text{in } \Omega, \\ B\phi = a(x')\dfrac{\partial \phi}{\partial \boldsymbol{\nu}} + b(x')\phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Here it should be noticed that we have, by Sobolev's imbedding theorem (see [1, Theorem 4.12]),

$$\phi(x) \in W^{2,p}(\Omega) \subset C^1(\overline{\Omega}), \quad N < p < \infty.$$

Moreover, it follows from an application of Proposition 3.12 with $v \equiv 1$ that the function $\phi(x) = K1(x)$ satisfies the conditions

$$\begin{cases} \phi(x) > 0 & \text{on } \overline{\Omega} \setminus M, \\ \phi(x') = 0 & \text{on } M, \\ \dfrac{\partial \phi}{\partial \boldsymbol{\nu}}(x') < 0 & \text{on } M, \end{cases}$$

where

$$M = \{x' \in \partial\Omega : a(x') = 0\}.$$

We define a subspace $C_\phi(\overline{\Omega})$ of $C(\overline{\Omega})$ by the formula

$$\begin{aligned} C_\phi(\overline{\Omega}) = &\{u \in C(\overline{\Omega}) : \text{there is a constant } \alpha > 0 \text{ such that} \\ &-\alpha\phi(x) \leq u(x) \leq \alpha\phi(x) \text{ in } \Omega\}, \end{aligned}$$

with the norm

$$\|u\|_e = \inf\{\alpha > 0 : -\alpha\phi(x) \leq u(x) \leq \alpha\phi(x) \text{ in } \Omega\}.$$

If we let

$$P_\phi = \{u \in C_\phi(\overline{\Omega}) : u \geq 0 \text{ on } \overline{\Omega}\},$$

then it is easy to verify that the space $C_\phi(\overline{\Omega})$ is an ordered Banach space having the positive cone P_ϕ with non-empty interior $\mathrm{Int}(P_\phi)$. Indeed, every function $u(x) \in C^1(\overline{\Omega})$ which satisfies the conditions

$$\begin{cases} u(x) > 0 & \text{on } \overline{\Omega} \setminus M, \\ u(x') = 0 & \text{on } M, \\ \dfrac{\partial u}{\partial \boldsymbol{\nu}}(x') < 0 & \text{on } M \end{cases}$$

belongs to the interior of the positive cone P_ϕ.

This setting has the advantages that it takes into consideration in an optimal way the *a priori* information given by the maximum principle and that it is amenable to the methods of abstract functional analysis (see [3]; [12]). More precisely, we can prove the following (see [19, Proposition 3.8]):

Proposition 4.1. *The resolvent K maps $C(\overline{\Omega})$ compactly into $C_\phi(\overline{\Omega})$. Moreover, the resolvent K, considered as an operator $K \colon C(\overline{\Omega}) \to C_\phi(\overline{\Omega})$, is strongly positive, that is, $Kv \in \mathrm{Int}(P_\phi)$ for all $v \in P \setminus \{0\}$.*

4.1. Eigenvalues of the Resolvent K

In this subsection we consider the resolvent K as an operator in the ordered Banach space $C_\phi(\overline{\Omega})$, and prove important results concerning its eigenvalues and corresponding eigenfunctions.

First, it follows from Proposition 4.1 that the resolvent

$$K \colon C_\phi(\overline{\Omega}) \longrightarrow C_\phi(\overline{\Omega})$$

is strongly positive and compact. Moreover, we find from Theorem 3.11 that all the eigenvalues of K are positive. Indeed, it suffices to note that

$$\begin{aligned} &Kv = \mu\, v, \quad v \in C_\phi(\overline{\Omega}),\ \mu \neq 0, \\ &\Longrightarrow \\ &\mathfrak{A}v = \frac{1}{\mu}\, v, \quad v \in W^{2,p}(\Omega),\ N < p < \infty. \end{aligned}$$

Therefore, we obtain that K has a countable number of positive eigenvalues, μ_j, which may accumulate only at 0. Hence they may be arranged in a decreasing sequence

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_j \geq \cdots \longrightarrow 0,$$

where each eigenvalue is repeated according to its geometric multiplicity.

