BOUNDARY VALUE PROBLEMS FOR ELLIPTIC INTEGRO-DIFFERENTIAL OPERATORS

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Dedicated to Professor Kiyosi Itô on his 80th birthday

Introduction and results.

This paper is a continuation of the previous note [T2] where we studied a class of degenerate boundary value problems for second-order elliptic differential operators and proved that this class of boundary value problems generates analytic semigroups both in the L^p topology and in the topology of uniform convergence. The purpose of this paper is to extend these results to the elliptic *integro-differential operator* case.

Let D be a bounded, *convex* domain of Euclidean space $\mathbf{R}^N$, with C^∞ boundary ∂D; its closure $\overline{D} = D \cup \partial D$ is an N-dimensional, compact C^∞ manifold with boundary.

Let W be a second-order, *elliptic* integro-differential operator with real coefficients such that

$$\begin{aligned} Wu(x) &= Au(x) + Su(x) \\ &:= \left(\sum_{i,j=1}^N a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^N b^i(x) \frac{\partial u}{\partial x_i}(x) + c(x)u(x) \right) \\ &\quad + \int_{\mathbf{R}^N \setminus \{0\}} \left(u(x+z) - u(x) - \sum_{j=1}^N z_j \frac{\partial u}{\partial x_j}(x) \right) s(x,z)\, m(dz). \end{aligned}$$

Here:

(1) $a^{ij} \in C^\infty(\overline{D})$, $a^{ij} = a^{ji}$ and there exists a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^N a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2, \quad x \in D,\ \xi \in \mathbf{R}^N.$$

(2) $b^i \in C^\infty(\overline{D})$.

(3) $c \in C^\infty(\overline{D})$, and $c \leq 0$ in D but $c \not\equiv 0$ in D.

(4) $s \in C(\overline{D} \times \mathbf{R}^N)$ and $0 \leq s \leq 1$ in $D \times \mathbf{R}^N$, and there exist constants $C_0 > 0$ and $0 < \theta_0 < 1$ such that

$$|s(x,z) - s(y,z)| \leq C_0|x-y|^{\theta_0}, \quad x, y \in D,\ z \in \mathbf{R}^N,$$

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and

$$s(x,z) = 0 \quad \text{if } x+z \notin \overline{D}. \tag{0.1}$$

Condition (0.1) implies that the integral operator S may be considered as an operator acting on functions u defined on the closure $\overline{D}$ (see [G-M, Chapter II, Remark 1.19]).

(5) The measure $m(dz)$ is a Radon measure on $\mathbf{R}^N \setminus \{0\}$ such that

$$\int_{\{|z|\le 1\}} |z|^2 \, m(dz) + \int_{\{|z|>1\}} |z| \, m(dz) < \infty. \tag{0.2}$$

The operator W is called a second-order *Waldenfels operator*. The differential operator A is called a diffusion operator which describes analytically a strong Markov process with continuous paths in the interior D. The integral operator S is called a second-order Lévy operator which is supposed to correspond to the jump phenomenon in the closure $\overline{D}$ (cf. [B-C-P], [T1]).

Let L be a first-order, boundary condition with real coefficients such that

$$Lu(x') = \mu(x') \frac{\partial u}{\partial \mathbf{n}}(x') + \gamma(x') u(x').$$

Here:

(1) $\mu \in C^\infty(\partial D)$ and $\mu \ge 0$ on ∂D.
(2) $\gamma \in C^\infty(\partial D)$ and $\gamma \le 0$ on ∂D.
(3) $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ is the unit interior normal to the boundary ∂D.

The boundary condition L is called a first-order *Ventcel' boundary condition*. The terms $\mu \partial u/\partial \mathbf{n}$ and γu of L are supposed to correspond to the reflection phenomenon and the absorption phenomenon, respectively.

Our fundamental hypothesis is the following:

(H) $\mu(x') - \gamma(x') > 0$ on ∂D.

The intuitive meaning of hypothesis (H) is that either the reflection phenomenon or the absorption phenomenon occurs at each point of the boundary ∂D.

The first purpose of this paper is to prove an existence and uniqueness theorem for the following nonhomogeneous boundary value problem in the framework of *Hölder spaces*:

$$\begin{cases} Wu = f & \text{in } D, \\ Lu = \varphi & \text{on } \partial D. \end{cases} \tag{$*$}$$

The crucial point is how to define a version of Hölder spaces in which problem $(*)$ is uniquely solvable.

We introduce a subspace of the Hölder space $C^{1+\theta}(\partial D)$, $0 < \theta < 1$, which is associated with the boundary condition L in the following way: We let

$$C_L^{1+\theta}(\partial D) = \left\{ \varphi = \mu\varphi_1 - \gamma\varphi_2 : \varphi_1 \in C^{1+\theta}(\partial D), \ \varphi_2 \in C^{2+\theta}(\partial D) \right\},$$

and define a norm

$$|\varphi|_{C_L^{1+\theta}(\partial D)} = \inf \left\{ |\varphi_1|_{C^{1+\theta}(\partial D)} + |\varphi_2|_{C^{2+\theta}(\partial D)} : \varphi = \mu\varphi_1 - \gamma\varphi_2 \right\}.$$

Then it is easy to verify that the space $C_L^{1+\theta}(\partial D)$ is a Banach space with respect to the norm $|\cdot|_{C_L^{1+\theta}(\partial D)}$. We remark that the space $C_L^{1+\theta}(\partial D)$ is an "interpolation space" between $C^{2+\theta}(\partial D)$ and $C^{1+\theta}(\partial D)$. More precisely, we have

$$\begin{cases} C_L^{1+\theta}(\partial D) = C^{2+\theta}(\partial D) & \text{if } \mu \equiv 0 \text{ on } \partial D, \\ C_L^{1+\theta}(\partial D) = C^{1+\theta}(\partial D) & \text{if } \mu > 0 \text{ on } \partial D. \end{cases}$$

Now we can state our existence and uniqueness theorem for problem $(*)$:

Theorem 1. *If hypothesis (H) is satisfied, then the mapping*

$$(W, L) : C^{2+\theta}(\overline{D}) \longrightarrow C^{\theta}(\overline{D}) \oplus C_L^{1+\theta}(\partial D)$$

is an algebraic and topological isomorphism *for all* $0 < \theta < \theta_0$. *In particular, for any* $f \in C^{\theta}(\overline{D})$ *and any* $\varphi \in C_L^{1+\theta}(\partial D)$, *there exists a unique solution* $u \in C^{2+\theta}(\overline{D})$ *of problem* $(*)$.

As an application of Theorem 1, we consider the problem of existence of Markov processes in probability theory. To do so, we let

$$M = \{x' \in \partial D : \mu(x') = 0\}.$$

Then, in view of condition (H) it follows that the boundary condition $Lu = 0$ on ∂D includes the condition $u = 0$ on M. With this fact in mind, we let

$$C_0(\overline{D} \setminus M) = \{u \in C(\overline{D}) : u = 0 \text{ on } M\}.$$

The space $C_0(\overline{D} \setminus M)$ is a closed subspace of $C(\overline{D})$; hence it is a Banach space.

A strongly continuous semigroup $\{T_t\}_{t \geq 0}$ on the space $C_0(\overline{D} \setminus M)$ is called a *Feller semigroup* on $\overline{D} \setminus M$ if it is non-negative and contractive on $C_0(\overline{D} \setminus M)$:

$$f \in C_0(\overline{D} \setminus M),\ 0 \leq f \leq 1 \quad \text{on } \overline{D} \setminus M \implies 0 \leq T_t f \leq 1 \quad \text{on } \overline{D} \setminus M.$$

It is known (cf. [T1, Chapter 9]) that if T_t is a Feller semigroup on $\overline{D} \setminus M$, then there exists a unique Markov transition function p_t on $\overline{D} \setminus M$ such that

$$T_t f(x) = \int_{\overline{D} \setminus M} p_t(x, dy) f(y), \quad f \in C_0(\overline{D} \setminus M),$$

and further p_t is the transition function of some *strong Markov process*.

