Quantum teleportation for continuous variables via a partially entangled state and nonorthogonal measurement

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Contents

Chapter 1

Introduction

1.1 Introduction

In 1980's quantum mechanics had matured and the information-oriented society was developed, then the research field of quantum information processing was produced by the fusion of physics and information science. This research field has potentials that we can practice new tasks, quantum computing, quantum communicating, and so on, which cannot be achieved by the conventional technology, and also has significance that we can deeply reconsider about quantum mechanics itself. Especially, 'quantum entangled states', of which theoretical basis was rigidly built by J. Bell at first, is the fundamentals of 'quantum teleportation' produced by C. H. Bennett and rest in 1993[1].

1.2 Squeezed State

In quantum mechanics, variables X and P ($[X, P] = i$) have finite variances to satisfy the uncertainty principle. For a coherent state which is a state of an ideal laser light, both the variance of X and P equal 1/2, which is called SNL (Shot Noise Level).

The variance of X can be *squeezed* to be lower than SNL by spreading that of P to conserve uncertainty principle (the role of X and P can be exchanged), then we call such a state squeezed state, which was theoretically proposed by H.P.Yuen in 1976[2]. At first it was expected that we can perform sensitive measurements or correct communications by using squeezed states with lower noise, and researches about it have proceeded. In the field of quantum information, squeezed states are considered important also, because we can produce entangled states with two squeezed lights passing through a beam splitter[3].

In experiment, a squeezed light is generated by $\chi^{(2)}$ or $\chi^{(3)}$ non-linear optical process. A squeezed state had been first generated by R.E.Slusher[4], by using Na atoms as a non-linear optical device. After that, researchers have made much effort to gain high level squeezing, and now they achieve 5.3

dB squeezing by OPA(Optical Parametric Amplifier)[10, 11], or 10 dB squeezing by OPO (Optical Parametric Oscillator).

1.3 Quantum Teleportation

Quantum teleportation is an information communication protocol by using quantum entangled states, and it was theoretically produced by C. H. Bennett and rest at 1993[1].

Alice (the sender) and Bob (the receiver) share an entangled state, and Alice conduct a Bell measurement with the state of Victor (the client). Since a Bell measurement is the projection to entangled basis, Victor and Bob are entangled by the Bell measurement. Then, Victor's original state is broken by the measurement, and it can be regenerated at Bob's space by a unitary transformation corresponding to the result of the Bell measurement, which needs to be informed from Alice by a classical way. An important requirement is that Alice cannot gain any information about Victor's original state in this process because the result of the Bell measurement does not depend on the state, and a copy of the state cannot be generated because of No-cloning theorem[5].

Bennett put forward quantum teleportation by qubit firstly, and after that, experiments of quantum teleportation had been performed with quantum states of discrete variables, i.e., experiments with polarization of lights by Zeiringer and rest[6]. However, the Bell measurement was not completely performed in their experiments because it was difficult to generate a single photon state and they could only provide one measurement base of four Bell basis[3].

Then, a proposal to use continuous variables of lights (Quadrature) was presented by Braunstein and Kimble[7], and it was firstly performed with squeezed lights in 1998 by Furusawa and rest[8].

1.4 Noise in Quantum channels

When we analyze quantum teleportation, we should consider the effect of noises at quantum channels, caused by interaction with environment. Noises usually cause decoherence, impair entanglement and decrease fidelity of teleportation. Especially in the case of continuous variables, it is an advantage that operation of quantum channels are simply practiced, but it is disadvantage that the fidelity of teleportation is easily decreased by quantum noises.

On teleportation for a one-half spin state, the effect of environment is estimated by representation of damping Bloch vector[13], and the master equation of the Lindblad type[14]. For general qubit teleportation, it has been shown that there are cases in which fidelity is enhanced depending on the result of the Bell measurement, even if the entangled state is dissipated [17].

For continuous variables, the effect of thermal noises on squeezed lights has been studied by the master equation of the Lindblad type[18] or by taking a Gaussian noise[19] in a transmission channel, and by putting arbitrary dissipative parameters to the variances of the Wigner function[20]. K. Park and H. Jeong had compared which of the quantum teleportation using a coherent state and a number state is more robust against the noises in the quantum channel

These days, a "hybrid" teleportation is demonstrated by Furusawa et al., in which the sending state is qubit and an entangled state is formed by continuous variable state[16], and they gained a high fidelity $F \approx 0.88$.

We should also consider measurement errors. A Bell measurement is usually formulated as the von-Neumann measurement for the Bell basis, which is an orthogonal measurement. However, in practice, there are measurement errors and they are represented by a nonorthogonal measurement[21, 22], especially for continuous variables.

1.5 Quantum Dissipative System and NETFD

Time-evolution of quantum dissipative systems can be described by a quantum master equation for the density operator[24]. In the case of no dissipation, this equation comes to the Liouville von-Noemann equation, but otherwise, it consists of intricate commutation relations of operators. In order to solve a master equation, usually we map it to a partial differential equation of c-number function.

Non-Equilibrium Thermo Field Dynamics (NETFD) was constructed to calculate quantum dissipative systems by canonical operator algebra[27, 28, 29, 30]. In formulation of NETFD, by introducing two kinds of operators (tilde-operator and non-tilde-operator) and a ket vacuum corresponding to a density operator, we can cancel the intricacy of commutation relation in the quantum master equation. Quantum master equations for density operators come to Schrödinger equations for ket vacua in NETFD, and the time-evolution is driven by hat-Hamiltonian $\hat{H}$. By defining annihilation and creation operators of the ket vacuum, the formalism based on the canonical operator algebra can be structured. NETFD provides us simpler and clearer procedures compared with projecting to the partial differential equation.

Quantum field theory is constructed upon the stabilities of a vacuum and one-particle state. For non-equilibrium and dissipative systems, both a vacuum and one-particle state are unstable. In order to construct quantum field theory describing this situation by the canonical operator formalism, we should define annihilation and creation operators for the vacuum at each time, and these are given by a time-dependent Bogoliubov transformation in NETFD. With NETFD, we obtain a viewpoint that the time-evolution of dissipative quantum systems is controlled by a condensation of particle-pairs into the thermal vacuum.

In NETFD, semi-free $\hat{H}$ can be derived in axiom, and the dissipative term $\hat{\Pi}$ in the case of semi-free has been applied to the case of the squeezing process[31], by supposing the properties of the thermal bath is the same. In this thesis, we show the conditions in which this assumption is established, by deriving the quantum master equation of the squeezing process in the thermal bath. Then, we analyze quantum teleportation for continuous variables with squeezed states generated under dissipative environment, within NETFD.