By applying Theorem 2.1 with

$$E := C_\phi(\overline{\Omega}), \quad P := P_\phi, \quad L := K,$$

we can characterize the eigenvalues and positive eigenfunctions of the resolvent K as follows (see [13]):

Theorem 4.2. *The resolvent K, considered as an operator*

$$K\colon C_\phi(\overline{\Omega}) \longrightarrow C_\phi(\overline{\Omega}),$$

has the following spectral properties:

(i) *The largest eigenvalue (principal eigenvalue)*

$$\mu_1 = r(K) = \lim_{n\to\infty} \sqrt[n]{\|K^n\|}$$

is algebraically simple and has a strictly positive eigenfunction $\phi_1(x)$.

(ii) *The other eigenvalues, μ_j, $j \geq 2$, do not possess positive eigenfunctions.*

As an application, we consider a special case of equation (1.6) with $m(x) \equiv 1$: For a given function $h \in P_\phi$, find a function $u(x)$ such that

$$\begin{cases} (A-\lambda)\, u = h & \text{in } \Omega, \\ Bu = a(x')\dfrac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Then, by combining Theorem 2.2 (with $\lambda := 1/\lambda$) and Proposition 4.1 we obtain the main result of this subsection (see [12, Theorem 16.6]):

Theorem 4.3. *If μ_1 is the principal eigenvalue of the operator K, then we have the following three assertions:*

(i) *If $0 < \lambda < 1/\mu_1$, then problem* (4.2) *has a unique positive solution $u \in \mathrm{Int}(P_\phi)$ for any given function $h \in P_\phi$.*

(ii) *If $\lambda > 1/\mu_1$, then problem* (4.2) *has no positive solution for any given function $h \in P_\phi$.*

(iii) *If $\lambda = 1/\mu_1$, then problem* (4.2) *has no solution for any given function $h \in P_\phi$.*

5. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2 which is inspired by Brown–Lin [8, Theorem 3.5] (see also [2]). The crucial point in the proof is how to controll the discontinuous term $m(x)$ in terms of Sobolev spaces.

5.1. Eigenvalue Problems with Indefinite Weight Function

This subsection is devoted to the study of the eigenvalue problem (1.3)

$$\begin{cases} Au = \lambda m(x) u & \text{in } \Omega, \\ Bu = a(x')\dfrac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = 0 & \text{on } \partial\Omega \end{cases}$$

with an indefinite weight function $m(x) \in L^\infty(\Omega)$. First, we recall (see Theorem 3.11 and Theorem 4.2) that the operator $\mathfrak{A}$ is a positive and selfadjoint operator from the Hilbert space $L^2(\Omega)$ into itself defined as follows:

(a) The domain of definition $D(\mathfrak{A})$ is the space

$$D(\mathfrak{A}) = \left\{ v \in W^{2,2}(\Omega) : Bv = 0 \quad \text{on } \partial\Omega \right\}.$$

(b) $\mathfrak{A}v = Av$, $v \in D(\mathfrak{A})$.

Then the next theorem is obtained by combining Theorem 4.2 and Theorem 3.11 (see [17, Theorem 0]):

Theorem 5.1. *Assume that condition (H.2) is satisfied. Then the spectrum of* $\mathfrak{A}$ *consists only of the discrete eigenvalues*

$$0 < \gamma_1 < \gamma_2 \leq \dots .$$

The first eigenvalue $\gamma_1 = 1/\mu_1$ *is algebraically simple and its corresponding eigenfunction* $\phi_1(x) \in W^{2,p}(\Omega)$, $N < p < \infty$, *may be chosen to be strictly positive in* Ω. *Namely, we have the assertions*

$$\begin{cases} A\phi_1 = \gamma_1 \phi_1 & \text{in } \Omega, \\ \phi_1 > 0 & \text{in } \Omega, \\ B\phi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, the other eigenvalues, γ_j, $j \geq 2$, *do not possess positive eigenfunctions.*

Remark 5.1. The first eigenvalue γ_1 is characterized by the variational formula

$$\gamma_1 = \inf \left\{ \frac{(\mathfrak{A}v, v)_{L^2(\Omega)}}{\int_\Omega v^2 \, dx} : v \in D(\mathfrak{A}), \ v \neq 0 \right\}.$$

Indeed, the L^p regularity for $p > N$ in Theorem 5.1 follows from the standard bootstrap procedure by repeatedly using the Sobolev imbeddings (see [1, Theorem 4.12]):

$$W^{2,p}(\Omega) \subset L^q(\Omega) \quad \text{if} \quad \begin{cases} 1 < p < \dfrac{N}{2}, \quad p \leq q \leq p^* =: \dfrac{pN}{N-2p}, \\ \dfrac{N}{2} = p \leq q < \infty, \\ \dfrac{N}{2} < p \leq q \leq \infty. \end{cases}$$

