GROWING-UP POSITIVE SOLUTIONS OF SEMILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

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Abstract. This paper is devoted to the study of the existence, uniqueness and asymptotic behavior of positive solutions of a class of *degenerate* boundary value problems for semilinear second-order elliptic differential operators which originates from the so-called Yamabe problem in Riemannian geometry. Our approach is based on the super-sub-solution method adapted to the degenerate case.

1. Introduction and Results

Let D be a bounded domain of Euclidean space $\mathbf{R}^N$, $N \geq 2$, with smooth boundary ∂D; its closure $\overline{D} = D \cup \partial D$ is an N-dimensional, compact smooth manifold with boundary. This paper is devoted to the study of the existence and uniqueness of positive solutions of the following semilinear elliptic boundary value problem:

$$\begin{cases} -\Delta u = \lambda u - h(x)\, u^p & \text{in } D, \\ Bu := a(x') \dfrac{\partial u}{\partial \mathbf{n}} + (1 - a(x'))u = 0 & \text{on } \partial D. \end{cases} \tag*{$(*)_\lambda$}$$

Here:

(1) $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \ldots + \partial^2/\partial x_N^2$ is the usual Laplacian.
(2) λ is a positive parameter.
(3) $h(x)$ is a real-valued function on the closure $\overline{D}$.
(4) $p > 1$.
(5) $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ is the unit exterior normal to the boundary ∂D.
(6) $a(x')$ is a real-valued function on the boundary ∂D.

A function $u(x) \in C^2(\overline{D})$ is called a *positive solution* of problem $(*)_\lambda$ if it satisfies problem $(*)_\lambda$ and is strictly positive everywhere in D.

It is worth pointing out here that the equation $-\Delta u - \lambda u + h(x)\, u^p = 0$ originates from the so-called Yamabe problem which is a basic problem in Riemannian geometry if we take $p = (N+2)/(N-2) > 1$ for $N \geq 3$ (see [7], [8]).

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Our fundamental conditions on the function $h(x)$ are the following:

$$h(x) \in C^{\theta}(\overline{D}), \quad 0 < \theta < 1; \tag{H.1}$$

$$h(x) \geq 0 \quad \text{on } \overline{D}. \tag{H.2}$$

We remark that Ouyang [9] and Korman-Ouyang [6] studied the case where the function $h(x)$ may change sign in D.

On the other hand, our boundary condition B is a linear combination of the Dirichlet and Neumann conditions. It is easy to see that the boundary condition B is non-degenerate (or coercive) if and only if either $a(x') \neq 0$ on ∂D or $a(x') \equiv 0$ on ∂D. Ouyang [8] and del Pino [3] studied the Dirichlet and Neumann cases, while Fraile et al. [4] studied the general non-degenerate case. For further studies of semilinear elliptic problems, we refer to Alama-Tarantello [1], Amann [2], Gámez [5] and Pao [10].

In this paper we study problem $(*)_\lambda$ in the *degenerate* case; more precisely, our fundamental condition on the function $a(x')$ is the following:

$$0 \leq a(x') \leq 1 \quad \text{on } \partial D. \tag{H.3}$$

Remark that the so-called Lopatinskii-Shapiro complementary condition is violated at the points $x' \in \partial D$ where $a(x') = 0$.

In order to formulate our results, let λ_1 be the first eigenvalue of the linearized eigenvalue problem

$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } D, \\ B\varphi = 0 & \text{on } \partial D. \end{cases} \tag{1.1}$$

It is known (see [11, Theorem 1]) that the first eigenvalue λ_1 is non-negative and simple and further that its associated eigenfunction $\varphi_1(x)$ can be chosen to be positive everywhere in D. By Green's formula, it is easy to see that a necessary condition on the parameter λ for the existence of positive solutions of problem $(*)_\lambda$ is that $\lambda > \lambda_1$.

Conversely, if $h(x) > 0$ on $\overline{D}$, then Taira–Umezu [14] proved that problem $(*)_\lambda$ has a unique positive solution $u_\lambda(x) \in C^{2+\theta}(\overline{D})$ for each $\lambda > \lambda_1$. Furthermore, the solution $u_\lambda(x)$ grows up as $\lambda \to \infty$, that is, the maximum norm $\|u_\lambda\|_\infty$ on $\overline{D}$ tends to infinity as $\lambda \to \infty$.

This paper is concerned with the case where the function $h(x)$ may vanish in D. More precisely, we assume that

The zero set $D(h) = \{x \in \overline{D} : h(x) = 0\}$ of the function $h(x)$ is bounded away from the boundary ∂D, (H.4)

and denote by $D_0(h)$ its *interior*. Following del Pino [3], we introduce a critical value $\lambda_1(D_0(h))$ in the following way: Let $\mathcal{B}$ be the set of all open subsets of D with smooth boundary. If $\Omega \in \mathcal{B}$, we denote by $\lambda_1(\Omega)$ the first eigenvalue of the Dirichlet problem

$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

By the celebrated Rayleigh theorem, we know that the first eigenvalue $\lambda_1(\Omega)$ is given by the variational formula

$$\lambda_1(\Omega) = \inf \left\{ \int_\Omega |\nabla u|^2 \, dx : u \in H_0^1(\Omega), \ \int_\Omega u^2 \, dx = 1 \right\},$$

where $H_0^1(\Omega)$ is the closure of the space $C_0^\infty(\Omega)$ of smooth functions with compact support in Ω in the Sobolev space $H^1(\Omega)$. Then we let

$$\lambda_1(D_0(h)) = \sup \{\lambda_1(\Omega) : \Omega \in \mathcal{B}, \ D_0(h) \subset \Omega\}. \tag{1.2}$$

We understand $\lambda_1(D_0(h)) = \infty$ in the case where the set $D_0(h)$ is empty. Remark that if the boundary $\partial D_0(h)$ is sufficiently regular, then the value $\lambda_1(D_0(h))$ coincides with the first eigenvalue of the Dirichlet eigenvalue problem

