SEMILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS IN CHEMICAL REACTOR THEORY

Kazuaki Taira and Kenichiro Umezu

Abstract. This paper is devoted to the study of semilinear elliptic boundary value problems arising in chemical reactor theory which obey the simple Arrhenius rate law and Newtonian cooling. We prove that ignition and extinction phenomena occur in the stable steady temperature profile at some critical values of a dimensionless heat evolution rate.

0. Introduction

Let D be a bounded domain of Euclidean space $\mathbf{R}^N$, $N \geq 2$, with smooth boundary ∂D; its closure $\overline{D} = D \cup \partial D$ is an N-dimensional, compact smooth manifold with boundary. We let

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{N} a^{ij}(x) \frac{\partial u}{\partial x_j}(x) \right) + c(x)u(x)$$

be a second-order, *elliptic* differential operator with real smooth coefficients on $\overline{D}$ such that:

(1) $a^{ij}(x) = a^{ji}(x)$, $1 \leq i, j \leq N$, and there exists a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2, \quad x \in \overline{D},\ \xi \in \mathbf{R}^N.$$

(2) $c(x) > 0$ in D.

In this paper we consider the following semilinear elliptic boundary value problem stimulated by a problem of chemical reactor theory (cf. [BGW]):

$$(*)_\lambda \qquad \begin{cases} Au = \lambda \exp\left[u/(1+\varepsilon u)\right] & \text{in } D, \\ Bu = a(x')\dfrac{\partial u}{\partial \boldsymbol{\nu}} + (1 - a(x'))u = 0 & \text{on } \partial D. \end{cases}$$

Here:

(1) λ and ε are positive parameters.

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(2) $a \in C^\infty(\partial D)$ and $0 \leq a(x') \leq 1$ on ∂D.
(3) $\partial/\partial\boldsymbol{\nu}$ is the conormal derivative associated with the operator A:

$$\frac{\partial}{\partial \boldsymbol{\nu}} = \sum_{i,j=1}^{N} a^{ij}(x') n_j \frac{\partial}{\partial x_i},$$

where $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ is the unit exterior normal to the boundary ∂D (see Figure 1).

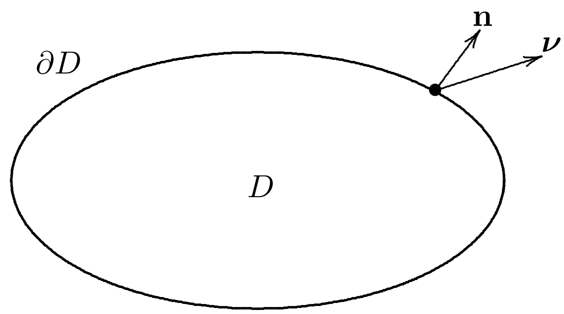

Figure 1

The nonlinear term

$$f(t) = \exp\left[\frac{t}{1+\varepsilon t}\right]$$

describes the temperature dependence of reaction rate for exothermic reactions obeying the simple *Arrhenius rate law* in circumstances in which heat flow is purely conductive. In this context the parameter ε is a dimensionless ambient temperature and the parameter λ is a dimensionless heat evolution rate. The equation $Au - \lambda f(u) = 0$ represents heat balance with reactant consumption ignored, where u is a dimensionless temperature excess, and the boundary condition $Bu = 0$ represents the exchange of heat at the surface of the reactant by *Newtonian cooling*. Moreover the boundary condition $Bu = a(x')(\partial u)/(\partial \boldsymbol{\nu}) + (1 - a((x'))u = 0$ is called the isothermal condition (or Dirichlet condition) if $a \equiv 0$ on ∂D, and is called the adiabatic condition (or Neumann condition) if $a \equiv 1$ on ∂D.

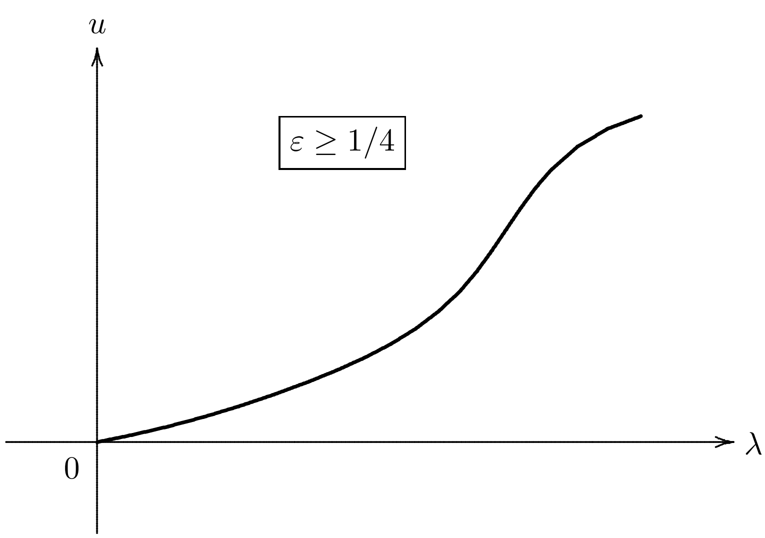

Figure 2

A function $u \in C^2(\overline{D})$ is called a *solution* of problem $(*)_\lambda$ if it satisfies the equation $Au - \lambda f(u) = 0$ in D and the boundary condition $Bu = 0$ on ∂D. A solution u is said to be *positive* if it is positive everywhere in D.

This paper is devoted to the study of the existence of positive solutions of problem $(*)_\lambda$. First, it follows from an application of [TU2, Theorem 1] that problem $(*)_\lambda$ has at least one positive solution for each $\lambda > 0$. Furthermore, by [Ta4, Example7] we know that problem $(*)_\lambda$ has a unique positive solution for each $\lambda > 0$ if $\varepsilon \geq 1/4$. In other words, if the activation energy is so low that the parameter ε exceeds the value $1/4$, then only a smooth progression of reaction rate with imposed ambient temperature can occur; such a reaction may be very rapid but it is only accelerating and lacks the discontinuous change associated with criticality and ignition. The situation may be represented schematically by Figure 2 (cf. [BGW, Figure 6]).

The purpose of this paper is to study the case where $0 < \varepsilon < 1/4$. First, in order to state our multiplicity theorem for problem $(*)_\lambda$, we define a function

$$\nu(t) = \frac{t}{f(t)} = \frac{t}{\exp\left[t/(1+\varepsilon t)\right]}, \quad t \geq 0.$$

It is easy to see (see Figure 3) that if $0 < \varepsilon < 1/4$, then the function $\nu(t)$ has a unique local maximum at $t = t_1(\varepsilon)$:

$$t_1(\varepsilon) = \frac{1 - 2\varepsilon - \sqrt{1 - 4\varepsilon}}{2\varepsilon^2},$$

and has a unique local minimum at $t = t_2(\varepsilon)$:

$$t_2(\varepsilon) = \frac{1 - 2\varepsilon + \sqrt{1 - 4\varepsilon}}{2\varepsilon^2}.$$

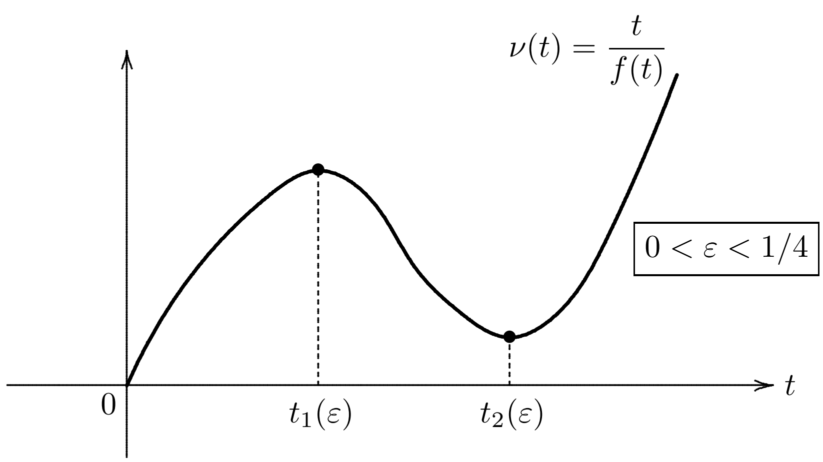

Figure 3

We remark that, as $\varepsilon \downarrow 0$, the local maximum $\nu(t_1(\varepsilon))$ and the local minimum $\nu(t_2(\varepsilon))$ behave respectively as follows:

$$\nu(t_1(\varepsilon)) \sim \exp\left[\frac{-1}{1+\varepsilon}\right],$$

$$\nu(t_2(\varepsilon)) \sim \frac{1}{\varepsilon^2} \exp\left[\frac{-1}{\varepsilon+\varepsilon^2}\right].$$