1.6 Structure of this Thesis

In this thesis we theoretically analyze quantum teleportation with continuous variables. Especially, we consider influences of dissipation in the squeezing process in non-linear optical devises and the measurement errors at the Bell measurement, then we quantify the fidelity of teleportation and the probability density function with the result of the Bell measurement.

In chapter 2, we explain the principle of quantum teleportation for continuous variables. In chapter 3, we explain the basis of NETFD, and derive a squeezed vacuum generated in dissipative environment, and we show derivation of the dissipative term $\hat{\Pi}$ in the squeezing process. In chapter 4, we formulate quantum teleportation for continuous variables via squeezed vacua generated under dissipative environment and via the Bell measurement with a measurement error, within NETFD, and analyze it. In chapter 5, we summarize the conclusions of this thesis. In Appendix, we show detailed calculations and background theorems.

Chapter 2

Quantum Teleportation for Continuous Variables

In this thesis, we use $|\,)$ as a ket state within quantum mechanics, and $|\,\rangle$ is a ket vacuum within NETFD, which we introduce in chapter 3. In this chapter, we describe quantum teleportation by the basic formalism of quantum mechanics.

a and $a^\dagger$ are the annihilation and creation operators of photon, satisfying

$$[a, a^\dagger] = 1, \tag{2.0.1}$$

and $|0)$ is the vacuum annihilated by a

$$a|0) = 0. \tag{2.0.2}$$

The coordinate operator X and the momentum operator P are defined by

$$\begin{pmatrix} X \\ P \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix}, \tag{2.0.3}$$

and satisfy the commutation relation

$$[X, P] = i. \tag{2.0.4}$$

2.1 Squeezed Vacuum

A squeezed vacuum is defined by

$$|\alpha, r) = D(\alpha)S(r)|0) \tag{2.1.1}$$

where $D(\alpha)$ and $S(r)$ are a displacement operator and a squeezing operator defined by

$$D(\alpha) = \mathrm{e}^{\alpha a^\dagger - \alpha^* a}, \tag{2.1.2}$$

$$S(r) = \mathrm{e}^{\frac{r}{2}(a^{\dagger 2} - a^2)}, \tag{2.1.3}$$

α is a complex variable representing the displacement in quadrature and r is a real variable called squeezing parameter. The expectation values and the variances of X and P for $|\alpha, r)$ are estimated as

$$\langle X \rangle = \sqrt{2}\alpha', \tag{2.1.4}$$

$$\langle P \rangle = \sqrt{2}\alpha'', \tag{2.1.5}$$

and

$$\langle (\Delta X)^2 \rangle = \langle X^2 \rangle - \langle X \rangle^2 = \frac{\mathrm{e}^r}{2}, \tag{2.1.6}$$

$$\langle (\Delta P)^2 \rangle = \langle P^2 \rangle - \langle P \rangle^2 = \frac{\mathrm{e}^{-r}}{2}. \tag{2.1.7}$$

where real variable α' and α'' are given by $\alpha = \alpha' + i\alpha''$. The variance of X is squeezed for $r > 0$ and that of P is spread, so that the uncertainly principle is satisfied

$$\langle (\Delta X)^2 \rangle \langle (\Delta P)^2 \rangle = \frac{1}{4}. \tag{2.1.8}$$

In the case of $r \to -\infty$, from (2.1.6), we see the variance of X is zero, then the squeezed vacuum reaches an eigenstate of X and the eigenvalue is $x = \sqrt{2}\alpha'$, i.e.

$$\lim_{r \to -\infty} |\alpha, r) = |x), \tag{2.1.9}$$

$$X|x) = x|x). \tag{2.1.10}$$

2.2 Entangled state

We consider a composite system AB, and prepare the squeezed vacua $|r)_A| - r)_B$, where

$$|r)_A = S_A(r)|0)_A, \tag{2.2.1}$$

$$| - r)_B = S_B(-r)|0)_B. \tag{2.2.2}$$

For this vacua, the variance of X_A and P_B are squeezed.

Here, we introduce a unitary operator V_{AB} defined by

$$V_{AB} = \mathrm{e}^{i\frac{\pi}{4}(a_A a_B^\dagger - a_A^\dagger a_B)}. \tag{2.2.3}$$

Applying V_{AB} to the squeezed vacua $|r)_A| - r)_B$, we obtain

$$|r, -r)_{AB} = V_{AB}|r)_A| - r)_B \tag{2.2.4}$$

$$= \frac{1}{\cosh r}\mathrm{e}^{\tanh r a_A^\dagger a_B^\dagger}|0)_A|0)_B, \tag{2.2.5}$$

where the second equality is from Appendix(A.1.22). The variances of $X_A - X_B$ and $P_A + P_B$ for $|r, -r)_{AB}$ satisfy

$$\langle \{\Delta(X_A - X_B)\}^2 \rangle + \langle \{\Delta(P_A + P_B)\}^2 \rangle = 2\mathrm{e}^{-2r}. \tag{2.2.6}$$

If A and B are separable, this function is bounded from below by 2[23] (see Appendix (A.2.7)), therefore, we see $|r, -r)_{AB}$ is entangled for $r > 0$

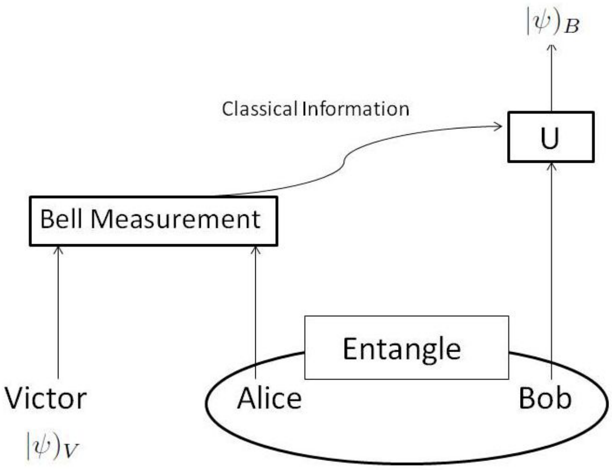

Figure 2.1: Schematic drawings of quantum teleportation. Victor sends a message $|\psi)_V$ to Bob.