$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } D_0(h), \\ \varphi = 0 & \text{on } \partial D_0(h). \end{cases}$$

Our first result is the following existence and uniqueness theorem of positive solutions of problem $(*)_\lambda$:

Theorem 1. *Assume that conditions (H.1) through (H.4) are satisfied. Then problem* $(*)_\lambda$ *has a unique positive solution* $u_\lambda(x) \in C^{2+\theta}(\overline{D})$ *for every* $\lambda_1 < \lambda < \lambda_1(D_0(h))$ *and no positive solution for all* $\lambda \geq \lambda_1(D_0(h))$. *Furthermore the solution* $u_\lambda(x)$ *grows up as* $\lambda \uparrow \lambda_1(D_0(h))$, *that is, the maximum norm* $\|u_\lambda\|_\infty$ *tends to infinity as* $\lambda \uparrow \lambda_1(D_0(h))$.

Theorem 1 is a generalization of del Pino [3, Theorem 2] where the Dirichlet and Neumann conditions are treated, and it is proved by Taira–Umezu [13, Theorem 3] under the condition that the boundary $\partial D_0(h)$ is sufficiently regular.

Secondly we study the asymptotic behavior of the unique positive solution $u_\lambda(x)$ as $\lambda \uparrow \lambda_1(D_0(h))$. To do so, we take a relatively compact, open subset Ω' of D with smooth boundary $\partial\Omega'$ which satisfies the conditions

$$\Omega' \supset D(h); \tag{1.3a}$$

The closure $\overline{\Omega'} = \Omega' \cup \partial\Omega'$ consists of a *finite* number of connected components. (1.3b)

Then we let

$$\Omega = D \setminus \overline{\Omega'},$$
$$\Gamma = \partial\Omega \cap D,$$

and introduce a non-negative smooth function $\rho(x)$ defined on the closure $\overline{\Omega}$ such that

$$\rho(x) = \begin{cases} \inf\{|x-y| : y \in \Gamma\} & \text{on a tubular neighborhood of the topological boundary } \Gamma \text{ of } \Omega \text{ in } D; \\ 1 & \text{on a tubular neighborhood of the boundary } \partial D. \end{cases} \tag{1.4}$$

Now we can state our second result which is a generalization of del Pino [3, Theorem 3] to the degenerate case:

Theorem 2. *Assume that conditions (H.1) through (H.4) are satisfied. If Ω' is a relatively compact, open subset of D with smooth boundary $\partial\Omega'$ which satisfies conditions (1.3a) and (1.3b) and if $\Omega = D \setminus \overline{\Omega'}$, then, for any bounded sub-interval I of the interval $(\lambda_1, \lambda_1(D_0(h)))$ and any $\alpha > 2/(p-1)$ there exists a constant $C > 0$ such that we have*

$$\sup_{\lambda \in I} u_\lambda(x) \leq C\, \rho(x)^{-\alpha}, \quad x \in \Omega. \tag{1.5}$$

Furthermore, if the interior $D_0(h)$ is connected and non-empty, then we have, for any compact subset K of $D_0(h)$,

$$\inf_{x \in K} u_\lambda(x) \longrightarrow \infty \quad \text{as } \lambda \uparrow \lambda_1(D_0(h)). \tag{1.6}$$

Rephrased, Theorem 2 asserts that the more the exponent p increases, the milder the solution $u_\lambda(x)$ behaves; while the more the set $D_0(h)$ enlarges, the wilder the solution $u_\lambda(x)$ behaves.

If the set $D_0(h)$ is equal to the unit open ball in $\mathbf{R}^N$, then we can give a precise description of the growing-up rate of the solution $u_\lambda(x)$ in $D_0(h)$ (see Theorem 5.1). For a similar description, we refer to del Pino [3, Remark 1] where the growing-up rate in the set $\overline{D} \setminus D(h)$ is given for a smooth function $h(x)$.

Finally, we discuss the behavior of the solution $u_\lambda(x)$ as $\lambda \uparrow \lambda_1(D_0(h))$ in the case where the zero set $D(h)$ is non-empty but its interior $D_0(h)$ is empty. Then Theorem 1 tells us that there exists a unique positive solution $u_\lambda(x)$ of problem $(*)_\lambda$ for each $\lambda > \lambda_1$ and that the maximum norm $\|u_\lambda\|_\infty$ tends to infinity as $\lambda \to \infty$.

Our third result generalizes assertion (1.6) of Theorem 2 to the case where the interior $D_0(h)$ is empty:

Theorem 3. *Assume that conditions (H.1) through (H.4) are satisfied, and further that there exists a sequence $\{\Omega_j\}_{j=1}^\infty$ of relatively compact, open subsets of D with smooth boundary such that the Ω_j contain the zero set $D(h)$ and satisfy the condition*

$$\lim_{j \to \infty} |\Omega_j| = 0,$$

where $|\cdot|$ denotes the Lebesgue measure of a measurable set of $\mathbf{R}^N$. Then problem $(*)_\lambda$ has a unique positive solution $u_\lambda(x)$ for each $\lambda > \lambda_1$ which tends to infinity as $\lambda \to \infty$, uniformly with respect to $x \in K$ for any compact subset K of D.

Example 1. If the zero set $D(h)$ consists of *finitely* many points in D, then Theorem 3 applies.

Example 2. If the zero set $D(h)$ consists of finitely many connected components of dimension m with $1 \leq m \leq N-1$, then Theorem 3 applies.

The rest of this paper is organized as follows.