If we introduce a linear operator

$$\mathcal{T}(\lambda) := \mathfrak{A} - \lambda m(x) I, \quad \lambda \geq 0, \tag{5.1}$$

then it follows that $\mathcal{T}(\lambda)$ is selfadjoint in $L^2(\Omega)$ and further that the eigenvalues and eigenfunctions of $\mathcal{T}(\lambda)$ correspond to those of the problem

$$\begin{cases} (A - \lambda m(x))\, v = \mu(\lambda)\, v & \text{in } \Omega, \\ Bv = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

Furthermore, by applying Theorem 4.2 to our situation we can obtain the following:

Theorem 5.2. *Assume that conditions (H.1) and (H.2) are satisfied. Then the spectrum of* $\mathcal{T}(\lambda)$ *consists only of the discrete eigenvalues*

$$\mu_1(\lambda) < \mu_2(\lambda) \leq \cdots . \tag{5.3}$$

The first eigenvalue $\mu_1(\lambda)$ is algebraically simple and its corresponding eigenfunction $\phi_1(x) \in W^{2,p}(\Omega)$, $N < p < \infty$, may be chosen to be strictly positive in Ω.

Proof. Indeed, by rescaling we may assume that

$$|m(x)| < 1 \quad \text{almost everywhere in } \Omega.$$

Then it is easy to see that the eigenvalue problem (5.2) is equivalent to the eigenvalue problem

$$\begin{cases} (A + \lambda\,(1 - m(x)))\, v = \gamma(\lambda) v & \text{in } \Omega, \\ Bv = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.4}$$

where

$$\lambda(1 - m(x)) \geq 0 \quad \text{almost everywhere in } \Omega,$$

and

$$\gamma(\lambda) = \mu(\lambda) + \lambda.$$

By applying Theorem 1.1 with $c(x) := c(x) + \lambda(1 - m(x)) \in L^\infty(\Omega)$, we find that the boundary value problem

$$\begin{cases} (A + \lambda\,(1 - m(x)))\, u = g & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in W^{2,p}(\Omega)$ for any $g \in L^p(\Omega)$. Hence, we can introduce a continuous linear operator (resolvent)

$$K_\lambda \colon L^p(\Omega) \longrightarrow W^{2,p}(\Omega)$$

by the formula $u = K_\lambda g$.

By applying Theorem 4.2 with $c(x) := c(x) + \lambda(1 - m(x))$, we can obtain the following two assertions:

(i) The spectrum of the eigenvalue problem (5.4) consists only of the discrete eigenvalues
$$0 < \gamma_1(\lambda) < \gamma_2(\lambda) \leq \cdots .$$

(ii) The first eigenvalue $\gamma_1(\lambda)$ is algebraically simple and its corresponding eigenfunction $\phi_1(x) \in W^{2,p}(\Omega)$ may be chosen to be strictly positive in Ω.

Here it should be noticed that the spectral radii $r(K_\lambda)$ and $r(K_0)$ are respectively given by the formulas

$$r(K_\lambda) := \lim_{n\to\infty} \sqrt[n]{\|K_\lambda^n\|} = \frac{1}{\gamma_1(\lambda)},$$
$$r(K_0) := r(K) = \lim_{n\to\infty} \sqrt[n]{\|K^n\|} = \mu_1 = \frac{1}{\gamma_1}.$$

Therefore, the desired assertion (5.3) follows by combining the above two assertions (i) and (ii) if we take

$$\mu_j(\lambda) = \gamma_j(\lambda) - \lambda, \quad j = 1, 2, \ldots .$$

The proof of Theorem 5.2 is complete. $\square$

Remark 5.2. The first eigenvalue $\mu_1(\lambda)$ is characterized by the variational formula

$$\begin{aligned}\mu_1(\lambda) &= \inf\left\{\frac{(\mathcal{T}(\lambda)v,v)_{L^2(\Omega)}}{\int_\Omega v^2\,dx} : v \in D(\mathfrak{A}),\ v \neq 0\right\}\\ &= \inf\left\{\frac{(\mathfrak{A}v,v)_{L^2(\Omega)} - \lambda\int_\Omega m(x)\,v^2\,dx}{\int_\Omega v^2\,dx} : v \in D(\mathfrak{A}),\ v \neq 0\right\}. \end{aligned} \tag{5.5}$$

5.2. Proof of Theorem 1.2

The proof of Theorem 1.2 is divided into six steps.