2.3 Bell Measurement

The operators $X_V - X_A$ and $P_V + P_A$ satisfy the commutation relation

$$[X_V - X_A, P_V + P_A] = 0, \tag{2.3.1}$$

therefore, $X_V - X_A$ and $P_V + P_A$ have simultaneous eigenstates $\{|\beta)_{VA}\}$, i.e.

$$(X_V - X_A)|\beta)_{VA} = x_-|\beta)_{VA}, \tag{2.3.2}$$

$$(P_V + P_A)|\beta)_{VA} = p_+|\beta)_{VA}, \tag{2.3.3}$$

where $\beta = x_- + ip_+$. $|\beta)_{VA}$ can be represented by

$$|\beta)_{VA} = \frac{1}{\sqrt{2\pi}} \int dx\, \mathrm{e}^{ip_+x} |x + x_-)_V |x)_A. \tag{2.3.4}$$

The projective measurement for basis $\{|\beta)_{VA}\}$ is called Bell Measurement for continuous variables.

2.4 Teleportation

In this section, we show the protocol of quantum teleportation with continuous variables. Victor is the originator of a message $|\psi)_V$, and Bob is the receiver. Alice, the secretary of Victor, shares an entangled state with Bob, and she conducts the Bell measurement with Victor (fig.2.1).

In order to find the principle of quantum teleportation, we consider the case of maximum squeezing $r \to \infty$. From (A.1.27), the entangled vacua between Alice and Bob are written by

$$\lim_{r\to\infty} |r, -r)_{AB} = \frac{1}{c} \int dx |x)_A |x)_B, \tag{2.4.1}$$

where c is the normalizing factor given by $c = \delta(0)$ at $r \to \infty$, and this state is maximally entangled.

$|\psi)_V$ is represented by the basis $\{|x)_V\}$ as

$$|\psi)_V = \int dx\psi(x)|x)_V \tag{2.4.2}$$

where

$$\psi(x) = {}_V(x|\psi)_V. \tag{2.4.3}$$

The global state after entangling Alice and Bob is written by

$$|in)_{VAB} = \frac{1}{c}\int dx\psi(x)|x)_V \int dx'|x')_A|x')_B. \tag{2.4.4}$$

By supposing Alice detects an outcome β by the measurement, Bob's state right after the Bell measurement is reduced to

$$\begin{aligned}
|\psi;\beta)_B &= \frac{1}{\sqrt{w(\beta)}}{}_{VA}(\beta|in)_{VAB} \\
&= \frac{1}{\sqrt{w(\beta)}c}{}_{VA}(\beta|\psi)_V \int dx'|x')_A|x')_B \\
&= \frac{1}{\sqrt{2\pi w(\beta)}c}\int dx \int dx' \int dx'' \mathrm{e}^{-ip_+x''}{}_V(x''+x_-|_A(x''|\psi(x)|x)_V|x')_A|x')_B \\
&= \frac{1}{\sqrt{2\pi w(\beta)}c}\int dx\ \mathrm{e}^{-ip_+x}\psi(x+x_-)|x)_B,
\end{aligned} \tag{2.4.5}$$

where $w(\beta)$ is the probability density to gain the outcome β by the measurement and estimated by

$$\begin{aligned}
w(\beta) &= {}_{VAB}(in|\beta)_{VAVA}(\beta|in)_{VAB} \\
&= \frac{1}{2\pi c^2}\int dx \int dx'\ \mathrm{e}^{-ip_+(x-x')}\psi^*(x'+x_-)\psi(x+x_-)_B(x'|x)_B \\
&= \frac{1}{2\pi c^2}\int dx|\psi(x+x_-)|^2 \\
&= \frac{1}{2\pi c^2},
\end{aligned} \tag{2.4.6}$$

therefore,

$$|\psi;\beta)_B = \int dx\ \mathrm{e}^{-ip_+x}\psi(x+x_-)|x)_B. \tag{2.4.7}$$

Here, Alice cannot gain any information about the original state $|\psi)$ from the Bell measurement, because, from (2.4.6) the probability to detect β does not depend on $\psi(x)$. After the measurement, Alice informs Bob about the outcome β by a classical way, and Bob operates the displacement $D(\beta)$, then gains the output

$$\begin{aligned}
|\psi_{out})_B &= D_B(\beta)|\psi')_B \\
&= \mathrm{e}^{\frac{1}{2}p_+x_-}\int dx\ \mathrm{e}^{-ip_+x}\psi(x+x_-)\mathrm{e}^{-x_-P_B}\mathrm{e}^{p_+X_B}|x)_B \\
&= \mathrm{e}^{\frac{1}{2}p_+x_-}\int dx\ \psi(x+x_-)|x+x_-)_B \\
&= \mathrm{e}^{\frac{1}{2}p_+x_-}|\psi)_B.
\end{aligned} \tag{2.4.8}$$

At the second equality, we used

$$\begin{aligned} D_J(\beta) &= \mathrm{e}^{ip_+ X_J - ix_- P_J} \\ &= \mathrm{e}^{x_- p_+/2} \mathrm{e}^{-ix_- P_J} \mathrm{e}^{ip_+ X_J}. \end{aligned} \tag{2.4.9}$$

Neglecting the global phase, we see the output is the same as Victor's original state.

In practice, the output is not exactly the same as $|\psi\rangle$ because squeezing parameter r is limited and there are noises in quantum channels.

2.5 Experiment

In experiments, a squeezed vacuum is generated by a non-linear optical process. When a pump light enters an optical cavity in which the second- harmonic generation is effective, the Hamiltonian of the second-harmonic mode in the cavity is given by

$$H_t = \omega a^\dagger a + \frac{\chi}{2}(a^{\dagger 2} \mathrm{e}^{-i2\omega t} - a^2 \mathrm{e}^{i2\omega t}). \tag{2.5.1}$$

The second term of (2.5.1) is th squeezing term, and the parameter χ is decided by[3]

$$\chi = \frac{\omega}{2n} |\chi^{(2)} E_p|, \tag{2.5.2}$$

where n is the index of refraction of the device, $\chi^{(2)}$ is the second non-linear optical constant, and $|E_p|$ is the intensity of the pump light. Through the time-evolution in time t_0 by this Hamiltonian, a squeezed vacuum with squeezing parameter $r = \chi t_0$ is generated.