On the other hand we let $\phi \in C^\infty(\overline{D})$ be the unique positive solution of the linear boundary value problem

$$\begin{cases} A\phi = 1 & \text{in } D, \\ B\phi = 0 & \text{on } \partial D, \end{cases} \tag{0.1}$$

and let

$$\|\phi\|_\infty = \max_{\overline{D}} \phi(x).$$

Now we can state our multiplicity theorem for problem $(*)_\lambda$:

Theorem 1. *We can find a constant* $\beta > 0$, *independent of* ε, *such that if* $0 < \varepsilon < 1/4$ *is so small that*

$$\frac{\nu(t_2(\varepsilon))}{\beta} < \frac{\nu(t_1(\varepsilon))}{\|\phi\|_\infty}, \tag{0.2}$$

then there exist at least three *distinct positive solutions of problem* $(*)_\lambda$ *for all* λ *satisfying the condition*

$$\frac{\nu(t_2(\varepsilon))}{\beta} < \lambda < \frac{\nu(t_1(\varepsilon))}{\|\phi\|_\infty}. \tag{0.3}$$

Theorem 1 is a generalization of [Wi, Theorem 4.3] to the degenerate case. The situation may be represented schematically by Figure 4 (cf. [BGW, Figure 6]).

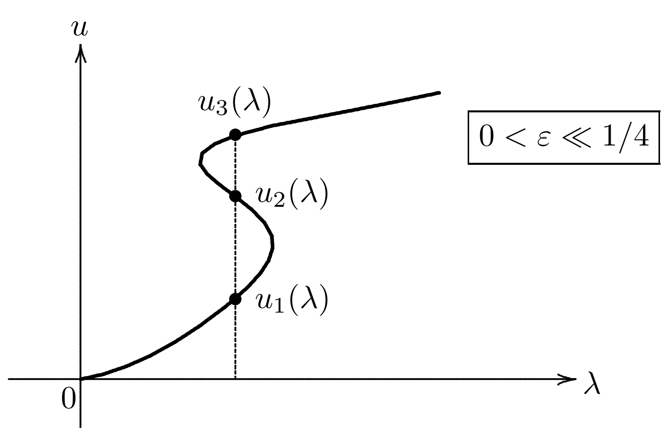

Figure 4

We remark that, as $\varepsilon \downarrow 0$,

$$\frac{\nu(t_2(\varepsilon))}{\beta} \sim \frac{1}{\varepsilon^2} \exp\left[\frac{-1}{\varepsilon+\varepsilon^2}\right],$$

$$\frac{\nu(t_1(\varepsilon))}{\|\phi\|_\infty} \sim \exp\left[\frac{-1}{1+\varepsilon}\right],$$

so that condition (0.2) makes sense.

Secondly, we state two existence and uniqueness theorems for problem $(*)_\lambda$. Let λ_1 be the first eigenvalue of the linear eigenvalue problem

$$\begin{cases} Au = \lambda u & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases}$$

The next two theorems assert that problem $(*)_\lambda$ is uniquely solvable for λ sufficiently small and sufficiently large if $0 < \varepsilon < 1/4$:

Theorem 2. *Let $0 < \varepsilon < 1/4$. If the parameter λ is so small that*

$$0 < \lambda < \frac{\lambda_1 \exp\left[\frac{2\varepsilon-1}{\varepsilon}\right]}{4\varepsilon^2}, \tag{0.4}$$

then problem $(*)_\lambda$ has a unique positive solution.

Theorem 3. *Let $0 < \varepsilon < 1/4$. We can find a constant $\Lambda > 0$, independent of ε, such that if the parameter λ is so large that $\lambda > \Lambda$, then problem $(*)_\lambda$ has a unique positive solution.*

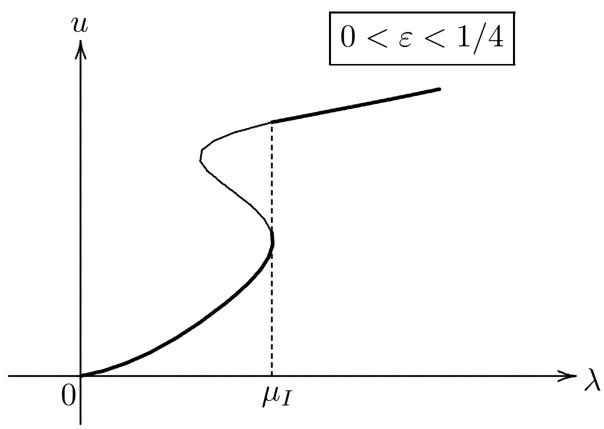

Figure 5

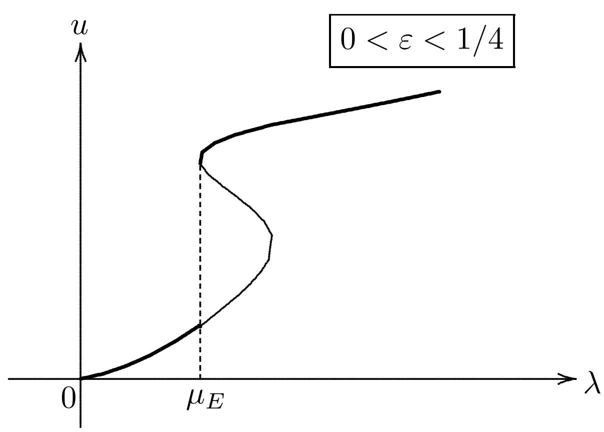

Figure 6

Theorems 2 and 3 are generalizations of [Wi, Theorems 2.9 and 2.6] to the degenerate case, respectively, although we only treat the nonlinear term $f(t) = \exp[t/(1+\varepsilon t)]$.

By virtue of Theorems 1, 2 and 3, we can define two positive numbers μ_I and μ_E by the formulas:

$$\begin{aligned} \mu_I &= \inf\{\mu > 0 : \text{ problem } (*)_\lambda \text{ is uniquely solvable for each } \mu < \lambda\}, \\ \mu_E &= \sup\{\mu > 0 : \text{ problem } (*)_\lambda \text{ is uniquely solvable for each } 0 < \lambda < \mu\}. \end{aligned}$$

Then it is easy to see that an *ignition* phenomenon occurs at $\lambda = \mu_I$ and an *extinction* phenomenon occurs at $\lambda = \mu_E$, respectively. In other words, a small increase in λ causes a large jump in the stable steady temperature profile at $\lambda = \mu_I$ and $\lambda = \mu_E$. More precisely the minimal positive solution $\underline{u}(\lambda)$ is continuous in $\lambda > \mu_I$ but is not continuous at $\lambda = \mu_I$, while the maximal positive solution $\overline{u}(\lambda)$ is continuous in $0 < \lambda < \mu_E$ but is not continuous at $\lambda = \mu_E$. The situation may be represented schematically by Figures 5 and 6 (cf. [BGW, Figure 6]).