Step 1: If $\lambda \geq 0$, we let

$$\begin{aligned} Q_\lambda(v) &:= (\mathcal{T}(\lambda)v,v)_{L^2(\Omega)}\\ &= (\mathfrak{A}v,v)_{L^2(\Omega)} - \lambda\int_\Omega m(x)v^2\,dx\\ &= \int_\Omega Av\cdot v\,dx - \lambda\int_\Omega m(x)v^2\,dx, \quad v \in D(\mathfrak{A}). \end{aligned}$$

Then the next lemma characterizes the range of possible eigenvalues corresponding to non-negative eigenfunctions of problem (1.3) (see [19, Lemma 4.3]):

Lemma 5.3. *If there exists a non-negative eigenfunction $\psi(x)$ corresponding to an eigenvalue λ of problem* (1.3)*, then we have, for all $v \in D(\mathfrak{A})$,*

$$Q_\lambda(v) \geq 0. \tag{5.6}$$

In particular, it follows from formula (5.5) *that*

$$\mu_1(\lambda) \geq 0.$$

Step 2: Now we let

$$\lambda_1(m) := \inf\left\{\frac{(\mathfrak{A}v,v)_{L^2(\Omega)}}{\int_\Omega m(x)v^2\,dx} : v \in D(\mathfrak{A}),\ \int_\Omega m(x)v^2\,dx > 0\right\}. \tag{5.7}$$

Since we have, by condition (H.2),

$$a(x') = 0 \implies v(x') = 0,$$

it follows from an application of the second Green formula (see [22, Theorem 14.8]) that

$$\begin{aligned} (\mathfrak{A}v,v)_{L^2(\Omega)} &= \int_\Omega Av\cdot v\,dx\\ &= \sum_{i,i=1}^N \int_\Omega a^{ij}(x)\frac{\partial v}{\partial x_i}\cdot\frac{\partial v}{\partial x_j}\,dx + \int_\Omega c(x)|v(x)|^2\,dx\\ &\quad - \int_{\partial\Omega}\frac{\partial v}{\partial \boldsymbol{\nu}}\cdot v\,d\sigma\\ &= \sum_{i,i=1}^N \int_\Omega a^{ij}(x)\frac{\partial v}{\partial x_i}\cdot\frac{\partial v}{\partial x_j}\,dx + \int_\Omega c(x)|v(x)|^2\,dx \end{aligned}$$

$$+\int_{\{a(x')\neq 0\}} \frac{b(x')}{a(x')} \cdot |v(x')|^2 \, d\sigma$$
$$\geq 0,$$

so that, by definition (5.7),

$$\lambda_1(m) \geq 0.$$

More precisely, we have the following (see [19, Lemma 4.4]):

Lemma 5.4. *The quantity $\lambda_1(m)$ can be estimated from below as follows:*

$$\lambda_1(m) \geq \frac{\gamma_1}{\|m^+\|_{L^\infty(\Omega)}}. \tag{5.8}$$

Remark 5.3. We recall that γ_1 is the first eigenvalue of $\mathfrak{A}$ given by the variational formula ($m(x) \equiv 1$)

$$\gamma_1 = \inf \left\{ \frac{(\mathfrak{A}v, v)_{L^2(\Omega)}}{\int_\Omega v^2 \, dx} : v \in D(\mathfrak{A}), \ \int_\Omega v^2 \, dx > 0 \right\},$$

and that

$$m^+(x) = \max\{m(x), 0\}, \quad x \in \Omega.$$

Step 3: We begin by considering the case where $\lambda > \lambda_1(m)$ (see [19, Lemma 4.5]):

Lemma 5.5. *If $\lambda > \lambda_1(m)$, then λ is not an eigenvalue of problem* (1.3) *possessing a non-negative eigenfunction. Moreover, it follows from formula* (5.5) *that*

$$\mu_1(\lambda) < 0, \quad \lambda > \lambda_1(m).$$

Step 4: Next we consider the case where $0 < \lambda < \lambda_1(m)$ (see [19, Lemma 4.6]):

