Energy from the gauge invariant observables

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Contents

1 Introduction

In this thesis, we consider the Witten's cubic string field theory [1], which is the field theory of bosonic open string. The action is expressed by

$$S = \frac{1}{g^2}\left[\frac{1}{2}\langle\Psi|Q|\Psi\rangle + \frac{1}{3}\langle\Psi|\Psi\star\Psi\rangle\right] ,$$

where $|\Psi\rangle$ is a string field, g is a coupling constant and Q is the BRST charge on world sheet theory. $\star$ denotes a star product which grew two string fields to one string field. The first term is kinetic term, and the second term is interaction term which corresponds to the three point vertex of string. Astonishingly, it has been shown that this simple action reproduces the result of the purtarbation theory of string [2], which corresponds to the first quantized theory of string.

One of the benefit of considering a string field theory is that the theory enables us to discuss the classical background of a string field. In the open string theory, one can find a background of open string as some objects which is spacially extended and on which the open string can be attached. These objects are called D_p-branes [3], where p denotes the spatial dimension of these object. Since the D_p-brane have the energy and is localized in spatially p dimensional space, it is thought as a soliton of string. In the open string field theory, a D_p-brane is expressed more simply, as a solution of the equation of motion of the open string field theory [4].

As the guidepost to discuss the nonperturbative aspect of string field theory, Sen gives a conjecture about the background of string field [5]. The Sen's conjecture states about the phenomenon called tachyon condensation. In the perturbative vacuum, which corresponds to the single D_p-brane background, there is a tachyon excitation. For example, we take $p = 25$ and consider bosonic open string. The existence of tachyon shows the unstability of the background. In string field theory, one can consider the potential of the tachyon field, and the perturbative vacuum will correspond to the local maximum of the potential. Sen predicted that there is a local minimum in the tachyon potential as dipcited in 1, and no tachyon excitation exists around there. This implies the vanishing of D_{25}-brane. This background is called a tachyon vacuum. Quantitatively, Sen's conjecture can be said that if one has a solution which corresponds to the tachyon vacuum, the energy of the solution lower than perturbative vacuum by the energy of the vanishing D_{25}-brane. Sen also gives two conjectures. Sen's second conjecture is that there is a solution which corresponds to the background with lower dimensional D_p-brane. This corresponds to the vacuum where the tachyon field has the configuration dipicted in Fig. 2. Since the configuration of the energy becomes a lump as depicted in Figure 3, the solution is called lump solution. Sen's third conjecture is that there is no open string excitation around tachyon vacuum.

The first and the second Sen's conjectures are about the energy, which is the one of the gauge invariant quantities. For the static solution, the energy is $-S$,

$$E\left[\Psi\right] = -\frac{1}{g^2}\left[\frac{1}{2}\langle\Psi|Q|\Psi\rangle + \frac{1}{3}\langle\Psi|\Psi\star\Psi\rangle\right] .$$

There is another simple gauge invariant quantities, which is called as the gauge invariant observables discovered in [6, 7]:

$$W\left[\mathcal{V},\Psi\right] = \langle I|\mathcal{V}\left(\xi = i\right)|\Psi\rangle ,$$

where $\mathcal{V}\left(\xi\right)$ is a on-shell closed string vertex operator, andξ is the coordinate of upper half plane. Similar to the Sen's conjecture, Ellwood gives a conjecture about the gauge invariant observables. It tells that the gauge invariant observables coincide with the difference of the one-point functions of an on-shell closed string state between the trivial vacuum and the one described by the solution $|\Psi\rangle$ [8, 9].

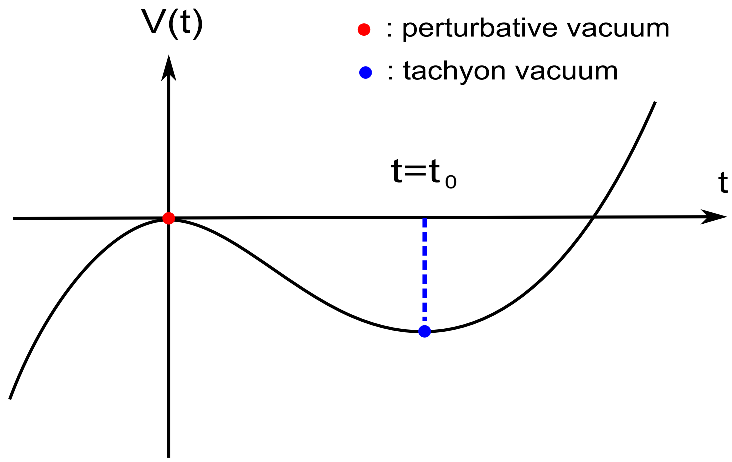

Figure 1: The potential of tachyon field

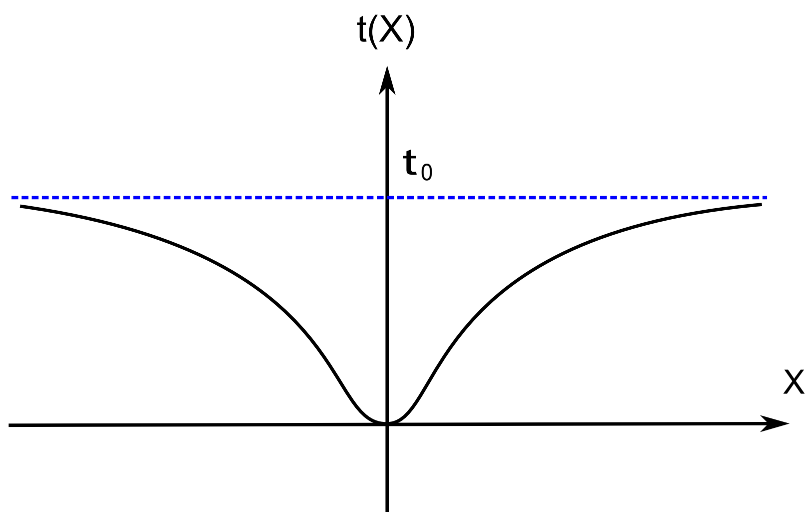

Figure 2: The lump solution

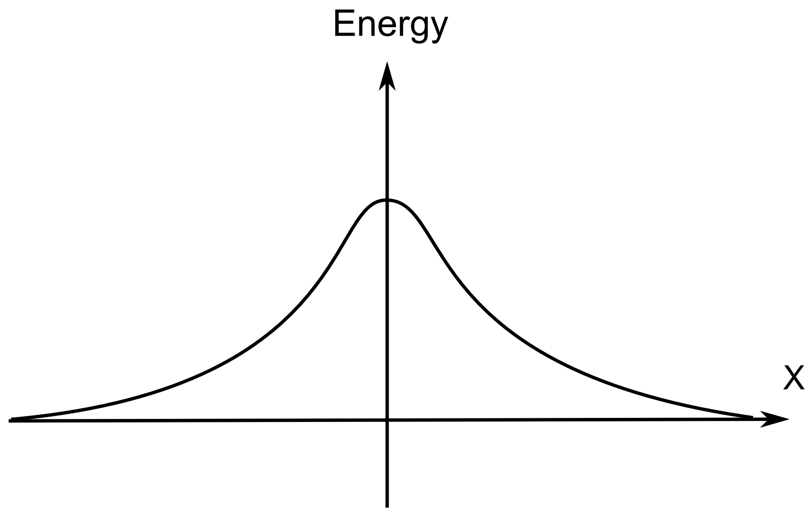

Figure 3: The energy of the lump solution

In [10], Schnabl gives the tachyon vacuum solution[1]. The energy of this solution are computed analytically, and coincides with Sen's first conjecture. More simple expression for tachyon vacuum solution is discovered by Erler and Schnabl in [12]. The solution is written by string fields K, B and c [13, 14]. K and B is defined by a line integral of energy momentum tensor and antighost on a specific frame, it is called sliver frame, respectively. The operators satisfy a simple algebra, which is called KBc algebra, and a simple transformation law under the action of BRST operator. Using this algebra and the transformation low, one can show the equation of motion algebraically. The gauge invariant observables are computed also, and the result coincides with Ellwood conjecture.

After the Erler-Schnabl solution of tachyon vacuum, some solutions have been constructed as a extension of it. In this thesis, we are interested in two specific solutions as a applications of our result. One is the Murata-Schnabl solution, which is suggested as the solution of multi-brane background [15, 16]. The other is the Bonora-Maccaferri-Tolla (BMT) solution, which is suggested as the lump solution [4]. Although one can show that these solutions satisfy equation of motion easily, the computation of the energy becomes difficult. Especially, the energy of the BMT solution is computed only numerically and there is no analytic result from the direct computation of its energy [17, 18]. On the other hand, the gauge invariant observables are computed analytically and easily in both of the solutions. This is because that the energy of the solution includes third power of $|\Psi\rangle$, while the gauge invariant observables is linear to $|\Psi\rangle$.

Even though the computation of the energy of the solution is difficult, we can compute the energy from the gauge invariant observable which seems to have the meaning of the energy. We consider the gauge invariant observable with the vertex operator

$$\mathcal{V} = \frac{2}{\pi i} c\bar{c}\partial X^0 \bar{\partial} X^0 \,,$$

which is the linear combination of a constant graviton and dilaton operator. Since this operator corresponds to the metric $g^{\mu\nu}$ with $\mu = \nu = 0$, the gauge invariant observable will be proportional to the expectation value of the energy momentum tensor $T_{\mu\nu}$ with $\mu = \nu = 0$. Therefore, it will equal to the energy of the

[1] For a review on these solutions, see [11].

system. Actually, the gauge invariant observables with this vertex operator give desired result for the energy in each solutions.

What we show in this thesis is to prove the relation between the energy and gauge invariant observable

$$E\left[\Psi\right]=\frac{1}{g^{2}}\left\langle I\right|\mathcal{V}\left(i\right)\left|\Psi\right\rangle \, .$$

This relation can make the computation of the energy easy little bit, because the energy can be computed from gauge invariant observable. As a application, we will compute the energy of Murata-Schnabl solution and BMT solution. Especially, it is useful to use this relation because the energy of the BMT solution can be computed analytically using this relation. The result coincides with the Sen's second conjecture.

Our thesis is constructed as follows.

In the section 2, we review the Witten's cubic string field theory and its gauge symmetry briefly. We also give the notations which we use in this thesis. In the section 3, we review the Sen's conjectures and Ellwood conjecture. We see the definition of the gauge invariant observable and the gauge invariance of it. In the section 4, the construction of the Erler-Schnabl tachyon vacuum solution is reviewed. On the way to construct, we review the definition of KBc algebra. We see that the energy and the gauge invariant observable are calculated analytically and coincide with Sen's first conjecture and Ellwood conjecture respectively. In the section 5, we review the Murata-Schnabl solution and BMT solution. The computations of the energy and gauge invariant observable of both solutions are shown in that section. One can see how the computations of the energy are difficult, while the computations of the gauge invariant observable are easy. In the section 6, we prove the relation between the energy and gauge invariant observable. We apply it to Murata-Schnabl solution and BMT solution. The section 7 is devoted to conclusion. The appendixes complement the computations in the section 6.

2 Witten's cubic string field theory

2.1 Notations and definitions of string theory

Let us define the notation of the string theory, which we use in this thesis.

We will consider flat 26 dimensional spacetime. The coordinate on spacetime $X^{\mu}\left(\sigma,\tau\right)$ $\left(\mu=0,1,\cdots,25\right)$ is described by the free worldsheet theory. The action of the woldsheet theory is given by

$$S_{W}[X,b,c]=\frac{1}{2\pi\alpha'}\int d^{2}\xi\partial X^{\mu}\bar{\partial}X_{\mu}+\frac{1}{2\pi}\int d^{2}\xi b\bar{\partial}c+\frac{1}{2\pi}\int d^{2}\xi\bar{b}\partial\bar{c}\,,$$

where c and b are ghost field and antighost field of conformal symmetry on worldsheet. The Rigge slope α' is taken to be 1 in this thesis. We assume that X^{μ} satisfy the Neumann boundary condition. The energy momentum tensors become

$$\begin{aligned}T\left(\xi\right) &= -:\partial X^{\mu}\partial X_{\mu}:+:\partial bc:-2\partial(:bc:)\,,\\ \bar{T}\left(\bar{\xi}\right) &= -:\bar{\partial}X^{\mu}\bar{\partial}X_{\mu}:+:\bar{\partial}\bar{b}\bar{c}:-2\bar{\partial}(:\bar{b}\bar{c}:)\,,\end{aligned}$$

where $::$ means normal ordering. ξ is the coordinate of upper half plane (U.H.P.), and we use this notation in the following. Since c, b and T satisfies the boundary condition on real axis

$$\begin{aligned}T\left(\xi\right) &= \bar{T}\left(\bar{\xi}\right)\,,\\ c\left(\xi\right) &= \bar{c}\left(\bar{\xi}\right)\,,\\ b\left(\xi\right) &= \bar{b}\left(\bar{\xi}\right)\,.\end{aligned}$$

It is useful to use doubling trick and define c, b and T on whole complex plane. It is given by

$$T(\xi) \;=\; \begin{cases} T(\xi) & , \Im\xi \geq 0 \\ \bar{T}(\bar{\xi}) & , \Im\xi < 0 \end{cases} , \tag{2.1}$$

the same extensions are applied to c and b.

BRST operator on the world-sheet of the open bosonic string Q is defined by

$$Q \;=\; \oint \frac{dz}{2\pi i}\left(cT^{(X)} + bc\partial c + \frac{3}{2}\partial^2 c\right) ,$$

where $T^{(X)}$ is the X part of energy momentum tensor T. We frequently use the nilpotency of Q

$$Q^2 = 0 \,,$$

and the invariance of the correlation function

$$\langle Q(\cdots)\rangle_R = 0 \,,$$

where R is an arbitrary complex plane. We also use the Virasolo generators, which are defined by

$$L_n = \frac{1}{2\pi i}\oint d\xi \xi^{n+1} T(\xi) \;.$$

2.2 Action of Witten's cubic string field theory

Witten's cubic string field theory [1] is the field theory of bosonic open string. The action is expressed by

$$S \;=\; -\frac{1}{2g^2}\langle\Psi|Q|\Psi\rangle + \frac{1}{3g^2}\langle\Psi|\Psi\star\Psi\rangle \,, \tag{2.2}$$

where $|\Psi\rangle$ is string field with worldsheet ghost number 1 and g is string coupling constant. The first term of left hand side corresponds to kinetic term, and the second term corresponds to interaction term. The string field $|\Psi\rangle$ is expanded by the basis of Fock space of worldsheet theory. In flat space, this expansion becomes

$$|\Psi\rangle \;=\; \int d^{26}k \left(T(k)\, c_1 + C(k)\, c_0 + A_\mu(k)\, \alpha^\mu_{-1} c_1 + \cdots\right)|k\rangle \,, \tag{2.3}$$

where

$$|k\rangle \;=\; e^{ikX(0)}|0\rangle \,,$$

with the $SL(2,\mathbb{R})$ invariant vacuum $|0\rangle$. Since k runs any value, the modes in the integrand are the off-shell extensions of the vertex operators. The terms inside of the expansion are characterized by their levels, which are defined by the eigenvalues of $L_0 + 1$. One can specify the coefficients of every levels as the fields of the corresponding string excitation. For example, since $T(k)$ corresponds to level 0, it will be specified as the Fourier mode of Tachyon field.

We will explain the kinetic term and the interaction term of the action (2.2), by defining the inner product $\langle\Psi|\Psi'\rangle$, and the star product $\star$. After these, we will see the gauge invariance of this action.

Inner product

On the upper half plane, the state-operator mapping gives the expression,

$$|\Psi\rangle \;\; = \;\; \mathcal{O}_\Psi\left(\xi = 0\right)|0\rangle\,,$$

using corresponding local operator $\mathcal{O}_\Psi$. To define the action, we need conjugation of $|\Psi\rangle$. It is called BPZ conjugation and defined by

$$\langle\Psi| \;\; \equiv \;\; \langle 0|I\circ\mathcal{O}_\Psi\left(0\right)\,,$$

where I is inversion:

$$I\circ\xi \;\; = \;\; -\frac{1}{\xi}\,.$$

This maps the operator on $\xi = 0$ to $\xi = \infty$. With this conjugation, the inner product of string fields $\langle\Psi|\Psi'\rangle$ is defined by expectation value

$$\langle\Psi|\Psi'\rangle \;\; = \;\; \langle 0|I\circ\mathcal{O}_\Psi\left(0\right)\mathcal{O}_{\Psi'}\left(0\right)|0\rangle\,, \tag{2.4}$$

where we took the expectation value of ghost sector as

$$\langle c\left(\xi_1\right)c\left(\xi_2\right)c\left(\xi_3\right)\rangle_{U.H.P.} \;\; = \;\; \left(\xi_1-\xi_2\right)\left(\xi_2-\xi_3\right)\left(\xi_3-\xi_1\right)\,. \tag{2.5}$$

Using (2.4), we can see the fields $T\left(k\right),\, A_\mu\left(k\right),\,\cdots$ have correct kinetic terms. Since the fields $T\left(k\right)$, $A_\mu\left(k\right)$, etc. have to be real fields, we need a condition imposing to string field. The condition is defined by

$$\left(\langle\Psi|\right)^\dagger \;\; = \;\; |\Psi\rangle\,, \tag{2.6}$$

where $\dagger$ denotes Hermitian conjugate. This condition is called the reality condition. For example, the part of $T(k)$ in the kinetic term of (2.2) becomes

$$-\frac{1}{2g^2}\langle\Psi|Q|\Psi\rangle \;\; = \;\; \int d^{26}k\frac{1}{2g^2}\left(1-k^2\right)T\left(k\right)^2+\cdots\,,$$

and it shows that $T\left(k\right)$ is tachyon field.

star product $\star$

To define interaction term of (2.2), we have to define the star product $\star$. In [19, 20], more general string vertices are given by CFT expectation values on the disk, as

$$\langle\Psi_1|\Psi_2\star\Psi_3\star\cdots\star\Psi_n\rangle \;\; = \;\; \langle f_1^{(n)}\circ\mathcal{O}_{\Psi_1}\left(0\right)\cdots f_n^{(n)}\circ\mathcal{O}_{\Psi_n}\left(0\right)\rangle_D\,, \tag{2.7}$$

where D denotes a disk and

$$f_k^{(n)} \;\; = \;\; \left(\frac{1+i\xi}{1-i\xi}\right)^{\frac{2}{n}}e^{\frac{2\pi ik}{n}}\,. \tag{2.8}$$

It is dipicted as Figure 4. When $n = 2$, this definition gives (2.4). From the form of expectation value on the disk, one can see that the star product glues the right half of the $|\Psi_i\rangle$ with the left half of $|\Psi_{i+1}\rangle$, and makes two string fields $|\Psi_i\rangle$ and $|\Psi_{i+1}\rangle$ to one string field $|\Psi_i\star\Psi_{i+1}\rangle$. Then, the inner product means gluing the right and left half of remaining string field after taking all star products of string fields.

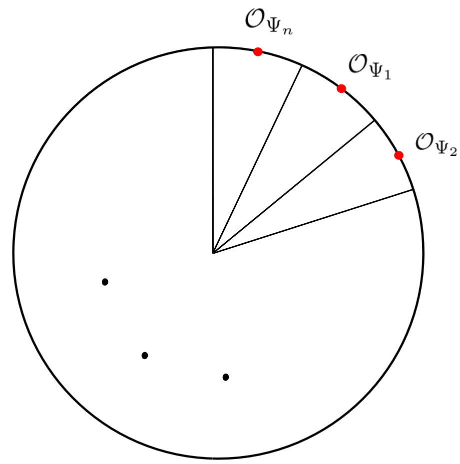

Figure 4: The difinition of the star product

From these definitions, the inner product and star product $\star$ have following properties:

$$\begin{aligned}
\langle A|B\rangle &= (-1)^{|A||B|}\,\langle B|A\rangle\,,\\
|A\star(B\star C)\rangle &= |\,(A\star B)\star C\rangle\,,\\
\langle A|B\star C\rangle &= \langle A\star B|C\rangle\,,\\
\langle QA|B\rangle &= -\,(-1)^{|A|}\,\langle A|QB\rangle\,,\\
Q|A\star B\rangle &= |QA\star B\rangle+(-1)^{|A|}\,|A\star QB\rangle\,,
\end{aligned} \tag{2.9}$$

where $|A\rangle$ and $|B\rangle$ are arbitrary string field with arbitrary ghost number. $|A|$ of $|A\rangle$ takes 0 when $|A\rangle$ is bosonic and 1 when $|A\rangle$ is fermionic. Since Q is defined by integral of conformal weight 1 primary field, these identities can hold on every coordinates. From these definitions, it has been shown that the action (2.2) reproduces Veneziano amplitude [21, 22, 23, 24] and more general string amplitudes [24].

Using these definition of the action (2.2), one can get the equation of motion of a open bosonic string field,

$$Q|\Psi\rangle+|\Psi\star\Psi\rangle=0\,. \tag{2.10}$$

These equations, which solution give extreme of action, include the information about classical background. One can discuss the classical background and nonperturbative aspect of string field by analyzing the solution of (2.10). As one can see from (2.3), the string field $|\Psi\rangle$ can be expressed by a summation of infinite number of particles. Thus, the equation of motion (2.10) becomes infinite number of equations and solving it is not easy. Actually, the solutions which have been found are written by using string fields which physical meaning is obscure. Since these string fields are not written in the language of particles, the physical meaning of the solutions is also obscure. Because of this, one has to compute the observables to confirm that the solution corresponds to which background D-brane.

Gauge symmetry

From the identities (2.9), one can show that the infinitesimal gauge transformation of (2.2) becomes

$$|\Psi\rangle \quad\rightarrow\quad |\Psi\rangle+Q|\Lambda\rangle+|\Psi\star\Lambda\rangle-|\Lambda\star\Psi\rangle\,, \tag{2.11}$$

where $|\Lambda\rangle$ is some string field with worldsheet ghost number 0. To see the finite gauge transform, let us consider the analogy between (2.2) and the Chern-Simons action:

$$S_{CS} = \frac{k}{2\pi}\int Tr\left(\frac{1}{2}AdA + \frac{1}{3}A^3\right), \tag{2.12}$$

where A is connection of gauge group 1-form and d is exterior derivative. Here we abreviate the wedge product $\wedge$. To see the analogy, we consider the identity state $|I\rangle$ by

$$|I \star \Psi\rangle = |\Psi \star I\rangle = |\Psi\rangle\,.$$

The explicit definition of identity state will be given later. From (2.9) and $|I\rangle$, we can express the inner product by

$$\langle\Psi|\Psi'\rangle = \langle I|\Psi \star \Psi'\rangle\,.$$

Using this expression, the analogy between (2.2) and (2.12) is seen by the replacements

$$\begin{aligned}\langle I|\cdot\rangle &\rightarrow \int \cdot\,,\\ \star &\rightarrow \wedge\,,\\ |\Psi\rangle &\rightarrow \Psi\,.\end{aligned}$$

The property that the integration of (2.12) will vanish when integrand is not 3-form, corresponds to the inner product will vanish when the sum of the ghost number of string fields is not 3. Besides, all of the property of d are satisfied by Q,

$$\begin{aligned}Q^2 &= 0\,,\\ \int Q(\Psi) &= \langle I|Q(\Psi)\rangle = 0\,.\end{aligned}$$

From the replacements, we can express (2.2) as the same form of (2.12):

$$S = \frac{1}{g^2}\int\left(\frac{1}{2}\Psi Q\Psi + \Psi^3\right),$$

where we abbreviate $\star$. Thus, the string field Ψ corresponds to A and the BRST charge Q corresponds to exterior derivative d. From the analogy with Chern-Simons theory, the finite gauge transformation becomes

$$\Psi \rightarrow U^{-1}QU + U^{-1}\Psi U\,, \tag{2.13}$$

with some string field U.

In addition to the gauge symmetry, this correspondence implies the form of solutions of equation of motion. Since the equation of motion of Chern Simons theory implies vanishing the field strength,

$$dA + A\wedge A = 0 \leftrightarrow F = 0\,,$$

where F is the field strength 2-form, if A is pure gauge form, it becomes equation of motion. Similary, the solution of equation of motion (2.10) will become pure gauge form:

$$\Psi = U^{-1}QU\,.$$

Since the pure gauge form with nonsingular U is trivial solution, one need singular gauge element U to describe a nontrivial solution.

The degree of exterior power in Chern-Simons theory corresponds to the ghost number. The difference from Chern-Simons theory comes from that the degree of ghost number can take minus. This makes gauge symmetry reducible. When $|\Psi\rangle$ is on-shell, the gauge transformation (2.11) is invariant under the transformation

$$\delta|\Lambda\rangle \to Q|\Lambda_{-1}\rangle + |\Psi \star \Lambda_{-1}\rangle + |\Lambda_{-1} \star \Psi\rangle \,,$$

where $|\Lambda_{-1}\rangle$ is a string field with ghost number -1. Similarly, Λ_{-n}, $(n = 0, 1, \cdots)$ have a gauge transformation:

$$\delta|\Lambda_{-n}\rangle \to Q|\Lambda_{-(n+1)}\rangle + |\Psi \star \Lambda_{-(n+1)}\rangle - (-1)^{n+1} |\Lambda_{-(n+1)} \star \Psi\rangle \,,$$

where $|\Lambda_{-n}\rangle$ is a string field with ghost number $-n$. Thus, the gauge symmetry of (2.2) becomes an infinitely reducible. One can fix this gauge symmetry using Batalin-Vilkovisky formalism [25, 26, 27, 28, 29] (see as reviews [30, 31] also).