The operator V_{AB} is performed by a half beam splitter (HBS), therefore, the entangled vacua are generated by passing squeezed vacua to HBS. The Bell measurement is taken by homodyne measurements, and the outcomes are transfered as electric current. The displacement is implemented by a coherent light and a beam splitter with high transmittance.

The first experiment of quantum teleportation for continuous values with squeezed lights is perfomed by Furusawa in 1998 (fig.2.2)[8].

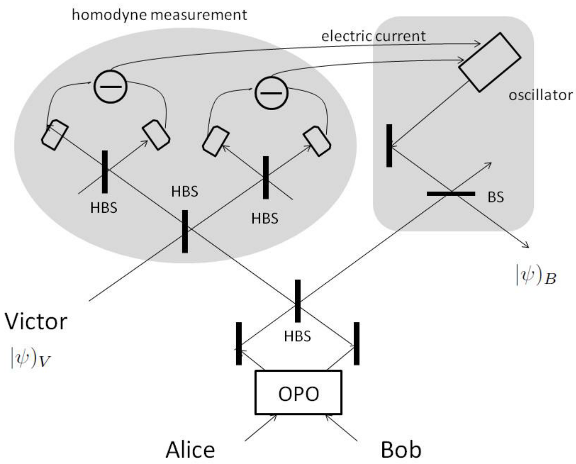

Figure 2.2: Experiment of quantum teleportation by Furusawa et al[8]. Squeezed vacua are generated by optical parametric oscillator (OPO), and they are entangled through HBS.

Chapter 3

NETFD

3.1 Formulation of NETFD

3.1.1 Basics of NETFD

An operator A in NETFD is accompanied by its tilde conjugate operator $\tilde{A}$. The tilde conjugate $\tilde{}$ for bosonic operators A and B is defined by

$$(AB)^{\sim} = \tilde{A}\tilde{B}, \tag{3.1.1}$$

$$(c_1 A + c_2 B)^{\sim} = c_1^* \tilde{A} + c_2^* \tilde{B}, \tag{3.1.2}$$

$$(\tilde{A})^{\sim} = A, \tag{3.1.3}$$

$$(A^\dagger)^{\sim} = \tilde{A}^\dagger. \tag{3.1.4}$$

Non-tilde and tilde operators are commutative at equal time

$$[A, \tilde{B}] = 0, \tag{3.1.5}$$

and satisfy thermal state condition for the thermal bra-vacuum $\langle 1|$

$$\langle 1|A^\dagger = \langle 1|\tilde{A}. \tag{3.1.6}$$

The ket-vacuum $|0(t)\rangle$ describes a state of a quantum system at time t including dissipative situations within NETFD. The time-evolution of $|0(t)\rangle$ is described by the dissipative Schrödinger equation ($\hbar = 1$)

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}|0(t)\rangle, \tag{3.1.7}$$

where the hat-Hamiltonian $\hat{H}$ is a time-evolution generator satisfying

$$(i\hat{H})^{\sim} = i\hat{H}, \tag{3.1.8}$$

$$\langle 1|\hat{H} = 0. \tag{3.1.9}$$

An operator satisfying the characteristic (3.1.8) is called *tildian* operator. The formal solution of (3.1.7) is

$$|0(t)\rangle = e^{-i\hat{H}t}|0\rangle, \tag{3.1.10}$$

where $|0\rangle$ is the ket-vacuum at the initial time. If the bra-vacuum $\langle 0|$ and the ket-vacuum $\|0\rangle$ are *tilde invariant*

$$\langle 1|^{\sim} = \langle 1|, \quad |0\rangle^{\sim} = |0\rangle, \tag{3.1.11}$$

and are normalized as

$$\langle 1|0\rangle = 1, \tag{3.1.12}$$

they extend to all time,i.e., (3.1.10)

$$\begin{aligned} |0(t)\rangle^{\sim} &= (\mathrm{e}^{-i\hat{H}t}|0\rangle)^{\sim} \\ &= \mathrm{e}^{-(i\hat{H})^{\sim}t}|0\rangle \\ &= \mathrm{e}^{-i\hat{H}t}|0\rangle \\ &= |0(t)\rangle \end{aligned} \tag{3.1.13}$$

and

$$\begin{aligned} \langle 1|0(t)\rangle &= \langle 1|\mathrm{e}^{-i\hat{H}t}|0\rangle \\ &= \langle 1|0\rangle \\ &= 1., \end{aligned} \tag{3.1.14}$$

where we used (3.1.8) and (3.1.9). The relation (3.1.14) means the conservation of probability.

The expectation value of an operator A at time t is given by

$$\begin{aligned} \langle A\rangle_t &\equiv \langle 1|A|0(t)\rangle \\ &= \langle 1|A\mathrm{e}^{-i\hat{H}t}|0\rangle \\ &= \langle 1|\mathrm{e}^{i\hat{H}t}A\mathrm{e}^{-i\hat{H}t}|0\rangle \\ &= \langle 1|A(t)|0\rangle, \end{aligned} \tag{3.1.15}$$

where we define the Heisenberg operator $A(t)$ as

$$A(t) = \mathrm{e}^{i\hat{H}t}A\mathrm{e}^{-i\hat{H}t}. \tag{3.1.16}$$

The time-evolution of $A(t)$ is described by the dissipative Heisenberg equation

$$\frac{d}{dt}A(t) = i[\hat{H}(t), A(t)]. \tag{3.1.17}$$

3.1.2 Semi-Free Hat-Hmiltonian

In this section, we derive the Hat-Hamiltonian of semi-free field. The semi-free hat-Hamiltonian is required as bilinear in $(a,\ \tilde{a},\ a^\dagger,\ \tilde{a}^\dagger)$ and invariant under the phase transition $a \to ae^{i\theta}$. Then we can write it as

$$\hat{H} = g_1 a^\dagger a + g_2 \tilde{a}^\dagger \tilde{a} + g_3 a\tilde{a} + g_4 a^\dagger \tilde{a}^\dagger + g_0, \tag{3.1.18}$$

where $g_j (j = 0, 1, 2, 3, 4)$ are complex c-numbers, and the operators a, $\tilde{a}^\dagger$,ets. satisfy the commutative relation

$$\left[\ a, a^\dagger\right] = 1, \left[\ \tilde{a}, \tilde{a}^\dagger\right] = 1. \tag{3.1.19}$$