By the maximum principle and the boundary point lemma, we can easily see from formula (3.2) below that the first eigenvalue $\lambda_1 = \lambda_1(a)$ satisfies the inequalities

$$\lambda_1(1) < \lambda_1(a) < \lambda_1(0),$$

and that the unique solution $\phi = \phi_{(a)}$ of problem (0.1) satisfies the inequalities

$$\phi_{(0)} < \phi_{(a)} < \phi_{(1)} \quad \text{in } D,$$

so that,

$$\frac{1}{\|\phi_{(1)}\|_\infty} < \frac{1}{\|\phi_{(a)}\|_\infty} < \frac{1}{\|\phi_{(0)}\|_\infty}.$$

Moreover, it follows from formula (2.11) below that the critical value $\beta = \beta(a)$ in Theorem 1 satisfies the inequalities

$$\frac{1}{\beta(1)} \leq \frac{1}{\beta(a)} \leq \frac{1}{\beta(0)},$$

and further from formula (4.14) below that the critical value $\Lambda = \Lambda(a)$ in Theorem 3 depends essentially on the first eigenvalue $\lambda_1 = \lambda_1(a)$.

Therefore, we find that the extinction phenomenon in the isothermal condition case occurs at the largest critical value $\mu_E(0)$, while the extinction phenomenon in the adiabatic condition case occurs at the smallest critical value $\mu_E(1)$. Similarly we find that the ignition phenomenon in the adiabatic condition case occurs at the smallest critical value $\mu_I(1)$, while the ignition phenomenon in the isothermal condition case occurs at the largest critical value $\mu_I(0)$.

The rest of this paper is organized as follows. Section 2 is devoted to the proof of Theorem 1. We reduce the study of problem $(*)_\lambda$ to the study of a nonlinear operator equation in an appropriate ordered Banach space as in Taira and Umezu [TU1] and [TU2]. Our proof of Theorem 1 may be carried out just as in the proof of [Wi, Theorem 4.3], by making use of the theory of positive mappings in ordered Banach spaces due to Amann [Am2]. In Section 3 we prove Theorem 2, by using a variant of variational method. In Section 4 we prove Theorem 3. Our proof of Theorem 3 is based on a method inspired by Wiebers [Wi, Theorems 2.9 and 2.6].

The authors are grateful to Kunimochi Sakamoto for fruitful conversations while working on this paper.

1. Ordered Banach spaces and the fixed point index

One of the most important tools in nonlinear functional analysis is the Leray-Schauder degree of a compact perturbation of the identity mapping of a Banach space into itself. In connection with nonlinear mappings in ordered Banach spaces, it is natural to consider mappings defined on open subsets of the positive cone. Since the positive cone is a retract of the Banach space, we can define a fixed point index for compact mappings defined on the positive cone (cf. [Am2, Section 11]).

1.1 Ordered Banach spaces.

Let X be a nonempty set. An ordering $\leq$ in X is a relation in X which is reflexive, transitive and antisymmetric. A nonempty set together with an ordering is called an ordered set.

Let V be a real vector space. An ordering $\leq$ in V is said to be *linear* if the following two conditions are satisfied:

(i) If x, $y \in V$ and $x \leq y$, then we have $x + z \leq y + z$ for all $z \in V$.
(ii) If x, $y \in V$ and $x \leq y$, then we have $\alpha x \leq \alpha y$ for all $\alpha \geq 0$.

A real vector space together with a linear ordering is called an *ordered vector space*.

If x, $y \in V$ and $x \leq y$, then the set $[x, y] = \{z \in X : x \leq z \leq y\}$ is called an *order interval*.

If we let

$$Q = \{x \in V : x \geq 0\},$$

then it is easy to verify that the set Q has the following two conditions:

(iii) If x, $y \in Q$, then $\alpha x + \beta y \in Q$ for all α, $\beta \geq 0$.
(iv) If $x \neq 0$, then at least one of x and $-x$ does not belong to Q.

The set Q is called the *positive cone* of the ordering $\leq$.

Let E be a Banach space E with a linear ordering $\leq$. The Banach space E is called an *ordered Banach space* if the positive cone Q is closed in E. It is to be expected that the topology and the ordering of an ordered Banach space are closely related if the norm is *monotone*: If $0 \leq u \leq v$, then $\|u\| \leq \|v\|$.

1.2 Retracts and retractions.

Let X be a metric space. A nonempty subset A of X is called a *retract* of X if there exists a continuous map $r : X \to A$ such that the restriction $r|_A$ to A is the identity map. The map r is called a *retraction*.

The next theorem due to Dugundji [Du1], [Du2] gives a sufficient condition in order that a subset of a Banach space be a retract:

Theorem 1.1. *Every nonempty closed convex subset of a Banach space E is a retract of E.*

1.3 The fixed point index.

Let E and F be Banach spaces, and let A be a nonempty subset of E. A map $f : A \to F$ is said to be *compact* if it is continuous and the image $f(A)$ is relatively compact in F.

Theorem 1.1 tells us that the positive cone Q of is a retract of the Banach space E. Therefore, we can define a fixed point index for compact mappings defined on the positive cone; more precisely the next theorem asserts that we can define a fixed point index for compact maps on closed subsets of a retract of E:

Theorem 1.2. *Let E be a Banach space and let X be a retract of E. If U is an open subset of X and if $f : \overline{U} \to X$ is a compact map such that $f(x) \neq x$ for all $x \in \partial U$, then we can define an integer $i(f, U, X)$ satisfying the following conditions:*

*(i) (*Normalization*): For every constant map $f : \overline{U} \to U$, we have*

$$i(f, U, X) = 1.$$

*(ii) (*Additivity*): For every pair (U_1, U_2), of disjoint open subsets of U such that $f(x) \neq x$ for all $x \in \overline{U} \setminus (U_1 \cup U_2)$, we have*

$$i(f, U, X) = i(f|_{\overline{U_1}}, U_1, X) + i(f|_{\overline{U_2}}, U_2, X).$$

*(iii) (*Homotopy invariance*): For every bounded, closed interval Λ and every compact map $h : \Lambda \times \overline{U} \to X$ such that $h(\lambda, x) \neq x$ for all $(\lambda, x) \in \Lambda \times \partial U$, the integer*

$$i(h(\lambda, \cdot), U, X)$$

is well-defined and independent of $\lambda \in \Lambda$.

*(iv) (*Permanence*): If Y is a retract of X and $f(\overline{U}) \subset Y$, then we have*

$$i(f, U, X) = i(f|_{\overline{U \cap Y}}, U \cap Y, Y).$$

The integer $i(f, U, X)$ is called the *fixed point index* of f over U with respect to X.

In fact, the integer $i(f, U, X)$ is defined by the formula

$$i(f, U, X) = \deg(I - f \circ r, r^{-1}(U), 0),$$

where $r : E \to X$ is an arbitrary retraction and $\deg(I - f \circ r, r^{-1}(U), 0)$ is the Leray-Schauder degree with respect to zero of the map $I - f \circ r$ defined on the closure of the open subset $r^{-1}(U)$ (see Figure 7).

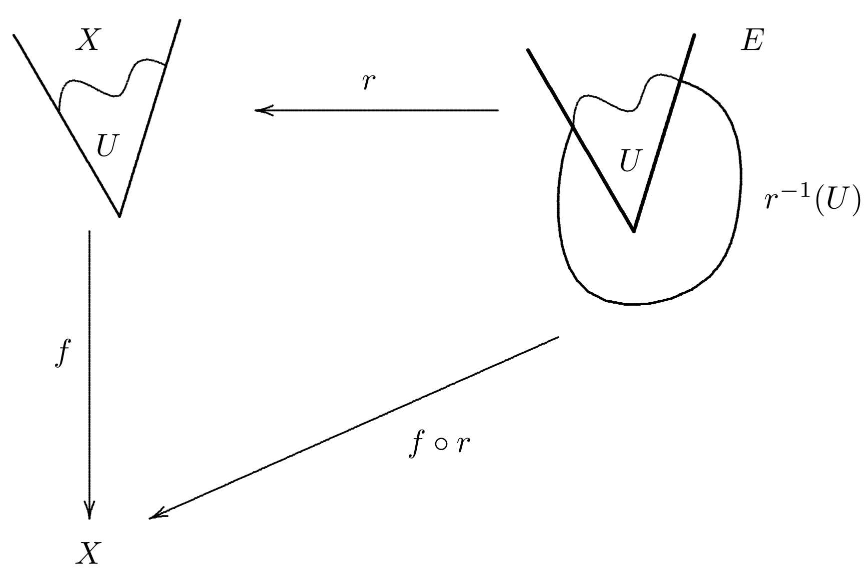

Figure 7

The fixed point index enjoys further important and useful properties.