In Section 2 we prove Theorem 1 by using the super-sub-solution method and comparison arguments with the Dirichlet and Neumann conditions. Section 3 is devoted to the proof of Theorem 2. Our approach is based on a modification of the variational technique of del Pino [3] adapted to the degenerate case. In Section 4 we prove Theorem 3. The essential step in the proof is how to construct a super-solution of problem $(*)_\lambda$ in order to prove the existence of a positive solution, while we construct a good sub-solution in order to study the behavior of the positive solution, by making use of the eigenfunction $\varphi_1(x)$ of problem (1.1). In Section 5 we consider the growing-up rate of the unique positive solution $u_\lambda(x)$ in the case where the interior $D_0(h)$ is the unit open ball in $\mathbf{R}^N$ and the function $h(x)$ satisfies a growth condition near the boundary $\partial D_0(h)$ (Theorem 5.1). In order to give a precise description of the growing-up rate of the solution $u_\lambda(x)$, we transpose problem $(*)_\lambda$ into an equivalent fixed point equation $(**)_\lambda$ for the resolvent K, and then apply the super-sub-solution method (Theorem 5.2).

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2. Proof of Theorem 1

In this section we prove Theorem 1 by using the super-sub-solution method and comparison arguments with the Dirichlet and Neumann conditions. By Taira–Umezu [13], we know that the problem $(*)_\lambda$ has at most one positive solution for every $\lambda > \lambda_1$.

(I) First, we prove that problem $(*)_\lambda$ has a positive solution $u_\lambda(x)$ for every $\lambda_1 < \lambda < \lambda_1(D_0(h))$, by using the super-sub-solution method.

Let $f(x,t)$ be a real-valued, Hölder continuous function with exponent $0 < \theta < 1$ on $\overline{D} \times [0,r]$ for any $r > 0$, and satisfy the following *slope condition* or *one-sided Lipschitz condition* (cf. [2], [10]): For any $r > 0$, there exists a constant $L > 0$ such that

$$f(x,t) - f(x,s) > -L(t-s), \quad x \in \overline{D}, \ 0 \le s < t \le r.$$

Now we consider the solvability of the semilinear elliptic problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } D, \\ Bu = a(x')\dfrac{\partial u}{\partial \mathbf{n}} + (1 - a(x'))u = 0 & \text{on } \partial D. \end{cases} \tag{2.1}$$

A non-negative function $\phi(x) \in C^2(\overline{D})$ is called a *sub-solution* of problem (2.1) if it satisfies the conditions

$$\begin{cases} -\Delta \phi \le f(x,\phi) & \text{in } D, \\ B\phi \le 0 & \text{on } \partial D. \end{cases}$$

Similarly, a non-negative function $\psi(x) \in C^2(\overline{D})$ is called a *super-solution* of problem (2.1) if it satisfies the conditions

$$\begin{cases} -\Delta \psi \ge f(x,\psi) & \text{in } D, \\ B\psi \ge 0 & \text{in } \partial D. \end{cases}$$

The next theorem, [13, Theorem 1], plays a fundamental role in the construction of positive solutions of problem (2.1) (cf. [2, Theorem 9.4], [10, Theorems 3.2.1 and 3.2.2] for the non-degenerate case):

Theorem 2.1. *Assume that condition (H.3) is satisfied. If there exist a sub-solution $\phi(x)$ and a super-solution $\psi(x)$ of problem (2.1) such that $\phi(x) \leq \psi(x)$ on $\overline{D}$, then problem (2.1) has a solution $u(x) \in C^{2+\theta}(\overline{D})$ such that $\phi(x) \leq u(x) \leq \psi(x)$ on $\overline{D}$.*

(I-a) We construct a super-solution of problem $(*)_\lambda$ for each $\lambda_1 < \lambda < \lambda_1(D_0(h))$, by using the following existence result for the Neumann problem due to del Pino [3] (see Ouyang [8] for the case where $\partial D_0(h)$ is sufficiently smooth):

Theorem 2.2. *Assume that conditions (H.1), (H.2) and (H.4) are satisfied. Then the homogeneous Neumann problem*

$$\begin{cases} -\Delta v = \lambda\, v - h(x)\, v^p & \text{in } D, \\ \dfrac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial D \end{cases} \tag{2.2}$$

has a unique positive solution $v_\lambda(x) \in C^{2+\theta}(\overline{D})$ for each $0 < \lambda < \lambda_1(D_0(h))$.

Let $\psi_\lambda(x)$ be a unique positive solution of problem (2.2) for $0 < \lambda < \lambda_1(D_0(h))$. Then it follows that the function ψ_λ is a super-solution of problem $(*)_\lambda$, since we have

$$B\psi_\lambda = a(x')\frac{\partial \psi_\lambda}{\partial \mathbf{n}} + (1 - a(x'))\psi_\lambda = (1 - a(x'))\psi_\lambda \geq 0 \quad \text{on } \partial D.$$

(I-b) Next we construct a sub-solution of problem $(*)_\lambda$. Let $\varphi_1(x) \in C^\infty(\overline{D})$ be the positive eigenfunction corresponding to the first eigenvalue λ_1 of problem (1.1), normalized as $\|\varphi_1\|_\infty = 1$. If $\lambda > \lambda_1$, then we have, for $\varepsilon > 0$,

$$-\Delta(\varepsilon\varphi_1) - \lambda\varepsilon\varphi_1 + h(x)\,(\varepsilon\varphi_1)^p \leq \left(\|h\|_\infty \varepsilon^{p-1} - (\lambda - \lambda_1)\right)\varepsilon\varphi_1 \quad \text{in } D.$$

This proves that the function $\varepsilon_\lambda\varphi_1(x)$ is a sub-solution of problem $(*)_\lambda$ if ε_λ is sufficiently small.