3 Observables and conjecture

Since one has known the action (2.2) and its gauge symmetry (2.13), one can consider observables which is invariant under the gauge transformation (2.11). We are interested in the nonperturbative information which comes from equation of motion (2.10) for now. Since the meanings of analytic solutions which has been found and we consider in this thesis are not clear, we will consider two observables which can indicate the physical meanings of the solutions. One is the classical energy. We will consider a static solution of (2.10) and denote it as $|\Psi\rangle$ in the following. The energy of static solution is just $-S$,

$$E[\Psi] = \frac{1}{g^2}\left(\frac{1}{2}\langle\Psi|Q|\Psi\rangle + \frac{1}{3}\langle\Psi|\Psi \star \Psi\rangle\right) . \tag{3.1}$$

Since it is the same form with the action, the gauge invariance is obvious. Another one is called a gauge invariant observable defined by

$$W(\Psi, \mathcal{V}) = \langle I|\mathcal{V}(i)|\Psi\rangle \,,$$

where $\mathcal{V}(i)$ is an on-shell closed-string vertex operator inserted at the midle point of the string as dipicted in Figure 5. Here, we used the coordinate of upper half plane. To distinguish the solutions, there are conjectures for these observables.

We will explain the gauge invariance of the observables and the conjectures about the observables.

3.1 Sen's conjecture

Sen's conjecture [5] is a conjecture about the vacuum of tachyon field t. Tachyon is negative mass particles living on unstable D-branes. Since the tachyon describes instability of the D-brane, an effective potential $V(t)$ of tachyon field has local maximum around $t = 0$ (perturbative vacuum for string theory), where the D-brane exists. Here, we denote the vev of tachyon field as just t. Corresponding to another classical solution of the equation of motion, the potential should also have a local minimum where $t = t_0$ (other fields also take vev). Since the local minimum is stable and there is no tachyon excitation, the unstable D-brane will vanish around the local minimum.

From these perspectives, Sen gives following 3 conjectures.

1. The depth of the local minimum equals the tension T_p of the original D_p-brane (with proper normalization of the space-time volume).

$$V(0) - V(t_0) = T_p \,,$$

This reflects the energy difference between the solutions with and without D-brane.

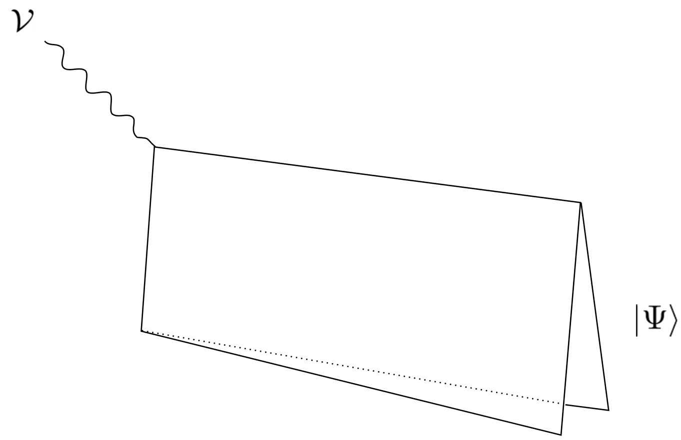

Figure 5: A gauge invariant observable with a closed string vertex operator $\mathcal{V}$ and classical solution $|\Psi\rangle$

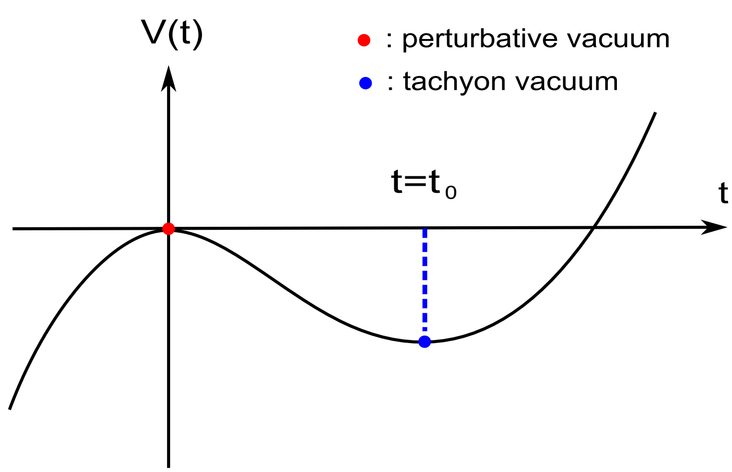

Figure 6: The potential of tachyon field

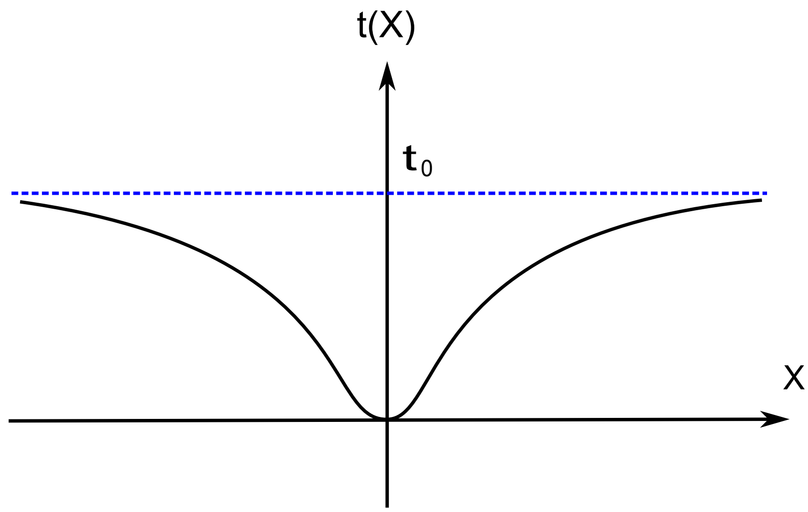

Figure 7: The lump solution

2. Other solutions exist, representing lower dimensional D-branes. When we consider a lower dimensional D-brane localizing in X direction, the configuration of tachyon field is dipicted in Figure 7. Since the energy distribution about X becomes lump as dipicted in Figure 8, these solutions are called lump solutions.

3. There are no perturbative states of open string around the tachyon solution, because perturbative states in open string field theory represent open string degrees of freedom and there are no open strings when the D-brane is absent.

These conjectures are called Sen's conjecture. Since we have to consider finite value of vev of tachyon field, these conjectures have to be shown in string field theory. In this thesis, we use the first and the second conjecture to support the identification of the solutions. In this thesis we consider D_{25}-brane as the background D-brane which exits at perturbative vacuum.

The first conjecture implies that when one computes the energy (3.1) of a solution $|\Psi\rangle$, one will get the energy measured from the tachyon vacuum solution,

$$E\left[\Psi\right] = E\left(D_{\Psi}\right) - T_{25}V_{25}\,,$$

where V_{25} is the volume of D_{25}-brane and T_p is the tension of D_p-brane:

$$T_{25} = \frac{1}{2\pi^2 g^2}\,.$$

$E\left(D_{\Psi}\right)$ is energy of D_{Ψ}-brane expressed by $|\Psi\rangle$. For example, $E\left(D_{\Psi}\right) = 0$ when $|\Psi\rangle$ expresses tachyon vacuum $t = t_0$, and $E\left(D_{\Psi}\right) = N \times \left(T_{25}V_{25}\right)$ when $|\Psi\rangle$ expresses N D-branes background.

The second conjecture means as follows. There is a solution which the tachyon field has a configuration $t = t\left(X\right)$ as dipicted in Figure 7 with some particular spacetime direction X. The solution $|\Psi\rangle$ which corresponds $t = t\left(X\right)$ expresses lower dimensional D-brane. For example, when X is one direction, $|\Psi\rangle$ expresses D_{24}-brane and the energy becomes

$$E\left[\Psi\right] = T_{24}V_{24} - T_{25}V_{25}\,,$$

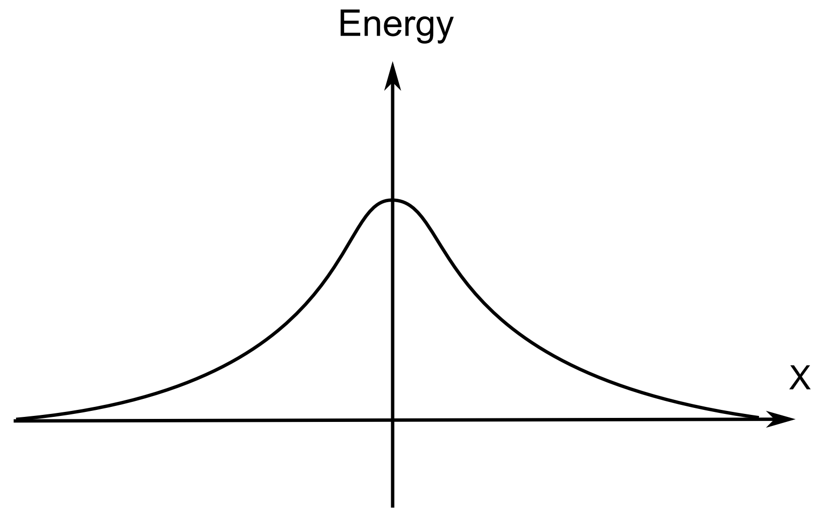

Figure 8: The energy of the lump solution

where

$$T_{24} = \frac{1}{\pi g^2}\,.$$

The third conjecture suggest that the tachyon vacuum solution supports no open string excitations. To consider this conjecture, let us expand the string field around the tachyon vacuum solution $|\Psi_0\rangle$

$$|\Psi\rangle \;=\; |\Psi_0\rangle + |\Psi'\rangle\,.$$

The action (2.2) becomes

$$S\left[\Psi_0,\Psi'\right] = S_0\left[\Psi_0\right] + \frac{1}{g^2}\left[-\frac{1}{2}\langle\Psi|Q_{\Psi_0}|\Psi\rangle + \frac{1}{3}\langle\Psi|\Psi\star\Psi\rangle\right]\,,$$

where

$$Q_{\Psi_0}|\Psi\rangle \equiv Q|\Psi\rangle + |\,\Big(\Psi_0\star\Psi - (-)^{|\Psi|}\,\Psi\star\Psi_0\Big)\rangle\,,$$

and $S_0\left[\Psi_0\right]$ is a constant

$$S_0\left[\Psi_0\right] = \frac{1}{g^2}\left[-\frac{1}{2}\langle\Psi_0|Q|\Psi_0\rangle + \frac{1}{3}\langle\Psi_0|\Psi_0\star\Psi_0\rangle\right]\,,$$

which takes the constant value predicted by the first conjecture. Following to [32, 33], the existence of open string excitations around $|\Psi_0\rangle$ can be checked by the existence of the homotopy operator of Q_{Ψ_0}:

$$Q_{\Psi_0}|A\rangle = 1\,.$$

If there is such a string field $|A\rangle$, every string field $|\Phi\rangle$ which is Q_{Ψ_0}-closed can be expressed Q_{Ψ_0}-exact form:

$$|\Phi\rangle = Q_{\Psi_0}|A\star\Phi\rangle\,.$$

Therefore the third conjecture means that there is welldefined string field corresponding homotopy operator of Q_{Ψ_0} around the tachyon vacuum solution $|\Psi_0\rangle$.

3.2 Gauge invariant observable and Ellwood's conjecture

A gauge invariant observable $W(\Psi,\mathcal{V})$ is defined as a closed string tadpole in open string field theory [6, 7].

$$W(\Psi,\mathcal{V})=\langle I|\mathcal{V}(i)|\Psi\rangle \tag{3.2}$$

Here $\mathcal{V}(i)$ is a vertex operator of on-shell closed string inserted at middle point of string. We will see the gauge invariance of $W(\Psi,\mathcal{V})$ and Ellwood's conjecture about the value of it.

gauge invariance

We will see that $W(\Psi,\mathcal{V})$ is invariant under the gauge transformation (2.11)

$$W(\Psi+Q\Lambda+\Psi\star\Lambda-\Lambda\star\Psi)=W(\Psi,\mathcal{V})\ .$$

Since $W(\Psi,\mathcal{V})$ is linear in $|\Psi\rangle$, this equation becomes

$$\begin{aligned}\langle I|\mathcal{V}(i)|Q\Lambda\rangle &= 0\,, &(3.3)\\ \langle I|\mathcal{V}(i)|\Psi\star\Lambda-\Lambda\star\Psi\rangle &= 0\,. &(3.4)\end{aligned}$$

The equation (3.3) is satisfied from the property $Q(\mathcal{V})=0$:

$$\langle I|\mathcal{V}(i)|Q\Lambda\rangle=\langle Q(\mathcal{V}(i)\,\mathcal{O}_\Lambda(0))\rangle_{U.H.P.}=0\,.$$

The second equation (3.4) is satisfied from the invariance of expectation value. The two terms on the left hand side of (3.4) becomes

$$\begin{aligned}\langle I|\mathcal{V}(i)|\Psi\star\Lambda\rangle &= \langle\Lambda|\mathcal{V}(i)|\Psi\rangle=\langle I\circ\mathcal{O}_\Lambda(0)\,\mathcal{V}(i)\,\mathcal{O}_\Psi(0)\rangle_{U.H.P.}\,,\\ \langle I|\mathcal{V}(i)|\Lambda\star\Psi\rangle &= \langle\Psi|\mathcal{V}(i)|\Lambda\rangle=\langle I\circ\mathcal{O}_\Psi(0)\,\mathcal{V}(i)\,\mathcal{O}_\Lambda(0)\rangle_{U.H.P.}\,.\end{aligned}$$

where I is inversion. Since the vertex operator $\mathcal{V}$ is conformal weight $(0,0)$ primary field and $\xi=i$ is invariant under the inversion I, the invariance of expectation value under $SL(2,\mathbb{R})$ transformation shows

$$\begin{aligned}\langle I\circ\mathcal{O}_\Lambda(0)\,\mathcal{V}(i)\,\mathcal{O}_\Psi(0)\rangle_{U.H.P.} &= \langle I\circ\mathcal{O}_\Psi(0)\,I\circ\mathcal{V}(i)\,I\circ(I\circ O_\Lambda(0))\rangle_{U.H.P.}\\ &= \langle I\circ\mathcal{O}_\Psi(0)\,\mathcal{V}(i)\,\mathcal{O}_\Lambda(0)\rangle_{U.H.P.}\,.\end{aligned}$$

Therefore (3.4) is satisfied and $W(\Psi,\mathcal{V})$ turns out to be gauge invariant.

Ellwood's conjecture

Compared with the energy (3.1), the meaning of the gauge invariant observable (3.2) is little bit subtle. Ellwood gave a conjecture about this quantity in terms of the quantities in CFT on worldsheet associated with the solution $|\Psi\rangle$ [8, 9].

- Let us denote the boundary CFT around perturbative vacuum as $BCFT_0$ and the one around the solution $|\Psi\rangle$ as $BCFT_\Psi$. Then,

$$W(\Psi,\mathcal{V})=\mathcal{A}^{disk}_\Psi(\mathcal{V})-\mathcal{A}^{disk}_0(\mathcal{V})\ , \tag{3.5}$$

  where $\mathcal{A}^{disk}_\Phi(\mathcal{V})$ is the disk amplitude with the vertex operator of closed string $\mathcal{V}$ and boundary conditions given by $BCFT_\Phi$.

Since the closed string vertex operator $\mathcal{V}$ take the form

$$\mathcal{V} = c\bar{c}\mathcal{O}^m \,,$$

where $\mathcal{O}^m$ is weight $(1,1)$ matter operator, the vacuum expectation value of $\mathcal{V}$ will vanish

$$\langle \mathcal{V}\,(z=0)\rangle_{disk} = 0\,,$$

where we use z as the disk coordinate. To get a non vanishing disk amplitude, we have to soak up three ghost zero mode. Therefore the $\mathcal{A}_{\Phi}^{disk}\,(\mathcal{V})$ is defined by

$$\mathcal{A}_{\Phi}^{disk}\,(\mathcal{V}) = -\frac{e^{-i\theta}}{2\pi i}\langle \mathcal{V}\,(0)\,c\left(e^{i\theta}\right)\rangle_{disk}^{BCFT_{\Phi}}\,.$$

The parameter θ is arbitrary. We will put it to 0.

In the next section, we will review the construction of analytic solution of tachyon vacuum. After the solution was constructed, one has to investigate which background corresponds to the solution. To see this, one computes the energy and gauge invariant observable. These conjecture are used to indicate the solutions from the value of the energy and gauge invariant observable.

4 Construction of analytic solution

First, in [10], Schnabl found an analytic solution of equation of motion (2.10), it was the solution for tachyon vacuum. After this, in [12], Erler and Schnabl found a simple analytic solution, it was constructed by string fields which satisfy a simple algebra. This algebra is called KBc subalgebra [13, 14]. Using this simple algebra and their BRST transformation (it is also simple), many solutions have been constructed. In this section, we will review the construction of Erler-Schnabl solution for tachyon vacuum, and the computation of the energy and the gauge invariant observables. In the next section, we will review the other solutions for multiple brane solution and for lump solution.

4.1 KBc algebra

Before the definition of KBc subalgebra, let us consider about the sliver frame which makes the algebraic properties of star product clear.

4.1.1 sliver frame

The coordinate on sliver frame z is defined from upper half plane ξ

$$z = \frac{1}{\pi i}\ln\frac{1+i\xi}{1-i\xi}\,.$$

The sliver frame maps the upper half plane to semi infinite stripe with width 1. Under the transformation, the right and left half of the arc $|\xi| = 1$ will be mapped to the left and right edges of stripe, $\left(\Re z = \frac{1}{2},\, \Im z = 0 \to \infty\right)$, and $\left(\Re z = -\frac{1}{2},\, \Im z = 0 \to \infty\right)$. Thus the point $\xi = i$ will go to $z = i\infty$. The origin is unchanged. This is dipicted in Figure 9. When we use sliver frame to express the star product (2.7), the position of stripe corresponds to $|\Psi_a\rangle$ is shifted by

$$\begin{aligned} z_0 &= z\,, \\ z_a &= z + a\,, \end{aligned}$$

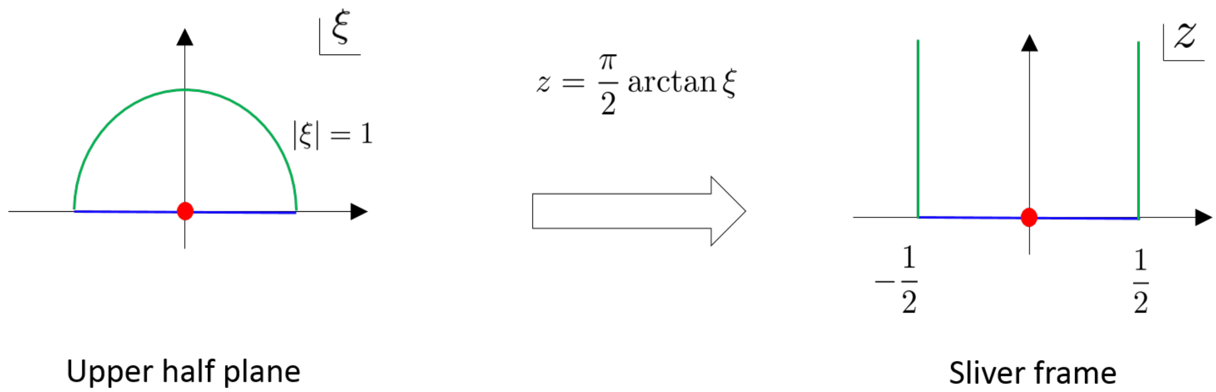

Figure 9: Sliver frame

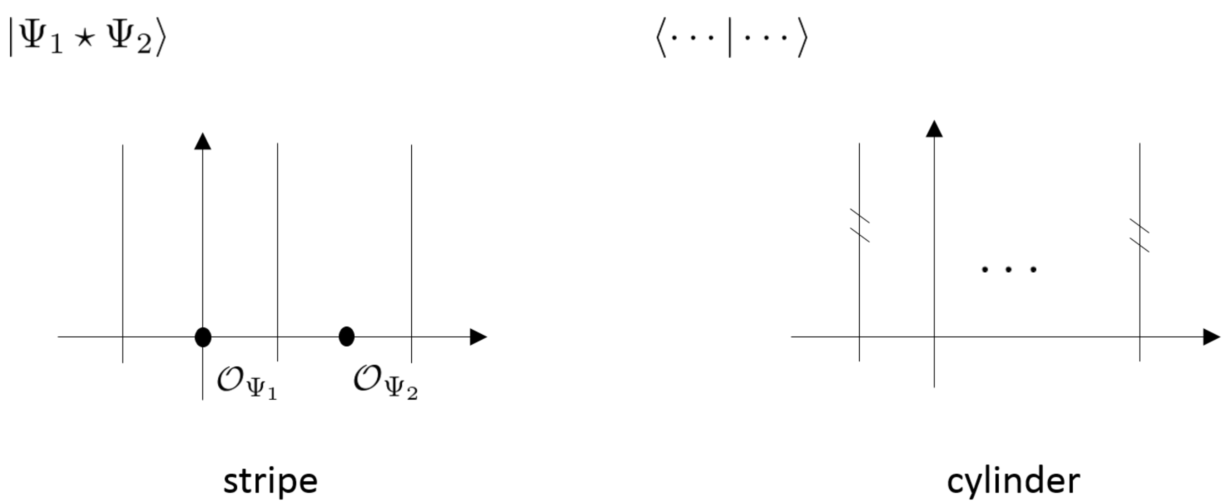

Figure 10: The star product and inner product on sliver frame

where $\Re z = -1 \sim 1$. Thus star product is expressed as multiplication of stripes corresponding to the string fields, and inner product identifies both edge of the remaining stripe, making semi infinite cylinder as dipicted in Figure 10.

To express algebraic properties, we express a string field with an operator inserted the cylinder with width 0. To define this operator, we introduce following string field

$$K = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} T\left(z\right) \, ,$$

where $T\left(z\right)$ is the energy momentum tensor defined by doubling trick (2.1). The commutator of some field $\phi\left(z\right)$ and K become

$$\left[K, \phi\left(z\right)\right] = -\partial\phi\left(z\right) = -\frac{\partial}{\partial\tau}\phi\left(z\right) \, ,$$

where $\tau = \Re z$. This shows that K is translation generator of the direction $\Re z$. Using K, we can express the stripe corresponds to the string field $|\Psi\rangle$ as

$$|\Psi\rangle \equiv e^{\frac{K}{2}} \mathcal{O}_{\Psi}\left(z=0\right) e^{\frac{K}{2}} |I\rangle \, ,$$

where $|I\rangle$ is identity state. Since $|I \star I\rangle = |I\rangle$, the star product of two string fields $|\Psi_1\rangle$ and $|\Psi_2\rangle$ become

$$\begin{aligned} |\Psi_1 \star \Psi_2\rangle &= \Psi_1 \Psi_2 |I\rangle \, , \\ \Psi_i &= e^{\frac{K}{2}} \mathcal{O}_{\Psi_i}\left(z=0\right) e^{\frac{K}{2}} \, . \end{aligned}$$

We could use the correspondence between $|\Psi\rangle$ and Ψ, instead of the ordinary state-operator correspondence. In this meaning, the operator K corresponds to the string field which is $K|I\rangle$. Then, star product becomes just multiplications of the operators.

$$|\Psi_1 \star \Psi_2\rangle \longleftrightarrow \Psi_1 \Psi_2 \, .$$

In the following, we call the string field $\Psi|I\rangle$ as Ψ. Using this expression, the algebraic structure become simple and clear.

identity state

One can define the identity state $|I\rangle$ using K. It is defined as string field $|W_0\rangle$ which corresponds to a stripe of width 0 without any operators insertion. Actually, from the definition of star product,

$$|A\rangle \star |W_0\rangle = |W_0\rangle \star |A\rangle = |A\rangle \, ,$$

with arbitrary $|A\rangle$. We will see the explicit definition of $|W_0\rangle$.