Corollary 1.3. *Let E be a Banach space and let X be a retract of E. If U is an open subset of X and if $f : \overline{U} \to X$ is a compact map such that $f(x) \neq x$ for all $x \in \partial U$, then the fixed point index $i(f, U, X)$ has the following properties:*

*(v) (*Excision*): For every open subset $V \subset U$ such that $f(x) \neq x$ for all $x \in \overline{U} \setminus V$, we have*

$$i(f, U, X) = i(f|_{\overline{V}}, V, X).$$

*(vi) (*Solution property*): If $i(f, U, X) \neq 0$, then the map f has at least one fixed point in U.*

2. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. First, we transpose the nonlinear problem $(*)_\lambda$ into an equivalent fixed point equation for the resolvent K in an appropriate ordered Banach space, just as in Taira and Umezu [TU1] and [TU2].

(I) If $1 < p < \infty$, we define a closed linear subspace of the Sobolev space $W^{2,p}(D)$ by the formula

$$W^{2,p}_B(D) = \{u \in W^{2,p}(D) : Bu = 0 \text{ on } \partial D\}.$$

By [TU1, Theorem 1.1], we can introduce a continuous linear operator

$$K : L^p(D) \longrightarrow W^{2,p}_B(D)$$

as follows: For any $g \in L^p(D)$, the function $u = Kg \in W^{2,p}(D)$ is the unique solution of the problem

$$\begin{cases} Au = g & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases} \tag{2.1}$$

Then, by the Ascoli-Arzelà theorem we find that the operator K, considered as

$$K : C(\overline{D}) \longrightarrow C^1(\overline{D}),$$

is *compact*. Indeed it follows from an application of Sobolev's imbedding theorem that $W^{2,p}(D)$ is continuously imbedded into $C^{2-N/p}(\overline{D})$ for all $N < p < \infty$.

For u, $v \in C(\overline{D})$, we write $u \preceq v$ if $u(x) \leq v(x)$ in $\overline{D}$. Then the space $C(\overline{D})$ is an ordered Banach space with the linear ordering $\preceq$, and with the positive cone

$$P = \left\{u \in C(\overline{D}) : u \succeq 0\right\}.$$

For u, $v \in C(\overline{D})$ the notation $u \prec v$ means that $v - u \in P \setminus \{0\}$. Then it is known (see [TU1, Lemma 2.1]) that K is *strictly positive*, that is, Kg is positive everywhere in D if $g \succ 0$. Moreover it is easy to verify that a function $u(x)$ is a solution of problem $(*)_\lambda$ if and only if it satisfies the equation

$$u = \lambda K(f(u)) \quad \text{in } C(\overline{D}). \tag{2.2}$$

(II) The proof of Theorem 1 is based on the following result on multiple positive fixed points of nonlinear operators on ordered Banach spaces essentially due to Legget and Williams [LW] (cf. [Wi, Lemma 4.4]):

Lemma 2.1. *Let $(X, Q, \preceq)$ be an ordered Banach space such that the positive cone Q has non-empty interior. Moreover let $\eta : Q \to [0, \infty)$ be a continuous and concave functional and let G be a compact mapping of $Q_\tau := \{w \in Q : \|w\| \leq \tau\}$ into Q for some constant $\tau > 0$ such that*

$$\|G(w)\| < \tau \quad \text{for all } w \in Q_\tau \text{ satisfying } \|w\| = \tau. \tag{2.3}$$

Assume that there exist constants $0 < \delta < \tau$ and $\sigma > 0$ such that the set

$$W = \left\{ w \in \overset{o}{Q}_\tau : \eta(w) > \sigma \right\} \tag{2.4}$$

is non-empty, where $\overset{o}{A}$ denotes the interior of a subset A of Q, and that

$$\|G(w)\| < \delta \quad \text{for all } w \in Q_\delta \text{ satisfying } \|w\| = \delta, \tag{2.5}$$

$$\eta(w) < \sigma \quad \text{for all } w \in Q_\delta, \tag{2.6}$$

and

$$\eta(G(w)) > \sigma \quad \text{for all } w \in Q_\tau \text{ satisfying } \eta(w) = \sigma. \tag{2.7}$$

Then the mapping G has at least three *distinct fixed points.*

Proof. Let $i(G, U, Q)$ denote the fixed point index of the mapping $G(\cdot)$ over an open subset U with respect to the positive cone Q as is stated in Theorem 1.2.

We let

$$\widetilde{G}(w) = tG(w) + (1 - t) \cdot 0 = tG(w), \quad 0 \leq t \leq 1.$$

Then we have, by condition (2.3),

$$\|\widetilde{G}(w)\| = t\|G(w)\| < \tau \quad \text{for all } \|w\| = \tau.$$

This implies that

$$w \neq \widetilde{G}(w) \quad \text{for all } w \in \partial \overset{o}{Q}_\tau.$$

Therefore, by the homotopy invariance (iii) and the normalization (i) of the index we obtain that

$$i(G, \overset{o}{Q}_\tau, Q) = i(0, \overset{o}{Q}_\tau, Q) = 1. \tag{2.8}$$

Similarly, by condition (2.4) it follows that

$$i(G, \overset{o}{Q}_\delta, Q) = 1. \tag{2.9}$$

Next we show that

$$i(G, W, Q) = 1. \tag{2.10}$$

By the continuity of η we find that the set W is open, so that the index $i(G, W, Q)$ is well-defined. Moreover, by condition (2.6) we can choose a point $w_0 \in W$. We remark that if $w \in \partial W$, then it follows that either $\|w\| = \tau$ or $\eta(w) = \sigma$.

(i) First, if $\|w\| = \tau$, we let

$$\widehat{G}(w) = tG(w) + (1-t)w_0, \quad 0 \leq t \leq 1.$$

Then we have, by condition (2.3),

$$\|\widehat{G}(w)\| \leq t\|G(w)\| + (1-t)\|w_0\| < \tau.$$

This implies that

$$w \neq \widehat{G}(w) \quad \text{for all } \|w\| = \tau.$$

(ii) Secondly, if $\eta(w) = \sigma$, it follows from condition (2.7) that

$$\begin{aligned} \eta(\widehat{G}(w)) &= \eta(tG(w) + (1-t)w_0) \\ &\geq t\eta(G(w)) + (1-t)\eta(w_0) \\ &> t\sigma + (1-t)\sigma = \sigma, \end{aligned}$$

since the functional η is concave. Hence we have

$$w \neq \widehat{G}(w) \quad \text{for all } \eta(w) = \sigma.$$

Summing up, we have proved that

$$w \neq \widehat{G}(w) \quad \text{for all } w \in \partial W.$$

Therefore, by the homotopy invariance (iii) and the normalization (i) of the index it follows that

$$i(G, W, Q) = i(w_0, W, Q) = 1.$$

Now, if we let

$$U = \left\{ w \in \overset{o}{Q}_\tau : \eta(w) < \sigma, \ \|w\| > \delta \right\},$$

then we find from condition (2.5) that the sets $\overset{o}{Q}_\delta$, U and W are disjoint (see Figure 8).