Lemma 5.6. *If $0 < \lambda < \lambda_1(m)$, then we have, for all $v \in D(\mathfrak{A})$,*

$$Q_\lambda(v) \geq \gamma_1 \left(1 - \frac{\lambda}{\lambda_1(m)}\right) \int_\Omega v^2 \, dx. \tag{5.9}$$

In particular, it follows from formula (5.5) *that*

$$\mu_1(\lambda) > 0, \quad 0 < \lambda < \lambda_1(m).$$

Step 5: By combining Lemmas 5.5 and 5.6, we have the following (see [19, Proposition 4.7]):

Proposition 5.7. *If $\lambda > 0$ and $\lambda \neq \lambda_1(m)$, then λ is not an eigenvalue of problem* (1.3) *possessing a non-negative eigenfunction.*

Step 6: Finally, the next theorem of the Kreĭn and Rutman type proves Theorem 1.2 (see [19, Theorem 4.8]):

Theorem 5.8. *Assume that condition (H.1) and (H.2) are satisfied. Then we have the following four assertions:*

(i) *$\lambda_1(m)$ is an eigenvalue of problem* (1.3).

(ii) $\lambda_1(m)$ *is algebraically simple.*
(iii) $\lambda_1(m)$ *admits a strictly positive eigenfunction* $\phi_1(x)$.
(iv) *No other eigenvalues,* $\lambda_j(m)$, $j \geq 2$, *have positive eigenfunctions.*

Proof. The proof of Theorem 5.8 is divided into four steps.

(1) We consider the following eigenvalue problem:

$$\begin{cases} Aw - \lambda_1(m)\, m(x)\, w = \mu w & \text{in } \Omega, \\ Bw = a(x')\dfrac{\partial w}{\partial \boldsymbol{\nu}} + b(x')w = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.10}$$

Then it is easy to see that $\lambda = \lambda_1(m)$ is an eigenvalue of problem (1.3) with corresponding eigenfunction $w(x)$

$$\begin{cases} Aw = \lambda m(x) w & \text{in } \Omega, \\ Bw = 0 & \text{on } \partial\Omega \end{cases}$$

if and only if $\mu = 0$ is an eigenvalue of problem (5.10) with corresponding eigenfunction $w(x)$, that is, $\mu(\lambda_1(m)) = 0$.

To prove assertion (i), we introduce a densely defined, selfadjoint operator $\mathcal{S}\colon L^2(\Omega) \to L^2(\Omega)$ by the formula

$$\mathcal{S} := \mathcal{T}(\lambda_1(m)) = \mathfrak{A} - \lambda_1(m)\, m(x) I.$$

It suffices to show that the first eigenvalue $\mu_1(\lambda_1(m))$ of the operator $\mathcal{S}$ is equal to zero, that is, $\mu_1(\lambda_1(m)) = 0$. Our situation may be represented schematically by the following figure:

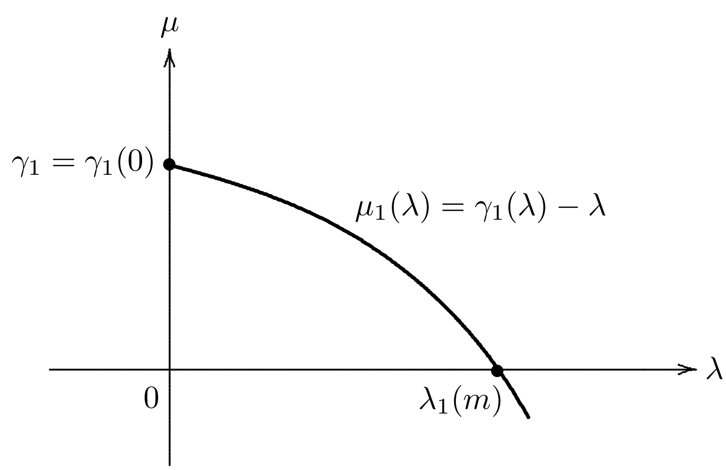

FIGURE 5.1

By the variational formula (5.5) with $\lambda := \lambda_1(m)$, it follows that

$\mu_1(\lambda_1(m))$

$$= \inf \left\{ \frac{(\mathfrak{A}v, v)_{L^2(\Omega)} - \lambda_1(m) \int_\Omega m(x) v^2\, dx}{\int_\Omega v^2\, dx} : v \in D(\mathfrak{A}),\ v \neq 0 \right\}$$
$$= \inf \left\{ \frac{Q_{\lambda_1(m)}(v)}{\int_\Omega v^2\, dx} : v \in D(\mathfrak{A}),\ v \neq 0 \right\}. \tag{5.11}$$

Since we have, for all $v \in D(\mathfrak{A})$,

$$Q_{\lambda_1(m)}(v) = (\mathfrak{A}v, v)_{L^2(\Omega)} - \lambda_1(m) \int_\Omega m(x)\, v^2\, dx \geq 0,$$

it follows from formula (5.11) that

$$\mu_1(\lambda_1(m)) \geq 0.$$

The next claim proves the desired assertion (i):

Claim 5.1. $\mu_1(\lambda_1(m)) = 0$.