From (3.1.8) and (3.1.18), we have

$$\begin{aligned} 0 &= i\hat{H} - (i\hat{H})^\sim \\ &= i(g_1 a^\dagger a + g_2 \tilde{a}^\dagger \tilde{a} + g_3 a\tilde{a} + g_4 a^\dagger \tilde{a}^\dagger + g_0) \\ &\quad +i(g_1^* \tilde{a}^\dagger \tilde{a} + g_2^* a^\dagger a + g_3^* \tilde{a} a + g_4^* \tilde{a}^\dagger a^\dagger + g_0^*) \\ &= i(g_1 + g_3^*) a^\dagger a + i(g_2 + g_1^*) \tilde{a}^\dagger \tilde{a} \\ &\quad +i(g_3 + g_3^*) a\tilde{a} + i(g_4 + g_4^*) a^\dagger \tilde{a}^\dagger + g_0 + g_0^*. \end{aligned} \tag{3.1.20}$$

Each term in the right-hand side of (3.1.20) is zero, so that the coefficients satisfy

$$\Re\mathrm{e}\ g_1 = -\Re\mathrm{e}\ g_2 \equiv \omega, \tag{3.1.21}$$

$$\Im\mathrm{m}\ g_1 = \Im\mathrm{m}\ g_2 = c_1, \tag{3.1.22}$$

$$\Re\mathrm{e}\ g_3 = \Re\mathrm{e}\ g_4 = \Re\mathrm{e}\ g_0 = 0, \tag{3.1.23}$$

$$\Im\mathrm{m}\ g_3 = c_2, \tag{3.1.24}$$

$$\Im\mathrm{m}\ g_4 = c_3, \tag{3.1.25}$$

$$\Im\mathrm{m}\ g_0 = c_4, \tag{3.1.26}$$

then $\hat{H}$ can be rewritten as

$$\begin{aligned} \hat{H} &= (\omega + ic_1) a^\dagger a + (-\omega + ic_1) \tilde{a}^\dagger \tilde{a} + ic_2 a\tilde{a} + ic_3 a^\dagger \tilde{a}^\dagger + ic_4 \\ &= \omega(a^\dagger a + \tilde{a}^\dagger \tilde{a}) + i\hat{\Pi}, \end{aligned} \tag{3.1.27}$$

where

$$\hat{\Pi} = c_1(a^\dagger a + \tilde{a}^\dagger \tilde{a}) + c_2 a\tilde{a} + c_3 a^\dagger \tilde{a}^\dagger + c_4. \tag{3.1.28}$$

Furthermore, from (3.1.9), we have

$$\begin{aligned}
0 &= \langle 1|\hat{H} \\
&= \langle 1|\{\omega(a^\dagger a + \tilde{a}^\dagger \tilde{a}) + c_1(a^\dagger a + \tilde{a}^\dagger \tilde{a}) + c_2 a\tilde{a} + c_3 a^\dagger \tilde{a}^\dagger + c_4\} \\
&= \langle 1|\{ic_1(\tilde{a}a + a\tilde{a}) + ic_2 a\tilde{a} + ic_3\tilde{a}\tilde{a}^\dagger + ic_4\} \\
&= \langle 1|\{i(2c_1 + c_2)a\tilde{a} + ic_3(a\tilde{a} + 1) + ic_4\} \\
&= i\langle 1|\{(2c_1 + c_2 + c_3)a\tilde{a} + c_3 + c_4\}
\end{aligned} \tag{3.1.29}$$

, where we used (3.1.6). (3.1.29) shows

$$2c_1 + c_2 + c_3 = 0, \qquad c_3 + c_4 = 0, \tag{3.1.30}$$

therefore the number of unknown quantities in (3.1.28) are reduced, and $\hat{\Pi}$ is written as

$$\hat{\Pi} = c_1(a^\dagger a + \tilde{a}^\dagger \tilde{a}) + c_2 a\tilde{a} - (2c_1 + c_2)a^\dagger \tilde{a}^\dagger + (2c_1 + c_2) \tag{3.1.31}$$

.

Here, let us consider the Heisenberg operators

$$a(t) = \mathrm{e}^{i\hat{H}t} a \mathrm{e}^{-i\hat{H}t}, \tag{3.1.32}$$

$$a^\ddagger(t) = \mathrm{e}^{i\hat{H}t} a^\dagger \mathrm{e}^{-i\hat{H}t}. \tag{3.1.33}$$

the equations of motion of them are

$$\begin{aligned}
\frac{d}{dt}a(t) &= i[\hat{H}(t), a(t)] \\
&= -i\omega a(t) + c_1 a(t) + (2c_1 + c_2)a^\ddagger(t),
\end{aligned} \tag{3.1.34}$$

$$\begin{aligned}
\frac{d}{dt}a^\ddagger(t) &= i[\hat{H}(t), a^\ddagger(t)] \\
&= i\omega a^\ddagger(t) - c_1 a^\ddagger(t) - c_2 a(t),
\end{aligned} \tag{3.1.35}$$

and they satisfy

$$\begin{aligned}
\langle 1|a^\ddagger(t) &= \langle 1|\mathrm{e}^{i\hat{H}t} a^\dagger \mathrm{e}^{-i\hat{H}t} \\
&= \langle 1|a^\dagger \mathrm{e}^{-i\hat{H}t} \\
&= \langle 1|\tilde{a}\mathrm{e}^{-i\hat{H}t} \\
&= \langle 1|\mathrm{e}^{i\hat{H}t}\tilde{a}\mathrm{e}^{-i\hat{H}t} \\
&= \langle 1|\tilde{a}(t).
\end{aligned} \tag{3.1.36}$$

Note that $\hat{H}$ is not Hermitian and $a^\ddagger(t)$ is not the Hermite conjugate of $a(t)$. In following, we will use $\dagger$ instead of $\ddagger$ for simplicity, unless it is confusing.