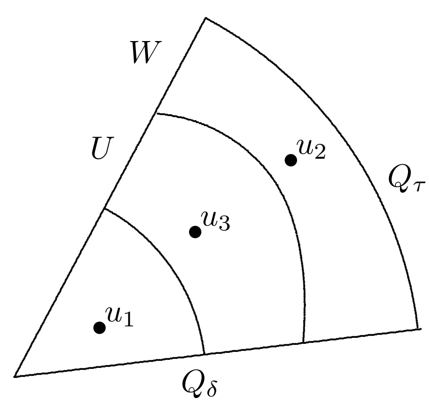

Figure 8

Thus, by the additivity (ii) of the index it follows from assertions (2.8), (2.9) and (2.10) that

$$i(G,U,Q) = i(G, \overset{o}{Q}_\tau, Q) - i(G, \overset{o}{Q}_\delta, Q) - i(G,W,Q) = -1.$$

Therefore, by the solution property (vi) of the index we can find *three* distinct fixed points u_1, u_2, u_3 of $G(\cdot)$ such that

$$u_1 \in \overset{o}{Q}_\delta, \quad u_2 \in W, \quad u_3 \in U.$$

The proof of Lemma 2.1 is now complete. □

(III) *End of Proof of Theorem 1*. The proof of Theorem 1 may be carried out just as in the proof of [Wi, Theorem 4.3].

Let $\mathcal{B}$ be the set of all subdomains Ω of D with smooth boundary such that $\operatorname{dist}(\Omega, \partial D) > 0$, and let

$$\beta = \sup_{\Omega \in \mathcal{B}} C_\Omega, \quad C_\Omega = \inf_{x \in \Omega} (K\chi_\Omega)(x), \tag{2.11}$$

where χ_A denotes the characteristic function of a set A. It is easy to see that the constant β is positive, since the resolvent K of problem (2.1) is strictly positive.

Since $\lim_{t\to\infty} \nu(t) = \lim_{t\to\infty} t/f(t) = \infty$, we can find a constant $\overline{t}_1(\varepsilon)$ such that (see Figure 9)

$$\overline{t}_1(\varepsilon) = \min\left\{t > t_2(\varepsilon) : \nu(t) = \nu(t_1(\varepsilon))\right\}.$$

Then we remark that

$$t_1(\varepsilon) < t_2(\varepsilon) < \overline{t}_1(\varepsilon),$$

and

$$\nu(t_1(\varepsilon)) = \nu(\overline{t}_1(\varepsilon)) = \frac{\overline{t}_1(\varepsilon)}{f\left(\overline{t}_1(\varepsilon)\right)}. \tag{2.12}$$

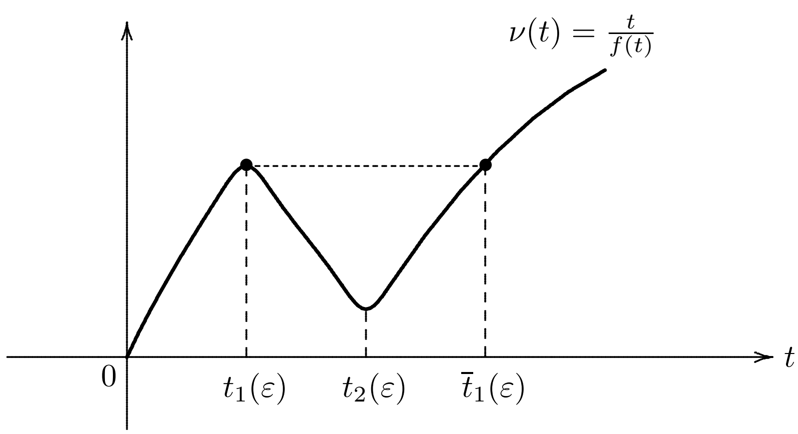

Figure 9

Now we shall apply Lemma 2.1 with

$$\begin{aligned}
&X := C(\overline{D}),\\
&Q := P = \{u \in C(\overline{D}) : u \succeq 0\},\\
&G(\cdot) := \lambda K(f(\cdot)),\\
&\delta := t_1(\varepsilon), \quad \sigma := t_2(\varepsilon), \quad \tau := \overline{t}_1(\varepsilon).
\end{aligned}$$

To do this, it suffices to verify that the conditions of Lemma 2.1 are fulfilled for all λ satisfying condition (0.4).

(III-a) If $t > 0$, we let

$$P(t) = \{u \in P : \|u\|_\infty \leq t\}.$$

If $u \in P(\overline{t}_1(\varepsilon))$ and $\|u\|_\infty = \overline{t}_1(\varepsilon)$ and if $\phi = K1$ is the unique solution of problem (0.1), then it follows from condition (0.3) and formula (2.12) that

$$\begin{aligned}
\|\lambda K(f(u))\|_\infty &< \frac{\nu(t_1(\varepsilon))}{\|\phi\|_\infty}\|K(f(u))\|_\infty\\
&\leq \frac{\nu(t_1(\varepsilon))}{\|\phi\|_\infty} f(\overline{t}_1(\varepsilon))\|K1\|_\infty\\
&= \nu(t_1(\varepsilon)) f(\overline{t}_1(\varepsilon))\\
&= \overline{t}_1(\varepsilon),
\end{aligned}$$

since $f(t)$ is increasing. This proves that the mapping $\lambda K(f(\cdot))$ satisfies condition (2.3) with $Q_\tau := P(\overline{t}_1(\varepsilon))$.

Similarly we can verify that if $u \in P(t_1(\varepsilon))$ and $\|u\|_\infty = t_1(\varepsilon)$, then we have

$$\|\lambda K(f(u))\|_\infty < t_1(\varepsilon).$$

This proves that the mapping $\lambda K(f(\cdot))$ satisfies condition (2.5) with $Q_\delta := P(t_1(\varepsilon))$.

(III-b) If $\Omega \in \mathcal{B}$, we let

$$\eta(u) = \inf_{x \in \Omega} u(x).$$

Then it is easy to see that η is a continuous and concave functional of P. If $u \in P(t_1(\varepsilon))$, then we have

$$\eta(u) \leq \|u\|_\infty \leq t_1(\varepsilon) < t_2(\varepsilon).$$

This verifies condition (2.6) for the functional η.

(III-c) If we let

$$W = \left\{ u \in \overset{o}{P}(\overline{t}_1(\varepsilon)) : \eta(u) > t_2(\varepsilon) \right\},$$

then we find that

$$W \supset \left\{ u \in P : \frac{\overline{t}_1(\varepsilon)}{2} \leq u < \overline{t}_1(\varepsilon) \quad \text{on } \overline{D}, \quad \eta(u) > t_2(\varepsilon) \right\} \neq \emptyset,$$

since $t_2(\varepsilon) < \bar{t}_1(\varepsilon)$. This verifies condition (2.4) for the functional η.

(III-d) Now, since $\lambda > \nu(t_2(\varepsilon))/\beta$, by formula (2.11) we can find a subdomain $\Omega \in \mathcal{B}$ such that

$$\lambda > \frac{\nu(t_2(\varepsilon))}{C_\Omega}.$$

If $u \in P(\bar{t}_1(\varepsilon))$ and $\eta(u) = t_2(\varepsilon)$, then we have

$$\begin{aligned} \eta(\lambda K(f(u))) &= \inf_{x\in\Omega} \lambda K(f(u))(x) \\ &\geq \inf_{x\in\Omega} \lambda K(f(u)\chi_\Omega)(x) \\ &> \frac{\nu(t_2(\varepsilon))}{C_\Omega} \inf_{x\in\Omega} K(f(u)\chi_\Omega)(x). \end{aligned} \tag{2.13}$$

However, since $\inf_\Omega u = \eta(u) = t_2(\varepsilon)$ and $f(t)$ is increasing, it follows that

$$\begin{aligned} \frac{\nu(t_2(\varepsilon))}{C_\Omega} \inf_{x\in\Omega} K(f(u)\chi_\Omega)(x) &\geq \frac{\nu(t_2(\varepsilon))}{C_\Omega} \inf_{x\in\Omega} K(f(t_2(\varepsilon))\chi_\Omega)(x) \\ &= \frac{\nu(t_2(\varepsilon))}{C_\Omega} f(t_2(\varepsilon)) \inf_{x\in\Omega} (K\chi_\Omega)(x) \\ &= \nu(t_2(\varepsilon)) f(t_2(\varepsilon)) \\ &= t_2(\varepsilon). \end{aligned} \tag{2.14}$$

Therefore, combining inequalities (2.13) and (2.14) we obtain that

$$\eta(\lambda K(f(u))) > t_2(\varepsilon).$$

This verifies condition (2.7) for the mapping $\lambda K(f(\cdot))$.