(I-c) By [12, Lemma 2.1], we see that the functions $\psi_\lambda(x)$ and $\varphi_1(x)$ are comparable. This implies that if ε_λ is sufficiently small, then it follows that $\varepsilon_\lambda\varphi_1(x) \leq \psi_\lambda(x)$ on $\overline{D}$. Therefore, by applying Theorem 2.1 we can find a positive solution $u_\lambda(x)$ of problem $(*)_\lambda$ for every $\lambda_1 < \lambda < \lambda_1(D_0(h))$ such that

$$\varepsilon_\lambda\varphi_1(x) \leq u_\lambda(x) \leq \psi_\lambda(x) \quad \text{on } \overline{D}.$$

(II) Secondly we prove a non-existence result for all $\lambda \geq \lambda_1(D_0(h))$. To do so, we need the following existence and non-existence results for the Dirichlet problem due to del Pino [3] (see Ouyang [8] for the case where $\partial D_0(h)$ is sufficiently smooth):

Theorem 2.3. *Assume that conditions (H.1), (H.2) and (H.4) are satisfied. Then the homogeneous Dirichlet problem*

$$\begin{cases} -\Delta w = \lambda\, w - h(x)\, w^p & \text{in } D, \\ w = 0 & \text{on } \partial D \end{cases} \tag{2.3}$$

has a unique positive solution $w_\lambda(x) \in C^{2+\theta}(\overline{D})$ *for each* $\lambda_1(D) < \lambda < \lambda_1(D_0(h))$, *and it has no positive solution for all* $\lambda \geq \lambda_1(D_0(h))$. *Here* $\lambda_1(D)$ *is the first eigenvalue of the Dirichlet eigenvalue problem*

$$\begin{cases} -\Delta u = \lambda\, u & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases}$$

Now assume to the contrary that problem $(*)_\lambda$ has a positive solution $u_\lambda(x)$ for some $\lambda \geq \lambda_1(D_0(h))$. Then it follows that the function $u_\lambda(x)$ is a super-solution of problem (2.3), since we have

$$u_\lambda \geq 0 \quad \text{on } \partial D.$$

On the other hand, if $\varphi_0(x)$ is a positive eigenfunction corresponding to the first eigenvalue $\lambda_1(D)$, then it is easy to verify that the function $\varepsilon_\lambda \varphi_0(x)$ is a sub-solution of problem (2.3) for ε_λ sufficiently small, since we have $\lambda > \lambda_1(D)$.

Therefore, by applying Theorem 2.1 to the Dirichlet case ($a(x') \equiv 0$ on ∂D) we can find a positive solution $w_\lambda(x) \in C^{2+\theta}(\overline{D})$ of problem (2.3) for $\lambda \geq \lambda_1(D_0(h))$ such that

$$\varepsilon_\lambda \varphi_0(x) \leq w_\lambda(x) \leq u_\lambda(x) \quad \text{on } \overline{D}.$$

However, this contradicts Theorem 2.3.

(III) Finally, since we have proved the existence and non-existence results for problem $(*)_\lambda$, we can prove just as in [13] that the maximum norm $\|u_\lambda\|_\infty$ tends to infinity as $\lambda \uparrow \lambda_1(D_0(h))$.

The proof of Theorem 1 is now complete. □

3. Proof of Theorem 2

This section is devoted to the proof of Theorem 2. Our approach is based on a modification of the variational technique of del Pino [3] adapted to the degenerate case. The proof is divided into four steps.

(I) We introduce a non-negative smooth function $\rho(x) \in C^\infty(\overline{\Omega})$ defined by formula (1.4), and consider a function

$$v(x) = C\,\rho(x)^{-\alpha}, \quad \alpha > 2/(p-1),$$

where C is a positive constant to be chosen later on. Then we have, by a direct computation,

$$-\Delta v = C\left(\alpha\rho^{-\alpha-1}\Delta\rho - \alpha(\alpha+1)\rho^{-\alpha-2}|\nabla\rho|^2\right).$$

Since $\rho(x) = 1$ in a tubular neighborhood of ∂D, by integration by parts it follows that we have, for all non-negative functions $\varphi(x) \in C^1(\Omega)$ having support away from $\Gamma = \partial\Omega \cap D$,

$$\begin{aligned} \int_\Omega \nabla v \cdot \nabla\varphi\, dx &= -\int_\Omega \Delta v \cdot \varphi\, dx + \int_{\partial\Omega} \frac{\partial v}{\partial \mathbf{n}}\, \varphi\, d\sigma \\ &= C\int_\Omega \left(\alpha\rho^{-\alpha-1}\Delta\rho - \alpha(\alpha+1)\rho^{-\alpha-2}|\nabla\rho|^2\right)\varphi\, dx, \end{aligned} \tag{3.1}$$

where $d\sigma$ is the surface element on $\partial\Omega$.

On the other hand we find that any positive solution $u(x)$ of problem $(*)_\lambda$ satisfies the formula

$$\int_\Omega (\nabla u \cdot \nabla \varphi + (h\, u^p - \lambda u)\, \varphi)\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}}\, \varphi\, d\sigma. \tag{3.2}$$

However, since we have

$$Bu = a(x')\frac{\partial u}{\partial \mathbf{n}} + (1 - a(x'))u = 0,$$

it follows that

$$\begin{cases} u(x') = 0 & \text{if } a(x') = 0, \\ \dfrac{\partial u}{\partial \mathbf{n}}(x') = -\dfrac{1 - a(x')}{a(x')} u(x') \leq 0 & \text{if } a(x') > 0, \end{cases}$$

so that

$$\frac{\partial u}{\partial \mathbf{n}} \leq 0 \quad \text{on } \partial D.$$

Therefore, combining inequalities (3.1) and (3.2) we have, for all non-negative functions $\varphi \in C^1(\Omega)$ having support away from Γ,

$$\begin{aligned} &\int_\Omega (\nabla (u - v) \cdot \nabla \varphi + (h\, u^p - \lambda u)\, \varphi)\, dx \\ &= C \int_\Omega \left(-\alpha \rho^{-\alpha - 1} \Delta \rho + \alpha(\alpha + 1)\rho^{-\alpha - 2} |\nabla \rho|^2 \right) \varphi\, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}}\, \varphi\, d\sigma \\ &\leq C \int_\Omega \left(-\alpha \rho^{-\alpha - 1} \Delta \rho + \alpha(\alpha + 1)\rho^{-\alpha - 2} |\nabla \rho|^2 \right) \varphi\, dx. \end{aligned} \tag{3.3}$$