We define a linear operator $\mathcal{W}$ from $C_0(\overline{D} \setminus M)$ into itself as follows:

(a) The domain of definition $D(\mathcal{W})$ is the set

$$D(\mathcal{W}) = \left\{u \in C^2(\overline{D}) \cap C_0(\overline{D} \setminus M) : Wu \in C_0(\overline{D} \setminus M), Lu = 0\right\}.$$

(b) $\mathcal{W}u = Wu$, $u \in D(\mathcal{W})$.

The next theorem is a generalization of Theorem 4 of [T2] to the integro-differential operator case:

Theorem 2. *If hypothesis (H) is satisfied, then the operator $\mathcal{W}$ is closable in the space $C_0(\overline{D}\setminus M)$, and its minimal closed extension $\overline{\mathcal{W}}$ is the infinitesimal generator of some Feller semigroup $\{T_t\}_{t\geq 0}$ on $\overline{D}\setminus M$.*

Theorem 2 asserts that there exists a Feller semigroup on $\overline{D}\setminus M$ corresponding to such a diffusion phenomenon that a Markovian particle moves both by jumps and continuously in the state space $\overline{D}\setminus M$ until it "dies" at the time when it reaches the set M where the particle is definitely absorbed (cf. [K, Theorem 5.2], [S, Theorem 2.2], [G-M, Chapter VIII, Theorem 3.3]).

The second purpose of this paper is to study problem $(*)$ from the point of view of analytic semigroup theory in functional analysis. The forthcoming two theorems generalize Theorems 2 and 3 of [T2] to the integro-differential operator case.

First we state a generation theorem of analytic semigroups in the the L^p topology. To do so, we associate with problem $(*)$ a unbounded linear operator W_p from $L^p(D)$ into itself as follows:

(a) The domain of definition $D(W_p)$ is the set

$$D(W_p) = \left\{ u \in H^{2,p}(D) : Lu = 0 \right\}.$$

(b) $W_p u = Wu$, $u \in D(W_p)$.

Then we can prove the following:

Theorem 3. *Let $1 < p < \infty$. Assume that hypothesis (H) is satisfied. Then we have the following:*

(i) For every $\varepsilon > 0$, there exists a constant $r_p(\varepsilon) > 0$ such that the resolvent set of W_p contains the set $\Sigma_p(\varepsilon) = \left\{ \lambda = r^2 e^{i\vartheta} : r \geq r_p(\varepsilon), -\pi + \varepsilon \leq \vartheta \leq \pi - \varepsilon \right\}$, and that the resolvent $(W_p - \lambda I)^{-1}$ satisfies the estimate

$$\left\| (W_p - \lambda I)^{-1} \right\| \leq \frac{c_p(\varepsilon)}{|\lambda|}, \quad \lambda \in \Sigma_p(\varepsilon), \tag{0.3}$$

where $c_p(\varepsilon) > 0$ is a constant depending on ε.

(ii) The operator W_p generates a semigroup e^{zW_p} on the space $L^p(D)$ which is analytic *in the sector $\Delta_\varepsilon = \{z = t + is : z \neq 0, |\arg z| < \pi/2 - \varepsilon\}$ for any $0 < \varepsilon < \pi/2$.*

Secondly we state a generation theorem of analytic semigroups in the topology of uniform convergence. We introduce a linear operator $\mathfrak{W}$ from $C_0(\overline{D}\setminus M)$ into itself as follows:

(a) The domain of definition $D(\mathfrak{W})$ is the set

$$D(\mathfrak{W}) = \left\{ u \in C_0(\overline{D}\setminus M) \cap H^{2,p}(D) : Wu \in C_0(\overline{D}\setminus M), Lu = 0 \right\}.$$

(b) $\mathfrak{W}u = Wu$, $u \in D(\mathfrak{W})$.

Here we remark that the domain $D(\mathfrak{W})$ is *independent* of $N < p < \infty$ (see the proof of Lemma 4.2).

Then Theorem 3 remains valid with $L^p(D)$ and W_p replaced by $C_0(\overline{D}\setminus M)$ and $\mathfrak{W}$, respectively:

Theorem 4. *If hypothesis (H) is satisfied, then we have the following:*

(i) For every $\varepsilon > 0$, there exists a constant $r(\varepsilon) > 0$ such that the resolvent set of $\mathfrak{W}$ contains the set $\Sigma(\varepsilon) = \{\lambda = r^2 e^{i\vartheta} : r \geq r(\varepsilon), -\pi + \varepsilon \leq \vartheta \leq \pi - \varepsilon\}$, and that the resolvent $(\mathfrak{W} - \lambda I)^{-1}$ satisfies the estimate

$$\|(\mathfrak{W} - \lambda I)^{-1}\| \leq \frac{c(\varepsilon)}{|\lambda|}, \quad \lambda \in \Sigma(\varepsilon), \tag{0.4}$$

where $c(\varepsilon) > 0$ is a constant depending on ε.

(ii) The operator $\mathfrak{W}$ generates a semigroup $e^{z\mathfrak{W}}$ on the space $C_0(\overline{D} \setminus M)$ which is analytic *in the sector $\Delta_\varepsilon = \{z = t + is : z \neq 0, |\arg z| < \pi/2 - \varepsilon\}$ for any $0 < \varepsilon < \pi/2$.*

Theorems 3 and 4 express a *regularizing effect* for the parabolic integro-differential operator $\partial/\partial t - W$ with homogeneous boundary condition L (cf. [G-M, Chapter VIII, Theorem 3.1]).

The rest of this paper is organized as follows. In Section 1 we study problem $(*)$ in the framework of Hölder spaces, and prove Theorem 1. The essential point in the proof is to estimate the integral operator S in terms of Hölder norms. We show that the operator (W, L) may be considered as a perturbation of a compact operator to the operator (A, L) in the framework of Hölder spaces. Thus the proof of Theorem 1 is reduced to the differential operator case which is studied in detail in [T2]. Section 2 is devoted to the proof of Theorem 2. The proof is based on a version of the Hille-Yosida theorem in semigroup theory in terms of the maximum principle. In Section 3 we prove Theorem 3. We estimate the integral operator S in terms of L^p norms, and show that S is an A_p-completely continuous operator in the sense of Gohberg and Kreĭn [G-K]. Section 4 is devoted to the proof of Theorem 4. Theorem 4 follows from Theorem 3 by using Sobolev's imbedding theorems and a λ-dependent localization argument, just as in [T2].

1 Proof of Theorem 1.

(I) First we prove Theorem 1 in the case where $S \equiv 0$:

Theorem 1.1. *If hypothesis (H) is satisfied, then the mapping*

$$(A, L) : C^{2+\theta}(\overline{D}) \longrightarrow C^{\theta}(\overline{D}) \oplus C_L^{1+\theta}(\partial D)$$

is an algebraic and topological isomorphism *for all $0 < \theta < 1$.*

Proof. The proof is divided into four steps.

(i) Let (f, φ) be an arbitrary element of $C^{\theta}(\overline{D}) \oplus C_L^{1+\theta}(\partial D)$ with $\varphi = \mu\varphi_1 - \gamma\varphi_2$. First we show that the boundary value problem

$$\begin{cases} Au = f & \text{in } D, \\ Lu = \varphi & \text{on } \partial D \end{cases} \tag{$**$}$$

can be reduced to the study of an operator on the boundary.