Proof. By definition (5.7) of $\lambda_1(m)$, we can find a sequence $\{v_j\} \subset D(\mathfrak{A})$ such that

$$\int_\Omega m(x)\, v_j^2\, dx = 1,$$
$$(\mathfrak{A}v_j, v_j)_{L^2(\Omega)} \longrightarrow \lambda_1(m) \quad \text{as } j \to \infty.$$

Then we have the assertion

$$\begin{aligned} Q_{\lambda_1(m)}(v_j) &= (\mathfrak{A}v_j, v_j)_{L^2(\Omega)} - \lambda_1(m) \int_\Omega m(x)\, v_j^2\, dx \\ &\longrightarrow 0 \quad \text{as } j \to \infty. \end{aligned} \tag{5.12}$$

On the other hand, it follows that

$$1 = \int_\Omega m(x)\, v_j^2\, dx \leq \|m^+\|_{L^\infty(\Omega)} \int_\Omega v_j^2\, dx,$$

so that

$$\int_\Omega v_j^2\, dx \geq \frac{1}{\|m^+\|_{L^\infty(\Omega)}}. \tag{5.13}$$

Therefore, by combining assertions (5.12) and (5.13) we obtain that

$$\frac{Q_{\lambda_1(m)}(v_j)}{\int_\Omega v_j^2\, dx} \longrightarrow 0 \quad \text{as } j \to \infty.$$

By formula (5.11), this proves that $\mu_1(\lambda_1(m)) = 0$.

The proof of Claim 5.1 is complete. □

(2) We recall that $\lambda_1(m)$ is an eigenvalue of problem (1.3) with corresponding eigenfunction $w(x)$ if and only if zero is an eigenvalue of the operator $\mathcal{S} = \mathcal{T}(\lambda_1(m))$ with corresponding eigenfunction $w(x)$:

$$\begin{cases} Aw = \lambda_1(m)\, m(x) w & \text{in } \Omega, \\ Bw = 0 & \text{on } \partial\Omega \end{cases}$$
$$\iff \mathcal{S}w = (\mathfrak{A} - \lambda_1(m)\, m(x))\, w = 0 \cdot w \quad \text{in } L^2(\Omega).$$

However, Claim 5.1 tells us that zero is the first eigenvalue of $\mathcal{S}$. Therefore, assertions (ii) and (iii) of Theorem 5.8 follow from an application of Theorem 5.2 with $\lambda := \lambda_1(m)$.

(3) Proposition 5.7 proves assertion (iv) of Theorem 5.8.

(4) Finally, the variational formula (1.5) is an immediate consequence of formula (5.7).

Now the proof of Theorem 5.8 and hence that of Theorem 1.2 is complete. □

6. Proof of Theorem 1.3

In this final section we prove Theorem 1.3, extending Theorem 4.3 to the general case. The proof is divided into two steps.

Step 1: First, we prove assertion (i) of Theorem 1.3. If $\lambda = 0$, then the desired assertion (i) follows from an application of Theorem 3.7. Hence we have only to consider the case where $\lambda > 0$.

Just as in the proof of Theorem 5.2, it is easy to see that

$$\begin{aligned} &\begin{cases} (A - \lambda\, m(x))\, u = h & \text{in } \Omega, \\ Bu = a(x')\dfrac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = 0 & \text{on } \partial\Omega \end{cases} \\ \Longleftrightarrow\ &\begin{cases} (A + \lambda(1 - m(x)))\, u = h + \lambda u & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega \end{cases} \\ \Longleftrightarrow\ & u = K_\lambda h + \lambda K_\lambda u \quad \text{in } C(\overline{\Omega}) \\ \Longleftrightarrow\ & \frac{1}{\lambda} u - K_\lambda u = \frac{1}{\lambda} K_\lambda h \quad \text{in } C(\overline{\Omega}). \end{aligned}$$