If we let

$$\underline{h} = \frac{1}{2} \inf_{x \in \Omega} h(x),$$

then we obtain from inequality (3.3) that

$$\begin{aligned} &\int_\Omega (\nabla (u - v) \cdot \nabla \varphi + (h\, u^p - \lambda u - \underline{h}\, v^p)\, \varphi)\, dx \\ &\leq C|\Omega|\, \|\varphi\|_{\infty, \overline{\Omega}}\, \|\rho\|_{\infty, \overline{\Omega}}^{-\alpha p} \Big(\alpha \|\rho\|_{\infty, \overline{\Omega}}^{-\alpha - 1 + \alpha p} \|\Delta \rho\|_{\infty, \overline{\Omega}} \\ &\qquad + \alpha(\alpha + 1) \|\rho\|_{\infty, \overline{\Omega}}^{-\alpha - 2 + \alpha p} \||\nabla \rho|\|_{\infty, \overline{\Omega}} - \underline{h}\, C^{p - 1} \Big), \end{aligned} \tag{3.4}$$

where $\|\cdot\|_{\infty, \overline{\Omega}}$ is the maximum norm of $C(\overline{\Omega})$. Since $\underline{h} > 0$ and $-\alpha - 2 + \alpha p > 0$, it follows from inequality (3.4) that

$$\int_\Omega (\nabla (u - v) \cdot \nabla \varphi + (h\, u^p - \lambda u - \underline{h}\, v^p)\, \varphi)\, dx \leq 0, \tag{3.5}$$

if we take the constant C (independent of λ) so large that

$$C^{p-1} > \frac{\alpha}{\underline{h}} \left(\|\rho\|_{\infty,\overline{\Omega}} \|\Delta\rho\|_{\infty,\overline{\Omega}} + (\alpha+1) \| |\nabla\rho| \|_{\infty,\overline{\Omega}} \right) \|\rho\|_{\infty,\overline{\Omega}}^{-\alpha-2+\alpha p}. \tag{3.6}$$

We remark that formula (3.5) remains valid for all non-negative functions φ in the Sobolev space $H^1(\Omega)$ having support away from Γ.

(II) Let I be a bounded sub-interval of the interval $(\lambda_1, \lambda_1(D_0(h)))$, and $u_\lambda(x)$ a positive solution of problem $(*)_\lambda$ for $\lambda \in I$. Then we have the assertion

$$x \in \Omega \text{ and } u_\lambda(x) \geq C\,\rho(x)^{-\alpha} \implies h(x)\,u_\lambda(x)^p - \lambda\,u_\lambda(x) \geq \underline{h}\,u_\lambda(x)^p, \tag{3.7}$$

if we take a constant $C > 0$ sufficiently large, independent of $\lambda \in I$.

Indeed, if we let

$$\overline{\lambda} = \sup_{\lambda \in I} \lambda,$$

then it follows that

$$\begin{aligned} h(x)\,u_\lambda(x)^p - \lambda\,u_\lambda(x) - \underline{h}\,u_\lambda(x)^p &\geq \underline{h}\,u_\lambda(x)^p - \overline{\lambda}\,u_\lambda(x) \\ &= u(x)\left(\underline{h}\,u_\lambda(x)^{p-1} - \overline{\lambda}\right) \\ &\geq u(x)\left(\underline{h}\,C^{p-1}\,\rho(x)^{-\alpha(p-1)} - \overline{\lambda}\right) \\ &\geq u(x)\left(\underline{h}\,C^{p-1}\,\|\rho\|_{\infty,\overline{\Omega}}^{-\alpha(p-1)} - \overline{\lambda}\right) \\ &> 0, \end{aligned}$$

if we take the constant $C > 0$ so large that

$$C^{p-1} > \frac{\overline{\lambda}}{\underline{h}}\,\|\rho\|_{\infty,\overline{\Omega}}^{\alpha(p-1)}. \tag{3.8}$$

(III) In order to prove assertion (1.5), assume to the contrary that one can find a bounded interval $I_0 \subset (\lambda_1, \lambda_1(D_0(h)))$ such that, for any constant $C > 0$ there exist a parameter $\lambda_0 \in I_0$ and a point $x_0 \in \Omega$ such that the unique positive solution $u_{\lambda_0}(x)$ of problem $(*)_{\lambda_0}$ satisfies the inequality

$$u_{\lambda_0}(x_0) > C\,\rho(x_0)^{-\alpha}.$$

We choose a large constant C satisfying inequalities (3.6) and (3.8) with $I := I_0$, and let

$$\begin{aligned} u_0(x) &= u_{\lambda_0}(x), \\ v_0(x) &= C\,\rho(x)^{-\alpha}, \\ w_0(x) &= \max\{u_0(x) - v_0(x), 0\}. \end{aligned}$$

Then it follows that the function $w_0(x)$ belongs to the Sobolev space $H^1(\Omega)$ having support away from Γ, since we have, for all x in a tubular neighborhood of Γ in Ω,

$$u_0(x) < v_0(x).$$

Hence, applying inequality (3.5) to the functions $u := u_0$, $v := v_0$ and $\varphi := w_0$ we obtain that

$$\int_\Omega (\nabla(u_0 - v_0) \cdot \nabla w_0 + (h\, u_0^p - \lambda_0 u_0 - \underline{h}\, v_0^p)\, w_0)\, dx \leq 0,$$

or equivalently,

$$\int_{\text{supp}\, w_0} (\nabla(u_0 - v_0) \cdot \nabla w_0 + (h\, u_0^p - \lambda_0\, u_0 - \underline{h}\, v_0^p)\, w_0)\, dx \leq 0.$$