First, we will define the string field $|W_\alpha\rangle$ which corresponds to a stripe of width α without any operator insertion. Since $|\Psi\rangle = \mathcal{O}_{\Psi}\left(0\right)|0\rangle$ corresponds to a stripe of width 1, $|W_\alpha\rangle$ becomes

$$|W_\alpha\rangle = e^{-(\alpha-1)K}|0\rangle \, .$$

$|W_\alpha\rangle$ is called a wedge state. The explicit form can be got by considering the inner product with arbitrary state $|\phi\rangle = \phi\left(0\right)|0\rangle$. From the definition, the inner product becomes

$$\langle W_\alpha | \phi\rangle = \langle f \circ \phi\left(0\right)\rangle_{C_{\alpha+1}} \, ,$$

where C_a is the cylinder with width a. On upper half plane, this becomes

$$\langle W_\alpha | \phi\rangle = \langle f_\alpha \circ \phi\left(0\right)\rangle_{U.H.P.} \, , \tag{4.1}$$

where

$$f_\alpha \circ \xi = \tan\left(\frac{2}{1+\alpha}\frac{1}{\pi i}\ln\frac{1+i\xi}{1-i\xi}\right) .$$

Since the generators of the conformal mappings are the Virasoro generators L_n, one can express $f_\alpha \circ \phi(\xi)$ by

$$f_\alpha \circ \phi(\xi) = U_{f_\alpha}\phi(\xi)\, U_{f_\alpha}^{-1} ,$$

where U_{f_α} is element of conformal mapping

$$U_{f_\alpha} = \exp\left(\sum v_n L_n\right) .$$

Since $f_\alpha(\xi)$ is regular at $\xi = 0$ and $f_\alpha(\xi = 0) = 0$, $v_n = 0$ $(n < 0)$. Since $L_n|0\rangle = 0$ $(n \geq 0)$, we can see

$$U_{f_\alpha}|0\rangle = U_{f_\alpha}^{-1}|0\rangle = |0\rangle .$$

Therefore, the inner product (4.1) becomes

$$\langle W_\alpha|\phi\rangle = \langle 0|U_{f_\alpha}\phi(0)\,|0\rangle .$$

This gives the definition of wedge state

$$\langle W_\alpha| = \langle 0|U_{f_\alpha} . \tag{4.2}$$

One can get the explicit form of a wedge state from the expression 4.2. Let us consider the case that $\phi(\xi)$ is weight 0 primary field.

$$U_{f_\alpha}\phi(\xi)\, U_{f_\alpha}^{-1} = \phi(f_\alpha(\xi)) .$$

Using the identities

$$\begin{aligned} [L_n, \phi(\xi)] &= \xi^{n+1}\partial\phi(\xi) , \\ e^{\sum_{n\geq 0} v_n L_n}\phi(\xi)\, e^{-\sum_{n\geq 0} v_n L_n} &= \phi\left(e^{\sum_{n\geq 0} v_n \xi^{n+1}\partial_\xi}\xi\right) , \end{aligned}$$

we can get the relation between v_ns and f_α

$$e^{\sum_{n\geq 0} v_n \xi^{n+1}\partial_\xi}\xi = f_\alpha(\xi) .$$

Since $L_0|0\rangle = 0$, we can scale f_α arbitrary. It is convenient that we take wedge state as

$$|W_\alpha\rangle = \langle 0|U_{\frac{1+\alpha}{2}f_\alpha} .$$

Let us define v_n as the coefficient for $U_{\frac{1+\alpha}{2}f_\alpha}$,

$$e^{\sum_{n\geq 0} v_n \xi^{n+1}\partial_\xi}\xi = \frac{1+\alpha}{2}f_\alpha(\xi) .$$

From this, wedge state can be written down recursively:

$$\begin{aligned} \langle W_\alpha| = \langle 0|\exp\Bigg[&-\frac{(1+\alpha)^2-4}{3(1+\alpha)^2}L_{-2} + \frac{(1+\alpha)^4-16}{30(1+\alpha)^4}L_{-4} \\ &-\frac{\left((1+\alpha)^2-4\right)\left(176+128(1+\alpha)^2+11(1+\alpha)^4\right)}{1890(1+\alpha)^4}L_{-6}+\cdots\Bigg] . \end{aligned}$$

From the explicit form of $|W_\alpha\rangle$, we can define the identity state $|I\rangle$ as $|W_{\alpha=0}\rangle$. Even though $|I\rangle$ can be expressed explicitly, the computations using $|I\rangle$ often diverge. Since the state $|W_{\alpha\to\infty}\rangle$ (which is called sliver state) exists, the eigenvalue of K is not negative and takes 0.

4.1.2 KBc algebra

Similarly to K, one can define following string fields

$$\begin{aligned} B &\equiv \int_{i\infty}^{-i\infty} \frac{dz}{2\pi i} b(z) \ , \\ c &\equiv c(z=0) \ , \end{aligned}$$

where z is the coordinate of sliver frame. K, B and c satisfy the following algebra

$$\begin{aligned} [K, B] = 0 \quad &, \quad [K, c] = \partial c \, , \\ \{B, c\} = 1 \quad &, \quad B^2 = c^2 = 0 \, , \end{aligned}$$

where the multiplication is the star product and we abbreviate that from now on. This algebra is called the KBc algebra [14, 13]. The BRST transformations of these string fields are

$$\begin{aligned} [Q, K] = 0 \quad &, \quad \{Q, B\} = K \, , \\ \{Q, c\} \quad &= \quad c\partial c = cKc \, . \end{aligned}$$

These string fields satisfy the reality condition (2.6).

4.2 Erler-Schnabl solution

The tachyon vacuum solution constructed from KBc algebra has been constructed by Erler and Schnabl [12]. The solution is given by

$$\Psi_0 = \frac{1}{\sqrt{1+K}} \left[c + cKBc\right] \frac{1}{\sqrt{1+K}} \, , \tag{4.3}$$

which satisfies the reality condition (2.6). To show that Ψ_0 satisfies equation of motion (2.10), one can express this as the pure gauge form:

$$\Psi_0 = \left(1 - \frac{1}{\sqrt{1+K}} Bc \frac{1}{\sqrt{1+K}}\right) Q \left(1 - \frac{1}{\sqrt{1+K}} Bc \frac{1}{\sqrt{1+K}}\right)^{-1} . \tag{4.4}$$

Since

$$\begin{aligned} \frac{1}{1 - F(K) BcF(K)} &= 1 + \sum_{n=1}^{\infty} F^{2n-1}(K) BcF(K) \ , \\ &= 1 + \frac{1}{1 - F^2(K)} F(K) BcF(K) \ , \end{aligned} \tag{4.5}$$

when we express $(1+K)^{-1/2} = F(K)$, (4.4) becomes

$$\begin{aligned} \Psi_0 &= (1 - F(K) BcF(K)) Q \left(\frac{1}{1 - F^2(K)} F(K) BcF(K)\right) \\ &= (1 - F(K) BcF(K)) \frac{1}{1 - F^2(K)} F(K) (cKBc) F(K) \\ &= \frac{1}{1 - F^2(K)} F(K) (cKBc) F(K) - F(K) Bc \left(-1 + \frac{1}{1 - F^2(K)}\right) (cKBc) F(K) \\ &= F(K) c \frac{1}{1 - F^2(K)} KBcF(K) \\ &= \frac{1}{\sqrt{1+K}} \left[c + cKBc\right] \frac{1}{\sqrt{1+K}} \, . \end{aligned} \tag{4.6}$$

Therefore the solution (4.3) is a pure gauge solution and satisfies equation of motion. Since the eigenvalues of K take 0 and positive numbers, the factor

$$\frac{1}{1-F^2(K)}=\frac{1+K}{K},$$

is singular, while $\sqrt{1+K}$ is welldefined. Thus the Erler-Schnabl solution expresses nontrivial solution.

4.2.1 Energy

To compute the energy, one can express the solution as the superposition of wedge states $e^{-\alpha K}$, $(\alpha>0)$. Using the gauge transformation, Erler-Schnabl solution (4.3) becomes

$$\Psi_0=[c+cKBc]\,\frac{1}{1+K}\,. \tag{4.7}$$

Since the eigenvalue of K is not negative, we can express $1/1+K$ by using Laplace transform

$$\frac{1}{1+K}\;=\;\int_0^\infty dt e^{-t(1+K)}\,.$$

Then, the solution (4.7) is expressed by

$$\Psi_0=\int_0^\infty dt e^{-t}\,[c+cKBc]\,e^{-tK}\,.$$

The energy of Ψ_0 is computed analytically. Since $cKBc=Q\,(Bc)$, the energy becomes,

$$\begin{aligned}E\,[\Psi_0]&=\frac{1}{6g^2}\langle\Psi_0|Q|\Psi_0\rangle\\&=\frac{1}{6g^2}\int dt_1dt_2e^{-(t_1+t_2)}\langle I|c\,(0)\,e^{-t_1K}cKc\,(0)\,e^{-t_2K}|I\rangle\\&=\frac{1}{6g^2}\int dt_1dt_2e^{-(t_1+t_2)}\langle e^{(t_1+t_2)K}c\,(0)\,e^{-t_1K}cKc\,(0)\,e^{-t_2K}\rangle_{C_{t_1+t_2}}\\&=\frac{1}{6g^2}\int dt_1dt_2e^{-(t_1+t_2)}\langle c\,(t_1+t_2)\,c\partial c\,(t_2)\rangle_{C_{t_1+t_2}}\,,\end{aligned}$$

where we denote C_L as a cylinder of circumference L and

$$e^{-tK}c\,(0)\,e^{tK}=c\,(t)\;. \tag{4.8}$$

Therefore the problem becomes to compute the correlation function on a cylinder. From the normalization of the expectation value of the ghost fields (2.5), the expectation value on sliver frame becomes,

$$\langle c\,(z_1)\,c\,(z_2)\,c\,(z_3)\rangle_{C_L}=\left(\frac{L^3}{\pi}\right)\sin\frac{\pi}{L}\,(z_1-z_2)\sin\frac{\pi}{L}\,(z_2-z_3)\sin\frac{\pi}{L}\,(z_3-z_1)\;. \tag{4.9}$$

Using this, one can compute the energy

$$\begin{aligned}E\,[\Psi_0]&=\frac{1}{6g^2}V_{25}\int dt_1dt_2e^{-(t_1+t_2)}\left(-\frac{(t_1+t_2)^2}{\pi^2}\sin^2\left(\frac{\pi t_2}{t_1+t_2}\right)\right)\\&=-\frac{1}{6\pi^2g^2}V_{25}\int_0^\infty dte^{-t}t^2\int_0^1 ds\sin^2(\pi s)\\&=-\frac{1}{2\pi^2g^2}V_{25}\,,\end{aligned}$$

where we took $t=t_1+t_2$ and $t_2=st$. V_{25} is the volume of D_{25}-brane which comes from the path integral of the zero mode of spacetime coordinate X^μ. Since the tension of D_{25}-brane is

$$T_{25}=\frac{1}{2\pi^2 g^2}\,,$$

the energy can be expressed by

$$E\left[\Psi_0\right]=-T_{25}V_{25}\,.$$

This shows the Sen's first conjecture, and Ψ_0 corresponds to tachyon vacuum.

We can see this solution have a homotopy operator

$$A=B\frac{1}{1+K}\,,$$

which is welldefined and satisfies $Q_{\Psi_0}A=1$. Because of this, the solution (4.7) satisfies Sen's third conjecture.

4.2.2 Gauge invariant observable

It is easy to show the Ellwood's conjecture about gauge invariant observables. Since $cKBc=Q\left(Bc\right)$ and $Q\left(\mathcal{V}\right)=0$,

$$\begin{aligned}
W\left[\Psi_0,\mathcal{V}\right] &= \langle I|\mathcal{V}\left(i\infty,-i\infty\right)|\Psi_0\rangle \\
&= \int dte^{-t}\langle I|\mathcal{V}\left(i\infty,-i\infty\right)c\left(0\right)e^{-tK}|I\rangle \\
&= \int dte^{-t}\langle\mathcal{V}\left(i\infty,-i\infty\right)c\left(0\right)\rangle_{C_t}\,,
\end{aligned} \tag{4.10}$$

where we used $z\simeq z+t$ on C_t. By a scale transformation, we can reduce the expectation value to the one on a cylinder of circumference 1, producing a factor of t for the c ghost.

$$\begin{aligned}
\therefore W\left[\Psi_0,\mathcal{V}\right] &= \langle\mathcal{V}\left(i\infty,-i\infty\right)c\left(0\right)\rangle_{C_1}\int dte^{-t}t \\
&= \langle\mathcal{V}\left(i\infty,-i\infty\right)c\left(0\right)\rangle_{C_1}
\end{aligned}$$

Under the transformation to a disk coordinate, this equals to the disk amplitude of free $BCFT_0$ (we denoted it as $BCFT_0$). Since the expectation value of closed string tadpole in tachyon vacuum is zero, this result can be expressed by

$$W\left[\Psi_0,\mathcal{V}\right]=\mathcal{A}^{disk}_{\Psi}\left(\mathcal{V}\right)-\mathcal{A}^{disk}_{0}\left(\mathcal{V}\right).$$

This shows the Ellwood's conjecture.

4.3 Okawa type solution

The derivation of (4.6) shows that the string field

$$\Psi=F\left(K\right)c\frac{1}{1-F^2\left(K\right)}KBcF\left(K\right) \tag{4.11}$$

with arbitrary function $F\left(K\right)$ can be expressed by pure gauge form and satisfy equation of motion. The solutions which take this form are called Okawa type solution [13, 14, 34]. When we take $F\left(K\right)=\left(1+K\right)^{-1/2}$, we get Erler-Schnabl tachyon vacuum solution.

The pure gauge form of Ψ is written in

$$\Psi = (1 - FBcF)\, Q \left(1 + \frac{F}{1-F^2} BcF\right) ,$$

where we used (4.5) instead of $(1 - FBcF)^{-1}$. Then, one can show the string field

$$A = \frac{1-F^2}{K} B$$

is homotopy operator of Q_Ψ, because

$$\begin{aligned} QA &= 1 - F^2 , \\ \Psi A + A\Psi &= F^2 . \end{aligned}$$

Similar to the classical solution of Chern-Simons theory, we need a singularity in the solution Ψ to get nontrivial solution. This singularity comes from the function of K which can take 0 or positive value as the eigenvalue. On the other hand, we need regular expression for homotopy operator of Q_Ψ because of Sen's third conjecture. Since the singularity of homotopy operator can come from

$$\frac{1-F^2}{K} ,$$

F has to be regular. Then, the singular part of solution can come from

$$\frac{F}{1-F^2} .$$

Therefore the conditions which gives nontrivial solution become

$$\begin{aligned} \frac{1-F^2}{K} &: \text{ regular in K} , \\ \frac{F}{1-F^2} &: \text{ singular in K} . \end{aligned}$$

For example, Erler-Schnabl solution is

$$\begin{aligned} \frac{1-F^2}{K} &= \frac{1}{1+K} , \\ \frac{F}{1-F^2} &= \frac{\sqrt{1+K}}{K} , \end{aligned}$$

where $F/\left(1-F^2\right)$ has singularity at $K = 0$ while $\left(1-F^2\right)/K$ is regular. Since the solution have a singularity, we often need to regulate the solution.

5 Other solutions

In this section, we will review two analytic solutions.

One is called Murata-Schnabl solution [15, 16], which is the one of Okawa-type solution. Murata-Schnabl solution is thought that it corresponds to multi-brane background. As we will see soon, the energy and gauge invariant observable is calculated and shows they satisfy Sen's conjecture and Ellwood's conjecture. However the regularizations of the energy and gauge invariant observable are different.

The other one is called Bonora-Maccaferri-Tolla (BMT) solution [4, 17], which is thought that it corresponds to a lump solution. While the computation of a gauge invariant observable is very easy, the computation of energy is very hard and there is only numerical result. Moreover, we need regularization to the solution and it causes anomaly to the equation of motion.

In the following, we will review the construction of Murata-Schnabl solution and BMT solution, the computations of the energy and gauge invariant observable, and the problems about regularizations.

5.1 Murata-Schnabl solution

Murata-Schnabl solution [16, 15] is the Okawa type solution (4.11) with

$$\begin{aligned}\Psi_{MS} &= F(K)\, cB\frac{B}{G(K)}cF(K)\,,\\ G(K) &\equiv 1-F^2(K)\\ &= \left(\frac{K+1}{K}\right)^{N-1}.\end{aligned} \tag{5.1}$$

We can use gauge equivalent form of this

$$\Psi_{MS} = (1-G)\, cB\frac{K}{G(K)}c\,.$$

Ψ_{MS} corresponds to a configuration with N D-branes. When $N=0$, this is equal to Erler-Schnabl solution.

5.1.1 Useful correlators

In the computation of the energy, we need to compute the quantity

$$\langle F_1, F_2, F_3, F_4\rangle \equiv \langle I|F_1(K)\, cF_2(K)\, cF_3(K)\, cF_4(K)\, cB|I\rangle\,,$$

where $F_i(K)\,,\ i=1,\cdots,4$ is a function of K. We assume that $F_i(K)$ can be written in a Laplace transform,

$$F_i(K) = \int_0^\infty dt_i f_i(t_i)\, e^{-t_iK}\,,$$

of arbitrary distributions f_i, which is called geometric string fields [35]. Similarly to the computation of the energy of Erler-Schnabl solution, the quantity can be expressed by

$$\langle F_1, F_2, F_3, F_4\rangle \equiv \int_0^\infty \left(\prod_{i=1}^4 dt_i f_i(t_i)\right)\langle c(t_2+t_3+t_4)\, c(t_3+t_4)\, c(t_4)\, c(0)\, B\rangle_{C_{\sum_{i=1}^4 t_i}}\,. \tag{5.2}$$

To commute the integrand, we need to eliminate B insertion. From the definition of B,

$$B = \frac{1}{L}\left(\int_{\delta-\epsilon-i\infty}^{\delta-\epsilon+i\infty}\frac{dz}{2\pi i}(z)_\delta\, b(z) - \int_{\delta+\epsilon-i\infty}^{\delta+\epsilon+i\infty}\frac{dz}{2\pi i}(z)_\delta\, b(z)\right)\,,$$

where $(z)_\delta$ is analytic function on C_L which is defined by

$$(z)_\delta = \begin{cases} z & \Re z>\delta \\ z+L & \Re z<\delta\end{cases}\,.$$

We can enclose the counter of B around $c(z_i)$ respectively, so that the integrand becomes

$$\begin{aligned}\langle c(t_2+t_3+t_4)\, c(t_3+t_4)\, c(t_4)\, c(0)\, B\rangle_{C_s} &= -\frac{t_4}{s}\langle c(t_2+t_3+t_4)\, c(t_3+t_4)\, c(0)\rangle_{C_s}\\ &\quad+\frac{t_3+t_4}{s}\langle c(t_2+t_3+t_4)\, c(t_4)\, c(0)\rangle_{C_s}\\ &\quad-\frac{t_s+t_3+t_4}{s}\langle c(t_3+t_4)\, c(t_4)\, c(0)\rangle_{C_s}\,,\end{aligned}$$

where $s = \sum_{i=1}^{4} t_i$. Using (4.9), this becomes

$$\begin{aligned}\langle c(t_2+t_3+t_4)\, c(t_3+t_4)\, c(t_4)\, c(0)\, B\rangle_{C_s} &= \frac{s^2}{\pi^3}\Big\{-t_4 \sin\frac{\pi}{s} t_2 \sin\frac{\pi}{s}(t_3+t_4)\sin\frac{\pi}{s}(t_2+t_3+t_4) \\ &\quad + (t_3+t_4)\sin\frac{\pi}{s}(t_2+t_3)\sin\frac{\pi}{s}(t_4)\sin\frac{\pi}{s}(t_2+t_3+t_4) \\ &\quad - (t_2+t_3+t_4)\sin\frac{\pi}{s}(t_3)\sin\frac{\pi}{s}(t_4)\sin\frac{\pi}{s}(t_3+t_4)\Big\}\,. \end{aligned} \tag{5.3}$$

We can reduce this to more useful form [10, 13, 34]

$$\begin{aligned}\langle c(t_2+t_3+t_4)\, c(t_3+t_4)\, c(t_4)\, c(0)\, B\rangle_{C_s} &= \frac{s^2}{4\pi^3}\Big\{t_4 \sin\frac{2\pi t_2}{s} - (t_3+t_4)\sin\frac{2\pi(t_2+t_3)}{s} \\ &\quad + t_2 \sin\frac{2\pi t_4}{s} - (t_2+t_3)\sin\frac{2\pi(t_3+t_4)}{s} \\ &\quad + t_3 \sin\frac{2\pi(t_2+t_3+t_4)}{s} + (t_2+t_3+t_4)\sin\frac{2\pi t_3}{s}\Big\}\,. \end{aligned}$$

To compute (5.2), they insert into the integral an identity in the form

$$1 = \int_0^\infty ds\delta\left(s - \sum_{i=1}^4 t_i\right) = \int_0^\infty ds \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} e^{sz} e^{-z\sum_{i=1}^4 t_i}\,.$$

The second equality is just the ordinary Fourier representation of the delta function with the i absorbed in the integration variable, so the contour runs along the imaginary axis. Since the integral of t_i in (5.2) can be computed using

$$\begin{aligned}\int_0^\infty dt_i f_i(t_i)\, e^{-zt_i} &= F_i(z)\,, \\ \int_0^\infty dt_i t_i f_i(t_i)\, e^{-zt_i} &= F_i'(z)\,, \\ \int_0^\infty dt_i f_i(t_i)\, e^{-t_i\left(z\pm\frac{2\pi i}{s}\right)} &= F_i\left(z \pm \frac{2\pi i}{s}\right)\,, \end{aligned}$$

where $F_i'(z) = \partial_z F(z)$, (5.2) becomes

$$\begin{aligned}\langle F_1, F_2, F_3, F_4\rangle &= \int_0^\infty ds \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i}\frac{s^2}{4\pi^2} e^{sz}\frac{1}{2i} \\ &\quad \times\big[-F_1\Delta F_2F_3F_4' + F_1\Delta\left(F_2F_3'\right)F_4 + F_1\Delta\left(F_2F_3\right)F_4' - F_1F_2'F_3\Delta F_4 \\ &\quad + F_1F_2'\Delta\left(F_3F_4\right) + F_1F_2\Delta\left(F_3'F_4\right) - F_1\Delta\left(F_2F_3'F_4\right) - F_1\left(F_2\Delta F_3F_4\right)'\big]\,, \end{aligned} \tag{5.4}$$

where all the arguments of function F_i are z and

$$\Delta_s F(z) \equiv F\left(z - \frac{2\pi i}{s}\right) - F\left(z + \frac{2\pi i}{s}\right)\,.$$

We abbreviated s of Δ_s in (5.4).

Let us consider some identities for $\langle F_1, F_2, F_3, F_4\rangle$. From the definition,

$$\begin{aligned}\langle F_1, 1, F_3, F_4\rangle &= 0\,, \\ \langle F_1, F_2, 1, F_4\rangle &= 0\,, \\ \langle F_1, F_2, F_3, 1\rangle &= 0\,, \\ \langle F_1, K, K, F_4\rangle &= 0\,, \\ \langle F_1, F_2, K, K\rangle &= 0\,, \end{aligned} \tag{5.5}$$

because $c^2 = cKcKc = 0$. In addition these identities, we consider the case

$$\langle K, F_2, K, F_4 \rangle = 0 \,. \tag{5.6}$$

This should be satisfied because the left hand side is expressed by

$$\langle K, F_2, K, F_4 \rangle = \langle I | Q \left(BcF_2\left(K\right) Q\left(c\right) F_4\left(K\right)\right) | I \rangle = 0 \,.$$

This relation can be broken by anomaly. To see the condition that this identity becomes correct, we will see explicit computation of $\langle K, F_2, K, F_4 \rangle$. When we took $F_1\left(K\right) = K$ and $F_3\left(K\right) = K$, the square-bracket part of the integrand of (5.4) becomes

$$\begin{aligned}&\frac{2\pi i}{s^2}\left(z\partial_z - s\partial_s\right)\left(-sF_2\Delta_{2s}^2 F_4\right)\\ &+\Delta_{2s}\left(F_2 \circ_s zF_4\right) - \Delta_{2s}\left(z \circ_s F_2 F_4\right) - \Delta_{2s}\left(F_2 \circ_s z^2 F_4'\right) + \Delta_{2s}\left(zF_2 \circ_s zF_4'\right) \,,\end{aligned} \tag{5.7}$$

where

$$\begin{aligned}\Delta_{2s}\left(f_1\left(z\right) \circ_s f_2\left(z\right)\right) &\equiv \Delta_{2s}\left(f\left(z - \frac{\pi i}{s}\right) g\left(z + \frac{\pi i}{s}\right) + f\left(z + \frac{\pi i}{s}\right) g\left(z - \frac{\pi i}{s}\right)\right)\\ &= \left(\Delta_s f\right) g + f\left(\Delta_s g\right) \,.\end{aligned}$$

In current situation F_2 and F_4 can be G, KG or K/G, and at most $\mathcal{O}\left(z\right)$ at infinity. Because of the factor e^{sz} in the integrand of (5.4), we can add a line integral along the arch at infinity in the left half plane $\Re z < 0$, and make the closed contour integral along the contour C_s. The contour C_s needs subscript s to indicate that the contour which encircles all of the poles which appear in (5.7) depends on s. Then the integral of the second line of (5.7) becomes zero because

$$\oint_{C_s} e^{sz}\Delta_{2s}\left(f_1 \circ_s f_2\right) = 0 \,. \tag{5.8}$$

Therefore $\langle K, F_2, K, F_4 \rangle$ reduces to the surface terms

$$\begin{aligned}\langle K, F_2, K, F_4 \rangle &= \int_0^\infty ds \oint_{C_s} \frac{dz}{2\pi i} e^{sz} \frac{1}{4\pi^2}\left(z\partial_z - s\partial_s\right)\left(-sF_2\Delta_{2s}^2 F_4\right)\\ &= \left(\lim_{s\to\infty} - \lim_{s\to 0}\right)\oint_{C_s} dz e^{sz}\frac{1}{8\pi^3 i}s^2 F_2 \Delta_{2s}^2 F_4 \,.\end{aligned}$$

The surface term at $s = 0$ vanishes if both F_2 and F_4 are at most $\mathcal{O}\left(z\right)$ at infinity, and it is the current situation. The one at $s = \infty$ vanishes if $F_2\partial^2 F_4$ does not have poles on the imaginary axis, because

$$\begin{aligned}F_2\Delta_{2s}^2 F_4 &= F_2\left(z\right)\left(F_4\left(z - \frac{2\pi i}{s}\right) - 2F_4\left(z\right) + F_4\left(z + \frac{2\pi i}{s}\right)\right)\\ &\sim \left(\frac{2\pi i}{s}\right)^2 F_2\left(z\right)\partial^2 F_4 + \mathcal{O}\left(s^{-3}\right) \,.\end{aligned}$$

When all of the assumptions we use are satisfied, $\langle K, F_2, K, F_4 \rangle = 0$.