The equation of motion of the bra vector $\langle 1|a^\dagger(t)a(t)$ is given by

$$\begin{aligned}
\frac{d}{dt}\langle 1|a^\dagger(t)a(t) &= \langle 1|\left\{\frac{da^\dagger(t)}{dt}a(t) + a^\dagger(t)\frac{da(t)}{dt}\right\} \\
&= \langle 1|\left\{\left[i(\omega + c_1)a^\dagger(t) - c_2\tilde{a}(t)\right]a(t) + a^\dagger(t)\left[-i(\omega + -c_1)a^\dagger(t) + (2c_1 + c_2)\tilde{a}(t)\right]\right\} \\
&= \langle 1|\left\{-c_2\tilde{a}(t)a(t) - (2c_1 + c_2)a^\dagger(t)\tilde{a}^\dagger(t)\right\} \\
&= \langle 1|\left\{-c_2 a^\dagger(t)a(t) - (2c_1 + c_2)a(t)a^\dagger(t)\right\} \\
&= \langle 1|\left\{-2(c_1 + c_2)a^\dagger(t)a(t) - (2c_1 + c_2)\right\} \\
&= \langle 1|\left\{-2\kappa a^\dagger(t)a(t) + i\Sigma^<\right\}, \qquad (3.1.37)
\end{aligned}$$

where κ and $\Sigma^<$ are defined as

$$\kappa = c_1 + c_2, \qquad (3.1.38)$$

$$i\Sigma^< = -(2c_1 + c_2) \qquad (3.1.39)$$

. Applying the thermal ket-vacuum $|0\rangle$ to (3.1.37), we obtain the equation of motion for the expectation value of the particle number (3.1.37) deals

$$n(t) = \langle 1|a^\dagger(t)a(t)|0\rangle, \qquad (3.1.40)$$

as

$$\begin{aligned}
\frac{d}{dt}n(t) &= \langle 1|\left\{-2\kappa a^\dagger(t)a(t) + i\Sigma^<\right\}|0\rangle \\
&= -2\kappa n(t) + i\Sigma^<. \qquad (3.1.41)
\end{aligned}$$

Assuming the system will reach to the thermal equilibrium state at $t \to \infty$, then $n(t \to \infty) = \bar{n}$, we have from (3.1.41)

$$i\Sigma^< = 2\kappa\bar{n}. \qquad (3.1.42)$$

Therefore, (3.1.41) is reduced to

$$\frac{d}{dt}n(t) = -2\kappa\left[n(t) - \bar{n}\right]. \qquad (3.1.43)$$

Solving this equation, we obtain

$$\begin{aligned}
n(t) &= \left[n(0) - \bar{n}\right]\mathrm{e}^{-2\kappa} + \bar{n} \\
&= n(0) - \left[n(0) - \bar{n}\right]\left(1 - \mathrm{e}^{-2\kappa t}\right). \qquad (3.1.44)
\end{aligned}$$

From (3.1.38), (3.1.39) and (3.1.42), c_1 and c_2 can be written as

$$c_1 = -\kappa - i\Sigma^< = -(1 + 2\bar{n})\kappa, \qquad (3.1.45)$$

$$c_2 = 2\kappa + i\Sigma^< = 2(1 + \bar{n})\kappa, \qquad (3.1.46)$$

then we finally gain the general form of semi-free hat-Hamiltonian

$$\hat{H} = \hat{H}_0 + i\hat{\Pi}, \tag{3.1.47}$$

$$\hat{H}_0 = \omega(a^\dagger a - \tilde{a}^\dagger \tilde{a}), \tag{3.1.48}$$

$$\hat{\Pi} = -\kappa \left[(1+2\bar{n})(a^\dagger a + \tilde{a}^\dagger \tilde{a}) - 2(1+\bar{n}) a\tilde{a} - 2\bar{n} a^\dagger \tilde{a}^\dagger \right] - 2\kappa\bar{n}. \tag{3.1.49}$$

3.1.3 The Principle of Correspondence

The ket vacuum and the bra vacuum correspond to the density operator and trace;

$$\rho(t) \qquad \longleftrightarrow \qquad |0(t)\rangle, \tag{3.1.50}$$

$$\mathrm{tr} \qquad \longleftrightarrow \qquad \langle 1|. \tag{3.1.51}$$

Tilde operators corresponds to Hermite conjugate operators taken on right side;

$$A\rho(t)B^\dagger \qquad \longleftrightarrow \qquad A\tilde{B}|0(t)\rangle \tag{3.1.52}$$

$$\left(\; |A\rho B^\dagger\rangle = A|\rho B^\dagger\rangle = A\tilde{B}|\rho\rangle \; \right). \tag{3.1.53}$$

The hat-Hamiltonian $\hat{H}$ corresponds to the super-operator L, for example, in the case of the master equation

$$LX = \omega_S[a^\dagger a, X] + i\left\{\kappa[aX, a^\dagger] + [a, Xa^\dagger]\right\} + i2\kappa\bar{n}[a, [X, a^\dagger]\,]. \tag{3.1.54}$$

Then the dissipative Schrödinger equation has the similar meaning of the master equation for the density operator.

$$\frac{\partial}{\partial t}\rho(t) = -iL\rho(t) \qquad \longleftrightarrow \qquad \frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}|0(t)\rangle. \tag{3.1.55}$$

3.2 Creation and Annihilation Operators and Bogoliubov Transformation

3.2.1 Creation Operator

From the thermal state condition for the bra-vacuum (3.1.6), we have

$$\langle 1|a^\dagger = \langle 1|\tilde{a}. \tag{3.2.1}$$

Then let us consider an operator $\gamma^{\mathring{\dagger}}$ as

$$\gamma^{\mathring{\dagger}} \equiv a^\dagger - \tilde{a}, \tag{3.2.2}$$

so that this operator satisfies

$$\langle 1|\gamma^{\mathring{\dagger}} = 0. \tag{3.2.3}$$

Because $\gamma^{\mathring{\dagger}}$ annihilate the bra-vacuum, it is a creation operator.

3.2.2 Thermal Vacuum

Assuming the initial ket-vacuum is specified by

$$a|0\rangle = f\tilde{a}^{\dagger}|0\rangle, \tag{3.2.4}$$

with a real number f. Here, we define an operator γ as

$$\gamma = c_1(a - f\tilde{a}^{\dagger}), \tag{3.2.5}$$

then it satisfies

$$\gamma|0\rangle = 0, \tag{3.2.6}$$

and commutative relation with $\gamma^{\ddagger}$ and $\tilde{\gamma}^{\ddagger}$

$$\begin{aligned} [\gamma, \gamma^{\ddagger}] &= c_1[a - f\tilde{a}^{\dagger}, a^{\dagger} - \tilde{a}] \\ &= c_1(1-f), \end{aligned} \tag{3.2.7}$$

$$\begin{aligned} [\gamma, \tilde{\gamma}^{\ddagger}] &= c_1[a - f\tilde{a}, \tilde{a}^{\dagger} - a] \\ &= 0. \end{aligned} \tag{3.2.8}$$