To do so, we consider the following Neumann problem:

$$\begin{cases} Av = f & \text{in } D, \\ \frac{\partial v}{\partial \mathbf{n}} = \varphi_1 & \text{on } \partial D. \end{cases} \tag{N}$$

Recall that the existence and uniqueness theorem for problem (N) is well established in the framework of Hölder spaces (see [G-T, Theorem 6.31]). Thus we find that a function $u \in C^{2+\theta}(\overline{D})$ is a solution of problem $(*)$ if and only if the function $w = u - v \in C^{2+\theta}(\overline{D})$ is a solution of the problem:

$$\begin{cases} Aw = 0 & \text{in } D, \\ Lw = \varphi - Lv & \text{on } \partial D. \end{cases}$$

Here we remark that

$$Lv = \mu \frac{\partial v}{\partial \mathbf{n}} + \gamma v = \mu \varphi_1 + \gamma v,$$

so that

$$Lw = -\gamma(\varphi_2 + v) \in C^{2+\theta}(\partial D).$$

However we know that every solution $w \in C^{2+\theta}(\overline{D})$ of the homogeneous equation: $Aw = 0$ in D can be expressed as follows (see [G-T, Theorem 6.14]):

$$w = \mathcal{P}\psi, \quad \psi \in C^{2+\theta}(\partial D).$$

Thus one can reduce the study of problem $(**)$ to that of the equation

$$T\psi := L\mathcal{P}\psi = -\gamma(\varphi_2 + v) \quad \text{on } \partial D. \tag{+}$$

More precisely we have the following:

Proposition 1.2. *For functions $f \in C^{\theta}(\overline{D})$ and $\varphi \in C_L^{1+\theta}(\partial D)$, there exists a solution $u \in C^{2+\theta}(\overline{D})$ of problem $(**)$ if and only if there exists a solution $\psi \in C^{2+\theta}(\partial D)$ of equation $(+)$.*

(ii) We study the operator T in question. It is known (cf. [H, Chapter XX]) that the operator

$$T\psi = L\mathcal{P}\psi = \mu \frac{\partial}{\partial \mathbf{n}} (\mathcal{P}\psi) + \gamma\psi$$

is a first-order, *pseudo-differential operator* on the boundary ∂D.

The next proposition is an essential step in the proof of Theorem 1.1:

Proposition 1.3. *If hypothesis (H) is satisfied, then there exists a* parametrix E *in the Hörmander class $L^0_{1,1/2}(\partial D)$ for T which maps $C^{k+\theta}(\partial D)$ continuously into itself for any integer $k \geq 0$.*

Proof. By making use of Theorem 22.1.3 of [H, Chapter XXII] just as in [T2, Lemma 4.2], one can construct a parametrix E in the Hörmander class $L^0_{1,1/2}(\partial D)$ for T:

$$ET \equiv TE \equiv I \mod L^{-\infty}(\partial D).$$

The boundedness of $E : C^{k+\theta}(\partial D) \to C^{k+\theta}(\partial D)$ follows from an application of [B, Theorem 1], since $C^{k+\theta}(\partial D) = B^{k+\theta}_{\infty,\infty}(\partial D)$. □

(iii) We consider problem $(**)$ in the framework of Sobolev spaces of L^p style, and prove an L^p version of Theorem 1.1.

If k is a positive integer and $1 < p < \infty$, we define the Sobolev space

$$\begin{aligned} H^{k,p}(D) = &\text{the space of (equivalence classes of) functions} \\ &u \in L^p(D) \text{ whose derivatives } D^\alpha u,\ |\alpha| \le k, \text{ in the} \\ &\text{sense of distributions are in } L^p(D), \end{aligned}$$

and the Besov space

$$\begin{aligned} B^{k-1/p,p}(\partial D) = &\text{the space of the boundary values } \varphi \text{ of functions} \\ &u \in H^{k,p}(D). \end{aligned}$$

In the space $B^{k-1/p,p}(\partial D)$, we introduce a norm

$$|\varphi|_{B^{k-1/p,p}(\partial D)} = \inf \|u\|_{H^{k,p}(D)}\,,$$

where the infimum is taken over all functions $u \in H^{k,p}(D)$ which equal φ on the boundary ∂D. The space $B^{k-1/p,p}(\partial D)$ is a Banach space with respect to this norm $|\cdot|_{B^{k-1/p,p}(\partial D)}$ (cf. [B-L]).

We introduce a subspace of $B^{1-1/p,p}(\partial D)$ which is an L^p version of $C_L^{1+\theta}(\partial D)$. We let

$$\begin{aligned} B_L^{1-1/p,p}(\partial D) = \Big\{ &\varphi = \mu\varphi_1 - \gamma\varphi_2 : \\ &\varphi_1 \in B^{1-1/p,p}(\partial D),\ \varphi_2 \in B^{2-1/p,p}(\partial D) \Big\}, \end{aligned}$$

and define a norm

$$|\varphi|_{B_L^{1-1/p,p}(\partial D)} = \inf \left\{ |\varphi_1|_{B^{1-1/p,p}(\partial D)} + |\varphi_2|_{B^{2-1/p,p}(\partial D)} : \varphi = \mu\varphi_1 - \gamma\varphi_2 \right\}.$$

Then it is easy to verify that the space $B_L^{1-1/p,p}(\partial D)$ is a Banach space with respect to the norm $|\cdot|_{B_L^{1-1/p,p}(\partial D)}$.

Then, arguing just as in the proof of [T2, Theorem 1] we can obtain the following L^p version of Theorem 1.1:

Theorem 1.4. *If hypothesis (H) is satisfied, then the mapping*

$$(A, L) : H^{2,p}(D) \longrightarrow L^p(D) \oplus B_L^{1-1/p,p}(\partial D)$$

is an algebraic and topological isomorphism.

(iv) Now we remark that

$$\begin{cases} C^\theta(\overline{D}) \subset L^p(D), \\ C_L^{1+\theta}(\partial D) \subset B_L^{1-1/p,p}(\partial D). \end{cases}$$

Thus we find from Theorem 1.4 that problem (∗∗) has a unique solution $u \in H^{2,p}(D)$ for any $f \in C^\theta(\overline{D})$ and any $\varphi \in C_L^{1+\theta}(\partial D)$. Furthermore, by virtue of Proposition 1.2 it follows that the solution u can be written in the form

$$u = v + \mathcal{P}\psi, \quad v \in C^{2+\theta}(\overline{D}), \ \psi \in B^{2-1/p,p}(\partial D).$$

However, Proposition 1.3 tells us that

$$\psi \in C^{2+\theta}(\partial D),$$

since we have $\psi \equiv E(T\psi) = -E\left(\gamma(\varphi_2 + v)\right) \bmod C^\infty(\partial D)$.

Therefore we obtain that

$$u = v + \mathcal{P}\psi \in C^{2+\theta}(\overline{D}).$$

The proof of Theorem 1.1 is complete. □

(II) Next we study the integral operator S in the framework of Hölder spaces. To do so, we need the following elementary estimates for the measure $m(dz)$:

Claim 1.5. *For $\varepsilon > 0$, we let*

$$\sigma(\varepsilon) = \int_{\{|z| \le \varepsilon\}} |z|^2 \, m(dz),$$
$$\delta(\varepsilon) = \int_{\{|z| > \varepsilon\}} |z| \, m(dz),$$
$$\tau(\varepsilon) = \int_{\{|z| > \varepsilon\}} m(dz).$$

Then we have, as $\varepsilon \downarrow 0$,

$$\sigma(\varepsilon) \to 0, \tag{1.1}$$
$$\delta(\varepsilon) \le \frac{C_1}{\varepsilon} + C_2, \tag{1.2}$$
$$\tau(\varepsilon) \le \frac{C_1}{\varepsilon^2} + C_2, \tag{1.3}$$

where

$$C_1 = \int_{\{|z| \le 1\}} |z|^2 \, m(dz), \quad C_2 = \int_{\{|z| > 1\}} |z| \, m(dz).$$

Proof. Assertion (1.1) follows immediately from condition (0.2).