5.1.2 Energy

Using the formula (5.4), we can compute the energy of Murata-Schnabl solution. From the equation of motion, energy becomes

$$\begin{aligned}E\left[\Psi_{MS}\right] &= \frac{1}{6g^2}\langle \Psi_{MS} | Q | \Psi_{MS} \rangle\\ &= \frac{1}{6g^2}\left[\langle \frac{K}{G}, \left(1-G\right), \frac{K}{G}, KG \rangle - \langle K, \left(1-G\right), \frac{K}{G}, K \rangle\right.\\ &\quad \left. - \langle \frac{K}{G}, \left(1-G\right), K, K \rangle + \langle K, \left(1-G\right), K, \frac{K}{G} \rangle\right] \,.\end{aligned} \tag{5.9}$$

Here we abbreviate the volume factor V_{25}. From (5.5), the third term will vanish. The forth term will vanish in current condition (5.1). However, to see the general expression of energy, it is better to keep it. Using (5.4), one can get

$$\begin{aligned} E\left[\Psi_{MS}\right] &= \frac{1}{6g^2}\int_0^\infty ds \int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{s^2}{8\pi^3 i}e^{sz}\left[\frac{16\pi i z^2}{s}\frac{G'}{G} - zG\Delta\left(z^2\frac{G'}{G^2}\right) + 2z\Delta\left(z^2\frac{G'}{G}\right)\right. \\ &\qquad \left. +2z^2\Delta\left(zG\right)\frac{G'}{G^2} - z\frac{\Delta\left(z^2G'\right)}{G} + 2z^2G'\Delta\left(\frac{z}{G}\right)\right]. \end{aligned}$$

To use (5.8), we can simplify this to

$$\begin{aligned} E\left[\Psi_{MS}\right] &= \frac{1}{6g^2}\int_0^\infty ds \int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{e^{sz}}{8\pi^3 i}\left[24\pi i s z^2\frac{G'}{G} - 3\left(z\partial_z - s\partial_s\right)\left(s^2 z\frac{\Delta\left(zG\right)}{G}\right)\right. \\ &\qquad \left. +2s^2\Delta_{2s}\left(z\circ\frac{z^2G'}{G}\right) - s^2\Delta_{2s}\left(zG\circ\frac{z^2G'}{G^2}\right) + 2s^2\Delta_{2s}\left(z^2G'\circ\frac{z}{G}\right)\right]. \end{aligned}$$

When we close the contour of z integral by adding sufficient large arch at the infinity of the left half plane $\Re z < 0$, the second line of the right hand side will vanish because of (5.8).

$$E\left[\Psi_{MS}\right] = \frac{1}{6g^2}\int_0^\infty ds \oint_{C_s}\frac{dz}{2\pi i}e^{sz}\left[\frac{3}{\pi^2}sz^2\frac{G'}{G} - \frac{3}{8\pi^3 i}\left(z\partial_z - s\partial_s\right)\left(s^2 z\frac{\Delta\left(zG\right)}{G}\right)\right] \tag{5.10}$$

Let us consider the second term explicitly, which are thought as the anomalous contribution. From the explicit form of G (5.1), the second term becomes

$$-\frac{3}{8\pi^3 i}\left(\lim_{s\to\infty} - \lim_{s\to 0}\right)\oint_{C_s}\frac{dz}{2\pi i}e^{sz}s^2 z\left(\frac{z}{z+1}\right)^{N-1}\left\{\frac{\left(z+1-\frac{2\pi i}{s}\right)^{N-1}}{\left(z-\frac{2\pi i}{s}\right)^{N-2}} - \frac{\left(z+1+\frac{2\pi i}{s}\right)^{N-1}}{\left(z+\frac{2\pi i}{s}\right)^{N-2}}\right\}.$$

The contour is taken to encircle all of the poles on imaginary axis. When $N = 0$, 1 or 2, the integral of z will vanish. When $N > 2$,

$$\begin{aligned} &-\frac{3}{8\pi^3 i}\left(\lim_{s\to\infty} - \lim_{s\to 0}\right)\oint_{C_s}\frac{dz}{2\pi i}e^{sz}s^2 z\left(\frac{z}{z+1}\right)^{N-1}\left\{\frac{\left(z+1-\frac{2\pi i}{s}\right)^{N-1}}{\left(z-\frac{2\pi i}{s}\right)^{N-2}} - \frac{\left(z+1+\frac{2\pi i}{s}\right)^{N-1}}{\left(z+\frac{2\pi i}{s}\right)^{N-2}}\right\} \\ \propto\ &\left(\lim_{s\to\infty} - \lim_{s\to 0}\right)\left[\partial_z^{N-3}\left(s^2 e^{sz}\frac{z^N}{\left(z+1\right)^{N-1}}\left(z+1-\frac{2\pi i}{s}\right)^{N-1}\right)\Bigg|_{z=\frac{2\pi i}{s}} - \left(s\to -s\right)\right]. \end{aligned}$$

Since this behaves as $\mathcal{O}\left(s\right)$ in the limit $s\to 0$ and $\mathcal{O}\left(s^{-1}\right)$ in the limit $s\to\infty$, this contribution vanishes when $N > 2$. When $N < 0$, this contribution proportional to

$$\left(\lim_{s\to\infty} - \lim_{s\to 0}\right)\left[\partial_z^{-N-1}\left\{s^2 e^{sz}\left(z+1\right)^{1-N}\left(\frac{\left(z-\frac{2\pi i}{s}\right)^{-N+2}}{\left(z+1-\frac{2\pi i}{s}\right)^{-N+1}} - \frac{\left(z+\frac{2\pi i}{s}\right)^{-N+2}}{\left(z+1+\frac{2\pi i}{s}\right)^{-N+1}}\right)\right\}_{z=0}\right].$$

Since this also behaves as $\mathcal{O}\left(s\right)$ in the limit $s\to 0$ and $\mathcal{O}\left(s^{-1}\right)$ in the limit $s\to\infty$, this contribution vanishes when $N < 0$. Therefore the second term vanishes at all N.

Then, the energy becomes

$$E\left[\Psi_{MS}\right] = \frac{1}{2\pi^2 g^2}\int_0^\infty ds \oint_C \frac{dz}{2\pi i}e^{sz}sz^2\frac{G'}{G}.$$

Here we remove the index s of C_s, because the position of pole of the integrand is independent from s. Thus, we can integrate about s before the integral of z and we get

$$E\left[\Psi_{MS}\right] = -\frac{1}{2\pi^2 g^2} \oint_C \frac{dz}{2\pi i} \frac{G'}{G}.$$

From (5.1), we get the energy of Murata-Schnabl solution

$$\begin{aligned} E\left[\Psi_{MS}\right] &= -\frac{1}{2\pi^2 g^2} \oint_C \frac{dz}{2\pi i} \left(\frac{z}{z+1}\right)^{N-1} (N-1) \left(\frac{(z+1)^{N-2}}{z^{N-1}} - \frac{(z+1)^{N-1}}{z^N}\right) \\ &= -\frac{1}{2\pi^2 g^2} (1-N) , \end{aligned}$$

which coincide with N D_{25}-brane configuration according to Sen's first conjecture. Note that we abbreviated the volume of the brane V_{25}.

In the way to compute the energy, the regularization problem arises in the choice of the contour of the integral (5.10). The way to enclose the contour is little bit obscure because the contour passing through the poles of integrand.

5.1.3 Gauge invariant observable

We will consider the gauge invariant observable

$$W\left(\Psi_{MS}, \mathcal{V}\right) = \langle I | \mathcal{V}\left(i\infty, -i\infty\right) cB\frac{K}{G} c\left(1-G\right) | I \rangle . \tag{5.11}$$

The Ellwood's conjecture indicate the gauge invariant observable of Murata-Schnabl solution becomes

$$W\left(\Psi_{MS}, \mathcal{V}\right) = (N-1)\, \mathcal{A}_0^{disk}\left(\mathcal{V}\right) ,$$

because $\mathcal{A}^{disk}_{\Psi_{MS}}\left(\mathcal{V}\right) = N\mathcal{A}_0^{disk}\left(\mathcal{V}\right)$ if Ψ_{MS} corresponds to N branes background.

Using the Laplace transform of K/G and $(1-G)$

$$\begin{aligned} \frac{K}{G} &\equiv \int_0^\infty dt_1 e^{-t_1 K} g_1\left(t_1\right) , \\ 1-G &\equiv \int_0^\infty dt_2 e^{-t_2 K} g_2\left(t_2\right) , \end{aligned}$$

the gauge invariant observable (5.11) becomes

$$\begin{aligned} W\left(\Psi_{MS}, \mathcal{V}\right) &= \int_0^\infty dt_1 dt_2 g_1 g_2 \langle \mathcal{V}\left(i\infty, -i\infty\right) c\left(0\right) Bc\left(t_2\right) \rangle_{C_{t_1+t_2}} \\ &= \int_0^\infty dt_1 dt_2 g_1 g_2 t_2 \langle \mathcal{V}\left(i\infty, -i\infty\right) c\left(0\right) \rangle_{C_1} , \end{aligned}$$

where we used the properties that B commutes with $\mathcal{V}$ and c is weight 1 primary. The integrals can be evaluated by

$$\begin{aligned} \int dt_1 g_1\left(t_1\right) e^{-\epsilon t_1} &= \left.\frac{z}{G\left(z\right)}\right|_{z=\epsilon} , \\ \int dt_1 g_2\left(t_2\right) t_2 e^{-\epsilon t_2} &= \left.\partial_z \left(1-G\left(z\right)\right)\right|_{z=\epsilon} , \end{aligned} \tag{5.12}$$

with taking the limit of $\epsilon \to 0$. Since $\langle \mathcal{V}(i\infty, -i\infty)\, c(0)\rangle_{C_1} = \mathcal{A}_0^{disk}(\mathcal{V})$, the gauge invariant observable becomes

$$\begin{aligned} W(\Psi_{MS}, \mathcal{V}) &= \lim_{\epsilon\to 0} \frac{z}{G(z)}\partial_z (1 - G(z))\Big|_{z=\epsilon} \mathcal{A}_0^{disk}(\mathcal{V}) \\ &= \lim_{\epsilon\to 0} \frac{\epsilon^N}{(\epsilon+1)^{N-1}} \left(-\frac{(\epsilon+1)^{N-2}}{\epsilon^{N-1}} + \frac{(\epsilon+1)^{N-1}}{\epsilon^N} \right) (N-1)\, \mathcal{A}_0^{disk}(\mathcal{V}) \\ &= (N-1)\, \mathcal{A}_0^{disk}(\mathcal{V}) \,, \end{aligned}$$

which support Ellwood conjecture too.

Let us comment to the results about Murata-Schnabl solution. Since both of the value of the energy and gauge invariant observable coincide with Sen's conjecture and Ellwood conjecture respectively, the Murata-Schnabl solution can be considered as the multi-brane solution. Compared with the Erler-Schnabl solution, the computation of the energy became complicated. On the other hand, the computation of the gauge invariant observable was not so complicated. This is because the energy is computed from three point function of string field, while the gauge invariant observable is computed from one point function of string field (even though there is on-shell closed string vertex operator).

Although the value of the energy and gauge invariant observable was acceptable, there is a problem about the regularization. In the computation of the energy, the authors started the computation from (5.10) and express it as the integrals of s and z,

$$\int_0^\infty ds \quad \text{and} \quad \int_{-i\infty}^{i\infty} dz \,.$$

Since the pole of the integrand of this integral placed on the imaginary axis of z plane, we have to regulate the z integral to state which poles we will encircle. On the other hand, the computation of gauge invariant observable needed the regularization (5.12). This regularization corresponds to the replacement of K by $K + \epsilon$ ($\epsilon \ll 1$) regulating the singularity from $K = 0$. Then the regularized Murata-Schnabl solution becomes

$$\Psi_{MS}^\epsilon = (1 - G(K+\epsilon))\, cB \frac{(K+\epsilon)}{G(K+\epsilon)} c \,. \tag{5.13}$$

However, Ψ_{MS}^ϵ does not satisfy the equation of motion and the anomaly term will arise:

$$\begin{aligned} Q\Psi_{MS}^\epsilon + (\Psi_{MS}^\epsilon)^2 &= \Gamma_\epsilon \,, \\ \Gamma_\epsilon &= \epsilon\,(1 - G_\epsilon(K))\, c\frac{K+\epsilon}{G_\epsilon(K)} c \,, \end{aligned} \tag{5.14}$$

where $G_\epsilon(K) = G(K+\epsilon)$. This causes anomaly term to the expression of energy (5.10)

$$\begin{aligned} E[\Psi_{MS}^\epsilon] &= \frac{1}{g^2}\left[\frac{1}{2}\langle \Psi_{MS}^\epsilon | Q | \Psi_{MS}^\epsilon\rangle + \frac{1}{3}\langle \Psi_{MS}^\epsilon | \Psi_{MS}^\epsilon \star \Psi_{MS}^\epsilon\rangle\right] \\ &= \frac{1}{6g^2}\langle \Psi_{MS}^\epsilon | Q | \Psi_{MS}^\epsilon\rangle + \frac{1}{3g^2}\langle \Psi_{MS}^\epsilon | \Gamma_\epsilon\rangle \,. \end{aligned}$$

Therefore, the computation of the energy will be different from the above computation. It is necessary to find a more solid way to define the solution, and there are many attempts to rectify the situation [36, 37, 38, 39, 40].

5.2 Bonora-Maccaferri-Tola solution

In [4], Bonora, Maccaferri, and Tolla construct solutions corresponding to a relevant deformations of BCFT, called BMT solution[2]. Before we see the construction of the solution, we review how the lump solution will be expressed by BCFT. Then, we will see how Erler-Schnabl solution can be extended to a lump solution.

5.2.1 BMT solution

In [42, 43], Witten makes so-called boundary string field theory (BSFT). Roughly speaking, the action of BSFT is a partition function of BCFT which is the theory of bulk free and interactive on the boundary.

$$S_{BCFT} = S_{bulk} + S_{boundary}\,,$$

where S_{bulk} is the action of free closed string of half infinite cylinder C_T,

$$S_{bulk} = \frac{1}{8\pi}\int d^2\sigma \partial X_\mu \bar{\partial} X^\mu$$

and $S_{boundary}$ is the interaction term on the boundary. In [43], Witten computes the partition function with Witten deformation

$$S_{boundary} = \frac{1}{8\pi}\int_0^T dsu\left(\frac{1}{2} : X^2 : (s) + \gamma - 1 + \ln(2\pi u)\right)\,, \tag{5.15}$$

where u is a coupling constant and X is a some specific direction of X^μ. The Witten deformation is valid when the volume of X is infinite. The constant terms $\gamma - 1 + \ln(2\pi u)$ are necessary to make the partition function converge and to make the preferable property of $S_{boundary}$ under the scale transformation,

$$S_{boundary}(u, C_T) = S_{boundary}(uT, C_1)\,.$$

Since the interaction term is just a mass term inserted on the boundary, one can compute the partition function explicitly [44]

$$\begin{aligned} Z(uT) &\equiv \langle e^{-S_{boundary}} \rangle_{C_T} \\ &= \frac{1}{\sqrt{2\pi}}\sqrt{2uT}\,\Gamma(2uT)\left(\frac{e}{2uT}\right)^{2u}\,, \end{aligned}$$

where $\Gamma(a)$ is Euler gamma function. We took the finite volume $V_{X^\mu \neq X}$ of the direction $X^\mu \neq X$ to 1 as the normalization. The partition function diverges in the UV limit $uT \to 0$ as $1/\sqrt{uT}$. We regulate this by

$$\lim_{uT\to 0} Z(uT) = \lim_{uT\to 0}\frac{1}{2\sqrt{\pi uT}} = \lim_{uT\to 0}\int \frac{dx}{2\pi} e^{-uTx^2} = \frac{V_X}{2\pi}\,.$$

On the other hand, in the IR limit $uT \to \infty$, the partition function becomes

$$\lim_{uT\to\infty} Z(uT) = 1\,.$$

In the UV limit, the interaction on the boundary will vanish and the BCFT becomes free theory on D_{25}-brane background. In the IR limit, the value of X at the boundary is suppressed to 0 by infinite mass term on the boundary. Therefore, the corresponding BCFT with $uT \to \infty$ is the theory with D_{24}-brane

[2] An earlier proposal for such solutions were made in [41]

background. Then, the ratio of the tensions of D_{25}-brane and D_{24}-brane is given by the ratio of the partition functions divided by volume.

$$\frac{\tau_{24}}{\tau_{25}} = \frac{\lim_{u\to\infty} Z(u)}{\lim_{u\to 0}\frac{Z(u)}{V_X}} = 2\pi\,. \tag{5.16}$$

This is expected value of the ratio of the tensions.

The result (5.16) needs the regularization of infinite volume of X in (5.15). For a finite volume of spacetime, the deformation has to be the cosine deformation:

$$S_{boundary} = \int_0^T dsu\left[-u^{-1/R^2} : \cos\left(\frac{1}{R}X\right) : (s) + A(R)\right], \tag{5.17}$$

where X direction is a circle of radius $R > \sqrt{2}$. $A(R)$ is a constant determined in [4]. The partition function of the cosine deformation is computed exactly in [45, 46]. The result is the same as (5.16).

Let us consider the case that $S_{boundary}$ is a constant u and the volume of X is finite. Then, the partition function becomes

$$Z(uT) = e^{-uT},$$

where we took the whole volume of D_{25}-brane to 1 by normalization. In this case, the partition function is just 1 in the UV limit, while the one is 0 in the IR limit. This implies vanishing D_{25}-brane in the IR limit. Actually, in the computation of Erler-Schnabl solution, we can find the same factor. For example, in (4.10) the factor e^{-t} corresponds this where t is the circumference of the cylinder in the integrand of the right hand side of (4.10). This factor comes from the Laplace transform:

$$\frac{1}{K+1} = \int_0^\infty dte^{-tK}e^{-t}\,.$$

From this, one can guess that the lump solution will be given by the operator which Laplace transform becomes

$$\int_0^\infty dte^{-tK}e^{-S_{boundary}}, \tag{5.18}$$

instead of $(1+K)^{-1}$. This operator is given by considering a relevant matter string field ϕ which satisfies

$$\begin{aligned}
\lim_{s\to 0} s\phi(s)\phi(0) &= 0\,,\\
[c,\phi] = [B,\phi] &= 0\,,\\
Q\phi &= c\partial\phi + \partial c\delta\phi\,.
\end{aligned}$$

ϕ is taken to be the Witten deformation

$$\phi(s) = u\left(\frac{1}{2} : X^2 : (s) + \gamma - 1 + \ln(2\pi u)\right), \tag{5.19}$$

when X is noncompact, or the cosine deformation

$$\phi(s) = u\left[-u^{-1/R^2} : \cos\left(\frac{1}{R}X\right) : (s) + A(R)\right], \tag{5.20}$$

when X is compact. The string field ϕ is called seed. Then, the string field which Laplace transform becomes (5.18) is given by

$$\frac{1}{K+\phi} = \int_0^\infty dte^{-t(K+\phi)}\,,$$

We will comment about the problem caused by $\lim_{t\to\infty} e^{-t(K+\phi)} \neq 0$ later. The factor $\exp\left(-t\left(K+\phi\right)\right)$ can be expressed by

$$\begin{aligned} e^{-t(K+\phi)} &= e^{-tK}\exp\left[-\left(t\phi\left(0\right)+\frac{t^2}{2}\left[K,\phi\left(0\right)\right]+\frac{t^3}{3!}\left[K,\left[K,\phi\left(0\right)\right]\right]+\cdots\right)\right] \\ &= e^{-tK}\exp\left(-\int_0^t ds\phi\left(s\right)\right), \end{aligned}$$

where we used Zassenhaus formula

$$e^{t(X+Y)} = e^{tX}e^{tY}e^{-\frac{t^2}{2}[X,Y]}e^{\frac{t^3}{3!}(2[Y,[X,Y]]+[X,[X,Y]])}\cdots,$$

and assumed that $\left[\left[K,\phi\right],\phi\right]=0$, $\left[\left[\left[K,\phi\right],K\right],\phi\right]=0$, etc. inside a correlator. We call the BCFT with boundary interaction $\int_0^t ds\phi$ as $BCFT_\phi$.

Using the relevant deformation matter operator ϕ, Bonora, Maccaferri and Tolla construct the BMT solution:

$$\Psi_{BMT} = c\phi - \frac{1}{K+\phi}\left(\phi-\delta\phi\right)Bc\partial c, \tag{5.21}$$

or its pure gauge form

$$\Psi_{BMT} = \left(1-\frac{1}{K+\phi}\phi Bc\right)Q\left(1+\frac{1}{K}\phi Bc\right).$$

Since the BMT solution is written in form of the pure gauge, the solution satisfy equation of motion. Let us denote that if ϕ is a constant u, the BMT solution becomes

$$\Psi_{BMT,0} = uc - \frac{1}{K+u}uBc\partial c.$$

As we expect, this is gauge equivalent to the Erler-Schnabl solution (4.3). We can see this from a scale transformation of sliver frame

$$z \to uz.$$

The operators K, B and c transform under this scale transformation as

$$c \to \frac{1}{u}c, \qquad \left(B,K\right) \to u\left(B,K\right).$$

Then the BMT solution with $\phi = u$ becomes

$$\Psi_{BMT,0} = c - \frac{1}{K+1}Bc\partial c,$$

and it is gauge equivalent to the Erler-Schnabl solution Ψ_0:

$$\Psi_{BMT,0} = \frac{1}{\sqrt{K+1}}\left(Q+\Psi_0\right)\sqrt{K+1}.$$

There is a problem coming from the regularization of the solution. This problem relates to the definition of $\frac{1}{K+\phi}$ which appears in the BMT solution as

$$\frac{1}{K+\phi} \equiv \int_0^\infty dt e^{-t(K+\phi)},$$

via the Schwinger parametrization. Since the deformed sliver state $\lim_{t\to\infty} e^{-t(K+\phi)}$ does not vanish, $K+\phi$ has zero or positive value as its eigenvalue. Thus, we have to regularize this expression. One way to regularize the divergence is the same as the regularization in the computation of gauge invariant observable of Murata-Schnabl solution. This regularization replaces $\frac{1}{K+\Phi}$ by $\frac{1}{K+\phi+\epsilon}$ with $1 \gg \epsilon > 0$ and consider

$$\Psi^{\epsilon}_{BMT} = c\phi - \frac{1}{K+\phi+\epsilon}\left(\phi - \delta\phi\right) Bc\partial c\,. \tag{5.22}$$

As we saw in the case of Murata-Schnabl solution (5.14), this regularization causes anomaly to the equation of motion [47].

$$Q\Psi^{\epsilon}_{BMT} + \left(\Psi^{\epsilon}_{BMT}\right)^2 = \Gamma_\epsilon \equiv \frac{\epsilon}{K+\phi+\epsilon}\left(\phi - \delta\phi\right) c\partial c\,.$$

In [17], the authors propose a way to deal with the problem using the distribution theory.

5.2.2 Energy

Since the BMT solution with cosine deformation has nontrivial interaction, the exact computation of correlation functions with ϕ in $BCFT_\phi$ with boundary is difficult. The computation of the energy, which includes three point function of ϕ, is hard to perform. Even though in the case of the Witten deformation, the computation of the energy is hard to get exact value and only performed numerically [18, 17]. Moreover, in the Witten deformation, the volume of D_{25}-brane is infinite and the energy is divergent. There are a problem about the regularization and anomaly also. On the other hand, the gauge invariant observable can be computed easily. The computation of the gauge invariant observable include the computation of one point function of ϕ in $BCFT_\phi$. This one point function can be computed by differentiating the partition function by the coupling constant.