Furthermore, in view of assertion (3.7) this implies that

$$\int_{\text{supp}\, w_0} (\nabla(u_0 - v_0) \cdot \nabla w_0 + \underline{h}\, ({u_0}^p - {v_0}^p)\, w_0)\, dx \leq 0.$$

Therefore, we conclude that

$$\begin{aligned} 0 &< \int_{\text{supp}\, w_0} \underline{h}\, ({u_0}^p - {v_0}^p)\, w_0\, dx \\ &\leq \int_{\text{supp}\, w_0} (\nabla(u_0 - v_0) \cdot \nabla w_0 + \underline{h}\, ({u_0}^p - {v_0}^p)\, w_0)\, dx \\ &\leq 0, \end{aligned}$$

since $w_0 = u_0 - v_0$ on $\text{supp}\, w_0$ and the Lebesgue measure of $\text{supp}\, w_0$ is positive. This contradiction proves assertion (1.5).

(IV) Finally, it remains to prove assertion (1.6). Let $u_\lambda(x)$ be a positive solution of problem $(*)_\lambda$ for any $\lambda_1(D) < \lambda < \lambda_1(D_0(h))$, and let $\varphi_0(x)$ be a positive eigenfunction corresponding to the first eigenvalue $\lambda_1(D)$ of problem (2.4). Then, just as in step (II) of the proof of Theorem 1 we find that the function $u_\lambda(x)$ is a super-solution of problem (2.3) and further that the function $\varepsilon_\lambda \varphi_0(x)$ is a sub-solution of problem (2.3) if ε_λ is sufficiently small.

Therefore, by applying Theorem 2.1 to the Dirichlet case ($a(x') \equiv 0$ on ∂D) we can find a positive solution $w_\lambda(x) \in C^{2+\theta}(\overline{D})$ of problem (2.3) such that

$$\varepsilon_\lambda \varphi_0(x) \leq w_\lambda(x) \leq u_\lambda(x), \quad x \in \overline{D}. \tag{3.9}$$

However, we know from part (ii) of [3, Theorem 3] that assertion (1.6) holds for the function $w_\lambda(x)$, that is,

$$\inf_{x \in K} w_\lambda(x) \longrightarrow \infty \quad \text{as } \lambda \uparrow \lambda_1(D_0(h)).$$

In view of inequalities (3.9), it follows that assertion (1.6) holds also for the solution $u_\lambda(x)$ of problem $(*)_\lambda$.

Now the proof of Theorem 2 is complete. □

4. Proof of Theorem 3

In this section we prove Theorem 3. The essential step in the proof is how to construct a super-solution of problem $(*)_\lambda$ in order to prove the existence of a positive solution, while we construct a good sub-solution in order to study the behavior of the positive solution, by making use of the eigenfunction $\varphi_1(x)$ of problem (1.1).

(I) Now take the positive eigenfunction $\varphi_1(x)$ corresponding to the first eigenvalue λ_1 of problem (1.1) such that $\|\varphi_1\|_\infty = 1$ and let

$$v(x) = \left(\frac{\lambda - \lambda_1}{\|h\|_\infty}\right)^{1/(p-1)} \varphi_1(x), \quad \lambda > \lambda_1.$$

Then we have

$$-\Delta v - \lambda v + h(x) v^p \leq 0 \quad \text{in } D.$$

This implies that $v(x)$ is a sub-solution of problem $(*)_\lambda$.

(II) Next we construct a super-solution of problem $(*)_\lambda$. Since the function $h(x)$ satisfies conditions (H.1), (H.2) and (H.4), for each $\lambda > \lambda_1$ we can choose a non-negative function $\tilde{h}(x) \in C^\theta(\overline{D})$ such that

1. The zero set $D(\tilde{h})$ of the function $\tilde{h}(x)$ is bounded away from ∂D;
2. The interior $D_0(\tilde{h})$ of $D(\tilde{h})$ is not empty;
3. The interior $D_0(\tilde{h})$ contains the zero set $D(h)$ of the function $h(x)$;
4. $\tilde{h}(x) \leq h(x), \ x \in \overline{D}$;
5. $\lambda < \lambda_1(D_0(\tilde{h}))$.

Here we have used the fact that the value $\lambda_1(D_0(h))$ defined by formula (1.2) tends to infinity as the Lebesgue measure $|D_0(h)|$ goes to zero.

Now we consider the following boundary value problem

$$\begin{cases} -\Delta u = \lambda u - \tilde{h}(x) u^p & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases} \tag{4.1}$$

Theorem 1 tells us that problem (4.1) has a unique positive solution $w(x) \in C^{2+\theta}(\overline{D})$. Then it follows that the function $Cw(x)$ is a super-solution of problem $(*)_\lambda$ for all $C \geq 1$. Indeed, we have

$$-\Delta(Cw) - \lambda Cw + h(x)(Cw)^p = Cw^p(C^{p-1}h(x) - \tilde{h}(x)) \geq 0 \quad \text{in } D,$$

since $h(x) \geq \tilde{h}(x)$ on $\overline{D}$ and $p > 1$.

(III) Applying Theorem 2.1 to the sub-solution $v(x)$ and the super-solution $Cw(x)$ for C sufficiently large, we can find a solution $u_\lambda(x)$ of problem $(*)_\lambda$ such that

$$v(x) = \left(\frac{\lambda - \lambda_1}{\|h\|_\infty}\right)^{1/(p-1)} \varphi_1(x) \leq u_\lambda(x) \leq Cw(x) \quad \text{on } \overline{D}.$$

This proves that the solution $u_\lambda(x)$ tends to infinity as $\lambda \to \infty$, uniformly with respect to $x \in K$ for any compact subset K of D, since we have $\varphi_1(x) > 0$ in D.