The proof of Theorem 1 is now complete. □

3. Proof of Theorem 2

We let

$$f(t) = \exp\left[\frac{t}{1+\varepsilon t}\right], \quad t \geq 0.$$

If $u_1 = u_1(\lambda)$ and $u_2 = u_2(\lambda)$ are two positive solutions of problem $(*)_\lambda$, then we have, by the mean value theorem,

$$\begin{aligned} \int_D A(u_1 - u_2)\cdot(u_1 - u_2)\,dx &= \int_D \lambda(f(u_1) - f(u_2))(u_1 - u_2)\,dx \\ &= \lambda \int_D G(x)(u_1 - u_2)^2\,dx, \end{aligned} \tag{3.1}$$

where

$$G(x) = \int_0^1 f'\left(u_2(x) + \theta(u_1(x) - u_2(x))\right) d\theta.$$

We shall prove Theorem 2 by using a variant of variational method. To do so, we introduce an unbounded linear operator $\mathfrak{A}$ from the Hilbert space $L^2(D)$ into itself as follows:

(a) The domain of definition $D(\mathfrak{A})$ of $\mathfrak{A}$ is the space

$$D(\mathfrak{A}) = \left\{u \in W^{2,2}(D) : Bu = 0\right\}.$$

(b) $\mathfrak{A}u = Au$, $u \in D(\mathfrak{A})$.

Then it is known (see [Ta1, Theorems 7.3 and 7.4], [Um, Theorem 2]) that the operator $\mathfrak{A}$ is a positive and self-adjoint operator in $L^2(D)$, and has a compact resolvent. Hence we obtain that the first eigenvalue λ_1 of $\mathfrak{A}$ is characterized by the following formula:

$$\lambda_1 = \min\left\{\int_D Au(x)\cdot\overline{u(x)}\,dx : u \in W^{2,2}(D),\ \int_D |u(x)|^2\,dx = 1,\ Bu = 0\right\}. \tag{3.2}$$

Thus it follows from formulas (3.2) and (3.1) that

$$\begin{aligned} \lambda_1 \int_D (u_1 - u_2)^2\,dx &\leq \int_D A(u_1 - u_2)\cdot(u_1 - u_2)\,dx \\ &= \lambda \int_D G(x)(u_1 - u_2)^2\,dx \\ &\leq \lambda \sup f'(t) \int_D (u_1 - u_2)^2\,dx. \end{aligned} \tag{3.3}$$

However, it is easy to see that

$$\sup f'(t) = f'\left(\frac{1-2\varepsilon}{2\varepsilon^2}\right) = 4\varepsilon^2 \exp\left[\frac{1-2\varepsilon}{\varepsilon}\right].$$

Hence, combining this fact with inequality (3.3) we obtain that

$$\lambda_1 \int_D (u_1 - u_2)^2\,dx \leq 4\lambda\varepsilon^2 \exp\left[\frac{1-2\varepsilon}{\varepsilon}\right] \int_D (u_1 - u_2)^2\,dx.$$

Therefore, we find that $u_1 \equiv u_2$ in D, if the parameter λ is so small that condition (0.3) is satisfied, that is, if we have

$$\lambda_1 - 4\lambda\varepsilon^2 \exp\left[\frac{1-2\varepsilon}{\varepsilon}\right] > 0.$$

The proof of Theorem 2 is complete. □

4. Proof of Theorem 3

This section is devoted to the proof of Theorem 3. Our proof of Theorem 3 is based on a method inspired by Wiebers [Wi, Theorems 2.9 and 2.6].

4.1. An a priori estimate.

In this subsection we shall establish an *a priori* estimate for positive solutions of problem $(*)_\lambda$ which will play an important role in the proof of Theorem 3.

First, we introduce another ordered Banach subspace of $C(\overline{D})$ for the fixed point equation (2.2) which combines the good properties of the resolvent K of problem (2.1) with the good properties of the natural ordering of $C(\overline{D})$.

Let $\phi = K1$ be the unique solution of problem (0.1). Then it follows from an application of [TU1, Lemma 2.1] that the function $\phi(x)$ belongs to $C^\infty(\overline{D})$ and satisfies the conditions

$$\phi(x) \begin{cases} > 0 & \text{if either } x \in D \text{ or } a(x) > 0, \\ = 0 & \text{if } a(x) = 0, \end{cases}$$

and

$$\frac{\partial \phi}{\partial \boldsymbol{\nu}}(x) < 0 \quad \text{if } a(x) = 0.$$

By using the function $\phi(x)$, we can introduce a subspace of $C(\overline{D})$ as follows:

$$C_\phi(\overline{D}) = \left\{ u \in C(\overline{D}) : \text{ there exists a constant } c > 0 \text{ such that } -c\phi \preceq u \preceq c\phi \right\}.$$

The space $C_\phi(\overline{D})$ is given a norm by the formula

$$\|u\|_\phi = \inf\{c > 0 : -c\phi \preceq u \preceq c\phi\}.$$

If we let

$$P_\phi = C_\phi(\overline{D}) \cap P = \{u \in C_\phi(\overline{D}) : u \succeq 0\},$$

then it is easy to see that the space $C_\phi(\overline{D})$ is an ordered Banach space having the positive cone P_ϕ with nonempty interior. For u, $v \in C_\phi(\overline{D})$, the notation $u \ll v$ means that $v - u$ is an interior point of P_ϕ. We know (see [TU1, Proposition 2.2]) that K maps $C_\phi(\overline{D})$ compactly into itself, and that K is *strongly positive*, that is, $Kg \gg 0$ for all $g \in P_\phi \setminus \{0\}$.

It is easy to see that a function $u(x)$ is a solution of problem $(*)_\lambda$ if and only if it satisfies the equation

$$u = \lambda K(f(u)) \quad \text{in } C_\phi(\overline{D}). \tag{4.1}$$

Recall (see [Ta3, Theorem 1]) that the first eigenvalue λ_1 of $\mathfrak{A}$ is positive and simple and that the corresponding eigenfunction $\varphi_1(x)$ is positive everywhere in D. Without loss of generality, we may assume that

$$\max_{\overline{D}} \varphi_1(x) = 1.$$

We let

$$\gamma = \min\left\{ \frac{f(t_1(\varepsilon))}{t_1(\varepsilon)} : 0 < \varepsilon < \frac{1}{4} \right\}. \tag{4.2}$$

Here we remark that $t_1(\varepsilon) \to 1$ as $\varepsilon \downarrow 0$, so that the constant γ is positive.