In this subsection, we review the analysis by Erler and Maccaferri [47] about the energy of regularized BMT solution (5.22). They show the energy of the BMT solution becomes the one of the lump solution, if the solution

$$\Psi^{\epsilon}_{0} = c\left(\phi + \epsilon\right) - \frac{1}{K+\phi+\epsilon}\left(\phi + \epsilon - \delta\phi\right) Bc\partial c\,, \tag{5.23}$$

has the energy of tachyon vacuum. Since this is the BMT solution with seed $\phi + \epsilon$, this solution has no anomaly in the equation of motion.

Since the regularized BMT solution (5.22) is static solution and does not satisfy equation of motion, the energy becomes

$$\begin{aligned}
\lim_{\epsilon\to 0} E\left[\Psi^{\epsilon}_{BMT}\right] &= \lim_{\epsilon\to 0}\frac{1}{g^2}\left[\frac{1}{2}\langle \Psi^{\epsilon}_{BMT}|Q|\Psi^{\epsilon}_{BMT}\rangle + \frac{1}{3}\langle \left(\Psi^{\epsilon}_{BMT}\right)^3\rangle\right]\,,\\
&= \lim_{\epsilon\to 0} -\frac{1}{g^2}\left[\frac{1}{6}\langle \left(\Psi^{\epsilon}_{BMT}\right)^3\rangle - \frac{1}{2}\langle \Psi^{\epsilon}_{BMT}|\Gamma_\epsilon\rangle\right]\,.
\end{aligned}$$

where $\langle(\Psi^{\epsilon}_{BMT})^3\rangle = \langle \Psi^{\epsilon}_{BMT}|\Psi^{\epsilon}_{BMT}\star\Psi^{\epsilon}_{BMT}\rangle$. Using Ψ^{ϵ}_{0}, the regularized BMT solution is expressed by

$$\begin{aligned}
\Psi^{\epsilon}_{BMT} &= \Psi^{\epsilon}_{0} + \Delta_\epsilon\,,\\
\Delta_\epsilon &\equiv -\epsilon c + B\frac{\epsilon}{K+\phi+\epsilon}c\partial c\,.
\end{aligned}$$

Note that there is no factor of ϕ except for $(K+\phi+\epsilon)^{-1}$. Then, the energy becomes

$$\begin{aligned}
E\left[\Psi^{\epsilon}_{BMT}\right] &= -\frac{1}{g^2}\left[\frac{1}{6}\langle(\Psi^{\epsilon}_{0})^3\rangle + \frac{1}{2}\langle\Delta_\epsilon|\left(\Psi^{\epsilon}_{0}\right)^2\rangle + \frac{1}{2}\langle\left(\Delta_\epsilon\right)^2|\Psi^{\epsilon}_{0}\rangle + \frac{1}{6}\langle(\Delta_\epsilon)^3\rangle - \frac{1}{2}\langle\Delta_\epsilon|\Gamma_\epsilon\rangle\right]\\
&= E\left[\Psi^{\epsilon}_{0}\right] - \frac{1}{g^2}\left[\frac{1}{2}\langle\Delta_\epsilon|\left(\Psi^{\epsilon}_{BMT}\right)^2\rangle - \frac{1}{2}\langle\left(\Delta_\epsilon\right)^2|\Psi^{\epsilon}_{BMT}\rangle + \frac{1}{6}\langle(\Delta_\epsilon)^3\rangle - \frac{1}{2}\langle\Psi^{\epsilon}_{BMT}|\Gamma_\epsilon\rangle\right]\,,
\end{aligned}$$

where we use the fact that Ψ_0^ϵ satisfies the equation of motion. Erler and Maccaferri assume that $E\left[\Psi_0^\epsilon\right]$ is the energy of tachyon vacuum. The second bracket seems to include the correlation function of ϕ and to hard to compute. However one can reduce this to more simple expression by using an identity:

$$\Delta_\epsilon \Psi_{BMT}^\epsilon = \Gamma_\epsilon \,.$$

Then the energy becomes

$$\lim_{\epsilon\to 0} E\left[\Psi_{BMT}^\epsilon\right] = \lim_{\epsilon\to 0} E\left[\Psi_0^\epsilon\right] + \lim_{\epsilon\to 0}\frac{1}{g^2}\left[\frac{1}{2}\langle \Delta_\epsilon|\Gamma_\epsilon\rangle - \frac{1}{6}\langle (\Delta_\epsilon)^3\rangle\right] . \tag{5.24}$$

We can see the second term can be computed from one point function of ϕ and the third term can be computed from partition function in the $BCFT_\phi$.

From the explicit definition, the second term of (5.24) becomes

$$\frac{1}{2g^2}\langle \Delta_\epsilon|\Gamma_\epsilon\rangle = \frac{1}{2g^2}\langle I|\frac{\epsilon}{K+\phi+\epsilon}Bc\partial c\frac{\epsilon}{K+\phi+\epsilon}\left(\phi-\delta\phi\right)c\partial c\rangle \,.$$

Since $\lim_{t\to\infty} e^{-t(K+\phi+\epsilon)}$ vanishes because of the regularization of ϵ, the factor $(K+\phi+\epsilon)^{-1}$ can be expressed by Laplace transform

$$\frac{1}{K+\phi+\epsilon} = \int_0^\infty dt e^{-t(K+\epsilon)}\exp\left(-\int_0^t ds\phi\left(s\right)\right) .$$

Using this, the second term of (5.24) becomes

$$\begin{aligned}\frac{1}{2g^2}\langle \Delta_\epsilon|\Gamma_\epsilon\rangle &= \frac{1}{2g^2}\int_0^\infty dt_1 dt_2 e^{-\epsilon(t_1+t_2)}\langle\left(\phi-\delta\phi\right)(0)\rangle_{C_{t_1+t_2}}^{BCFT_\phi,m}\langle Bc\partial c\left(t_2\right)c\partial c\left(0\right)\rangle_{C_{t_1+t_2}}^{gh}\\ &= \frac{1}{2g^2}\int_0^\infty ds s^2 e^{-\epsilon s}\langle\left(\phi-\delta\phi\right)(0)\rangle_{C_s}^{BCFT_\phi,m}\int_0^1 dq\langle Bc\partial c\left(q\right)c\partial c\left(0\right)\rangle_{C_1}^{gh} ,\end{aligned}$$

where we separated the expectation value into matter part $\langle\cdot\rangle^{BCFT_\phi,m}$ in the $BCFT_\phi$ and ghost part $\langle\cdot\rangle^{gh}$. The variable s and q is defined by $s=t_1+t_2$ and $t_2=sq$. Using (5.3), this becomes

$$\begin{aligned}\frac{1}{2g^2}\langle \Delta_\epsilon|\Gamma_\epsilon\rangle &= -\frac{1}{4\pi^2 g^2}\int_0^\infty ds s^2 e^{-\epsilon s}\langle\left(\phi-\delta\phi\right)(0)\rangle_{C_s}^{BCFT_\phi,m}\\ &= -\frac{1}{4\pi^2 g^2}\frac{1}{\epsilon}\int_0^\infty d\alpha\alpha^2 e^{-\alpha}\langle\left(\phi-\delta\phi\right)_u(0)\rangle_{C_{\alpha/\epsilon}}^{BCFT_{\phi_u},m}\\ &= -\frac{1}{4\pi^2 g^2}\int_0^\infty d\alpha\alpha e^{-\alpha}\left(\frac{\partial}{\partial\left(L\right)}Z_{BCFT_\phi}\left(L\right)\right)\Big|_{L=\frac{\alpha}{\epsilon}u} ,\end{aligned}$$

where $\alpha=\epsilon s$ and we used

$$\phi-\delta\phi = u\frac{\partial}{\partial u}\phi \,. \tag{5.25}$$

u is the coupling constant of the deformation (5.19) or (5.20). Since the partition function $Z_{BCFT_\phi}\left(L\right)$ is finite in the limit $L\to\infty$, the differential of $Z\left(L\right)$ by L will vanish faster than $1/L$. The contribution from where v goes to 0 faster than ϵ is evaluated as follows. When ϕ is the Witten deformation, the behavior of the partition function in the UV limit is $\lim_{L\to 0} Z\left(L\right)\propto 1/\sqrt{L}$. Then the contribution becomes

$$\lim_{L\to 0}\int_0^L d\alpha\alpha e^{-\alpha}\left(\frac{\partial}{\partial\left(L'\right)}Z_{BCFT_\phi}\left(L'\right)\right)\Big|_{L=\frac{\alpha}{\epsilon}u} \propto \lim_{L\to 0}\sqrt{L} ,$$

and vanish. When ϕ is the cosine deformation, the contribution from where α goes to 0 faster than ϵ will vanish because $\lim_{L\to 0} Z(L)$ is finite. Therefore, we can put $v/\epsilon \to \infty$ inside the integral and get

$$\frac{1}{2g^2}\langle \Delta_\epsilon | \Gamma_\epsilon \rangle \underset{\epsilon\to 0}{\longrightarrow} 0\,.$$

The third term of the right hand side of (5.24) can be computed in the similar way.

$$\begin{aligned}
-\frac{1}{6g^2}\langle (\Delta_\epsilon)^3 \rangle &= -\frac{1}{6g^2}\langle I| \left(\left(\frac{\epsilon}{K+\phi+\epsilon}\right) Bc\partial c\right)^3 |I\rangle \\
&= -\frac{1}{6g^2}\int d\alpha \alpha^2 e^{-\alpha} Z_{BCFT_\phi}\left(\frac{\alpha}{\epsilon}\right)\int_0^1 dq \int_0^q dr \langle Bc\partial c(q)\, Bc\partial c(r)\, Bc\partial c(0)\rangle_{C_1}^{gh} \\
&= \frac{1}{4\pi^2 g^2}\int d\alpha \alpha^2 e^{-\alpha} Z_{BCFT_\phi}\left(\frac{\alpha}{\epsilon}\right).
\end{aligned}$$

Since the contribution to the integral where α falls faster than ϵ is at most $\mathcal{O}\left(\alpha^{5/2}\right)$, we can take $\epsilon \to 0$ before the integral.

$$-\frac{1}{6g^2}\langle (\Delta_\epsilon)^3 \rangle \underset{\epsilon\to 0}{\longrightarrow} \frac{1}{2\pi^2 g^2}\lim_{L\to\infty} Z_{BCFT_\phi}(L)\,.$$

Therefore, the energy of the regularized BMT solution becomes

$$\lim_{\epsilon\to 0} E\left[\Psi^\epsilon_{BMT}\right] = \lim_{\epsilon\to 0} E\left[\Psi^\epsilon_0\right] + \frac{1}{2\pi^2} Z^{IR}_{BCFT_\phi}\,,$$

where $Z^{IR}_{BCFT_\phi} = \lim_{L\to\infty} Z_{BCFT_\phi}(L)$. When we normalize the energy of tachyon vacuum solution as $-T_{25}V_{25}$, the second term of the right hand side of the energy becomes

$$\frac{1}{2\pi^2} Z^{IR}_{BCFT_\phi} = T_{24}V_{24}$$

from (5.16). If the term $\lim_{\epsilon\to 0} E\left[\Psi^\epsilon_0\right]$ is the energy of tachyon vacuum, the energy of regularized BMT solution becomes

$$\lim_{\epsilon\to 0} E\left[\Psi^\epsilon_{BMT}\right] = -T_{25}V_{25} + T_{24}V_{24}\,. \tag{5.26}$$

This satisfies the Sen's second conjecture.

5.2.3 Gauge invariant observable

The gauge invariant observable of the BMT solution is computed in [4]. From the definition,

$$\lim_{\epsilon\to 0} W\left(\Psi^\epsilon_{BMT}, \mathcal{V}\right) = \lim_{\epsilon\to 0}\langle I|\mathcal{V}(i\infty, -i\infty)\left(c\phi_u - \frac{1}{K+\phi+\epsilon} u\partial_u\phi_u Bc\partial c\right)(0)|I\rangle\,,$$

where we wrote the coupling constant u explicitly and used (5.25). The contribution from the first term vanishes because

$$\begin{aligned}
\langle I|\mathcal{V}(i\infty, -i\infty)\, c\phi_u|I\rangle &= \lim_{\xi\to 0}\langle \mathcal{V}(i\infty, -\infty)\, c\phi_u\rangle_{C_\xi} \\
&= \lim_{\xi\to 0}\langle \mathcal{V}(i\infty, -\infty)\, c\phi_{u\xi}\rangle_{C_1} \\
&= 0\,.
\end{aligned}$$

Therefore the gauge invariant observable becomes

$$\begin{aligned} W\left(\Psi_{BMT}^{\epsilon}, \mathcal{V}\right) &= -\langle I|\mathcal{V}\left(i\infty, -i\infty\right) \frac{1}{K+\phi+\epsilon} u\partial_u \phi_u Bc\partial c\left(0\right)|I\rangle \\ &= -\int_0^\infty dte^{-\epsilon t} \langle \mathcal{V}\left(i\infty, -\infty\right) u\partial_u\phi_u Bc\partial c\left(0\right)\rangle_{C_t}^{BCFT_{\phi_u}} \\ &= \int_0^\infty dte^{-\epsilon t}\frac{1}{t}\langle \mathcal{V}\left(i\infty, -i\infty\right) c\left(0\right) u\partial_u\phi_u\rangle_{C_t}^{BCFT_{\phi_u}} \\ &= \int_0^\infty dye^{-\epsilon y/u}\frac{\partial}{\partial y}\langle \mathcal{V}\left(i\infty, -i\infty\right) c\left(0\right)\rangle_{C_1}^{BCFT_{\phi_y}}, \end{aligned}$$

where $y = ut$. The integral can be performed and we get

$$W\left(\Psi_{BMT}^{\epsilon}, \mathcal{V}\right) = -\mathcal{A}_0^{disk}\left(\mathcal{V}\right) + \frac{\epsilon}{u}\int_0^\infty dye^{-\epsilon y/u}\langle \mathcal{V}\left(i\infty, -i\infty\right) c\left(0\right)\rangle_{C_1}^{BCFT_{\phi_y}},$$

where we used same notation in (3.5). The second term of the right hand side becomes

$$\frac{1}{u}\int_0^\infty dy' e^{-y'/u}\langle \mathcal{V}\left(i\infty, -i\infty\right) c\left(0\right)\rangle_{C_1}^{BCFT_{\phi_{y'/\epsilon}}},$$

where $y' = \epsilon y$. In Witten deformation, the contribution from the region of $y'/\epsilon \sim 0$ vanishes because of

$$\lim_{L\to 0}\int_0^L dy' Z_{BCFT_\phi}\left(\phi_{y'/\epsilon}\right) \propto \lim_{L\to 0}\sqrt{L}\,.$$

In cosine deformation, the contribution from the same region vanishes also. Because of these, we can take $\epsilon \to 0$ before the integral giving

$$\frac{1}{u}\int_0^\infty dy' e^{-y'/u}\langle \mathcal{V}\left(i\infty, -i\infty\right) c\left(0\right)\rangle_{C_1}^{BCFT_{\phi_{y'/\epsilon}}} \underset{\epsilon\to 0}{\longrightarrow} \mathcal{A}_*^{disk}\left(\mathcal{V}\right),$$

where we called the $BCFT$ in the IR limit as $BCFT_*$. Therefore, the gauge invariant observables becomes

$$\lim_{\epsilon\to 0} W\left(\Psi_{BMT}^{\epsilon}, \mathcal{V}\right) = \mathcal{A}_*^{disk}\left(\mathcal{V}\right) - \mathcal{A}_0^{disk}\left(\mathcal{V}\right).$$

This satisfies Ellwood conjecture.

As we saw, the computation of the energy of the BMT solution is difficult. The assumption that Ψ_0^ϵ is the tachyon vacuum solution is reasonable expectation. This is based on the fact the constant seed $\phi = a$ has the coupling constant a which have mass dimension 1. It is more relevant in the IR limit than other operators in the seed of Ψ_0^ϵ. We can compute the correlation functions of ϕs in $BCFT_\phi$ in such a limit. However, we have to compute the correlation functions of ϕs with finite coupling constant, because the computation of the energy of Ψ_0^ϵ includes the integral of the coupling constant multiplied the length of cylinder. The computation is performed numerically in [18] using the Witten's deformation. Since there is a divergence of the volume of D_{25}-brane in the Witten's deformation, the regularization problem arise also.

On the other hand, the computation of the gauge invariant observable is computed analytically. Moreover, the computation is easier than that of the energy. This is because that the gauge invariant observable can be computed by evaluating the one point function of ϕ in $BCFT_\phi$. This one point function is written in the differential of the partition function $\partial_L Z_{BCFT_\phi}\left(L\right)$, and the gauge invariant observable becomes the difference of the $\mathcal{A}_{BCFT}^{disk}\left(\mathcal{V}\right)$ between IR limit and UV limit.

6 Energy from gauge invariant observable

From the examples in the previous section, we can see how much difficult the computation of the energy is. In the case of Murata-Schnabl solution, the computations of the energy and the gauge invariant observable need different regularization. In the case of BMT solution, the computation of energy is hard to perform analytically while the computation of gauge invariant observable is performed analytically. On the other hand, the gauge invariant observables are able to be computed analytically. Moreover, we may be able to compute the energy from the gauge invariant observable with the vertex operator

$$\mathcal{V} = \frac{2}{\pi i} c\bar{c}\partial X^0 \bar{\partial} X^0 \,. \tag{6.1}$$

The gauge invariant observable with this vertex operator corresponds to the one point function of the linear combination of graviton and dilaton $g^{\mu\nu}|_{\mu,\nu=0}$ with zero momentum. Since g^{00} couples to the energy momentum tensor T_{00}, this one point function is expected to proportional to the energy of the system. Actually, the one point function

$$\mathcal{A}_a(\mathcal{V}) = \langle \mathcal{V}(i\infty, -i\infty)\, c(0)\rangle_{C_1}^{BCFT_a} \,,$$

becomes

$$\mathcal{A}_a(\mathcal{V}) = \frac{1}{2\pi^2} V_a \,,$$

where we used the expectation value of ghosts (4.9) and that of X^0s (C.1) in the appendix (C). The volume factor V_a varies as follows

$$V_a = \begin{cases} V_{25} & , BCFT_a = BCFT_0 \\ 0 & , \text{the vanishing } D\text{-brane background} \\ V_{24} & , BCFT_a = BCFT_* \end{cases} ,$$

where we used the notation of the discussion of the BMT solution. This gives the expected values of the energy of the solutions.

In this chapter, we will show that the energy can be expressed by gauge invariant observable with vertex operator (6.1) by

$$E = \frac{1}{g^2} \langle I|\mathcal{V}(i)\,|\Psi\rangle \,. \tag{6.2}$$

Using this relation, we will compute the energy of the Murata-Schnabl solution regularized by the same way of the computation of gauge invariant observable. In addition, we will compute the energy of the BMT solution, showing that the energy of Ψ_0^ϵ defined by (5.23) becomes the one of the tachyon vacuum solution.

In the section (6.1), we will prove (6.2) in the case of that the operator $\mathcal{O}_\Psi$ is local operator, where $\mathcal{O}_\Psi$ corresponds to the solution $|\Psi\rangle$ with ordinary state-operator mapping

$$|\Psi\rangle = \mathcal{O}_\Psi(\xi = 0)\,|0\rangle \,,$$

where $|0\rangle$ is $SL(2,\mathbb{R})$ invariant vacuum and ξ is the complex coordinate on the upper half plane. This gives formal proof of (6.2). This proof can be applied to the analytic solution constructed by KBc algebra, which does not satisfy the assumption that $\mathcal{O}_\Psi$ is local. In the section (6.2), we will show the relation (6.2) to the Okawa type solution (4.11). Since the regularization of the solution can cause anomaly to the equation of motion, we will consider (6.2) with anomalous contribution. In the section (6.3), we will show the relation (6.2) can hold to the Murata-Schnabl solution and the BMT solution. Using this relation, we will compute the energy of the Murata-Schnabl solution with the same regularization of that of the gauge invariant observable, and we can compute the energy of the BMT solution with cosine deformation analytically.

6.1 A proof of (6.2) for local $\mathcal{O}_\Psi$

As we emphasized in the previous chapter, the energy of the solutions is proportional to Ψ^3, while the gauge invariant observable is linear to Ψ. To transform the gauge invariant observable to the energy of the solution, we will use the equation of motion and increase the degree of Ψ. Because of this, we need some quantity which produce BRST operator Q from gauge invariant observable. What we will see in this section is that when there is no other vertex operator, there is a nonlocal operator $\chi\left(X^0\right)$ and $\mathcal{G}\left(X^0\right)$ which satisfy

$$\begin{aligned} \mathcal{V}(i) &= \{Q,\chi\} , \\ [Q,\mathcal{G}] &= \chi-\chi^\dagger , \end{aligned}$$

where $\chi^\dagger$ is the BPZ conjugate of χ. Using these identity and equation of motion, the gauge invariant observable will be transformed to the term proportional to the third power of Ψ in which $\mathcal{G}$ is inserted. When the contribution from $\mathcal{G}$ can be computed independently, the gauge invariant observable become to be proportional to Ψ^3, thus to the energy of the solution. Because of this, we assume that $\mathcal{O}_\Psi$ does not involve X^0 variable.

6.1.1 Open string field theory in a weak gravitational background

In this subsection, we will consider the string field expressed on upper half plane ξ.

It also serves as a review of [48] to consider the string field theory with closed string background:

$$S_h=-\frac{1}{g^2}\left[\frac{1}{2}\langle\Psi|Q|\Psi\rangle+\frac{1}{3}\langle\Psi|\Psi\star\Psi\rangle+h\langle I|\mathcal{V}(i)|\Psi\rangle\right] , \tag{6.3}$$

where $h\ll 1$. It has been shown in [49] that such a string field action describes string theory in a closed string background, for general on-shell $\mathcal{V}$. The vertex operator (6.1) is a linear combination of the constant graviton and dilaton. Therefore the action (6.3) should be the open string field theory in a constant metric and dilaton background. However, the constant metric can be transformed to flat metric $\eta^{\mu\nu}$ and the effect of dilaton background causes the change of the coupling constant $g\to g'$. Because of this, the action (6.3) can be transformed to the action of the ordinary cubic string field theory (2.2) with coupling constant g'.

To see the transformation of the action (6.3), let us define the nonlocal operator χ:

$$\mathcal{V}(i)=\{Q,\chi\} , \tag{6.4}$$

where

$$\begin{aligned} \chi &\equiv \lim_{\delta\to 0}\left[\int_{P_1}\frac{d\xi}{2\pi i}j\left(\xi,\bar\xi\right)-\int_{\bar P_1}\frac{d\bar\xi}{2\pi i}\bar j\left(\xi,\bar\xi\right)+\frac{c(1)}{2\pi\delta}\right] , \\ & \quad j\left(\xi,\bar\xi\right)\equiv 4\partial X^0(\xi)\,\bar c\bar\partial X^0\left(\bar\xi\right) , \\ & \quad \bar j\left(\xi,\bar\xi\right)\equiv 4\bar\partial X^0\left(\bar\xi\right)c\partial X^0(\xi) . \end{aligned} \tag{6.5}$$

The contour P_1 is depicted in the Figure (11) and $\bar P_1$ is its complex conjugate. We took the normal ordering for operators implicitly. Since the correlation function $\langle\partial X^0(\xi)\,\bar\partial X^0\left(\bar\xi\right)\rangle_{U.H.P.}$ diverges on the real axis we regularized the edge of contour by δ. Because of the third term of the right hand side of (6.5), χ is not singular in the limit of $\delta\to 0$. We give the details of the definition of χ and the derivation of (6.4)(6.6) in appendix A. Using (6.4) and equation of motion, the action (6.3) becomes

$$S_h=-\frac{1}{g^2}\left[\frac{1}{2}\left\langle\Psi'|Q'|\Psi'\right\rangle+\frac{1}{3}\left\langle\Psi'|\Psi'*\Psi'\right\rangle\right]+\mathcal{O}\left(h^2\right) , \tag{6.6}$$

where

$$\begin{aligned} |\Psi'\rangle &\equiv |\Psi\rangle + h\chi\,|I\rangle \ , \\ Q' &\equiv Q - h\left(\chi - \chi^\dagger\right) . \end{aligned} \tag{6.7}$$

$\chi^\dagger$ denotes the BPZ conjugate of χ and

$$\begin{aligned} \chi - \chi^\dagger &= \lim_{\delta\to 0}\left[\int_{P_1+P_2}\frac{d\xi}{2\pi i} j\left(\xi,\bar\xi\right) - \int_{\bar P_1+\bar P_2}\frac{d\bar\xi}{2\pi i}\bar j\left(\xi,\bar\xi\right)\right. \\ &\qquad\qquad \left. + \frac{c\,(1)}{2\pi\delta} - \frac{c\,(-1)}{2\pi\delta}\right] , \end{aligned}$$

where P_2 is depicted in (11).