The term $\delta(\varepsilon)$ can be estimated as follows:

$$\begin{aligned}\delta(\varepsilon) &= \int_{\{|z|>1\}} |z| \, m(dz) + \int_{\{\varepsilon < |z| \le 1\}} |z| \, m(dz) \\ &\le \int_{\{|z|>1\}} |z| \, m(dz) + \frac{1}{\varepsilon} \int_{\{\varepsilon < |z| \le 1\}} |z|^2 \, m(dz) \\ &\le \int_{\{|z|>1\}} |z| \, m(dz) + \frac{1}{\varepsilon} \int_{\{|z| \le 1\}} |z|^2 \, m(dz).\end{aligned}$$

The term $\tau(\varepsilon)$ is estimated in a similar way. □

By virtue of Claim 1.5, we can estimate the term Su in terms of Hölder norms, just as in [G-M, Chapter II, Lemmas 1.2 and 1.5]:

Lemma 1.6. *For every $\eta > 0$, there exists a constant $C_\eta > 0$ such that we have, for all $u \in C^2(\overline{D})$,*

$$\|Su\|_\infty \leq \eta \|\nabla^2 u\|_\infty + C_\eta \left(\|u\|_\infty + \|\nabla u\|_\infty\right).$$

Here

$$\|u\|_\infty = \sup_{x \in D} |u(x)|.$$

Lemma 1.7. *For every $\eta > 0$, there exists a constant $C_\eta > 0$ such that we have, for all $u \in C^{2+\theta_0}(\overline{D})$,*

$$\|Su\|_{C^{\theta_0}(\overline{D})} \leq \eta \|\nabla^2 u\|_{C^{\theta_0}(\overline{D})} + C_\eta \left(\|u\|_{C^{\theta_0}(\overline{D})} + \|\nabla u\|_{C^{\theta_0}(\overline{D})}\right).$$

Here

$$\|u\|_{C^{\theta_0}(\overline{D})} = \|u\|_\infty + [u]_{\theta_0}, \quad [u]_{\theta_0} = \sup_{\substack{x,y \in D \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^{\theta_0}}.$$

(III) *End of Proof of Theorem 1.* First, Theorem 1.1 implies that

$$\operatorname{ind}(A, L) = 0.$$

On the other hand, Lemma 1.7 tells us that the operator S maps $C^{2+\theta_0}(\overline{D})$ continuously into $C^{\theta_0}(\overline{D})$. Hence it follows from an application of [B-C-P, Théorème XXII] that S is a *compact* operator from $C^{2+\theta}(\overline{D})$ into $C^{\theta}(\overline{D})$ for all $0 < \theta < \theta_0$. This implies that the operator (W, L) is a perturbation of a compact operator to the operator (A, L).

Hence we find that

$$\operatorname{ind}(W, L) = \operatorname{ind}(A, L) = 0.$$

Therefore, in order to show the bijectivity of (W, L) it suffices to prove its *injectivity*:

$$\begin{cases} u \in C^{2+\theta}(\overline{D}), Wu = 0 \text{ in } D, Lu = 0 \text{ on } \partial D \\ \Longrightarrow \ u = 0 \text{ in } D. \end{cases}$$

However, this is an immediate consequence of the following *maximum principle*:

Proposition 1.8. *If hypothesis (H) is satisfied, then we have:*

$$\begin{cases} u \in C^2(\overline{D}), Wu \geq 0 \textit{ in } D, Lu \geq 0 \textit{ on } \partial D \\ \Longrightarrow \ u \leq 0 \textit{ on } \overline{D}. \end{cases}$$

Proof. If u is a constant m, then we have $0 \leq Wu = m\, c$ in D. This implies that $u \equiv m$ is non-positive, since $c \leq 0$ and $c \not\equiv 0$ in D.

Now we consider the case where u is not a constant. Assume to the contrary that:

$$m = \max_{\overline{D}} u > 0.$$

Then, applying the strong maximum principle (see [B-C-P, Théorème VII]) to the operator W we obtain that there exists a point x_0' of ∂D such that

$$\begin{cases} u(x_0') = m, \\ u(x) < u(x_0') \quad \text{for all } x \in D. \end{cases}$$

Furthermore it follows from an application of the boundary point lemma (see [B-C-P, Théorème VIII]) that

$$\frac{\partial u}{\partial \mathbf{n}}(x_0') < 0.$$

Hence we have

$$\mu(x_0') = 0, \quad \gamma(x_0') = 0,$$

since $Lu(x_0') \geq 0$. This contradicts hypothesis (H). □

The proof of Theorem 1 is now complete. □

2 Proof of Theorem 2.

The proof of Theorem 2 is based on the following version of the Hille-Yosida theorem in terms of the maximum principle (see [B-C-P, Théorème de Hille-Yosida-Ray]):

Theorem 2.1. *Let $\mathcal{A}$ be a linear operator from the space $C_0(\overline{D} \setminus M)$ into itself, and assume that:*

(α) The domain $D(\mathcal{A})$ is dense in the space $C_0(\overline{D} \setminus M)$.

(β) For any $u \in D(\mathcal{A})$ such that $\sup u > 0$, there exists a point $x \in \overline{D} \setminus M$ such that $u(x) = \sup u$ and $\mathcal{A}u(x) \leq 0$.

(γ) For all $\alpha > 0$, the range $R(\mathcal{A} - \alpha I)$ is dense in the space $C_0(\overline{D} \setminus M)$.

Then the operator $\mathcal{A}$ is closable in the space $C_0(\overline{D} \setminus M)$, and its minimal closed extension $\overline{\mathcal{A}}$ generates a Feller semigroup $\{T_t\}_{t \geq 0}$ on $\overline{D} \setminus M$.

Proof of Theorem 2. We have only to verify conditions (α), (β) and (γ) in Theorem 2.1 for the operator $\mathcal{W}$.

(γ) We obtain from Theorem 1 (and its proof) that the mapping

$$(W - \alpha, L) : C^{2+\theta}(\overline{D}) \longrightarrow C^{\theta}(\overline{D}) \oplus C_L^{1+\theta}(\partial D)$$

is an algebraic and topological isomorphism for all $\alpha > 0$. This verifies condition (γ), since the range $R(\mathcal{W} - \alpha I)$ contains the space $C^{\theta}(\overline{D}) \cap C_0(\overline{D} \setminus M)$ which is dense in $C_0(\overline{D} \setminus M)$.

(β) First let x_0 be a point of D such that $u(x_0) = \sup u$. Then it follows from an application of [B-C-P, Théorème V] that

$$\mathcal{W}u(x_0) = Wu(x_0) \leq 0.$$

Next let x_0' be a point of $\partial D \setminus M$ such that $u(x_0') = \sup u$. Assume to the contrary that

$$\mathcal{W}u(x_0') = Wu(x_0') > 0.$$

We have only to consider the case where u is not a constant. Then it follows from an application of the boundary point lemma that $(\partial u/\partial \mathbf{n})(x_0') < 0$. Hence we have

$$\mu(x_0') = 0,$$

since $Lu(x_0') = 0$. This contradicts the hypothesis: $x_0' \in \partial D \setminus M$, that is, $\mu(x_0') > 0$.

(α) The density of the domain $D(\mathcal{W})$ can be proved just as in the proof of [T2, Theorem 8.20], by using [B-C-P, Proposition III.1.6].

The proof of Theorem 2 is complete. □

3 Proof of Theorem 3.

The next theorem, which is a generalization of [T2, Theorem 6.1] to the integro-differential operator case, proves Theorem 3:

Theorem 3.1. *If hypothesis (H) is satisfied, then, for every $0 < \varepsilon < \pi/2$, there exists a constant $r_p(\varepsilon) > 0$ such that the resolvent set of W_p contains the set $\Sigma_p(\varepsilon) = \{\lambda = r^2 e^{i\vartheta} : r \geq r_p(\varepsilon), -\pi + \varepsilon \leq \vartheta \leq \pi - \varepsilon\}$, and that the resolvent $(W_p - \lambda I)^{-1}$ satisfies estimate (0.3).*

Proof. The proof is divided into three steps.

(i) We show that there exist constants $r_p(\varepsilon)$ and $c_p(\varepsilon)$ such that we have, for all $\lambda = r^2 e^{i\vartheta}$ satisfying $r \geq r_p(\varepsilon)$ and $-\pi + \varepsilon \leq \vartheta \leq \pi + \varepsilon$,

$$|u|_{2,p} + |\lambda|^{1/2} |u|_{1,p} + |\lambda| \|u\|_p \leq c_p(\varepsilon) \|(W_p - \lambda I)u\|_p. \tag{3.1}$$

Here

$$\|u\|_p = \|u\|_{L^p(D)}, \quad |u|_{1,p} = \|\nabla u\|_{L^p(D)}, \quad |u|_{2,p} = \|\nabla^2 u\|_{L^p(D)}.$$

First we recall (see [T2, formula (6.2)]) that estimate (3.1) is proved for the differential operator A:

$$|u|_{2,p} + |\lambda|^{1/2} |u|_{1,p} + |\lambda| \|u\|_p \leq c_p'(\varepsilon) \|(A_p - \lambda I)u\|_p. \tag{3.2}$$

Here the operator A_p is a unbounded linear operator from $L^p(D)$ into itself defined by the following:

(a) The domain of definition $D(A_p)$ is the set

$$D(A_p) = \left\{ u \in H^{2,p}(D) : Lu = 0 \right\}.$$

(b) $A_p u = Au$, $u \in D(A_p)$.