Then we have the following *a priori* estimate for all positive solutions u of problem $(*)_\lambda$:

Proposition 4.1. *We can find a constant* $0 < \varepsilon_0 \leq 1/4$ *such that if* $\lambda > \lambda_1/\gamma$ *and* $0 < \varepsilon \leq \varepsilon_0$, *then we have, for all positive solutions* u *of problem* $(*)_\lambda$,

$$u \succeq \lambda \varepsilon^{-2} \varphi_1.$$

Proof. (i) Let c be a parameter satisfying $0 < c < 1$. Then we have

$$A\left(\lambda c \varepsilon^{-2} \varphi_1\right) - \lambda f\left(\lambda c \varepsilon^{-2} \varphi_1\right) = \lambda c \varepsilon^{-2} \varphi_1 \left(\lambda_1 - \lambda \frac{f(\lambda c \varepsilon^{-2} \varphi_1)}{\lambda c \varepsilon^{-2} \varphi_1}\right) \quad \text{in } D.$$

However, since we have (see Figure 10)

$$\frac{f(t)}{t} \longrightarrow 0 \quad \text{as } t \to \infty,$$
$$\frac{f(t)}{t} \longrightarrow \infty \quad \text{as } t \to 0,$$

it follows that

$$\frac{f\left(\lambda c \varepsilon^{-2} \varphi_1\right)}{\lambda c \varepsilon^{-2} \varphi_1} \geq \min \left\{ \frac{f(t_1(\varepsilon))}{t_1(\varepsilon)}, \frac{f\left(\lambda \varepsilon^{-2}\right)}{\lambda \varepsilon^{-2}} \right\}. \tag{4.3}$$

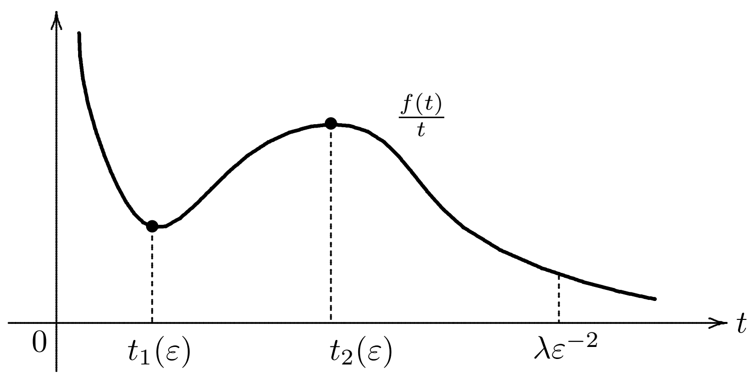

Figure 10

First, we obtain from formula (4.2) that, for all $\lambda > \lambda_1/\gamma$ and $0 < \varepsilon < 1/4$,

$$\lambda_1 - \lambda \frac{f(t_1(\varepsilon))}{t_1(\varepsilon)} \leq \lambda_1 - \lambda\gamma < 0. \tag{4.4}$$

Secondly, we have, for all $\lambda > \lambda_1/\gamma$,

$$\begin{aligned} \lambda_1 - \lambda \frac{f\left(\lambda \varepsilon^{-2}\right)}{\lambda \varepsilon^{-2}} &= \lambda_1 - \varepsilon^2 \exp\left[\frac{1}{\varepsilon + \varepsilon^2/\lambda}\right] \\ &\leq \lambda_1 - \varepsilon^2 \exp\left[\frac{1}{\varepsilon + \varepsilon^2 \gamma/\lambda_1}\right]. \end{aligned}$$

However, we can find a constant $\varepsilon_0 \in (0, 1/4]$ such that, for all $0 < \varepsilon \leq \varepsilon_0$,

$$\lambda_1 - \varepsilon^2 \exp\left[\frac{1}{\varepsilon + \varepsilon^2\gamma/\lambda_1}\right] < 0.$$

Hence it follows that, for all $\lambda > \lambda_1/\gamma$ and $0 < \varepsilon \leq \varepsilon_0$,

$$\lambda_1 - \lambda \frac{f\left(\lambda\varepsilon^{-2}\right)}{\lambda\varepsilon^{-2}} < 0. \tag{4.5}$$

Therefore, combining inequalities (4.3), (4.4) and (4.5) we obtain that, for all $\lambda > \lambda_1/\gamma$ and $0 < \varepsilon \leq \varepsilon_0$,

$$\begin{aligned} A\left(\lambda c\varepsilon^{-2}\varphi_1\right) - \lambda f\left(\lambda c\varepsilon^{-2}\varphi_1\right) &= \lambda c\varepsilon^{-2}\varphi_1\left(\lambda_1 - \lambda\frac{f\left(\lambda c\varepsilon^{-2}\varphi_1\right)}{\lambda c\varepsilon^{-2}\varphi_1}\right) \\ &\leq \lambda c\varepsilon^{-2}\varphi_1\left(\lambda_1 - \lambda\min\left\{\frac{f(t_1(\varepsilon))}{t_1(\varepsilon)}, \frac{f\left(\lambda\varepsilon^{-2}\right)}{\lambda\varepsilon^{-2}}\right\}\right) \\ &< 0 \quad \text{in } D. \end{aligned}$$

By applying the resolvent K to the both sides, we have, for all $\lambda > \lambda_1/\gamma$ and $0 < \varepsilon \leq \varepsilon_0$,

$$\lambda K\left(f(c\lambda\varepsilon^{-2}\varphi_1)\right) \gg c\lambda\varepsilon^{-2}\varphi_1. \tag{4.6}$$

(ii) Now we need the following lemma:

Lemma 4.2 ([Wi, Lemma 1.3]). *If there exist a function $\tilde{u} \gg 0$ and a constant $s_0 > 0$ such that $\lambda K(f(s\tilde{u})) \gg s\tilde{u}$ for all $0 \leq s < s_0$, then we have, for each fixed point u of the mapping $\lambda K(f(u))$,*

$$u \succeq s_0\tilde{u}.$$

(iii) Since $0 \ll \lambda K(f(0))$ and estimate (4.6) holds for all $0 < c < 1$, it follows from an application of Lemma 4.2 with $\tilde{u} := \lambda\varepsilon^{-2}\varphi_1$, $s_0 := 1$ and $s := c$ (and also equation (4.1)) that every positive solution u of problem $(*)_\lambda$ satisfies the estimate

$$u \succeq \lambda\varepsilon^{-2}\varphi_1$$

for all $\lambda > \lambda_1/\gamma$ and $0 < \varepsilon \leq \varepsilon_0$.

The proof of Proposition 4.1 is complete. □

4.2. End of Proof of Theorem 3.

(I) First, we define a function

$$F(t) = f(t) - f'(t)t = \frac{\varepsilon^2 t^2 + (2\varepsilon - 1)t + 1}{(1 + \varepsilon t)^2}\exp\left[\frac{t}{1 + \varepsilon t}\right] \quad \text{for } t \geq 0.$$

Then we have the following:

Lemma 4.3. *Let $0 < \varepsilon < 1/4$. Then the function $F(t)$ has the following properties (see Figure 11):*

$$F(t) \begin{cases} > 0 & \text{if either } 0 \le t < t_1(\varepsilon) \text{ or } t > t_2(\varepsilon), \\ = 0 & \text{if } t = t_1(\varepsilon) \text{ and } t = t_2(\varepsilon), \\ < 0 & \text{if } t_1(\varepsilon) < t < t_2(\varepsilon). \end{cases}$$

Moreover, the function $F(t)$ is decreasing in the interval $(0, (1-2\varepsilon)/2\varepsilon^2)$ and is increasing in the interval $((1-2\varepsilon)/2\varepsilon^2, \infty)$, and has a minimum at $t = (1-2\varepsilon)/2\varepsilon^2$.

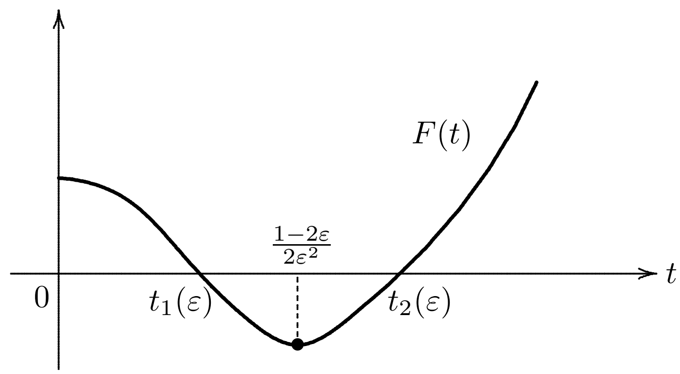

Figure 11

(II) The next proposition is an essential step in the proof of Theorem 3:

Proposition 4.4. *Let $0 < \varepsilon < 1/4$. Then there exists a constant $\alpha > 0$, independent of ε, such that we have, for all $u \succeq \alpha\varepsilon^{-2}\varphi_1$,*

$$K(F(u)) \gg 0. \tag{4.7}$$

Proof. Our proof mimics that of [Am1, Lemma 7.8].