The string field theory (6.6) is similar to the one considered in [48] as the open string field theory in the soft dilaton background. They have shown that the effect of such a background corresponds to a rescaling of the string coupling constant g. To see this, let us define $\mathcal{G}$ which satisfies

$$[Q, \mathcal{G}] = \chi - \chi^\dagger \, . \tag{6.8}$$

This is given by

$$\begin{aligned} \mathcal{G} &\equiv \lim_{\delta\to 0}\left[\int_{P_1+P_2}\frac{d\xi}{2\pi i} g_\xi\left(\xi,\bar\xi\right) - \int_{\bar P_1+\bar P_2}\frac{d\bar\xi}{2\pi i} g_{\bar\xi}\left(\xi,\bar\xi\right)\right] , \\ & \quad g_\xi\left(\xi,\bar\xi\right) \equiv 2\left(X^0\left(\xi,\bar\xi\right) - X^0\left(i,-i\right)\right)\partial X^0\left(\xi\right) \ , \\ & \quad g_{\bar\xi}\left(\xi,\bar\xi\right) \equiv 2\left(X^0\left(\xi,\bar\xi\right) - X^0\left(i,-i\right)\right)\bar\partial X^0\left(\bar\xi\right) \ . \end{aligned} \tag{6.9}$$

Since g_ξ, $g_{\bar\xi}$ have singularity at $\xi = i$, the contour $P_1 + P_2$ is deformed to the contour depicted in Figure (12). The $X^0\left(i,-i\right)$ is necessary for welldefined g_ξ and $g_{\bar\xi}$. $g_\xi, g_{\bar\xi}$ are defined with the usual normal ordering prescription (C.2) and under a conformal transformation $\xi \to \xi'\left(\xi\right)$, g_ξ transforms as

$$g_{\xi'}\left(\xi',\bar\xi'\right) = \frac{\partial\xi}{\partial\xi'} g_\xi\left(\xi,\bar\xi\right) + \frac{1}{2}\partial_{\xi'}\ln\frac{\partial\xi}{\partial\xi'} \, . \tag{6.10}$$

The singularities comes from real axis are canceled between the first term and the second term of the right hand side of (6.9). We give the derivation of (6.8) in A.

When the string fields does not include X^0, we can compute the contribution of $\mathcal{G}$. There are useful identities which can be shown from the definition of $\mathcal{G}$:

$$\langle\mathcal{G}\Psi_1|\Psi_2\rangle + \langle\Psi_1|\mathcal{G}\Psi_2\rangle = \langle\Psi_1|\Psi_2\rangle \ , \tag{6.11}$$

$$\langle\mathcal{G}\Psi_1|\Psi_2 * \Psi_3\rangle + \langle\Psi_1|\mathcal{G}\Psi_2 * \Psi_3\rangle + \langle\Psi_1|\Psi_2 * \mathcal{G}\Psi_3\rangle = \langle\Psi_1|\Psi_2 * \Psi_3\rangle \ . \tag{6.12}$$

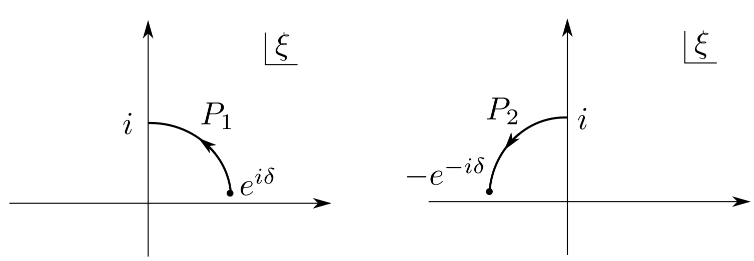

Figure 11: Contours P_1, P_2

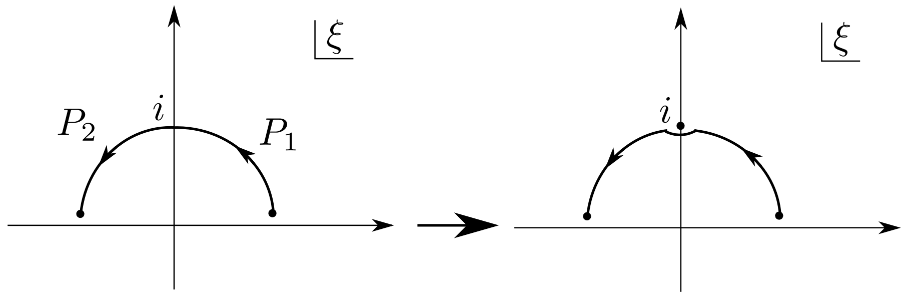

Figure 12: the contour to define $\mathcal{G}$

Using these identities and (6.8), we can transform the action (6.6) to

$$S_h = -\frac{1+h}{g^2}\left[\frac{1}{2}\left\langle\Psi''|Q|\Psi''\right\rangle + \frac{1}{3}\left\langle\Psi''|\Psi''*\Psi''\right\rangle\right] + \mathcal{O}\left(h^2\right) \ , \tag{6.13}$$

where

$$\left|\Psi''\right\rangle \equiv (1-h\mathcal{G})\left|\Psi'\right\rangle \ .$$

Therefore, the action (6.3) with the vertex operator (6.1) corresponds to the original action (2.2) for the string field $|\Psi''\rangle$ with the coupling constant g':

$$g' = \frac{1}{\sqrt{1+h}}g \, .$$

$\mathcal{G}$ can be regarded as the generator of general coordinate transformation.

6.1.2 Derivation of (6.2)

From the two expression of S_h (6.3) and (6.13), we can deduce the relation (6.2). When the solution $|\Psi\rangle$ is a static solution, the action (6.3) can be expressed by

$$S_h = -E\left[\Psi\right] - \frac{h}{g^2}\left\langle I|\mathcal{V}\left(i\right)|\Psi\right\rangle \ , \tag{6.14}$$

where $E\left[\Psi\right]$ is the energy of $|\Psi\rangle$. On the other hand, the string field $|\Psi''\rangle$ in (6.13) is related to $|\Psi\rangle$ by

$$\left|\Psi''\right\rangle = |\Psi\rangle + \left|\delta''\Psi\right\rangle \ ,$$

where $|\delta''\Psi\rangle$ is $\mathcal{O}\left(h\right)$. Using this expression, the action (6.13) becomes

$$\begin{aligned} S_h \quad = \quad & -\frac{1+h}{g^2}\left[\frac{1}{2}\left\langle\Psi|Q|\Psi\right\rangle + \frac{1}{3}\left\langle\Psi|\Psi*\Psi\right\rangle\right. \\ & \left. + \left\langle\delta''\Psi\right|\left(Q\left|\Psi\right\rangle + \left|\Psi*\Psi\right\rangle\right)\right] + \mathcal{O}\left(h^2\right) \ . \end{aligned}$$

Since $|\Psi\rangle$ satisfy the equation of motion, the first term of the second line vanish. Therefore (6.13) becomes

$$S_h = -\left(1+h\right)E\left[\Psi\right] + \mathcal{O}\left(h^2\right) \ . \tag{6.15}$$

Comparing the terms of order h in (6.14) and (6.15), we get the relation between energy and gauge invariant observable

$$E = \frac{1}{g^2}\left\langle I|\mathcal{V}\left(i\right)|\Psi\right\rangle \ . \tag{6.16}$$

We can show this relation in more direct way. From (6.11) and (6.12), we can show the identities

$$\begin{aligned}\frac{1}{3}\langle\Psi|\Psi*\Psi\rangle &= \langle\mathcal{G}\Psi|\Psi*\Psi\rangle\;, \\ \frac{1}{2}\langle\Psi|\,Q\,|\Psi\rangle &= \langle\mathcal{G}\Psi|\,Q\,|\Psi\rangle-\frac{1}{2}\langle\Psi|\,[Q,\mathcal{G}]\,|\Psi\rangle\;.\end{aligned} \tag{6.17}$$

Using these identities, equation of motion and

$$[Q,\mathcal{G}]\,|\Psi\rangle=\left(\chi-\chi^{\dagger}\right)|\Psi\rangle\;, \tag{6.18}$$

we can obtain

$$\begin{aligned}E &= \frac{1}{g^2}\left[\frac{1}{2}\langle\Psi|Q|\Psi\rangle+\frac{1}{3}\langle\Psi|\Psi*\Psi\rangle\right] \\ &= \frac{1}{g^2}\left[\langle\mathcal{G}\Psi|\left\{Q\,|\Psi\rangle+|\Psi*\Psi\rangle\right\}-\frac{1}{2}\langle\Psi|\,[Q,\mathcal{G}]\,|\Psi\rangle\right] \\ &= -\frac{1}{2g^2}\langle\Psi|\left(\chi-\chi^{\dagger}\right)|\Psi\rangle \\ &= -\frac{1}{g^2}\langle I|\,\chi\,|\Psi*\Psi\rangle \\ &= \frac{1}{g^2}\langle I|\,\chi Q\,|\Psi\rangle \\ &= \frac{1}{g^2}\langle I|\mathcal{V}\,(i)\,|\Psi\rangle\;.\end{aligned} \tag{6.19}$$

Before closing this section, a few comments are in order:

- The vertex operator $\mathcal{V}$ is expressed in a BRST exact form (6.4), with χ being a completely legal operator. This fact may appear odd because it implies that all the amplitudes involving $\mathcal{V}$ vanish[3]. Actually (6.4) holds on the assumption that there exists no operators around $\xi=1$. In the derivation of (6.4) in appendix A, we use (A.5) which is valid only when such a condition is satisfied, which is the case in our setup. However, in calculating amplitudes, this is not guaranteed because of the existence of other vertex operators and (6.4) cannot be used in such a situation.
- It is also possible to use
$$\mathcal{V}=c\bar{c}\partial X^{\mu}\bar{\partial}X^{\nu}h_{\mu\nu}\,,$$
with $h^{\mu}_{\mu}=-1$ and derive (6.2), provided the variables X^{μ} are described by the free worldsheet theory with the Neumann boundary condition.
- Suppose that $|\Psi\rangle$ does not satisfy the equation of motion:
$$Q\,|\Psi\rangle+|\Psi*\Psi\rangle\equiv|\Gamma\rangle\neq 0\,. \tag{6.20}$$
It is easy to see that the relation (6.19) is modified as
$$E=\frac{1}{g^2}\langle I|\mathcal{V}\,(i)\,|\Psi\rangle-\frac{1}{g^2}\langle I|\,\chi\,|\Gamma\rangle+\frac{1}{g^2}\langle\mathcal{G}\Psi|\Gamma\rangle\;. \tag{6.21}$$

[3]This question was raised by M. Schnabl.

6.2 Derivation of (6.2) for Okawa type solutions

The proof in the previous section based on the assumption that the solution $|\Psi\rangle = \mathcal{O}_\Psi|0\rangle$ is written by a local operator $\mathcal{O}_\Psi$ which is located away from the arch $|\xi| = 1$. However, the analytic solutions which we are interested in are written by non local operators K and B, which is the integral of the energy momentum tensor T and antighost b from $\xi = -i$ to $\xi = i$. These contour of K and B across the contour $P_1 + P_2$ and do not commute with g_ξ and j_ξ. Because of this, we need to check that the proof of the previous section can be held to such solutions. In the following, we will consider the string field expressed on sliver frame. Therefore, we use the state-operator mapping in the form of

$$|\Psi\rangle = \Psi|I\rangle\,,$$

where $|I\rangle$ is identity state.

As the model, we consider the Okawa-type solution (4.11),

$$\Psi = F(K)\,c\frac{KB}{1-F(K)^2}cF(K)\;.$$

We assume that the $F(K)$ and $K/\left(1-F^2\right)$ can be expressed by Laplace transform,

$$\begin{aligned} F(K) &= \int_0^\infty dLe^{-LK}f(L)\;,\\ \frac{K}{1-F^2} &= \int_0^\infty dLe^{-LK}\tilde{f}(L)\;. \end{aligned}$$

We will consider the case later, that the regularization is necessary to define the Laplace transform. Using these Laplace transforms and (4.8), the string field itself can be expressed by Laplace transform

$$\Psi = \int_0^\infty dLe^{-LK}\psi(L)\;, \tag{6.22}$$

where

$$\begin{aligned} \psi(L) &= \int dL_1dL_2dL_3\delta(L-L_1-L_2-L_3)\\ &\quad\times c(L_2+L_3)\,Bc(L_3)\,f(L_1)\,\tilde{f}(L_2)\,f(L_3)\;. \end{aligned} \tag{6.23}$$

We express (4.2) as

$$\Psi = \mathcal{L}\{\psi\}\;,$$

where $\mathcal{L}$ denotes the operation of the Laplace transform. When we define an inverse Laplace transform by $\mathcal{L}^{-1}$, ψ is expressed by

$$\psi(L) = \mathcal{L}^{-1}\{\Psi\}(L)\;.$$

In order to trace the computation (6.19), we have to show the identities (6.17) and (6.18) about the Okawa type solution. We will define $\mathcal{G}$ and χ on sliver frame, and show how the operators K and B affect the calculation of (6.17) and (6.18). Actually, as we will see, the effects of nonlocal operators are canceled each other and the identities are also hold on the Okawa type solution.

6.2.1 Definition of $\mathcal{G}$

The solution Ψ is expressed by the sum of wedge state e^{-LK} with insertion $\psi(L)$ in (4.2). Similarly, $|\mathcal{G}\Psi\rangle$ is defined by the sum of wedge state with insertion of $\psi(L)$ and $\mathcal{G}(L)$. $\mathcal{G}(L)$ is defined by

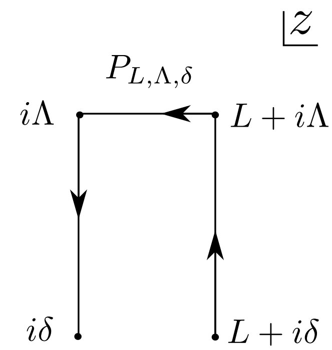

Figure 13: $P_{L,\Lambda,\delta}$

$$\begin{aligned}
\mathcal{G}(L,\Lambda,\delta) &\equiv \lim_{z_0\to i\infty}\left[\int_{P_{L,\Lambda,\delta}}\frac{dz}{2\pi i}g_z(z,\bar{z})-\int_{\bar{P}_{L,\Lambda,\delta}}\frac{d\bar{z}}{2\pi i}g_{\bar{z}}(z,\bar{z})\right], \\
& g_z(z,\bar{z}) = 2\left(X^0\left(z,\bar{z}\right)-X^0(z_0,\bar{z}_0)\right)\partial X^0(z)\,, \\
& g_{\bar{z}}(z,\bar{z}) = 2\left(X^0\left(z,\bar{z}\right)-X^0(z_0,\bar{z}_0)\right)\bar{\partial} X^0(z)\,,
\end{aligned}$$

where the contour is depicted in Figure (13). We can define $\mathcal{G}$ as the operation of $\mathcal{G}\left(L,\Lambda,\delta\right)$ to $e^{-LK}\psi\left(L\right)$ of every L. Using a test state $|\phi\rangle=\mathcal{O}_\phi\left(\xi=0\right)|0\rangle$, the definition of $|\mathcal{G}\Psi\rangle$ is given by

$$\langle\phi|\mathcal{G}\Psi\rangle = \lim_{(\Lambda,\delta)\to(\infty,0)}\int_0^\infty dL\left\langle e^{\left(L+\frac{1}{2}\right)K}f\circ\mathcal{O}_\phi\left(0\right)e^{-\left(L+\frac{1}{2}\right)K}\mathcal{G}(L,\Lambda,\delta)\psi\left(L\right)\right\rangle_{C_{L+1}}. \qquad (6.24)$$

Here, we used $f\left(\xi\right)\equiv\frac{\pi}{2}\arctan\xi$ and the fact that $|0\rangle=e^{-K}|I\rangle$. z which appears in the definition of $\mathcal{G}(L,\Lambda,\delta)$ is the complex coordinate on C_{L+1} such that $e^{-LK}\psi\left(L\right)$ corresponds to the region $0\leq \mathrm{Re}z\leq L$.

From this definition of $\mathcal{G}$, we can deduce the identities (6.11) and (6.12). The first term of the left hand side of (6.12) becomes

$$\begin{aligned}
\langle\mathcal{G}\Psi_i|\Psi_i\star\Psi_i\rangle &= \lim_{(\Lambda,\delta)\to(\infty,0)}\int_0^\infty dL_1\int_0^\infty dL_2\int_0^\infty dL_3 \\
&\qquad \times\left\langle e^{(L_2+L_3)K}\mathcal{G}(L_1,\Lambda,\delta)\psi_1(L_1)e^{-L_2K}\psi_2(L_2)e^{-L_3K}\psi_3(L_3)\right\rangle_{C_{L_1+L_2+L_3}},
\end{aligned} \qquad (6.25)$$

where $\psi_i\left(L_i\right)=\mathcal{L}^{-1}\left\{\Psi_i\right\}\left(L_i\right)$. When we assume that every Ψ_is do not include X^0, we can separate the contribution of $\mathcal{G}\left(L,\Lambda,\delta\right)$ from other expectation values:

$$\begin{aligned}
&\left\langle e^{(L_2+L_3)K}\mathcal{G}(L_1,\Lambda,a)\psi_1(L_1)e^{-L_2K}\psi_2(L_2)e^{-L_3K}\psi_3(L_3)\right\rangle_{C_{L_1+L_2+L_3}} \\
&= \langle\mathcal{G}(L_1,\Lambda,a)\rangle^{X^0}_{C_{L_1+L_2+L_3}} \\
&\quad\times\left\langle e^{(L_2+L_3)K}\psi_1(L_1)e^{-L_2K}\psi_2(L_2)e^{-L_3K}\psi_3(L_3)\right\rangle_{C_{L_1+L_2+L_3}},
\end{aligned} \qquad (6.26)$$

where $\langle\cdot\rangle^{X^0}$ is the expectation value of X^0. This expectation value of X^0 can be computed by using (C.3). Since the expectation value of g_ξ and $g_{\bar{\xi}}$ does not depend on $\Re z$ of the integral in the limit $z_0\to\infty$ and constant $\Im z\to\infty$, it becomes

$$\lim_{(\Lambda,\delta)\to(\infty,0)}\langle\mathcal{G}(L_1,\Lambda,\delta)\rangle^{X^0}_{C_{L_1+L_2+L_3}} = \frac{L_1}{L_1+L_2+L_3}\,. \qquad (6.27)$$

Therefore, the first term of the left hand side of (6.12) becomes

$$\begin{aligned}\langle \mathcal{G}\Psi_1|\Psi_2 * \Psi_3\rangle &= \int dL_1 dL_2 dL_3 \frac{L_1}{L_1+L_2+L_3} \\ &\qquad \times \left\langle e^{(L_2+L_3)K}\psi_1(L_1)e^{-L_2K}\psi_2(L_2)e^{-L_3K}\psi_3(L_3)\right\rangle_{C_{L_1+L_2+L_3}} .\end{aligned}$$

From the same computation, the second and the third term of the left hand side of (6.12) becomes

$$\begin{aligned}\langle \Psi_1|\mathcal{G}\Psi_2 * \Psi_3\rangle &= \int dL_1 dL_2 dL_3 \frac{L_2}{L_1+L_2+L_3} \\ &\qquad \times \left\langle e^{(L_2+L_3)K}\psi_1(L_1)e^{-L_2K}\psi_2(L_2)e^{-L_3K}\psi_3(L_3)\right\rangle_{C_{L_1+L_2+L_3}} , \\ \langle \Psi_1|\Psi_2 * \mathcal{G}\Psi_3\rangle &= \int dL_1 dL_2 dL_3 \frac{L_3}{L_1+L_2+L_3} \\ &\qquad \times \left\langle e^{(L_2+L_3)K}\psi_1(L_1)e^{-L_2K}\psi_2(L_2)e^{-L_3K}\psi_3(L_3)\right\rangle_{C_{L_1+L_2+L_3}} .\end{aligned}$$

From this, we can deduce the relation (6.12). Similarly, (6.11) and (6.17) can be shown from the definition.

6.2.2 (6.18) for Okawa type solutions

Let us consider the identity (6.18). For an arbitrary test state $|\phi\rangle = |\phi\rangle$, we can deduce

$$\begin{aligned}\langle\phi|\,[Q,\mathcal{G}]\,|\Psi\rangle &= \lim_{(\Lambda,\delta)\to(\infty,0)} \left[\int_0^\infty dL \left\langle e^{\left(L+\frac{1}{2}\right)K} f\circ\phi\,(0)\, e^{-\left(L+\frac{1}{2}\right)K} Q\mathcal{G}(L,\Lambda,\delta)\psi\,(L)\right\rangle_{C_{L+1}}\right. \\ &\qquad \left. -\int_0^\infty dL \left\langle e^{\left(L+\frac{1}{2}\right)K} f\circ\phi\,(0)\, e^{-\left(L+\frac{1}{2}\right)K} \mathcal{G}(L,\Lambda,\delta)\mathcal{L}^{-1}\left\{Q\Psi\right\}(L)\right\rangle_{C_{L+1}}\right] \\ &= \mathcal{A}_1 + \mathcal{A}_2 ,\end{aligned}$$

where

$$\mathcal{A}_1 \equiv \lim_{(\Lambda,\delta)\to(\infty,0)} \int_0^\infty dL \left\langle e^{\left(L+\frac{1}{2}\right)K} f\circ\phi\,(0)\, e^{-\left(L+\frac{1}{2}\right)K}\left[Q,\mathcal{G}(L,\Lambda,\delta)\right]\psi\,(L)\right\rangle_{C_{L+1}} , \tag{6.28}$$

$$\begin{aligned}\mathcal{A}_2 &\equiv \lim_{(\Lambda,\delta)\to(\infty,0)} \int_0^\infty dL \left\langle e^{\left(L+\frac{1}{2}\right)K} f\circ\phi\,(0)\, e^{-\left(L+\frac{1}{2}\right)K}\right. \\ &\qquad\qquad \left.\times\, \mathcal{G}(L,\Lambda,\delta)\left[Q\psi\,(L) - \mathcal{L}^{-1}\left\{Q\Psi\right\}(L)\right]\right\rangle_{C_{L+1}} .\end{aligned} \tag{6.29}$$

One can guess that $\mathcal{A}_1$ gives the contribution of $\left(\chi-\chi^\dagger\right)|\Psi\rangle$. As we will see soon, there is an additional contribution which comes from B in ψ. Similarly, one may guess that $\mathcal{A}_2$ will vanish. However because

$$Q\mathcal{L}^{-1}\left\{\Psi\right\}(L) - \mathcal{L}^{-1}\left\{Q\Psi\right\}(L) \neq 0 ,$$

there is an additional contribution. This contribution comes from K and cancels to the additional contribution of $\mathcal{A}_1$. We will see this cancellation explicitly.

Using the expression (4.2) and (6.23), $\mathcal{A}_1$ becomes

$$\begin{aligned}\mathcal{A}_1 &= \int dL \int dL_1 dL_2 dL_3 \delta\left(L-L_1-L_2-L_3\right) \\ &\qquad \times f\left(L_1\right)\tilde{f}\left(L_2\right)f\left(L_3\right) \\ &\qquad \times \left\langle e^{\left(L+\frac{1}{2}\right)K} f\circ\phi\,(0)\, e^{-\left(L+\frac{1}{2}\right)K}\left[Q,\mathcal{G}(L,\Lambda,\delta)\right] c\left(L_2+L_3\right)Bc\left(L_3\right)\right\rangle_{C_{L+1}} .\end{aligned} \tag{6.30}$$

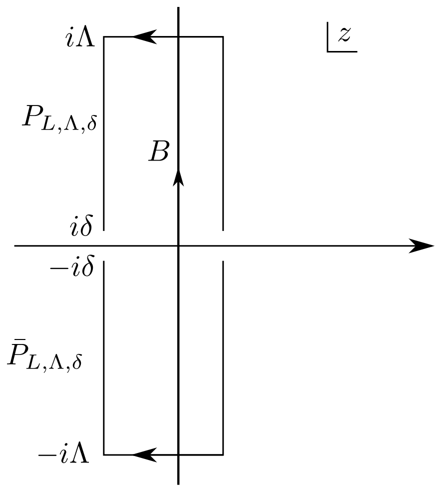

Figure 14: $P_{L,\Lambda,\delta}$ and B in $\mathcal{A}_1$

From the definition $[Q, \mathcal{G}(L,\Lambda,\delta)]$ becomes

$$\begin{aligned}
[Q, \mathcal{G}(L,\Lambda,\delta)] &= \int_{P_{L,\Lambda,\delta}} \frac{dz}{2\pi i} 4\partial X^0(z)\, \bar{c}\bar{\partial} X^0(\bar{z}) - \int_{\bar{P}_{L,\Lambda,\delta}} \frac{d\bar{z}}{2\pi i} 4\bar{\partial} X^0(\bar{z})\, c\partial X^0(z) \\
&\quad -2\left(c\partial X^0(i\infty) + \bar{c}\bar{\partial} X^0(-i\infty)\right)\left(\int_{P_{L,\Lambda,\delta}} \frac{dz}{2\pi i}\partial X^0(z) - \int_{\bar{P}_{L,\Lambda,\delta}} \frac{d\bar{z}}{2\pi i}\bar{\partial} X^0(\bar{z})\right) \\
&\quad + \int_{P_{L,\Lambda,\delta}} \frac{dz}{2\pi i}\frac{1}{2}\partial^2 c - \int_{\bar{P}_{L,\Lambda,\delta}} \frac{d\bar{z}}{2\pi i}\frac{1}{2}\bar{\partial}^2 \bar{c} \\
&\quad + \int_{P_{L,\Lambda,\delta}} dz \partial\kappa(z,\bar{z}) + \int_{\bar{P}_{L,\Lambda,\delta}} d\bar{z}\bar{\partial}\kappa(z,\bar{z}) \ , \qquad (6.31) \\
\kappa(z,\bar{z}) &\equiv \frac{1}{\pi i}\left(X^0(z,\bar{z}) - X^0(i\infty,-i\infty)\right)\left(c\partial X^0(z) - \bar{c}\bar{\partial} X^0(\bar{z})\right) .
\end{aligned}$$

Since for $\mathrm{Im}z$, $\mathrm{Im}z' \sim \infty$,

$$\begin{aligned}
\left\langle \partial X^0(z)\, \bar{\partial} X^0(\bar{z}')\right\rangle_{C_L}^{X^0} &\sim -2\left(\frac{\pi}{L}\right)^2 \exp\left(\frac{2\pi i}{L}(z - \bar{z}')\right) , \\
c(z) &\propto \exp\left(-\frac{2\pi i}{L} z\right) ,
\end{aligned}$$

we can ignore the $\mathrm{Im}z = \Lambda$ part of the contours $P_{L,\Lambda,\delta}, \bar{P}_{L,\Lambda,\delta}$ in the first and the second terms of (6.31), in the limit $\Lambda \to \infty$. The second and the third lines are integrated to the surface term of the contour $P_{L,\Lambda,\delta}$, $\bar{P}_{L,\Lambda,\delta}$. For the second line, one can integrate it explicitly using (C.1) and see that it will vanish in the limit of $\delta \to 0$. The third line vanishes because of the boundary conditions of $c, \bar{c}$.