Moreover, we recall the formula

$$r(K_\lambda) := \lim_{n\to\infty} \sqrt[n]{\|K_\lambda^n\|} = \frac{1}{\gamma_1(\lambda)} = \frac{1}{\lambda + \mu_1(\lambda)}.$$

Therefore, by applying Theorem 2.2 with

$$\begin{aligned} &E := C_\phi(\overline{\Omega}), \quad P := P_\phi, \\ &K := K_\lambda, \quad r(K) := r(K_\lambda) = \frac{1}{\gamma_1(\lambda)}, \\ &\lambda := \frac{1}{\lambda}, \quad h := \frac{1}{\lambda} K_\lambda h, \end{aligned}$$

we obtain that

(1) If $0 < \lambda < \gamma_1(\lambda)$, then problem (1.6) has a unique positive solution $u \in \operatorname{Int}(P_\phi)$ for any given function $h(x) \in L^p(\Omega)$ such that $h(x) \geq 0$ almost everywhere in Ω.

(2) If $\lambda > \gamma_1(\lambda)$, then problem (1.6) has no positive solution for any given function $h(x) \in L^p(\Omega)$ such that $h(x) \geq 0$ almost everywhere in Ω.

(3) If $\lambda = \gamma_1(\lambda)$, then problem (1.6) has no solution for any given function $h(x) \in L^p(\Omega)$ such that $h(x) \geq 0$ almost everywhere in Ω.

However, by Figure 5.1 we find that

$$\begin{aligned}
&0 < \lambda < \lambda_1(m) \implies \mu_1(\lambda) = \gamma_1(\lambda) - \lambda > 0 \implies 0 < \lambda < \gamma_1(\lambda); \\
&\lambda > \lambda_1(m) \implies \mu_1(\lambda) = \gamma_1(\lambda) - \lambda < 0 \implies \lambda > \gamma_1(\lambda); \\
&\lambda = \lambda_1(m) \implies \mu_1(\lambda) = \gamma_1(\lambda) - \lambda = 0 \implies \lambda = \gamma_1(\lambda).
\end{aligned}$$

Summing up, we have proved that if $0 \leq \lambda < \lambda_1(m)$, then problem (1.6) has a unique positive solution $u(x) \in W^{2,p}(\Omega)$ for any given function $h(x) \in L^p(\Omega)$ such that $h(x) \geq 0$ almost everywhere in Ω.

Step 2: Secondly, we prove assertion (ii) of Theorem 1.3. To do this, by Step 1 we may assume that $\lambda = \lambda_1(m)$. Then it follows that

$$\begin{cases} (A - \lambda_1(m)\, m(x))\, u = h & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega \end{cases}$$

$$\iff \begin{cases} u - \lambda_1(m) K_{\lambda_1(m)} u = K_{\lambda_1(m)} h \quad \text{in } C(\overline{\Omega}), \\ u > 0 \quad \text{in } \Omega. \end{cases}$$

However, by applying Theorem 5.8 and Theorem 2.1 with

$$\begin{aligned}
&E := C_\phi(\overline{\Omega}), \quad P := P_\phi, \quad L := K_{\lambda_1(m)}, \\
&\lambda_0 := \frac{1}{\lambda_1(m)}, \quad x_0 := \phi_1, \quad x_0^* := \psi_1,
\end{aligned}$$

we obtain that

$$\begin{cases} K_{\lambda_1(m)} \phi_1 = \dfrac{1}{\lambda_1(m)} \phi_1, \\ \phi_1 \gg 0, \end{cases}$$

and further that

$$\begin{cases} K^*_{\lambda_1(m)} \psi_1 = \dfrac{1}{\lambda_1(m)} \psi_1, \\ \psi_1 \gg 0. \end{cases}$$

Hence we have the formula

$$\begin{aligned}
\frac{1}{\lambda_1(m)} \left\langle K_{\lambda_1(m)} h, \psi_1 \right\rangle &= \left\langle \frac{1}{\lambda_1(m)} u - K_{\lambda_1(m)} u, \psi_1 \right\rangle \\
&= \frac{1}{\lambda_1(m)} \langle u, \psi_1 \rangle - \left\langle u, K^*_{\lambda_1(m)} \psi_1 \right\rangle \\
&= \frac{1}{\lambda_1(m)} \langle u, \psi_1 \rangle - \left\langle u, \frac{1}{\lambda_1(m)} \psi_1 \right\rangle \\
&= 0. \qquad (6.1)
\end{aligned}$$