In order to replace the last term $\|(A_p - \lambda I)u\|_p$ by the term $\|(W_p - \lambda I)u\|_p$, we need the following L^p-estimate for the operator S:

Lemma 3.2. *For every $\eta > 0$, there exists a constant $C_\eta > 0$ such that we have, for all $u \in H^{2,p}(D)$,*

$$\|Su\|_p \leq \eta |u|_{2,p} + C_\eta \left(\|u\|_p + |u|_{1,p} \right). \tag{3.3}$$

Proof. We decompose the term Su into the following three terms:

$Su(x)$

$$\begin{aligned}
&= \int_0^1 (1-t)\,dt \int_{\{|z|\leq\varepsilon\}} z\cdot\nabla^2 u(x+tz)z\, s(x,z)\, m(dz) \\
&\quad + \int_{\{|z|>\varepsilon\}} (u(x+z)-u(x))s(x,z)\, m(dz) - \int_{\{|z|>\varepsilon\}} z\cdot\nabla u(x)\, s(x,z)\, m(dz) \\
&:= S_1 u(x) + S_2 u(x) - S_3 u(x).
\end{aligned}$$

First we estimate the L^p norm of the term $S_3 u$. By using estimate (1.2), we obtain that

$$\left| \int_{\{|z|>\varepsilon\}} z\cdot\nabla u(x)\, s(x,z)\, m(dz) \right| \leq \delta(\varepsilon)\, |\nabla u(x)| \leq \left(\frac{C_1}{\varepsilon} + C_2 \right) |\nabla u(x)|.$$

Hence we have the L^p estimate of the term $S_3 u$:

$$\|S_3 u\|_p \leq \left(\frac{C_1}{\varepsilon} + C_2 \right) \|\nabla u\|_p.$$

Secondly we have

$$\left\| \int_{\{|z|>\varepsilon\}} u(\cdot)\, s(\cdot,z)\, m(dz) \right\|_p \leq \left(\frac{C_1}{\varepsilon^2} + C_2 \right) \|u\|_p.$$

Furthermore, by using Hölder's inequality and Fubini's theorem we obtain from condition (0.1) that

$$\begin{aligned}
&\int_{\mathbf{R}^N} \left| \int_{\{|z|>\varepsilon\}} u(x+z)\, s(x,z)\, m(dz) \right|^p dx \\
&\leq \int_{\mathbf{R}^N} \left(\int_{\{|z|>\varepsilon\}} |u(x+z)|\, s(x,z)\, m(dz) \right)^p dx \\
&\leq \int_{\mathbf{R}^N} \left(\int_{\{|z|>\varepsilon\}} |u(x+z)|^p s(x,z)^p m(dz) \right) \left(\int_{\{|z|>\varepsilon\}} m(dz) \right)^{p/q} dx \\
&= \tau(\varepsilon)^{p/q} \int_{\mathbf{R}^N} \int_{\{|z|>\varepsilon\}} |u(x+z)|^p s(x,z)^p\, m(dz)\, dx \\
&= \tau(\varepsilon)^{p/q} \int_{\{|z|>\varepsilon\}} \left(\int_{\mathbf{R}^N} |u(x+z)|^p s(x,z)^p\, dx \right) m(dz) \\
&\leq \tau(\varepsilon)^{p/q} \left(\int_D |u(y)|^p\, dy \right) \left(\int_{\{|z|>\varepsilon\}} m(dz) \right) \\
&= \tau(\varepsilon)^p \|u\|_p^p.
\end{aligned}$$

By estimate (1.3), we have the L^p estimate of the term $S_2 u$:

$$\|S_2 u\|_p \leq \left(\frac{C_1}{\varepsilon^2} + C_2 \right) \|u\|_p.$$

Similarly, by using Hölder's inequality and Fubini's theorem we find that

$$\begin{aligned}
&\int_{\mathbf{R}^N} \left| \int_0^1 (1-t)\,dt \int_{\{|z|\le\varepsilon\}} z\cdot\nabla^2 u(x+tz) z\, s(x,z)\, m(dz)\right|^p dx \\
&\le \int_{\mathbf{R}^N} \left(\int_0^1 dt \int_{\{|z|\le\varepsilon\}} |z|^2 |\nabla^2 u(x+tz)|\, s(x,z)\, m(dz)\right)^p dx \\
&\le \int_{\mathbf{R}^N} \int_0^1 dt \left(\int_{\{|z|\le\varepsilon\}} |z|^2 |\nabla^2 u(x+tz)|^p\, s(x,z)^p\, m(dz)\right) \\
&\quad \times \left(\int_{\{|z|\le\varepsilon\}} |z|^2\, m(dz)\right)^{p/q} dx \\
&= \sigma(\varepsilon)^{p/q} \int_{\mathbf{R}^N} \int_0^1 dt \left(\int_{\{|z|\le\varepsilon\}} |z|^2 |\nabla^2 u(x+tz)|^p\, s(x,z)^p\, m(dz)\right) dx \\
&= \sigma(\varepsilon)^{p/q} \int_0^1 dt \int_{\{|z|\le\varepsilon\}} |z|^2 \left(\int_{\mathbf{R}^N} |\nabla^2 u(x+tz)|^p\, s(x,z)^p\, dx\right) m(dz) \\
&\le \sigma(\varepsilon)^{p/q} \left(\int_D |\nabla^2 u(y)|^p\, dy\right) \left(\int_{\{|z|\le\varepsilon\}} |z|^2\, m(dz)\right) \\
&\le \sigma(\varepsilon)^p \left(\int_D |\nabla^2 u(y)|^p\, dy\right).
\end{aligned}$$

Hence we have the L^p estimate of the term $S_1 u$:

$$\|S_1 u\|_p \le \sigma(\varepsilon)\, \|\nabla^2 u\|_p.$$

Summing up, we have proved that

$$\begin{aligned}
\|Su\|_p &\le \|S_1 u\|_p + \|S_2 u\|_p + \|S_3 u\|_p \\
&\le \sigma(\varepsilon)\, |u|_{2,p} + \left(\frac{C_1}{\varepsilon} + C_2\right) |u|_{1,p} + \left(\frac{C_1}{\varepsilon^2} + C_2\right) \|u\|_p.
\end{aligned}$$

In view of assertion (1.1), this proves estimate (3.3) if we choose ε sufficiently small. □

Since we have

$$(A-\lambda)u = (W-\lambda)u - Su,$$

it follows from estimate (3.3) that

$$\|(A_p-\lambda)u\|_p \le \|(W_p-\lambda)u\|_p + \eta |u|_{2,p} + C_\eta \left(|u|_{1,p} + \|u\|_p\right).$$

Thus, carrying this estimate into estimate (3.2) we obtain that

$$\begin{aligned}
&|u|_{2,p} + |\lambda|^{1/2} |u|_{1,p} + |\lambda| \|u\|_p \\
&\le c_p'(\varepsilon) \|(W_p-\lambda)u\|_p + \eta c_p'(\varepsilon) |u|_{2,p} + C_\eta c_p'(\varepsilon) \left(|u|_{1,p} + \|u\|_p\right). \qquad (3.4)
\end{aligned}$$

Therefore, the desired estimate (3.1) follows from estimate (3.4) if we take the constant η so small that

$$\eta c'_p(\varepsilon) < 1$$

and the parameter λ so large that

$$|\lambda|^{1/2} > C_\eta c'_p(\varepsilon).$$

(ii) By estimate (3.1), we find that the operator $W_p - \lambda I$ is injective and its range $R(W_p - \lambda I)$ is closed in $L^p(D)$, for all $\lambda \in \Sigma_p(\varepsilon)$.