The proof of Theorem 3 is now complete. □

5. Growing-up rate of positive solutions

In this section we study the growing-up rate of the unique positive solution $u_\lambda(x)$ of problem $(*)_\lambda$ as $\lambda \to \lambda_1(D_0(h))$ under some further restrictions on the function $h(x)$, which generalizes assertion (1.6) of Theorem 2 (Theorem 5.1). In order to give a precise description of the growing-up rate of the solution $u_\lambda(x)$, we transpose problem $(*)_\lambda$ into an equivalent fixed point equation $(**)_\lambda$ for the resolvent K, and then apply the super-sub-solution method (Theorem 5.2).

We assume that the zero set $D(h)$ of the function $h(x)$ is given by the formula

$$D(h) = \left\{ x \in \mathbf{R}^N : |x| \leq 1 \right\}, \tag{5.1}$$

and we let

$$\begin{cases} D_1 := D_0(h) = \left\{ x \in \mathbf{R}^N : |x| < 1 \right\}, \\ D_r := \left\{ x \in \mathbf{R}^N : |x| < r \right\}, \quad r > 1. \end{cases}$$

Concerning the growth rate of $h(x)$ near $\partial D_0(h)$, we assume that there exist constants $\sigma > 1$, $C_1 > 0$ and $\delta_0 > 0$ such that

$$\sup_{x \in D_r} h(x) \leq C_1 (r-1)^\sigma, \quad r \in [1, 1+\delta_0]. \tag{5.2}$$

Example 5.1. If $h(x)$ is a function in $C^1(\mathbf{R}^N)$ given by the formula

$$h(x) = \begin{cases} 0 & |x| < 1, \\ (|x|-1)^2 & |x| \geq 1, \end{cases}$$

then $h(x)$ satisfies conditions (5.1) and (5.2) with $\sigma = 2$.

Now let $\phi_r(x)$ be a positive eigenfunction associated with the first eigenvalue $\lambda_1(D_r)$ of the Dirichlet problem

$$\begin{cases} -\Delta\varphi = \mu\varphi & \text{in } D_r, \\ \varphi = 0 & \text{on } \partial D_r, \end{cases} \tag{5.3}$$

where the eigenfunction $\phi_r(x)$ is normalized as $\|\phi_r\|_{\infty, \overline{D_r}} = 1$.

The next theorem gives a precise description of the growing-up rate of the positive solution $u_\lambda(x)$ in assertion (1.6) of Theorem 2:

Theorem 5.1. *Assume that the function $h(x)$ satisfies conditions (5.1) and (5.2) and that its zero set $D(h)$ is bounded away from the boundary ∂D. If $u_\lambda(x)$ is a unique positive solution of problem $(*)_\lambda$ for $\lambda_1 < \lambda < \lambda_1(D_0(h))$, then, for any compact subset K of $D_0(h)$ there exists a constant $C > 0$, independent of λ, such that*

$$u_\lambda(x) \geq C \left(\lambda_1(D_0(h)) - \lambda\right)^{(1-\sigma)/(p-1)} \phi_1(x) \quad \text{for all } x \in K, \tag{5.4}$$

for λ sufficiently close to $\lambda_1(D_0(h))$.

Proof. The proof is divided into four steps.

(I) First, we transpose problem $(*)_\lambda$ into an equivalent fixed point equation for the resolvent of the linearized boundary value problem. By using [13, Theorem 1.1], for a given constant $d > 0$ we can associate with the boundary value problem

$$\begin{cases} (-\Delta + d)u = f & \text{in } D, \\ Bu = 0 & \text{on } \partial D, \end{cases} \tag{5.5}$$

a linear operator

$$K_d : C^\theta(\overline{D}) \longrightarrow C^{2+\theta}(\overline{D})$$

in the following way: For any function $f \in C^\theta(\overline{D})$, the function $u = K_d f \in C^{2+\theta}(\overline{D})$ is the unique solution of problem (5.5). Then it is easy to verify that the operator K is uniquely extended to a strictly positive, compact linear operator K from the ordered Banach space $C(\overline{D})$ into itself ([12, Lemma 2.1]). Furthermore, we find that problem $(*)_\lambda$ is equivalent to a nonlinear operator equation

$$u = K_d((\lambda + d)u - hu^p) \quad \text{in } C(\overline{D}). \tag{$**$}_\lambda$$

(II) Secondly, we apply the super-sub-solution method to solve equation $(**)_\lambda$.

A non-negative function $\phi(x) \in C(\overline{D})$ is said to be a *super-solution* of equation $(**)_\lambda$ if it satisfies the condition

$$\phi(x) \geq K_d((\lambda + d)\phi - h\phi^p)(x) \quad \text{for } x \in \overline{D}.$$

Similarly, a non-negative function $\psi(x) \in C(\overline{D})$ is said to be a *sub-solution* of equation $(**)_\lambda$ if it satisfies the condition

$$\psi(x) \leq K_d((\lambda + d)\psi - h\psi^p)(x) \quad \text{for } x \in \overline{D}.$$

The next existence theorem for problem $(*)_\lambda$ is implicitly proved in the proof of [13, Theorem 1]:

Theorem 5.2. *Let $\phi(x)$ and $\psi(x)$ be respectively a sub-solution and a super-solution of equation $(**)_\lambda$ such that $\phi(x) \leq \psi(x)$ on $\overline{D}$. If the function*

$$g_d(x, t) = (\lambda + d)t - h(x)t^p$$

is monotonically increasing in t, that is, if we have

$$g_d(x, s) < g_d(x, t) \quad \textit{for all } x \in \overline{D} \textit{ and } 0 \leq s < t \leq \|\psi\|_\infty,$$

then equation $(**)_\lambda$ has a fixed point $u(x) \in C(\overline{D})$ such that

$$\phi(x) \leq u(x) \leq \psi(x) \quad \textit{on } \overline{D}.$$

In this case, the function $u(x)$ is a solution of problem $(*)_\lambda$ in the space $C^{2+\theta}(\overline{D})$.