Since $\psi_1 \gg 0$ and $K_{\lambda_1(m)} h \geq 0$, it follows from formula (6.1) that

$$K_{\lambda_1(m)} h = 0,$$

so that

$$h = (A - \lambda_1(m)\, m(x))\, u = 0 \quad \text{in } \Omega.$$

This proves that

$$\begin{cases} Au = \lambda_1(m)\, m(x) u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, by applying Theorem 1.2 to our situation we obtain that $h(x) = 0$ in Ω and that $u(x) = t\, \phi_1(x)$ for some constant $t > 0$.

The proof of Theorem 1.3 is now complete.

References

[1] R.A. Adams and J.J.F. Fournier, Sobolev spaces, second edition, Academic Press, Amsterdam Heidelberg New York Oxford, 2003.

[2] G.A. Afrouzi and K.J. Brown, *On principal eigenvalues for boundary value problems with indefinite weight and Robin boundary conditions*, Proc. Amer. Math. Soc. **127** (1999), 125–130.

[3] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. **18** (1976), 620–709.

[4] A. Ambrosetti and G. Prodi, *On the inversion of some differentiable mappings with singularities between Banach spaces*, Ann. Mat. Pura Appl. **93** (1972), 231–247.

[5] A. Ambrosetti and G. Prodi, A primer of nonlinear analysis, Cambridge University Press, Cambridge, 1993.

[6] J. Bergh and J. Löfström, Interpolation spaces, an introduction, Springer-Verlag, Berlin Heidelberg New York, 1976.

[7] J.-M. Bony, *Principe du maximum dans les espaces de Sobolev*, C. R. Acad. Sc. Paris **265** (1967), 333–336.

[8] K.J. Brown and S.S. Lin, *On the existence of positive eigenfunctions for an eigenvalue problem with indefinite weight function*, J. Math. Anal. Appl. **75** (1980), 112–120.

[9] R.S. Cantrell and C. Cosner, *Diffusive logistic equations with indefinite weights: population models in disrupted environments*, Proc. Roy. Soc. Edinburgh **112A** (1989), 293–318.

[10] K.C. Chang, Methods in nonlinear analysis, Springer-Verlag, Berlin Heidelberg New York, 2005.

[11] D.G. de Figueiredo, *Positive solutions of semilinear elliptic problems*, Lecture Notes in Mathematics, No. 957. Springer-Verlag, Berlin Heidelberg New York, (1982), 34–87.

[12] P. Hess, Periodic-parabolic boundary value problems and positivity, Pitman Research Notes in Mathematical Series **247**, Longman Scientific & Technical, Harlow, Essex, 1991.

[13] M.A. Krasnosel'skii, Positive solutions of operator equations, P. Noordhoff, Groningen, 1964.

[14] M.G. Kreĭn and M.A. Rutman, *Linear operators leaving invariant a cone in a Banach space*, Amer. Math. Soc. Transl. **10** (1962) 199–325.

[15] A. Manes and A.M. Micheletti, *Un'estensione della teoria variazionale classica degli autovalori per operatori ellitici del secondo ordine*, Boll. Un. Mat. Ital. **7** (1973), 285–301.

[16] K. Taira, Analytic semigroups and semilinear initial-boundary value problems, London Mathematical Society Lecture Note Series, vol. 223. Cambridge University Press, Cambridge, 1995.

[17] K. Taira, *Introduction to semilinear elliptic boundary value problems*, Taiwanese J. Math. **2** (1998), 127–172.

[18] K. Taira, *Logistic Dirichlet problems with discontinuous coefficients*, J. Math. Pures Appl. **82** (2003), 1137–1190.

[19] K. Taira, *Diffusive logistic equations with degenerate boundary conditions*, Mediterranean J. Math. **1** (2004), 315–365.

[20] K. Taira, Semigroups, boundary value problems and Markov processes, Springer-Verlag, Berlin Heidelberg New York, 2004.

[21] G.M. Troianiello, Elliptic differential equations and obstacle problems, Plenum Press, New York, 1987.

[22] J. Wloka, Partial differential equations, Cambridge University Press, Cambridge, 1987.

[23] K. Yosida, Functional analysis, sixth edition, Springer-Verlag, Berlin Heidelberg New York, 1980.

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