We show that the operator $W_p - \lambda I$ is surjective for all $\lambda \in \Sigma_p(\varepsilon)$:

$$R(W_p - \lambda I) = L^p(D), \quad \lambda \in \Sigma_p(\varepsilon).$$

To do so, it suffices to show that the operator $W_p - \lambda I$ is a Fredholm operator with

$$\operatorname{ind}(W_p - \lambda I) = 0, \quad \lambda \in \Sigma_p(\varepsilon), \tag{3.5}$$

since $W_p - \lambda I$ is injective for all $\lambda \in \Sigma_p(\varepsilon)$.

In order to prove assertion (3.5), we need the following:

Lemma 3.3. *The operator S is A_p-*completely continuous*, that is, the operator $S : D(A_p) \to L^p(D)$ is completely continuous where the domain $D(A_p)$ is endowed with the graph norm of A_p.*

Proof. Let $\{u_j\}$ be an arbitrary bounded sequence in the domain $D(A_p)$; hence there exists a constant $K > 0$ such that

$$\|u_j\|_p \leq K, \quad \|A_p u_j\|_p \leq K.$$

Then we have, by [T2, estimate (0.1)],

$$\|u_j\|_{2,p} \leq C\left(\|A_p u_j\|_p + \|u_j\|_p\right) \leq 2CK. \tag{3.6}$$

Therefore, by Rellich's theorem one may assume that the sequence $\{u_j\}$ itself is a Cauchy sequence in the space $H^{1,p}(D)$. Then, applying estimate (3.3) to the sequence $\{u_j - u_k\}$ and using estimate (3.6), we obtain that

$$\begin{aligned} \|Su_j - Su_k\|_p &\leq \eta |u_j - u_k|_{2,p} + C_\eta\left(\|u_j - u_k\|_p + |u_j - u_k|_{1,p}\right) \\ &\leq 4\eta CK + C_\eta \|u_j - u_k\|_{1,p}. \end{aligned}$$

Hence we have

$$\limsup_{j,k\to\infty} \|Su_j - Su_k\|_p \leq 4\eta CK.$$

This proves that the sequence $\{Su_j\}$ is a Cauchy sequence in the space $L^p(D)$, since η is arbitrary. $\square$

In view of Lemma 3.3, assertion (3.5) follows from an application of [G-K, Theorem 2.6]. Indeed we have, by [T2, Theorem 6.1],

$$\operatorname{ind}(W_p - \lambda I) = \operatorname{ind}(A_p - \lambda I + S) = \operatorname{ind}(A_p - \lambda I) = 0.$$

(iii) Summing up, we have proved that the operator $W_p - \lambda I$ is bijective for all $\lambda \in \Sigma_p(\varepsilon)$ and its inverse $(W_p - \lambda I)^{-1}$ satisfies estimate (0.3).

The proof of Theorem 3.1 is now complete. $\square$

4 Proof of Theorem 4.

The proof is carried out in a chain of auxiliary lemmas.

(I) We begin with a version of estimate (3.1):

Lemma 4.1. *Let $N < p < \infty$. If hypothesis (H) is satisfied, then, for every $\varepsilon > 0$, there exists a constant $r_p(\varepsilon) > 0$ such that if $\lambda = r^2 e^{i\vartheta}$ with $r \geq r_p(\varepsilon)$ and $-\pi + \varepsilon \leq \vartheta \leq \pi - \varepsilon$, we have, for all $u \in D(W_p)$,*

$$|\lambda|^{1/2} \|u\|_{C^1(\overline{D})} + |\lambda| \|u\|_{C(\overline{D})} \leq C_p(\varepsilon) |\lambda|^{N/2p} \|(W - \lambda)u\|_p, \tag{4.1}$$

with a constant $C_p(\varepsilon) > 0$.

Proof. First it follows from an application of the Gagliardo-Nirenberg inequality (see [F, Part I, Theorem 10.1] that

$$\|u\|_{C(\overline{D})} \leq C |u|_{1,p}^{N/p} \|u\|_p^{1-N/p}, \quad u \in H^{1,p}(D). \tag{4.2}$$

Here and in the following the letter C denotes a generic positive constant depending on p and ε, but independent of u and λ.

Combining inequality (4.2) with inequality (3.1), we obtain that

$$\begin{aligned} \|u\|_{C(\overline{D})} &\leq C \left(|\lambda|^{-1/2} \|(W - \lambda)u\|_p \right)^{N/p} \left(|\lambda|^{-1} \|(W - \lambda)u\|_p \right)^{1-N/p} \\ &= C |\lambda|^{-1+N/2p} \|(W - \lambda)u\|_p, \end{aligned}$$

so that

$$|\lambda| \|u\|_{C(\overline{D})} \leq C |\lambda|^{N/2p} \|(W - \lambda)u\|_p, \quad u \in D(W_p). \tag{4.3}$$

Similarly, applying inequality (4.2) to the functions $D_i u \in H^{1,p}(D)$ $(1 \leq i \leq n)$ we obtain that

$$\begin{aligned} \|\nabla u\|_{C(\overline{D})} &\leq C |\nabla u|_{1,p}^{N/p} \|\nabla u\|_p^{1-N/p} \\ &\leq C |u|_{2,p}^{N/p} |u|_{1,p}^{1-N/p} \\ &\leq C \left(\|(W - \lambda)u\|_p \right)^{N/p} \left(|\lambda|^{-1/2} \|(W - \lambda)u\|_p \right)^{1-N/p} \\ &= C |\lambda|^{-1/2+N/2p} \|(W - \lambda)u\|_p. \end{aligned}$$

This proves that

$$|\lambda|^{1/2} \|u\|_{C^1(\overline{D})} \leq C |\lambda|^{N/2p} \|(W - \lambda)u\|_p, \quad u \in D(W_p). \tag{4.4}$$

Therefore, the desired inequality (4.1) follows from inequalities (4.3) and (4.4).

(II) The next lemma proves estimate (0.4):

Lemma 4.2. *Let $N < p < \infty$. If hypothesis (H) is satisfied, then, for every $\varepsilon > 0$, there exists a constant $r(\varepsilon) > 0$ such that if $\lambda = r^2 e^{i\vartheta}$ with $r \geq r(\varepsilon)$ and $-\pi + \varepsilon \leq \vartheta \leq \pi - \varepsilon$, we have, for all $u \in D(\mathfrak{W})$,*

$$|\lambda|^{1/2} \|u\|_{C^1(\overline{D})} + |\lambda| \|u\|_{C(\overline{D})} \leq c(\varepsilon) \|(\mathfrak{W} - \lambda I)u\|_{C(\overline{D})}, \tag{4.5}$$

with a constant $c(\varepsilon) > 0$.

Proof. (1) First we show that the domain

$$D(\mathfrak{W}) = \left\{u \in C_0(\overline{D} \setminus M) \cap H^{2,p}(D) : Wu \in C_0(\overline{D} \setminus M), Lu = 0\right\}$$

is independent of $N < p < \infty$.

We let

$$\mathcal{D}_p = \left\{u \in H^{2,p}(D) \cap C_0(\overline{D} \setminus M) : Wu \in C_0(\overline{D} \setminus M), Lu = 0\right\}.$$

Since we have $L^{p_1}(D) \subset L^{p_2}(D)$ for $p_1 > p_2$, it follows that

$$\mathcal{D}_{p_1} \subset \mathcal{D}_{p_2} \quad \text{if } p_1 > p_2.$$

Conversely, let v be an arbitrary element of $\mathcal{D}_{p_2}$:

$$v \in H^{2,p_2}(D) \cap C_0(\overline{D} \setminus M), \quad Wv \in C_0(\overline{D} \setminus M), \quad Lv = 0.$$

Then, since we have v, $Wv \in C_0(\overline{D} \setminus M) \subset L^{p_1}(D)$, it follows from an application of Theorem 3.1 with $p = p_1$ that there exists a unique function $u \in H^{2,p_1}(D)$ such that

$$\begin{cases} (W - \lambda)u = (W - \lambda)v & \text{in } D, \\ Lu = 0 & \text{on } \partial D, \end{cases}$$

if we choose λ sufficiently large. Hence we have $u - v \in H^{2,p_2}(D)$ and

$$\begin{cases} (W - \lambda)(u - v) = 0 & \text{in } D, \\ L(u - v) = 0 & \text{on } \partial D. \end{cases}$$

Therefore, by applying again Theorem 3.1 with $p = p_2$ we obtain that $u - v = 0$, so that $v = u \in H^{2,p_1}(D)$. This proves that $v \in \mathcal{D}_{p_1}$.