The nontrivial effect of B comes from the fourth line of (6.31). Since the operators in the expectation value in (6.30) is time ordered by time variable $\Re z$ and B across the contour $P_{L,\Lambda,\delta}$ and $\bar{P}_{L,\Lambda,\delta}$, the contribution from the fourth line of (6.31) becomes

$$\left\langle e^{\left(L+\frac{1}{2}\right)K} f\circ\phi\left(0\right) e^{-\left(L+\frac{1}{2}\right)K}\left(\int_{P_{L,\Lambda,\delta}} dz\partial\kappa\left(z,\bar{z}\right)+\int_{\bar{P}_{L,\Lambda,\delta}} d\bar{z}\bar{\partial}\kappa\left(z,\bar{z}\right)\right) c\left(L_2+L_3\right)Bc\left(L_3\right)\right\rangle_{C_{L+1}}$$
$$= -\mathrm{Tr}\left[e^{-\frac{1}{2}K} f\circ\phi\left(0\right) e^{-\left(L+\frac{1}{2}\right)K} c\left(L_2+L_3\right)Bc\left(L_3\right)\kappa\left(i\delta,-i\delta\right)\right.$$
$$+ e^{-\frac{1}{2}K} f\circ\phi\left(0\right) e^{-\left(L+\frac{1}{2}\right)K}\kappa\left(L_1+i\delta,L_1-i\delta\right)c\left(L_2+L_3\right)Bc\left(L_3\right)$$
$$\left.+e^{-\frac{1}{2}K} f\circ\phi\left(0\right) e^{-\left(L+\frac{1}{2}\right)K} c\left(L_2+L_3\right)\left\{B,\kappa\left(a+i\Lambda,a-i\Lambda\right)\right\}c\left(L_3\right)\right] .$$

where we denote a as the position of B insertion. Therefore (6.28) becomes

$$\mathcal{A}_1 = \int dL\mathrm{Tr}\left[e^{-\frac{1}{2}K} f\circ\phi\left(0\right) e^{-\frac{1}{2}K}\left(\chi e^{-LK}\psi\left(L\right)+e^{-LK}\psi\left(L\right)\chi\right)\right]$$
$$+\int dL\frac{1}{L+1}\mathrm{Tr}\left[e^{-\frac{1}{2}K} f\circ\phi\left(0\right) e^{-\frac{1}{2}K} e^{-LK}\alpha\left(L\right)\right] , \tag{6.32}$$

where $\alpha\left(L\right)$ is defined in (B.5) and χ is given as

$$\chi = \lim_{(\Lambda,\delta)\to(\infty,0)}\left[\int_{i\delta}^{i\Lambda}\frac{dz}{2\pi i}4\partial X^0\left(z\right)\bar{c}\bar{\partial}X^0\left(\bar{z}\right)\right.$$
$$-\int_{-i\delta}^{-i\Lambda}\frac{d\bar{z}}{2\pi i}4\bar{\partial}X^0\left(\bar{z}\right)c\partial X^0\left(z\right)$$
$$\left.+\frac{c\left(0\right)}{2\pi\delta}\right] .$$

As we expected, $\mathcal{A}_1$ gives the contribution $\langle\phi|\left(\chi-\chi^\dagger\right)|\Psi\rangle$ which is given by the first line of (6.32). The second line is the effect of non local operator B.

In order to show (6.18), the second line of (6.32) have to be canceled with $\mathcal{A}_2$. Using (B.7) and $\mathcal{G}\left(0,\Lambda,\delta\right)=0$, $\mathcal{A}_2$ becomes

$$\mathcal{A}_2 = \lim_{(\Lambda,\delta)\to(\infty,0)}\int_0^\infty dL\left\langle e^{\left(L+\frac{1}{2}\right)K} f\circ\phi\left(0\right) e^{-\left(L+\frac{1}{2}\right)K}\right.$$
$$\left.\times\ \mathcal{G}(L,\Lambda,\delta)e^{LK}\partial_L\left(e^{-LK}\alpha\left(L\right)\right)\right\rangle_{C_{L+1}} . \tag{6.33}$$

The expectation value of integrand is computed by

$$\partial_t\left\langle e^{\left(L+\frac{1}{2}\right)K} f\circ\phi\left(0\right) e^{-\left(L+\frac{1}{2}\right)K}\mathcal{G}(L,\Lambda,\delta)e^{-tK}\alpha\left(L+t\right)\right\rangle_{C_{L+1}}\Bigg|_{t=0} .$$

Using (6.27), this becomes

$$\partial_t\left[\frac{L}{L+t+1}\left\langle e^{\left(L+\frac{1}{2}\right)K} f\circ\phi\left(0\right) e^{-\left(L+\frac{1}{2}\right)K}\alpha\left(L+t\right)\right\rangle_{C_{L+t+1}}\right]\Bigg|_{t=0}$$
$$=\frac{L}{L+1}\left\langle e^{\left(L+\frac{1}{2}\right)K} f\circ\phi\left(0\right) e^{-\left(L+\frac{1}{2}\right)K} e^{LK}\partial_L\left(e^{-LK}\alpha\left(L\right)\right)\right\rangle_{C_{L+1}}$$
$$-\frac{L}{\left(L+1\right)^2}\left\langle e^{\left(L+\frac{1}{2}\right)K} f\circ\phi\left(0\right) e^{-\left(L+\frac{1}{2}\right)K}\alpha\left(L\right)\right\rangle_{C_{L+1}} .$$

Substituting this to (6.33), we get

$$\begin{aligned}\mathcal{A}_2 &= \int_0^\infty dL \left\{ \frac{L}{L+1} \mathrm{Tr}\left[e^{-\frac{1}{2}K} f \circ \phi(0)\, e^{-\frac{1}{2}K} \partial_L \left(e^{-LK} \alpha(L) \right) \right] \right. \\ &\qquad \left. - \frac{L}{(L+1)^2} \mathrm{Tr}\left[e^{-\frac{1}{2}K} f \circ \phi(0)\, e^{-\frac{1}{2}K} e^{-LK} \alpha(L) \right] \right\} \\ &= -\int_0^\infty dL \frac{1}{L+1} \mathrm{Tr}\left[e^{-\frac{1}{2}K} f \circ \phi(0)\, e^{-\frac{1}{2}K} e^{-LK} \alpha(L) \right].\end{aligned}$$

The result is minus sign of the second line of (6.32). Therefore, there is no contribution from non local operator in Ψ and we get

$$\mathcal{A}_1 + \mathcal{A}_2 = \int dL \mathrm{Tr}\left[e^{-\frac{1}{2}K} f \circ \phi(0)\, e^{-\frac{1}{2}K} \left(\chi e^{-LK} \psi(L) + e^{-LK} \psi(L) \chi \right) \right].$$

This shows that the relation (6.18)

$$[Q, \mathcal{G}] |\Psi\rangle = \left(\chi - \chi^\dagger \right) |\Psi\rangle ,$$

is also held to Okawa type solution.

6.2.3 (6.2) for Okawa type solutions

Since the identities (6.17) and (6.18) have been shown, we can apply the formal proof (6.19) to the Okawa type solution. In summary, we have proved (6.2) for Okawa type solutions Ψ assuming the following conditions:

- Ψ satisfies the equation of motion.
- $\alpha(\infty) = 0$ and $\alpha(0)$ is well-defined for $\alpha(L)$ defined in (B.5).

In addition to these, it is implicitly assumed that all the quantities which appear in the course of the calculations are finite[4]. Conditions other than the equation of motion are concerning the regularity of the solution. If the equation of motion is not satisfied, we obtain (6.21) with $|\Gamma\rangle$ given in (6.20).

6.3 Other solutions

The computation in the previous section depends on that the solution satisfies the assumptions or not. Because of this, we can extend the computation to the other solutions [5]. We will see the applications to BMT solution (5.21) and Murata-Schnabl solution (5.1). What we have to do are define $\alpha(L)$ for each solution and check the assumptions we used in previous section.

6.3.1 BMT solution

As we saw in 5.2, the direct computation of the energy of regularized BMT solution (5.22) is difficult, while the computation of the gauge invariant observable is easy. Thus, the relation (6.2) can make the computation of the energy easy a bit. From the review in (5.2.2) about [47], we will show the relation (6.2) to the BMT solution (5.23) which is considered as the tachyon vacuum. From the definition (5.23), we get the Laplace transform of Ψ_0^ϵ:

[4] This is also assumed in section 6.1.

[5] Our results will not be useful for the marginal deformation solutions, for which it is trivial to calculate the energy, but may be relevant [50] in the context of the discussions in Ref. [51].

$$
\begin{aligned}
\Psi_0^\epsilon &= \int_0^\infty dL e^{-LK} \psi_0^\epsilon(L) \, , \\
\psi_0^\epsilon(L) &= \delta(L) c(\phi+\epsilon) - e^{-\epsilon L - \int_0^L ds \phi(s)} (\phi - \delta\phi + \epsilon) Bc\partial c \, ,
\end{aligned}
$$

where we assume that X in ϕ is different direction from X^0. From the same discussion in (B), the difference between $\mathcal{L}^{-1}\{Q\Psi_0^\epsilon\}$ and $Q\mathcal{L}^{-1}\{\Psi_0^\epsilon\}$ becomes

$$
\mathcal{L}^{-1}\{Q\Psi_0^\epsilon\}(L) = Q\mathcal{L}^{-1}\{\Psi_0^\epsilon\}(L) - e^{LK}\partial_L\left(e^{-LK}\alpha_0^\epsilon(L)\right) - \delta(L)\alpha_0^\epsilon(0) \, ,
$$

with

$$
\alpha_0^\epsilon(L) = e^{-\epsilon L - \int_0^L ds \phi(s)} (\phi - \delta\phi) c\partial c \, .
$$

One can see $\alpha_0^\epsilon(L)$ satisfies the conditions that $\alpha_0^\epsilon(L \to \infty) = 0$ and $\alpha_0^\epsilon(0)$ is welldefined. Since there is divergence in the Witten deformation which comes from noncompactness of X, the relation (6.2) can not be held in this case. The cosine deformation does not seem to have such problem [6]. Thus, the relation (6.2) can be held to the cosine deformation and one can see the energy of Ψ_0^ϵ from the gauge invariant observable $W[\mathcal{V}, \Psi_0^\epsilon]$ with the vertex operator (6.1). It shows correct energy of tachyon vacuum. Using this, one can get the energy of regularized BMT solution (5.22) analytically, and get the preferable result (5.26) which coincide with Sen's second conjecture.

It may be possible to calculate the energy of Ψ_{BMT}^ϵ directly for the cosine deformation. Since Ψ_{BMT}^ϵ has an anomaly in equation of motion, we need to evaluate the second and the third terms of (6.21). In order to do so, we need to know the IR behavior of some correlation functions of ϕ.

6.4 Murata-Schnabl solution

When we consider the regularized Murata-Schnabl solution (5.13), we can see the factor $e^{-\epsilon L}$ in $\alpha(L)$ for Murata-Schnabl solution. This satisfies the assumption that $\alpha(L \to \infty)$ and we can show the formal proof. Since there is an anomaly in the equation of motion, the relation between the energy and gauge invariant observable becomes (6.21).

As we saw in (5.1), the computations of the energy and gauge invariant observable of the Murata-Schnabl solution (5.1) are performed with different regularizations. As an application of our results, let us calculate the energy of the regularized Murata-Schnabl solution (5.13). Since it is regularized by the same way to compute the gauge invariant observable, we can get the energy with the same regularization with gauge invariant observable.

Since Ψ_{MS}^ϵ has an anomaly in equation of motion (5.14) the relation we have is

$$
E = \frac{1}{g^2}\left[\langle I|\mathcal{V}(i)|\Psi_{MS}^\epsilon\rangle - \langle I|\chi|\Gamma_\epsilon\rangle + \langle\mathcal{G}\Psi_{MS}^\epsilon|\Gamma_\epsilon\rangle\right] . \tag{6.34}
$$

After some calculations, details of which are presented in appendix D, we obtain in the limit $\epsilon \to 0$

$$
\begin{aligned}
\langle I|\mathcal{V}(i)|\Psi_{MS}^\epsilon\rangle &= \frac{N-1}{2\pi^2} \\
\langle I|\chi|\Gamma_\epsilon\rangle &\to R_N \, , \qquad (6.35)\\
\langle\mathcal{G}\Psi_{MS}^\epsilon|\Gamma_\epsilon\rangle &\to 0 \, , \qquad (6.36)
\end{aligned}
$$

[6] The partition function

$$
g(uT) \equiv \mathrm{Tr} e^{-T(K+\phi)} \, ,
$$

can be calculated perturbatively [45] and is finite for $0 \leq uT < \infty$. The UV and IR behaviors of the correlation functions of ϕ's are harmless.

where

$$R_N \equiv \begin{cases} -\frac{i}{8\pi^3}\sum_{k=0}^{N-2}\frac{N!}{k!(k+2)!(N-2-k)!}\left((2\pi i)^{k+2}-(-2\pi i)^{k+2}\right) & ,\ (N\geq 1)\ , \\ \frac{i}{8\pi^3}\sum_{k=0}^{-N-1}\frac{(1-N)!}{k!(k+2)!(-N-1-k)!}\left((2\pi i)^{k+2}-(-2\pi i)^{k+2}\right) & ,\ (N\leq 0)\ . \end{cases}$$

Therefore we get the energy

$$E=\frac{1}{g^2}\left(\frac{N-1}{2\pi^2}-R_N\right).$$

This coincides with the desired value $\frac{N-1}{2\pi^2}$ for $N=-1,0,1,2$. Thus, for these N, the anomaly Γ_ϵ is harmless at least in the calculation of energy, although we do not know the reason why this is so for $N=-1,2$ [7].

7 Conclusion

We showed the relation (6.2) between the energy and the gauge invariant observable with the static solution of equation of motion in Witten's cubic string field theory. The vertex operator we used is the linear combination of a constant graviton and dilaton operator (6.1). We also showed the relation in the case that the solution is written by using KBc algebra. In a recent paper [52], it is found that the boundary states can also be constructed from the gauge invariant observables. Therefore now we possess a more efficient way to study the physical properties of solutions which have been or will be discovered.

Recently in [40] the authors propose several new types of solutions made from K, B, c. It seems that our method can be applied to these solutions and derive (6.2) if the solutions are sufficiently regular. One particularly interesting solution mentioned in [40] is the one due to Masuda, which is claimed to have the energy of the double brane configuration but the gauge invariant observables of the perturbative vacuum. It would be intriguing to check how our derivation of (6.2) fails for this solution.

Interrelationship between energy and the gauge invariant observable will be important in exploring various aspects of string fields. For example, in the case of the BMT solution, the calculation of gauge invariant observables reduces to the integral of total derivative. This implies that these gauge invariant observables may have some topological nature. On the other hand, in [37], the energy is interpreted to be the winding number in string field theory. Our results may shed some light on the study of the topological invariants of the space of string fields.

In this paper, we consider the solutions which have a singularity in $K=0$. In [53], the author shows there is a transformation which sends $K\to 1/K$, and the energy is not changed under this transformation. On the other hand, the gauge invariant observable on the right hand side of (6.2) does not have such a symmetry. In [54], the author show that the relation (6.2) is modified to include the contribution from the singularity in $K=\infty$, which has a symmetry under the transformation.

This Doctorial thesis is based on the paper [55] which we have submitted in Journal of High Energy Physics.

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[7] $N=-1,2$ may be argued to be special in the following sense. Ψ_ε is gauge equivalent to

$$\epsilon\left(\frac{1}{G_\epsilon}-1\right)cBG_\epsilon c\,,$$

which is regular in the limit $\epsilon\to 0$ for $N=-1,2$. Another reason for $N=-1$ may be because there exists a regular solution [40].

other researchers of theoritical particle physics of the Tsukuba University.

A Derivations of (6.4), (6.6) and (6.8)

Since the quantities which appear in section 6.1 involve unusual combinations of operators, some explanation is necessary about the definitions and the treatment of them. In this appendix, we present the details of the definition of $\chi, \mathcal{G}$ and the derivation of (6.4)(6.6)(6.8).

$\{Q, \chi\} = \mathcal{V}(i, -i)$

Introducing θ such that $\xi = e^{i\theta}$, the contour integral on the right hand side of (6.5) is expressed as

$$\begin{aligned}
&\int_{P_1} \frac{d\xi}{2\pi i} j\left(\xi, \bar{\xi}\right) - \int_{\bar{P}_1} \frac{d\bar{\xi}}{2\pi i} \bar{j}\left(\xi, \bar{\xi}\right) \\
&\quad = \int_{\delta}^{\frac{\pi}{2}} \frac{d\theta}{2\pi i} i e^{i\theta} j\left(e^{i\theta}, e^{-i\theta}\right) - \int_{\delta}^{\frac{\pi}{2}} \frac{d\theta}{2\pi i} \left(-i e^{-i\theta}\right) \bar{j}\left(e^{i\theta}, e^{-i\theta}\right) . \qquad \text{(A.1)}
\end{aligned}$$

In calculating the BRST variation of this quantity, it is useful to notice

$$\frac{1}{2\pi i} j\left(\xi, \bar{\xi}\right) = \oint_{\xi} \frac{d\xi'}{2\pi i} b\left(\xi'\right) \mathcal{V}\left(\xi, \bar{\xi}\right) , \qquad \text{(A.2)}$$

$$-\frac{1}{2\pi i} \bar{j}\left(\xi, \bar{\xi}\right) = \oint_{\bar{\xi}} \frac{d\bar{\xi}'}{2\pi i} \bar{b}\left(\bar{\xi}'\right) \mathcal{V}\left(\xi, \bar{\xi}\right) , \qquad \text{(A.3)}$$

where $\mathcal{V}\left(\xi, \bar{\xi}\right)$ is the vertex operator defined in (6.1). Since $\mathcal{V}$ is BRST invariant, it is straightforward to show

$$\begin{aligned}
&\left\{ Q, \int_{P_1} \frac{d\xi}{2\pi i} j\left(\xi, \bar{\xi}\right) - \int_{\bar{P}_1} \frac{d\bar{\xi}}{2\pi i} \bar{j}\left(\xi, \bar{\xi}\right) \right\} \\
&\quad = \int_{\delta}^{\frac{\pi}{2}} d\theta \left(\frac{de^{i\theta}}{d\theta} \partial_{\xi} \mathcal{V}\left(e^{i\theta}, e^{-i\theta}\right) + \frac{de^{-i\theta}}{d\theta} \partial_{\bar{\xi}} \mathcal{V}\left(e^{i\theta}, e^{-i\theta}\right) \right) \\
&\quad = \mathcal{V}(i, -i) - \mathcal{V}\left(e^{i\delta}, e^{-i\delta}\right) . \qquad \text{(A.4)}
\end{aligned}$$

Assuming that there are no other operators around $\xi = 1$, the OPE's of $c, \bar{c}, X^0$ imply

$$\mathcal{V}\left(e^{i\delta}, e^{-i\delta}\right) = \frac{c\partial c(1)}{2\pi\delta} + \mathcal{O}(\delta) = \left\{ Q, \frac{c(1)}{2\pi\delta} \right\} + \mathcal{O}(\delta) , \qquad \text{(A.5)}$$

for $\delta \sim 0$. The assumption is valid in the setup of this paper. Using (A.5), we obtain

$$\{Q, \chi\} = \mathcal{V}(i, -i) .$$

It is possible to generalize our construction here to other closed string vertex operators. For any BRST invariant closed string vertex operator $\mathcal{V}\left(\xi, \bar{\xi}\right)$, one can define $j, \bar{j}$ as in (A.2)(A.3), and one can prove (A.4). If $\mathcal{V}\left(e^{i\delta}, e^{-i\delta}\right)$ can be expressed as

$$\mathcal{V}\left(e^{i\delta}, e^{-i\delta}\right) = \{Q, \mathcal{U}\} + \mathcal{O}(\delta) , \qquad \text{(A.6)}$$

in the limit $\delta \to 0$ as in (A.5), we obtain $\mathcal{V}(i, -i) = \{Q, \chi\}$ with

$$\chi \equiv \lim_{\delta \to 0} \left[\int_{P_1} \frac{d\xi}{2\pi i} j\left(\xi, \bar{\xi}\right) - \int_{\bar{P}_1} \frac{d\bar{\xi}}{2\pi i} \bar{j}\left(\xi, \bar{\xi}\right) + \mathcal{U} \right] .$$

(A.6) holds if there exists no on-shell open string vertex operator V_o such that

$$\langle \mathcal{V} V_o \rangle_{\text{disk}} \neq 0 .$$

(6.6)

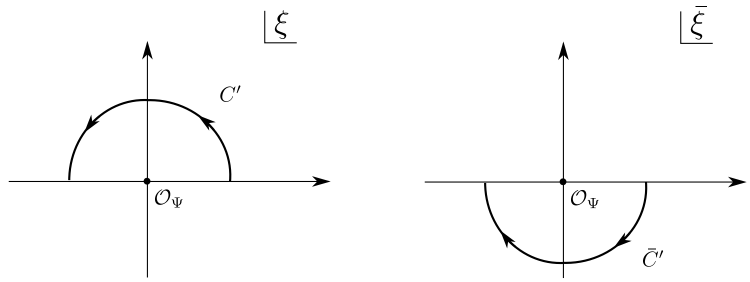

Figure 15: C'

Substituting (6.7) into (6.3), we obtain

$$\begin{aligned} S_h &= -\frac{1}{g^2}\left[\frac{1}{2}\left\langle\Psi'|Q|\Psi'\right\rangle+\frac{1}{3}\left\langle\Psi'|\Psi'*\Psi'\right\rangle+h\left\langle I|\mathcal{V}(i)|\Psi\right\rangle\right. \\ &\qquad \left.-h\left\langle I\right|\chi Q\left|\Psi\right\rangle-\frac{h}{2}\left\langle\Psi'\right|\left(\chi-\chi^\dagger\right)\left|\Psi'\right\rangle\right], \end{aligned}$$

where we have used

$$\begin{aligned} \chi\left|I\right\rangle &= \chi^\dagger\left|I\right\rangle, \\ \left\langle\Psi\right|\chi\left|\Psi\right\rangle &= -\left\langle\Psi\right|\chi^\dagger\left|\Psi\right\rangle. \end{aligned}$$

Since $Q\left|I\right\rangle=0$,

$$\left\langle I\right|\chi Q\left|\Psi\right\rangle=\left\langle I\right|\{Q,\chi\}\left|\Psi\right\rangle,$$

and we may be able to use (6.4) to show (6.6). We should check if the Q in the open string field action yields the BRST variation of χ as an operator in the bulk. The BRST operator acting on a string field $\left|\Psi\right\rangle=\mathcal{O}_\Psi(0)\left|0\right\rangle$ is given as

$$Q\left|\Psi\right\rangle = \left(\int_{C'}\frac{d\xi}{2\pi i}J_{\mathrm{B}}-\int_{\bar{C}'}\frac{d\bar{\xi}}{2\pi i}\bar{J}_{\mathrm{B}}\right)\mathcal{O}_\Psi(0)\left|0\right\rangle,$$

where $J_{\mathrm{B}},\bar{J}_{\mathrm{B}}$ are the BRST current and $C',\bar{C}'$ are depicted in the figure 15. Since $J_{\mathrm{B}}(\xi)=\bar{J}_{\mathrm{B}}\left(\bar{\xi}\right)$ for real ξ the contour integral can be expressed as

$$\oint_0\frac{d\xi}{2\pi i}J_{\mathrm{B}},$$

on the doubled Riemann surface. $\left(Q\chi(i,-i)+\chi(i,-i)Q\right)\left|\Psi\right\rangle$ in the open string field theory is given as

$$\left(\oint_{C''}\frac{d\xi}{2\pi i}J_{\mathrm{B}}-\oint_{\bar{C}''}\frac{d\bar{\xi}}{2\pi i}\bar{J}_{\mathrm{B}}\right)\chi\left(\xi,\bar{\xi}\right)\mathcal{O}_\psi\left|0\right\rangle,$$

where the contours $C'',\bar{C}''$ are the one which surrounds $P_1\bar{P}_1$ as depicted in figure 16. Hence the contour integral yields the BRST variation of χ and we obtain $\mathcal{V}(i,-i)\left|\Psi\right\rangle$.