(III) We construct a sub-solution of equation $(**)_\lambda$. To do so, we need the following elementary results:

Lemma 5.1. *If $r_1 > r_2 \geq 1$, then the first eigenvalue $\lambda_1(D_r)$ of problem (5.3) and its associated eigenfunction $\phi_r(x)$ satisfy respectively the conditions*

$$\lambda_1(D_{r_1}) = \left(\frac{r_2}{r_1}\right)^2 \lambda_1(D_{r_2}), \tag{5.6}$$

$$\phi_{r_1}(x) = \phi_{r_2}\left(\frac{r_2}{r_1}x\right), \quad x \in D_{r_1}. \tag{5.7}$$

Since the eigenvalue $\lambda_1(D_r)$ depends continuously on r, it follows that, for each $\lambda < \lambda_1(D_0(h))$ close to $\lambda_1(D_0(h))$ there exists a constant $r > 1$ such that

$$\lambda = \lambda_1(D_r).$$

If we let

$$\lambda' = \lambda_1(D_{2r-1}),$$

then we obtain that its associated eigenfunction $\phi_{2r-1}(x)$ satisfies the conditions

$$\begin{cases} (-\Delta + d)(\varepsilon\phi_{2r-1}) \leq (\lambda + d)\varepsilon\phi_{2r-1} - h(x)(\varepsilon\phi_{2r-1})^p & \text{in } D_{2r-1}, \\ \varepsilon\phi_{2r-1} = 0 & \text{on } \partial D_{2r-1}, \end{cases} \tag{5.8}$$

if ε may be chosen to be so small that

$$0 < \varepsilon \leq \left(\frac{\lambda - \lambda'}{\sup_{D_{2r-1}} h}\right)^{1/(p-1)}. \tag{5.9}$$

However, by using condition (5.2) and formula (5.6) we can prove that

$$\left(\frac{\lambda - \lambda'}{\sup_{D_{2r-1}} h}\right)^{1/(p-1)} \geq \left\{\frac{\lambda_1(D_r)(3r-1)}{2^\sigma C_1(2r-1)^2}\right\}^{1/(p-1)} (r-1)^{(1-\sigma)/(p-1)}.$$

This implies that there exists a constant $\tilde{C} > 0$, independent of r close to 1, such that condition (5.9) is valid for

$$\varepsilon = \tilde{C}(r-1)^{(1-\sigma)/(p-1)}.$$

Now we define a continuous function $v_r(x) \in C(\overline{D})$ as

$$v_r(x) = \begin{cases} \tilde{C}(r-1)^{(1-\sigma)/(p-1)}\phi_{2r-1}(x) & \text{in } D_{2r-1}, \\ 0 & \text{on } \overline{D} \setminus D_{2r-1}. \end{cases}$$

Then, by assertion (5.8) it is easy to see that the function $v_r(x)$ is a sub-solution of equation $(**)_\lambda$. In fact, we have the following:

Lemma 5.2. *There exists a constant $d_1 > 0$ such that, for all $d > d_1$ the function $v_r(x)$ satisfies the condition*

$$v_r(x) \le K_d((\lambda + d)v_r - h{v_r}^p) \quad \text{on } \overline{D}.$$

(IV) *End of Proof of Theorem 5.1*. First, by [5, Theorem 3.2] it follows that there exists a super-solution $w_\lambda(x) \in C^2(\overline{D})$ of problem $(*)_\lambda$ for each $\lambda_1 < \lambda < \lambda_1(D_0(h))$. Then we remark that the functions $Rw_\lambda(x)$ are super-solutions of equation $(**)_\lambda$ for all $R > 1$. Moreover we can choose constants $R_0 > 1$ and $d > d_1$ so large that

$$v_r(x) \le R_0 w_\lambda(x) \quad \text{on } \overline{D}$$

and

$$g_d(x, s) < g_d(x, t) \quad \text{for all } x \in \overline{D} \text{ and } 0 \le s < t < R_0\|w_\lambda\|_\infty.$$

Hence it follows from an application of Theorem 5.2 that problem $(*)_\lambda$ has a solution $u(x) \in C^{2+\theta}(\overline{D})$ such that

$$v_r(x) \le u(x) \le R_0 w_\lambda(x) \quad \text{on } \overline{D}.$$

However, by the uniqueness theorem for problem $(*)_\lambda$ (Theorem 1) we obtain that $u(x) = u_\lambda(x)$ in D, so that

$$v_r(x) = \tilde{C}(r-1)^{(1-\sigma)/(p-1)}\phi_{2r-1}(x) \le u_\lambda(x), \quad x \in D_{2r-1}. \tag{5.10}$$

Furthermore, we have, by formula (5.6),

$$\begin{aligned} r - 1 &= \frac{(2r-1)^2}{3r-1}\left(\frac{\lambda_1(D_r) - \lambda_1(D_{2r-1})}{\lambda_1(D_r)}\right) \\ &= \frac{r^2(2r-1)}{r+1}\left(\frac{\lambda_1(D_0(h)) - \lambda}{\lambda_1(D_0(h))}\right). \end{aligned}$$

Summing up, we can rewrite inequality (5.10) in the form

$$C(\lambda_1(D_0(h)) - \lambda)^{(1-\sigma)/(p-1)}\phi_{2r-1}(x) \le u_\lambda(x), \quad x \in D_{2r-1}, \tag{5.11}$$

where C is a positive constant independent of λ.

On the other hand, by formula (5.7) it follows that, as $r \downarrow 1$

$$\phi_{2r-1}(x) \longrightarrow \phi_1(x) \quad \text{in } C(\overline{D_0(h)}). \tag{5.12}$$

Therefore, the desired assertion (5.4) follows by combining inequality (5.11) and assertion (5.12).

Now the proof of Theorem 5.1 is complete. □

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