Since $t_2(\varepsilon) < 2\varepsilon^{-2}$, we find from Lemma 4.3 that

$$F(t) \ge F\left(2\varepsilon^{-2}\right) > 0, \quad t \ge 2\varepsilon^{-2}.$$

We define two functions

$$z_-(u)(x) = \begin{cases} -F(u(x)) & \text{if } u(x) \ge 2\varepsilon^{-2}, \\ 0 & \text{if } u(x) < 2\varepsilon^{-2}, \end{cases}$$

and

$$z_+(u)(x) = F(u(x)) + z_-(u)(x).$$

Moreover, we define two sets

$$M = \left\{ x \in \overline{D} : \varphi_1(x) > \frac{1}{2} \right\},$$

and

$$L = \left\{x \in \overline{D} : u(x) \geq 2\varepsilon^{-2}\right\}.$$

Then we have $M \subset L$ for all $u \succeq 4\varepsilon^{-2}\varphi_1$, and so

$$z_-(u) \leq -F\left(2\varepsilon^{-2}\right)\chi_L \leq -F\left(2\varepsilon^{-2}\right)\chi_M.$$

By using Friedrichs' mollifiers, we can construct a function $v \in C^\infty(\overline{D})$ such that $v \succ 0$ and

$$z_-(u) \leq -F\left(2\varepsilon^{-2}\right)v. \tag{4.8}$$

On the other hand, by Lemma 4.3 we remark that

$$\min\left\{F(t) : 0 \leq t \leq 2\varepsilon^{-2}\right\} = F\left(\frac{1-2\varepsilon}{2\varepsilon^2}\right) < 0.$$

Since $z_+(u)(x) = 0$ if $x \in L$ and $z_+(u)(x) = F(u(x))$ if $x \notin L$, it follows that

$$z_+(u) \geq F\left(\frac{1-2\varepsilon}{2\varepsilon^2}\right)\chi_{\overline{D}\setminus L}.$$

If α is a constant such that $\alpha > 4$, we define a set

$$M_\alpha = \left\{x \in \overline{D} : \varphi_1(x) < \frac{2}{\alpha}\right\}.$$

Then we have, for all $u \succeq \alpha\varepsilon^{-2}\varphi_1$,

$$\overline{D} \setminus L = \left\{x \in \overline{D} : u(x) < 2\varepsilon^{-2}\right\} \subset M_\alpha,$$

and so

$$z_+(u) \geq F\left(\frac{1-2\varepsilon}{2\varepsilon^2}\right)\chi_{M_\alpha}. \tag{4.9}$$

Hence, combining inequalities (4.8) and (4.9) we obtain that, for all $u \succeq \alpha\varepsilon^{-2}\varphi_1$,

$$K(F(u)) = K\left(z_+(u) - z_-(u)\right) \geq F\left(\frac{1-2\varepsilon}{2\varepsilon^2}\right)K(\chi_{M_\alpha}) + F\left(2\varepsilon^{-2}\right)Kv. \tag{4.10}$$

However, by [TU1, estimate (2.4)] it follows that there exists a constant $c > 0$ such that

$$Kv \succeq c\varphi_1. \tag{4.11}$$

Furthermore, since $\chi_{M_\alpha} \to 0$ in $L^p(D)$ as $\alpha \to \infty$, it follows that $K(\chi_{M_\alpha}) \to 0$ in $C^1(\overline{D})$ and so $K(\chi_{M_\alpha}) \to 0$ in $C_\phi(\overline{D})$. Hence, for any positive integer k we can choose the constant α so large that

$$K(\chi_{M_\alpha}) \preceq \frac{c}{k}\varphi_1. \tag{4.12}$$

Thus, carrying inequalities (4.11) and (4.12) into the right-hand side of inequality (4.10) we obtain that, for all $u \succeq \alpha\varepsilon^{-2}\varphi_1$,

$$K(F(u)) = K\left(z_+(u) - z_-(u)\right) \geq F\left(\frac{1-2\varepsilon}{2\varepsilon^2}\right)\frac{c}{k}\varphi_1 + F\left(2\varepsilon^{-2}\right)c\varphi_1 = F\left(2\varepsilon^{-2}\right)c\varphi_1\left(1 + \frac{F\left(\frac{1-2\varepsilon}{2\varepsilon^2}\right)}{F\left(2\varepsilon^{-2}\right)}\frac{1}{k}\right). \tag{4.13}$$

However, we have, as $\varepsilon \downarrow 0$,

$$\frac{F\left(\frac{1-2\varepsilon}{2\varepsilon^2}\right)}{F\left(2\varepsilon^{-2}\right)} = \frac{(4\varepsilon-1)(\varepsilon+2)^2}{\varepsilon^2+4\varepsilon+2}\exp\left[\frac{-2\varepsilon-3}{\varepsilon+2}\right] \longrightarrow -2e^{-3/2}.$$

Therefore, inequality (4.7) follows from inequality (4.13) if we take the positive integer k so large that

$$k > -\min_{0<\varepsilon<1/4}\frac{F\left(\frac{1-2\varepsilon}{2\varepsilon^2}\right)}{F\left(2\varepsilon^{-2}\right)}.$$

The proof of Proposition 4.4 is complete. □

Proposition 4.4 implies the following important property of the mapping $K(f(\cdot))$:

Proposition 4.5 ([Wi, Lemma 2.2])**.** *Let $0 < \varepsilon < 1/4$ and let α be the same constant as in Proposition 4.4. Then we have, for all $u \succeq \alpha\varepsilon^{-2}\varphi_1$ and all $s > 1$,*

$$sK(f(u)) \gg K(f(su)).$$

(III) Now we let

$$\Lambda = \max\left\{\frac{\lambda_1}{\gamma}, \alpha\right\}. \tag{4.14}$$

If u_1 and u_2 are two positive solutions of $(*)_\lambda$ with $\lambda > \Lambda$ and $0 < \varepsilon \leq \varepsilon_0$, then combining Propositions 4.1 and 4.5 we find that, for all $s > 1$,

$$sK(f(u_i)) \gg K(f(su_i)), \quad i = 1, 2,$$

so that

$$su_i = s\lambda K(f(u_i)) \gg \lambda K(f(su_i)), \quad i = 1, 2.$$

Hence we obtain that $u_1 = u_2$, by applying the following lemma:

Lemma 4.6 ([Wi, Lemma 1.3]). *If there exists a function $\tilde{u} \gg 0$ such that $s\tilde{u} \gg \lambda K(f(s\tilde{u}))$ for all $s > 1$, then $\tilde{u} \succeq u$ for each fixed point u of the mapping $\lambda K(f(u))$.*

Finally, it remains to consider the case where $\varepsilon_0 < \varepsilon < 1/4$. If u is a positive solution of problem $(*)_\lambda$, then we have

$$A\left(u - \frac{\lambda}{\lambda_1}\varphi_1\right) = \lambda f(u) - \lambda\varphi_1 \geq \lambda(1-\varphi_1) \geq 0 \quad \text{in } D.$$

By the strong maximum principle and the boundary point lemma (see [PW]), it follows that

$$u \succeq \frac{\lambda}{\lambda_1}\varphi_1.$$

By combining this assertion with Proposition 4.5, we can prove that the uniqueness result holds for all

$$\lambda \geq \frac{\alpha\lambda_1}{\varepsilon^2},$$

just as in the case $0 < \varepsilon \leq \varepsilon_0$.

The proof of Theorem 3 is now complete. □

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