(2) We shall make use of a λ-dependent localization argument in order to adjust the term $\|(W - \lambda)u\|_p$ in inequality (4.1) to obtain inequality (4.5), just as in [T2].

(2-a) If x_0' is a point of ∂D and if χ is a C^∞ coordinate transformation such that χ maps $B(x_0', \eta_0) \cap D$ into $B(0, \delta) \cap \mathbf{R}^N_+$ and flattens a part of the boundary ∂D into the plane $x_N = 0$, then we let

$$\begin{aligned} G_0 &= B(x_0', \eta_0) \cap D, \\ G' &= B(x_0', \eta) \cap D, \ 0 < \eta < \eta_0, \\ G'' &= B(x_0', \eta/2) \cap D, \ 0 < \eta < \eta_0. \end{aligned}$$

Here and in the following $B(x, \eta)$ denotes the ball of radius η about x.

Similarly, if x_0 is a point of D and if χ is a C^∞ coordinate transformation such that χ maps $B(x_0,\eta_0)$ into $B(0,\delta)$, then we let

$$\begin{aligned} G_0 &= B(x_0,\eta_0), \\ G' &= B(x_0,\eta),\ 0<\eta<\eta_0, \\ G'' &= B(x_0,\eta/2),\ 0<\eta<\eta_0. \end{aligned}$$

(2-b) We take a function $\varPhi \in C_0^\infty(\mathbf{R})$ such that $\varPhi$ equals 1 near the origin, and define

$$\varphi(x) = \varPhi(|x'|^2)\,\varPhi(x_N), \quad x=(x',x_N).$$

Here one may assume that the function φ is chosen so that

$$\begin{cases} \operatorname{supp}\varphi \subset B(0,1), \\ \varphi(x) = 1 \text{ on } B(0,1/2). \end{cases}$$

We introduce a localizing function

$$\varphi_0(x,\eta) := \varphi\left(\frac{x-x_0}{\eta}\right) = \varPhi\left(\frac{|x'-x_0'|^2}{\eta^2}\right)\varPhi\left(\frac{x_N-t}{\eta}\right), \quad x_0=(x_0',t).$$

We remark that

$$\begin{cases} \operatorname{supp}\varphi_0 \subset B(x_0,\eta), \\ \varphi_0(x,\eta) = 1 \text{ on } B(x_0,\eta/2). \end{cases}$$

Then it is easy to verify the following (cf. [T2, Claim 7.5]):

Claim 4.3. *If $u \in D(\mathfrak{W})$, then we have $\varphi_0 u \in \mathcal{D}(W_p)$.*

(3) Now let u be an arbitrary element of $D(\mathfrak{W})$. Then, by Claim 4.3 we can apply inequality (4.1) to the function $\varphi_0 u$ to obtain that

$$\begin{aligned} |\lambda|^{1/2}\|u\|_{C^1(\overline{G''})} + |\lambda|\|u\|_{C(\overline{G''})} &\le |\lambda|^{1/2}\,\|\varphi_0 u\|_{C^1(\overline{G'})} + |\lambda|\,\|\varphi_0 u\|_{C(\overline{G'})} \\ &= |\lambda|^{1/2}\,\|\varphi_0 u\|_{C^1(\overline{D})} + |\lambda|\,\|\varphi_0 u\|_{C(\overline{D})} \\ &\le C|\lambda|^{N/2p}\,\|(W-\lambda)(\varphi_0 u)\|_{L^p(D)}\,. \end{aligned} \tag{4.6}$$

(3-a) We estimate the last term $\|(W-\lambda)(\varphi_0 u)\|_{L^p(D)}$ in terms of the supremum norm of $C(\overline{D})$.

First we write the term $(W-\lambda)(\varphi_0 u)$ in the following form:

$$(W-\lambda)(\varphi_0 u) = \varphi_0\left((W-\lambda)u\right) + [A,\varphi_0]u + [S,\varphi_0]u,$$

where $[A,\varphi_0]$ and $[S,\varphi_0]$ are the commutators of A and φ_0 and of S and φ_0, respectively:

$$\begin{aligned} [A,\varphi_0]u &= A(\varphi_0 u) - \varphi_0 A u, \\ [S,\varphi_0]u &= S(\varphi_0 u) - \varphi_0 S u. \end{aligned}$$

Now we need the following elementary inequality:

Claim 4.4. *We have, for all $v \in C^j(\overline{G'})$ $(j = 0, 1, 2)$,*

$$\|v\|_{H^{j,p}(G')} \le |G'|^{1/p} \|v\|_{C^j(\overline{G'})},$$

where $|G'|$ is the measure of G'.

Since we have, for some constant $c > 0$,

$$|G'| \le |B(x_0, \eta)| \le c\eta^N,$$

it follows from an application of Claim 4.4 that

$$\begin{aligned} \|\varphi_0(W - \lambda)u\|_{L^p(D)} &= \|\varphi_0(W - \lambda)u\|_{L^p(G')} \\ &\le c^{1/p}\eta^{N/p} \|(W - \lambda)u\|_{C(\overline{G'})} \\ &\le c^{1/p}\eta^{N/p} \|(W - \lambda)u\|_{C(\overline{D})} . \end{aligned} \tag{4.7}$$

On the other hand we can estimate the commutators $[A, \varphi_0]u$ and $[S, \varphi_0]u$ as follows:

Claim 4.5. *We have, as $\eta \downarrow 0$,*

$$\|[A, \varphi_0]u\|_{L^p(D)} \le C\left(\eta^{-1+N/p}\|u\|_{C^1(\overline{D})} + \eta^{-2+N/p}\|u\|_{C(\overline{D})}\right), \tag{4.8}$$

$$\|[S, \varphi_0]u\|_{L^p(D)} \le C\left(\eta^{-1+N/p}\|u\|_{C^1(\overline{D})} + \eta^{-2+N/p}\|u\|_{C(\overline{D})}\right). \tag{4.9}$$

Proof. Estimate (4.8) is proved in [T2, inequality (7.9)].

In order to prove estimate (4.9), we remark that

$$\begin{aligned} &S(\varphi_0 u)(x) \\ &= \int_{\mathbf{R}^N\setminus\{0\}} (\varphi_0(x+z)u(x+z) - \varphi_0(x)u(x) - z \cdot \nabla(\varphi_0 u)(x))\, s(x,z)\, m(dz) \\ &= \varphi_0(x) \int_{\mathbf{R}^N\setminus\{0\}} (u(x+z) - u(x) - z \cdot \nabla u(x))\, s(x,z)\, m(dz) \\ &\quad + \left(\int_{\mathbf{R}^N\setminus\{0\}} (u(x+z) - u(x)) z\, s(x,z)\, m(dz)\right) \cdot \nabla\varphi_0(x) \\ &\quad + \int_{\mathbf{R}^N\setminus\{0\}} (\varphi_0(x+z) - \varphi_0(x) - z \cdot \nabla\varphi_0(x))\, u(x+z)\, s(x,z)\, m(dz) \\ &= \varphi_0(x) Su(x) + \left(\int_{\mathbf{R}^N\setminus\{0\}} (u(x+z) - u(x)) z\, s(x,z)\, m(dz)\right) \cdot \nabla\varphi_0(x) \\ &\quad + \int_{\mathbf{R}^N\setminus\{0\}} (\varphi_0(x+z) - \varphi_0(x) - z \cdot \nabla\varphi_0(x))\, u(x+z)\, s(x,z)\, m(dz). \end{aligned}$$

Hence we can write the commutator $[S, \varphi_0]u$ in the following form:

$$[S, \varphi_0]u(x)$$

$$= \left(\int_{\mathbf{R}^N \setminus \{0\}} (u(x+z) - u(x)) z\, s(x,z)\, m(dz) \right) \cdot \nabla \varphi_0(x)$$
$$+ \int_{\mathbf{R}^N \setminus \{0\}} (\varphi_0(x+z) - \varphi_0(x) - z \cdot \nabla \varphi_0(x))\, u(x+z)\, s(x,z)\, m(dz)$$
$$:= S_0^{(1)} u(x) + S_0^{(2)} u(x).$$