Here, we take c_1

$$c_1 = \frac{1}{1-f}, \tag{3.2.9}$$

so that γ and $\gamma^{\ddagger}$ satisfies the canonical commutative relation, and γ is the annihilation operator for the thermal vacuum $|0\rangle$. The Bogoliubov transformation in this case is represented by

$$\begin{pmatrix} \gamma \\ \tilde{\gamma}^{\ddagger} \end{pmatrix} = \begin{pmatrix} \frac{1}{1-f} & -\frac{f}{1-f} \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ \tilde{a}^{\dagger} \end{pmatrix}, \tag{3.2.10}$$

and the inverse transformation is

$$\begin{pmatrix} a \\ \tilde{a}^{\dagger} \end{pmatrix} = \begin{pmatrix} 1 & \frac{f}{1-f} \\ 1 & \frac{1}{1-f} \end{pmatrix} \begin{pmatrix} \gamma \\ \tilde{\gamma}^{\ddagger} \end{pmatrix}. \tag{3.2.11}$$

The expectation for the particle number is

$$\begin{aligned} \langle 1|a^{\dagger}a|0\rangle &= \langle 1| \left(\tilde{\gamma} + \frac{1}{1-f}\gamma^{\ddagger}\right) \left(\gamma + \frac{f}{1-f}\tilde{\gamma}^{\ddagger}\right)|0\rangle \\ &= \frac{f}{1-f} \qquad (3.2.12) \\ &\equiv n, \qquad (3.2.13) \end{aligned}$$

then (3.2.10) can be rewritten by using n as

$$\begin{pmatrix} \gamma \\ \tilde{\gamma}^{\ddagger} \end{pmatrix} = \begin{pmatrix} 1+n & -n \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ \tilde{a}^{\dagger} \end{pmatrix}. \tag{3.2.14}$$

3.2.3 Squeezed Vacuum

Let us consider the squeezed vacuum

$$|0_r\rangle = \hat{S}(r)|0\rangle, \tag{3.2.15}$$

where $\hat{S}(r)$ is defined as

$$\hat{S}(r) = S(r)\tilde{S}(r), \tag{3.2.16}$$

$$S(r) = \mathrm{e}^{-\frac{r}{2}\left(a^2 - a^{\dagger 2}\right)}, \tag{3.2.17}$$

and $|0\rangle$ satisfies the thermal state condition (3.2.4).

Here, we consider the transformation given by

$$a(r) = \hat{S}(r) a \hat{S}(r)^{-1}, \tag{3.2.18}$$

$$a^\dagger(r) = \hat{S}(r) a^\dagger \hat{S}(r)^{-1}. \tag{3.2.19}$$

Taking the first derivatives of $a(r)$ and $a^\dagger(r)$, we have

$$\begin{aligned} \frac{d}{dr}a(r) &= \hat{S}(r)\left[-\frac{1}{2}(a^2 - a^{\dagger 2}), a\right]\hat{S}(r)^{-1} \\ &= \hat{S}(r)(-a^\dagger)\hat{S}(r)^{-1} \\ &= -a^\dagger(x\chi), \end{aligned} \tag{3.2.20}$$

and

$$\begin{aligned} \frac{d}{dr}a^\dagger(r) &= \hat{S}(r)\left[-\frac{1}{2}(a^2 - a^{\dagger 2}), a^\dagger\right]\hat{S}(r)^{-1} \\ &= \hat{S}(r)(-a)\hat{S}(r)^{-1} \\ &= -a(r), \end{aligned} \tag{3.2.21}$$

then they can be put up in one equation such as

$$\frac{d}{dx}\begin{pmatrix} a(r) \\ a^\dagger(r) \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a(r) \\ a^\dagger(r) \end{pmatrix}. \tag{3.2.22}$$

Solving this differential equation with the initial condition $a(0) = a$ and $a^\dagger(0) = a^\dagger$, we have

$$\begin{aligned} \begin{pmatrix} a(r) \\ a^\dagger(r) \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \mathrm{e}^r & 0 \\ 0 & \mathrm{e}^{-r} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} a \\ a^\dagger \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{ch}r & \mathrm{sh}r \\ \mathrm{sh}r & \mathrm{ch}r \end{pmatrix}\begin{pmatrix} a \\ a^\dagger \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{ch}r & \mathrm{sh}r \\ \mathrm{sh}r & \mathrm{ch}r \end{pmatrix}\begin{pmatrix} a \\ a^\dagger \end{pmatrix}. \end{aligned} \tag{3.2.23}$$

The inverse transformation is

$$\begin{pmatrix} a \\ a^{\dagger} \end{pmatrix} = \begin{pmatrix} \mathrm{ch}r & -\mathrm{sh}r \\ -\mathrm{sh}r & \mathrm{ch}r \end{pmatrix} \begin{pmatrix} a(r) \\ a^{\dagger}(r) \end{pmatrix}. \tag{3.2.24}$$

We will derive the annihilation operator for the squeezed thermal vacuum (3.2.15). At first let us consider an operator $\gamma(r)$ given by

$$\begin{aligned} \gamma(r) &\equiv \hat{S}(r)\gamma\hat{S}(r)^{-1} \\ &= \hat{S}(r)\frac{1}{1-f}\left(a - f\hat{a}^{\dagger}\right)\hat{S}(r)^{-1} \\ &= \frac{1}{1-f}a(r) - \frac{f}{1-f}\hat{a}^{\dagger}(r) \\ &= \frac{1}{1-f}\mathrm{ch}ra + \frac{1}{1-f}\mathrm{sh}ra^{\dagger} - \frac{f}{1-f}\mathrm{sh}r\tilde{a} - \frac{f}{1-f}\mathrm{ch}r\tilde{a}^{\dagger}, \end{aligned} \tag{3.2.25}$$

which satisfies

$$\begin{aligned} \gamma(r)|0_r\rangle &= \hat{S}(r)\gamma\hat{S}(r)^{-1}\hat{S}(r)|0\rangle \\ &= \hat{S}(r)\gamma|0\rangle \\ &= 0. \end{aligned} \tag{3.2.26}$$