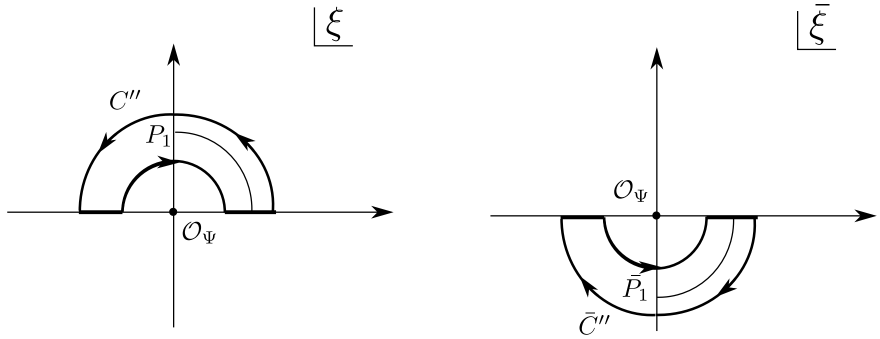

Figure 16: Contour which surrounds P_1

$\{Q, \mathcal{G}\} = \chi - \chi^\dagger$

The contour integral on the right hand side of (6.9) is defined in the same way as in (A.1). It is straightforward to calculate the BRST variations of $g_\xi, g_{\bar{\xi}}$ as

$$\begin{aligned}\left[Q, g_\xi\left(\xi,\bar{\xi}\right)\right] &= \frac{1}{2}\partial^2 c\left(\xi\right) + \partial_\xi\left(2\left(X^0\left(\xi,\bar{\xi}\right) - X^0\left(i,-i\right)\right)c\partial X^0\left(\xi\right)\right) \\ &\quad +2\bar{c}\bar{\partial}X^0\partial X^0\left(\xi,\bar{\xi}\right) - 2\left(c\partial X^0\left(i\right) + \bar{c}\bar{\partial}X^0\left(-i\right)\right)\partial X^0\left(\xi\right)\ , \\ \left[Q, g_{\bar{\xi}}\left(\xi,\bar{\xi}\right)\right] &= \frac{1}{2}\bar{\partial}^2 \bar{c}\left(\bar{\xi}\right) + \partial_{\bar{\xi}}\left(2\left(X^0\left(\xi,\bar{\xi}\right) - X^0\left(i,-i\right)\right)\bar{c}\bar{\partial} X^0\left(\bar{\xi}\right)\right) \\ &\quad +2c\partial X^0\bar{\partial}X^0\left(\xi,\bar{\xi}\right) - 2\left(c\partial X^0\left(i\right) + \bar{c}\bar{\partial}X^0\left(-i\right)\right)\bar{\partial}X^0\left(\bar{\xi}\right)\ ,\end{aligned}$$

and we find $[Q, \mathcal{G}]$ is equal to

$$\begin{aligned}\lim_{\delta\to 0}\Bigg[&\frac{1}{4\pi i}\left(\partial c\left(-e^{-i\delta}\right) - \bar{\partial}c\left(-e^{i\delta}\right) - \partial c\left(e^{i\delta}\right) + \bar{\partial}\bar{c}\left(e^{-i\delta}\right)\right) \\ &+\frac{1}{2\pi i}\left(\int d\xi\partial_\xi + \int d\bar{\xi}\partial_{\bar{\xi}}\right)\left(2\left(X^0\left(\xi,\bar{\xi}\right) - X^0\left(i,-i\right)\right)c\partial X^0\left(\xi\right)\right) \\ &-\frac{1}{2\pi i}\left(\int d\xi\partial_\xi + \int d\bar{\xi}\partial_{\bar{\xi}}\right)\left(2\left(X^0\left(\xi,\bar{\xi}\right) - X^0\left(i,-i\right)\right)\bar{c}\bar{\partial}X^0\left(\bar{\xi}\right)\right) \\ &+\int_{P_1+P_2}\frac{d\xi}{2\pi i}4\partial X^0\bar{c}\bar{\partial}X^0\left(\xi,\bar{\xi}\right) - \int_{\bar{P}_1+\bar{P}_2}\frac{d\bar{\xi}}{2\pi i}4\bar{\partial}X^0 c\partial X^0\left(\xi,\bar{\xi}\right) \\ &-2\left(c\partial X^0\left(i\right) + \bar{c}\bar{\partial}X^0\left(-i\right)\right)\left(\int_{P_1+P_2}\frac{d\xi}{2\pi i}\partial X^0\left(\xi\right) - \int_{\bar{P}_1+\bar{P}_2}\frac{d\bar{\xi}}{2\pi i}\bar{\partial}X^0\left(\bar{\xi}\right)\right)\Bigg]\ .\end{aligned}$$

The terms on the first line cancel with each other in the limit $\delta \to 0$ because of the boundary conditions of $c, \bar{c}$. Those on the fifth vanish if $\mathcal{O}_\Psi$ does not involve X^0. The second and the third lines yield in the limit $\delta \to 0$

$$\begin{aligned}&\frac{1}{\pi i}\left(X^0\left(\xi,\bar{\xi}\right) - X^0\left(i,-i\right)\right)\left(c\partial X^0\left(\xi\right) - \bar{c}\bar{\partial}X^0\left(\bar{\xi}\right)\right)\Bigg|^{\left(-e^{-i\delta},-e^{i\delta}\right)}_{\left(\xi,\bar{\xi}\right)=\left(e^{i\delta},e^{-i\delta}\right)} \\ &\sim -\frac{c\left(-1\right)}{2\pi\delta} + \frac{c\left(1\right)}{2\pi\delta}\ .\end{aligned}$$

Thus we get

$$[Q, \mathcal{G}] = \chi - \chi^\dagger\ .$$

B Laplace transformed form of the string field

We derive two formulas (B.1) (B.7) concerning the Laplace transform of the string field defined in section 6.2.

For two string fields A_1, A_2, which can be expressed as a sum of wedge states with insertions, it is easy to show

$$\mathcal{L}^{-1}\{A_1A_2\}(L) = \int_0^L dL' e^{L'K}\mathcal{L}^{-1}\{A_1\}\left(L-L'\right)e^{-L'K}\mathcal{L}^{-1}\{A_2\}\left(L'\right) . \tag{B.1}$$

The right hand side can be regarded as an operator version of convolution.

For $\psi(L)$ in (6.23),

$$\begin{aligned} Q\psi(L) &= Q\mathcal{L}^{-1}\{\Psi\}(L) \\ &= \int dL_1dL_2dL_3\delta(L-L_1-L_2-L_3) \\ &\quad \times\left[c\partial c(L_2+L_3)Bc(L_3) - c(L_2+L_3)Kc(L_3) + c(L_2+L_3)Bc\partial c(L_3)\right] \\ &\quad \times f(L_1)\tilde{f}(L_2)f(L_3) , \end{aligned} \tag{B.2}$$

which is not equal to

$$\begin{aligned} \mathcal{L}^{-1}\{Q\Psi\}(L) &= \int dL_1dL_2dL_3\delta(L-L_1-L_2-L_3) \\ &\quad \times\Bigg[\{\partial c(L_2+L_3)Bc(L_3) + c(L_2+L_3)Bc\partial c(L_3)\} \\ &\qquad \times f(L_1)\tilde{f}(L_2)f(L_3) \\ &\qquad -c(L_2+L_3)c(L_3)f(L_1)\mathcal{L}^{-1}\left\{\frac{K^2}{1-F^2}\right\}(L_2)f(L_3)\Bigg] . \end{aligned} \tag{B.3}$$

Therefore the BRST transformation and $\mathcal{L}^{-1}$ do not commute with each other. Comparing (B.2) and (B.3), assuming $\alpha(0) = \alpha(\infty) = 0$, we obtain

$$\mathcal{L}^{-1}\{Q\Psi\}(L) = Q\mathcal{L}^{-1}\{\Psi\}(L) - e^{LK}\partial_L\left(e^{-LK}\alpha(L)\right) , \tag{B.4}$$

where

$$\alpha(L) \equiv \mathcal{L}^{-1}\left\{Fc\frac{K}{1-F^2}cF\right\}(L) . \tag{B.5}$$

We expect $\alpha(\infty) = 0$ for regular solutions. $\alpha(0)$ is related to the behavior of $F(K), \frac{K}{1-F^2}$ for $K \sim \infty$ and may not vanish even if Ψ is regular. For example, the Erler-Schnabl solution [12] has

$$\begin{aligned} f(L) &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}L^{-\frac{1}{2}}e^{-L} , \\ \alpha(L) &= e^{-L}\frac{1}{\left(\Gamma\left(\frac{1}{2}\right)\right)^2}\int_0^L dL'\left(L-L'\right)^{-\frac{1}{2}}L'^{-\frac{1}{2}}c\partial c\left(L'\right) , \end{aligned}$$

and

$$\alpha(0) = c\partial c(0) ,$$

With $\alpha(0) \neq 0$, (B.4) cannot be valid for such solutions.

In order to get an identity similar to (B.4) for the solutions with $\alpha(\infty) = 0, \alpha(0) \neq 0$, we regularize Ψ and consider

$$\Psi_\eta \equiv F(K)e^{-\eta K}c\frac{BK}{1-F^2(K)}e^{-\eta K}cF(K)e^{-\eta K} ,$$

for $\eta > 0$. Ψ_η coincides with the original one in the limit $\eta \to 0$ and

$$\begin{aligned}\mathcal{L}^{-1}\{\Psi_\eta\}(L) &= \int dL_1 dL_2 dL_3 \delta(L - L_1 - L_2 - L_3) \\ &\qquad \times c(L_2 + L_3) Bc(L_3) \mathcal{L}^{-1}\{F_\eta\}(L_1) \mathcal{L}^{-1}\left\{\tilde{F}_\eta\right\}(L_2) \mathcal{L}^{-1}\{F_\eta\}(L_3) \,,\end{aligned}$$

where

$$\begin{aligned} F_\eta(K) &\equiv F(K) e^{-\eta K} \,, \\ \tilde{F}_\eta(K) &\equiv \frac{K}{1 - F^2(K)} e^{-\eta K} \,. \end{aligned}$$

$\mathcal{L}^{-1}\{F_\eta\}(L), \mathcal{L}^{-1}\left\{\tilde{F}_\eta\right\}(L)$ vanish for $L < \eta$ and we do not encounter any problem in deriving

$$\mathcal{L}^{-1}\{Q\Psi_\eta\}(L) = Q\mathcal{L}^{-1}\{\Psi_\eta\}(L) - e^{LK}\partial_L\left(e^{-LK}\alpha_\eta(L)\right) \,, \tag{B.6}$$

where

$$\alpha_\eta(L) \equiv \mathcal{L}^{-1}\left\{F_\eta c \tilde{F}_\eta c F_\eta\right\}(L) \,.$$

$\alpha_\eta(L) \sim \alpha(L)$ for $L \gg \eta$ and $\alpha_\eta(L) = 0$ for $L < 3\eta$. Therefore, in the limit $\eta \to 0$,

$$\partial\alpha_\eta(L) \to \partial\alpha(L) + \delta(L)\alpha(0) \,,$$

and (B.6) becomes

$$\mathcal{L}^{-1}\{Q\Psi\}(L) = Q\mathcal{L}^{-1}\{\Psi\}(L) - e^{LK}\partial_L\left(e^{-LK}\alpha(L)\right) - \delta(L)\alpha(0) \,, \tag{B.7}$$

which can be used for solutions with $\alpha(\infty) = 0, \alpha(0) \neq 0$, provided $\alpha(0)$ is well-defined. One can check that the Laplace transform of the right hand side yields $Q\Psi$.

C Correlation functions of X variables

In the calculations in section 6.2, we need the correlation functions of X variables, which are described by the free worldsheet theory with the Neumann boundary condition, on C_L. A conformal transformation which maps C_L to the upper half plane is given as

$$\begin{aligned} C_L &\to \text{UHP} \\ z &\to \xi = \tan\frac{\pi z}{L} \,. \end{aligned}$$

From the correlation functions

$$\begin{aligned} \left\langle \partial X^\mu(\xi)\, \partial X^\nu(\xi') \right\rangle_{\text{UHP}} &= \frac{-\frac{1}{2}\eta^{\mu\nu}}{(\xi - \xi')^2} \,, \\ \left\langle \partial X^\mu(\xi)\, \bar{\partial} X^\nu(\bar{\xi}') \right\rangle_{\text{UHP}} &= \frac{-\frac{1}{2}\eta^{\mu\nu}}{\left(\xi - \bar{\xi}'\right)^2} \,, \end{aligned}$$

we can get

$$\begin{aligned} \left\langle \partial X^\mu(z)\, \partial X^\nu(z') \right\rangle_{C_L} &= -\frac{1}{2}\eta^{\mu\nu}\left(\frac{\pi}{L}\right)^2 \frac{1}{\sin^2\frac{\pi(z - z')}{L}} \,, \\ \left\langle \partial X^\mu(z)\, \bar{\partial} X^\nu(\bar{z}') \right\rangle_{C_L} &= -\frac{1}{2}\eta^{\mu\nu}\left(\frac{\pi}{L}\right)^2 \frac{1}{\sin^2\frac{\pi(z - \bar{z}')}{L}} \,. \end{aligned} \tag{C.1}$$

We are interested in the correlation function of the form $\left\langle \left(X^0(z,\bar z) - X^0(z_0,\bar z_0)\right)\partial X^0(z)\right\rangle_{C_L}$. Since the difference $X^0(z,\bar z) - X^0(z_0,\bar z_0)$ for some $z_0, \bar z_0$ can be written as

$$X^0(z,\bar z) - X^0(z_0,\bar z_0) = \int_{z_0}^{z} dz'\partial X^0(z') + \int_{\bar z_0}^{\bar z} d\bar z'\bar\partial X^0(\bar z') \,,$$

using $\partial X^0, \bar\partial X^0$, the correlation function $\left\langle \left(X^0(z,\bar z) - X^0(z_0,\bar z_0)\right)\partial X^0(z)\right\rangle_{C_L}$ is well-defined. Here it is assumed that the operators are normal ordered as

$$: X^0\partial X^0 : (z,\bar z) \equiv \lim_{z'\to z}\left[X^0(z,\bar z)\,\partial X^0(z') - \frac{1}{2}\frac{1}{z'-z}\right] . \tag{C.2}$$

From (C.1) we obtain

$$\begin{aligned}&\left\langle \left(X^0(z,\bar z) - X^0(z_0,\bar z_0)\right)\partial X^0(z)\right\rangle_{C_L}\\ &\quad = \frac{\pi}{2L}\left[\cot\frac{\pi(z-\bar z)}{L} - \cot\frac{\pi(z-z_0)}{L} - \cot\frac{\pi(z-\bar z_0)}{L}\right] .\end{aligned} \tag{C.3}$$

If one chooses the reference point z_0 to be $i\infty$, we get

$$\left\langle \left(X^0(z,\bar z) - X^0(i\infty,-i\infty)\right)\partial X^0(z)\right\rangle_{C_L} = \frac{\pi}{2L}\cot\frac{\pi(z-\bar z)}{L} .$$

D Derivation of (6.35)(6.36)

We would like to calculate the second and the third terms on the right hand side of (6.34) in the limit $\epsilon\to 0$. These can be calculated basically using the s-z trick [16, 15].

Using

$$\mathcal{L}^{-1}\{\Gamma_\epsilon\}(L) \;=\; \int_0^\infty dL_1 dL_2\,\delta\left(L - \sum_i L_i\right) c(L_2)c(0)\mathcal{L}^{-1}\{F_\epsilon^2\}(L_1)\mathcal{L}^{-1}\left\{\frac{K+\epsilon}{G_\epsilon}\right\}(L_2)\,,$$

and

$$\begin{aligned}&\langle c(L_2)\,c(0)\,c(z)\rangle_{C_L}\\ &\quad = -\frac{1}{2}\left(\frac{L}{\pi}\right)^3\left[\left(\sin\left(\frac{\pi z}{L}\right)\right)^2\sin\frac{2\pi L_2}{L} - \left(\sin\left(\frac{\pi L_2}{L}\right)\right)^2\sin\frac{\pi z}{L}\right] ,\\ &\left\langle c(L_2)\,c(0)\left(\int_{i\delta}^{i\Lambda}\frac{dz}{2\pi i}4\partial X^0(z)\,\bar c\bar\partial X^0(\bar z) - \int_{-i\delta}^{-i\Lambda}\frac{d\bar z}{2\pi i}4\bar\partial X^0(\bar z)\,c\partial X^0(z)\right)\right\rangle_{C_L}\\ &\quad \underset{(\delta,\Lambda)\to(0,\infty)}{\longrightarrow} \frac{1}{4\pi}\left(\frac{L}{\pi}\right)^2\sin\frac{2\pi L_2}{L} ,\\ &\langle c(L_2)\,c(0)\,\kappa(i\delta,-i\delta)\rangle_{C_L} \underset{\delta\to 0}{\longrightarrow} 0 ,\end{aligned}$$

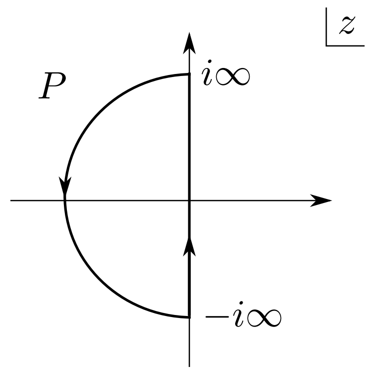

Figure 17: contour P

$\langle I|\,\chi\,|\Gamma_\epsilon\rangle$ becomes

$$\begin{aligned}
\langle I|\,\chi\,|\Gamma_\epsilon\rangle &= \frac{-1}{4\pi^3}\epsilon\int_0^\infty ds s^2\int_0^\infty dL_1 dL_2\delta\left(s-\sum_i L_i\right) \\
&\qquad\qquad \times\mathcal{L}^{-1}\left\{G_\epsilon\right\}(L_1)\mathcal{L}^{-1}\left\{\frac{K+\epsilon}{G_\epsilon}\right\}(L_2)\sin\frac{2\pi}{s}L_2 \\
&= \frac{-1}{4\pi^3}\epsilon\int_0^\infty ds s^2\int_0^\infty dL_1 dL_2\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}e^{\left(s-\sum_i L_i\right)z} \\
&\qquad\qquad \times\mathcal{L}^{-1}\left\{G_\epsilon\right\}(L_1)\mathcal{L}^{-1}\left\{\frac{K+\epsilon}{G_\epsilon}\right\}(L_2)\sin\frac{2\pi}{s}L_2 \\
&= \frac{i}{8\pi^3}\epsilon\int_0^\infty ds s^2\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}e^{sz}G_\epsilon(z)\Delta\left(\frac{z+\epsilon}{G_\epsilon}\right) \\
&= \frac{i}{8\pi^3}\epsilon\int_0^\infty ds s^2\oint_P\frac{dz}{2\pi i}e^{sz}G_\epsilon(z)\Delta\left(\frac{z+\epsilon}{G_\epsilon}\right). \qquad \text{(D.1)}
\end{aligned}$$

Here P is contour on the z plane shown in figure 17 and Δ is defined as [16, 15]

$$\Delta F(z) = F\left(z-\frac{2\pi i}{s}\right) - F\left(z+\frac{2\pi i}{s}\right).$$

For the Murata-Schnabl solution (5.1), (D.1) is evaluated as

$$\begin{aligned}
\langle I|\,\chi\,|\Gamma_\epsilon\rangle &= R_N + \mathcal{O}(\epsilon)\,, \qquad \text{(D.2)} \\
R_N &\equiv \begin{cases} -\frac{i}{8\pi^3}\sum_{k=0}^{N-2}\frac{N!}{k!(k+2)!(N-2-k)!}\left((2\pi i)^{k+2}-(-2\pi i)^{k+2}\right) & ,\ (N\geq 1)\,, \\ \frac{i}{8\pi^3}\sum_{k=0}^{-N-1}\frac{(1-N)!}{k!(k+2)!(-N-1-k)!}\left((2\pi i)^{k+2}-(-2\pi i)^{k+2}\right) & ,\ (N\leq 0)\,, \end{cases}
\end{aligned}$$

for $\epsilon \ll 1$.

The third term on the right hand side of (6.34) becomes

$$\int dL_1 dL_2 \frac{L_1}{L_1+L_2} \langle e^{L_2 K} \mathcal{L}^{-1}\{\Psi_\epsilon\}(L_1)\, e^{-L_2 K}\mathcal{L}^{-1}\{\Gamma_\epsilon\}(L_2)\rangle_{C_{L_1+L_2}}$$
$$= \epsilon \int_0^\infty ds \prod_{i=1}^4 dL_i \delta\left(s - \sum_{i=1}^4 L_i\right) \frac{L_1+L_2}{s}$$
$$\times \operatorname{Tr}\left[e^{-L_1 K}\mathcal{L}^{-1}\{G_\epsilon\}(L_1)\, cB e^{-L_2 K}\mathcal{L}^{-1}\left\{\frac{K+\epsilon}{G_\epsilon}\right\}(L_2)\, c\right.$$
$$\left. \times\, e^{-L_3 K}\mathcal{L}^{-1}\{G_\epsilon\}(L_3)\, c e^{-L_4 K}\mathcal{L}^{-1}\left\{\frac{K+\epsilon}{G_\epsilon}\right\}(L_4)\, c\right] .$$

Using

$$L\mathcal{L}^{-1}\{f\}(L) = \mathcal{L}^{-1}\{\partial f\}(L) ,$$

and eq.(2.5) in [15], we obtain

$$\int dL_1 dL_2 \frac{L_1}{L_1+L_2} \langle e^{L_2 K} \mathcal{L}^{-1}\{\Psi_\epsilon\}(L_1)\, e^{-L_2 K}\mathcal{L}^{-1}\{\Gamma_\epsilon\}(L_2)\rangle_{C_{L_1+L_2}}$$
$$= \frac{i}{8\pi^3}\epsilon \int_0^\infty ds s \oint_C \frac{dz}{2\pi i} e^{sz} \frac{1}{2i}$$
$$\times \left\{\left[\frac{z+\epsilon}{G_\epsilon}, G_\epsilon, \frac{z+\epsilon}{G_\epsilon}, G'_\epsilon\right] + \left[\left(\frac{z+\epsilon}{G_\epsilon}\right)', G_\epsilon, \frac{z+\epsilon}{G_\epsilon}, G_\epsilon\right]\right\},$$

where

$$\begin{aligned}[F_1,F_2,F_3,F_4] \equiv\ & \left[-F_1\Delta F_2 F_3 F'_4 + F_1 \Delta\left(F_2 F'_3\right) F_4 + F_1\Delta\left(F_2 F_3\right) F'_4 - F_1 F'_2 F_3 \Delta F_4\right. \\ & \left. + F_1 F'_2 \Delta\left(F_3 F_4\right) + F_1 F_2 \Delta\left(F'_3 F_4\right) - F_1 \Delta\left(F_2 F'_3 F_4\right) - F_1\left(F_2 \Delta F_3 F_4\right)'\right] .\end{aligned}$$

The contribution of $\mathcal{O}\left(\epsilon^0\right)$ is given by the following replacements

$$\begin{aligned} G'(z) &\to -(N-1)\, G(z) , \\ G''(z) &\to N(N-1)\frac{1}{z^2} G(z) , \\ \left(\frac{z}{G}\right)'(z) &\to N G^{-1}(z) , \end{aligned}$$

and one can see

$$\int dL_1 dL_2 \frac{L_1}{L_1+L_2} \langle e^{L_2 K} \mathcal{L}^{-1}\{\Psi_\epsilon\}(L_1)\, e^{-L_2 K}\mathcal{L}^{-1}\{\Gamma_\epsilon\}(L_2)\rangle_{C_{L_1+L_2}} \sim \mathcal{O}(\epsilon) .$$

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