First, just as in Lemma 1.6 we can estimate the term $S_0^{(1)} u$ as follows:

$$\begin{aligned} \|S_0^{(1)} u\|_{L^p(D)} &= \|S_0^{(1)} u\|_{L^p(G')} \\ &\leq 2 \left(\sigma(\eta) \|u\|_{C^1(\overline{D})} + \delta(\eta) \|u\|_{C(\overline{D})} \right) \|\nabla \varphi_0\|_{L^p(G')} \\ &\leq 2 \left(\sigma(\eta) \|u\|_{C^1(\overline{D})} + \left(\frac{C_1}{\eta} + C_2 \right) \|u\|_{C(\overline{D})} \right) \|\nabla \varphi_0\|_{L^p(G')}. \end{aligned}$$

However it follows from an application of Claim 4.4 that

$$\begin{aligned} \|\nabla \varphi_0\|_{L^p(G')} &\leq C \eta^{N/p} \|\nabla \varphi_0\|_{C(\overline{G'})} \leq C' \eta^{-1+N/p}, \\ \|\nabla^2 \varphi_0\|_{L^p(G')} &\leq C \eta^{N/p} \|\nabla^2 \varphi_0\|_{C(\overline{G'})} \leq C' \eta^{-2+N/p}, \end{aligned}$$

since we have, as $\eta \downarrow 0$,

$$|\nabla \varphi_0| = O(\eta^{-1}), \quad |\nabla^2 \varphi_0| = O(\eta^{-2}).$$

Therefore we obtain that

$$\|S_0^{(1)} u\|_{L^p(D)} \leq C \left(\eta^{-1+N/p} \|u\|_{C^1(\overline{D})} + \eta^{-2+N/p} \|u\|_{C(\overline{D})} \right). \tag{4.10}$$

Similarly, arguing as in the proof of Lemma 3.2 we can estimate the term $S_0^{(2)} u$ as follows:

$$\begin{aligned} \|S_0^{(2)} u\|_{L^p(D)} &\leq C \|u\|_{C(\overline{D})} \|\nabla^2 \varphi_0\|_{L^p(G')} \\ &\leq C \|u\|_{C(\overline{D})}\ \eta^{N/p} \|\nabla^2 \varphi_0\|_{C(\overline{G'})} \\ &\leq C \eta^{-2+N/p} \|u\|_{C(\overline{D})}. \end{aligned} \tag{4.11}$$

Thus, the desired estimate (4.9) follows by combining estimates (4.10) and (4.11). □

Therefore, combining estimates (4.6), (4.7), (4.8) and (4.9) we obtain that

$$\begin{aligned} &|\lambda|^{1/2} \|u\|_{C^1(\overline{G''})} + |\lambda| \|u\|_{C(\overline{G''})} \\ &\leq C |\lambda|^{N/2p} \|(W - \lambda)(\varphi_0 u)\|_{L^p(D)} \\ &= C |\lambda|^{N/2p} \|\varphi_0 ((W - \lambda) u) + [A, \varphi_0] u + [S, \varphi_0] u\|_{L^p(D)} \\ &\leq C |\lambda|^{N/2p} \left(\eta^{N/p} \|(W - \lambda) u\|_{C(\overline{G'})} + \eta^{-1+N/p} \|u\|_{C^1(\overline{G'})} + \eta^{-2+N/p} \|u\|_{C(\overline{G'})} \right) \\ &\leq C |\lambda|^{N/2p} \left(\eta^{N/p} \|(W - \lambda) u\|_{C(\overline{D})} + \eta^{-1+N/p} \|u\|_{C^1(\overline{D})} + \eta^{-2+N/p} \|u\|_{C(\overline{D})} \right). \end{aligned} \tag{4.12}$$

(3-b) We remark that the closure $\overline{D} = D \cup \partial D$ can be covered by a finite number of sets of the forms:

$$\begin{cases} B(x_0, \eta/2), & x_0 \in D, \\ B(x_0', \eta/2) \cap \overline{D}, & x_0' \in \partial D. \end{cases}$$

Therefore, taking the supremum of inequality (4.12) over $x \in \overline{D}$ we find that

$$\begin{aligned} &|\lambda|^{1/2} \|u\|_{C^1(\overline{D})} + |\lambda| \|u\|_{C(\overline{D})} \\ &\leq C |\lambda|^{N/2p} \eta^{N/p} \left(\|(W - \lambda)u\|_{C(\overline{D})} + \eta^{-1} \|u\|_{C^1(\overline{D})} + \eta^{-2} \|u\|_{C(\overline{D})} \right). \end{aligned} \tag{4.13}$$

(4) We now choose the localization parameter η. We let

$$\eta = \frac{\eta_0}{|\lambda|^{1/2}} K,$$

where K is a positive constant (to be chosen later) satisfying

$$0 < \eta = \frac{\eta_0}{|\lambda|^{1/2}} K < \eta_0,$$

that is,

$$0 < K < |\lambda|^{1/2}.$$

Then we obtain from inequality (4.13) that

$$\begin{aligned} &|\lambda|^{1/2} \|u\|_{C^1(\overline{D})} + |\lambda| \|u\|_{C(\overline{D})} \\ &\leq C\, \eta_0^{N/p} K^{N/p} \|(W - \lambda)u\|_{C(\overline{D})} + \left(C\, \eta_0^{N/p-1} K^{-1+N/p} \right) |\lambda|^{1/2} \|u\|_{C^1(\overline{D})} \\ &\quad + \left(C\, \eta_0^{N/p-2} K^{-2+N/p} \right) |\lambda| \|u\|_{C(\overline{D})}. \end{aligned} \tag{4.14}$$

However, since the exponents $-1 + N/p$ and $-2 + N/p$ are negative, we can choose the constant K so large that

$$C\, \eta_0^{N/p-1} K^{-1+N/p} < 1,$$

and

$$C\, \eta_0^{N/p-2} K^{-2+N/p} < 1.$$

Then, the desired inequality (4.5) follows from inequality (4.14).

The proof of Lemma 4.2 is complete. □

(III) The next lemma, together with Lemma 4.2, proves that the resolvent set of $\mathfrak{W}$ contains the set $\Sigma(\varepsilon) = \left\{ \lambda = r^2 e^{i\vartheta} : r \geq r(\varepsilon), \ -\pi + \varepsilon \leq \vartheta \leq \pi - \varepsilon \right\}$:

Lemma 4.6. *If $\lambda \in \Sigma(\varepsilon)$, then, for any $f \in C_0(\overline{D} \setminus M)$, there exists a unique function $u \in D(\mathfrak{W})$ such that $(\mathfrak{W} - \lambda I)u = f$.*

Proof. Since we have, for all $1 < p < \infty$,

$$f \in C_0(\overline{D} \setminus M) \subset L^p(D),$$

it follows from an application of Theorem 3 that if $\lambda \in \Sigma_p(\varepsilon)$, there exists a unique function $u \in H^{2,p}(D)$ such that

$$(W - \lambda)u = f \quad \text{in } D, \tag{4.15}$$

and

$$Lu = \mu \frac{\partial u}{\partial \mathbf{n}} + \gamma u = 0 \quad \text{on } \partial D. \tag{4.16}$$

However, by Sobolev's imbedding theorem it follows that

$$u \in H^{2,p}(D) \subset C^{2-N/p}(\overline{D}) \subset C^1(\overline{D}) \quad \text{if} \quad N < p < \infty.$$

Hence we have, by formula (4.16) and condition (H),

$$u = 0 \quad \text{on} \quad M = \{x' \in \partial D : \mu(x') = 0\},$$

so that

$$u \in C_0(\overline{D} \setminus M).$$

Further, in view of equation (4.15) we find that

$$Wu = f + \lambda u \in C_0(\overline{D} \setminus M).$$

Summing up, we have proved that

$$\begin{cases} u \in D(\mathfrak{W}), \\ (\mathfrak{W} - \lambda I)u = f. \end{cases}$$

Now the proof of Theorem 4 is complete. □

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