However, $\gamma(r)$ is not appropriate as the annihilation operator , because the commutation relation between $\gamma^{\ddagger}$ and $\tilde{\gamma}^{\ddagger}$ is

$$\begin{aligned} [\gamma(r), \gamma^{\ddagger}] &= \frac{1}{1-f}\mathrm{ch}r - \frac{f}{1-f}\mathrm{ch}r \\ &= \mathrm{ch}r, \end{aligned} \tag{3.2.27}$$

$$\begin{aligned} [\gamma(r), \tilde{\gamma}^{\ddagger}] &= \frac{1}{1-f}\mathrm{sh}r - \frac{f}{1-f}\mathrm{sh}r \\ &= \mathrm{sh}r \\ &\neq 0. \end{aligned} \tag{3.2.28}$$

Then let us consider γ_r as

$$\gamma_r \equiv c_2\gamma(r) + c_3\tilde{\gamma}(r). \tag{3.2.29}$$

It annihilates the thermal squeezed vacuum,i.e.,

$$\gamma_r|0_r\rangle = 0 \tag{3.2.30}$$

and satisfies the commutation relation

$$[\gamma_r, \gamma^{\ddagger}] = c_2\mathrm{ch}r + c_3\mathrm{sh}r, \tag{3.2.31}$$

$$[\gamma_r, \tilde{\gamma}^{\ddagger}] = c_2\mathrm{sh}r + c_3\mathrm{ch}r. \tag{3.2.32}$$

Here, we take c_2 and c_3 as

$$\begin{cases} c_2 = \mathrm{ch}r, \\ c_3 = -\mathrm{sh}r \end{cases}, \tag{3.2.33}$$

so that γ_r satisfies the canonical commutation relation with $\gamma^{\text{♀}}$, and is appropriate as the annihilation operator. γ_r is described by a,$\tilde{a}$,$a^\dagger$ and $\tilde{a}^\dagger$ as

$$\begin{aligned} \gamma_r &= \mathrm{ch}r\gamma(r) - \mathrm{sh}r\gamma(r) \\ &= \mathrm{ch}r\frac{1}{1-f}\left[a(r) - f\hat{a}^\dagger(r)\right] - \mathrm{sh}r\left[\tilde{a}(r) - fa^\dagger(r)\right] \\ &= \frac{1}{1-f}\left[1+(1+f)\mathrm{sh}^2r\right]a - \frac{1+f}{1-f}\mathrm{ch}r\mathrm{sh}r\tilde{a} \\ &\quad + \frac{1+f}{1-f}\mathrm{ch}r\mathrm{sh}ra^\dagger - \frac{1}{1-f}\left[1+(1+f)\mathrm{sh}^2r\right]\tilde{a}^\dagger. \end{aligned} \tag{3.2.34}$$

From (3.2.14),(3.2.34) and these tilde conjugate, we obtain the Bogoliubov transformation

$$\begin{pmatrix} \gamma_r \\ \tilde{\gamma}_r \\ \tilde{\gamma}^{\text{♀}} \\ \gamma^{\text{♀}} \end{pmatrix} = \begin{pmatrix} \frac{1}{1-f} + \frac{1+f}{1-f}\mathrm{sh}^2r & -\frac{1+f}{1-f}\mathrm{ch}r\mathrm{sh}r & -\frac{f}{1-f} - \frac{1+f}{1-f}\mathrm{sh}^2r & \frac{1+f}{1-f}\mathrm{ch}r\mathrm{sh}r \\ -\frac{1+f}{1-f}\mathrm{ch}r\mathrm{sh}r & \frac{1}{1-f} + \frac{1+f}{1-f}\mathrm{sh}^2r & \frac{1+f}{1-f}\mathrm{ch}r\mathrm{sh}r & -\frac{f}{1-f} - \frac{1+f}{1-f}\mathrm{sh}^2r \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ \tilde{a} \\ \tilde{a}^\dagger \\ a^\dagger \end{pmatrix}, \tag{3.2.35}$$

and inverse transformation

$$\begin{pmatrix} a \\ \tilde{a} \\ \tilde{a}^\dagger \\ a^\dagger \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{f}{1-f} + \frac{1+f}{1-f}\mathrm{sh}^2r & -\frac{1+f}{1-f}\mathrm{ch}r\mathrm{sh}r \\ 0 & 1 & -\frac{1+f}{1-f}\mathrm{ch}r\mathrm{sh}r & \frac{f}{1-f} + \frac{1+f}{1-f}\mathrm{sh}^2r \\ 1 & 0 & \frac{1}{1-f} + \frac{1+f}{1-f}\mathrm{sh}^2r & -\frac{1+f}{1-f}\mathrm{ch}r\mathrm{sh}r \\ 0 & 1 & -\frac{1+f}{1-f}\mathrm{ch}r\mathrm{sh}r & \frac{1}{1-f} + \frac{1+f}{1-f}\mathrm{sh}^2r \end{pmatrix} \begin{pmatrix} \gamma_r \\ \tilde{\gamma}_r \\ \tilde{\gamma}^{\text{♀}} \\ \gamma^{\text{♀}} \end{pmatrix}. \tag{3.2.36}$$

Calculating the expectation for $a^\dagger a$ and a^2, we see

$$\begin{aligned} \langle 1|a^\dagger a|0_r\rangle &= \langle 1|\left[\tilde{\gamma}_r - \frac{1+f}{1-f}\mathrm{ch}r\mathrm{sh}r\tilde{\gamma}^{\text{♀}} + \left(\frac{1}{1-f} + \frac{1+f}{1-f}\mathrm{sh}^2r\right)\gamma^{\text{♀}}\right] \\ &\qquad \left[\gamma_r + \left(\frac{f}{1-f} + \frac{1+f}{1-f}\mathrm{sh}^2r\right)\tilde{\gamma}^{\text{♀}} - \frac{1+f}{1-f}\mathrm{ch}r\mathrm{sh}r\gamma^{\text{♀}}\right]|0_r\rangle \\ &= \frac{f}{1-f} + \frac{1+f}{1-f}\mathrm{sh}^2r \qquad (3.2.37) \\ &\equiv n_{f,r}, \qquad (3.2.38) \end{aligned}$$

and

$$\begin{aligned} \langle 1|a^2|0_r\rangle &= \langle 1|\left[\gamma_r + \left(\frac{f}{1-f} + \frac{1+f}{1-f}\mathrm{sh}^2r\right)\tilde{\gamma}^{\text{♀}} - \frac{1+f}{1-f}\mathrm{ch}r\mathrm{sh}r\gamma^{\text{♀}}\right]^2|0_r\rangle \\ &= -\frac{1+f}{1-f}\mathrm{ch}r\mathrm{sh}r \qquad (3.2.39) \\ &\equiv m_{f,r}. \qquad (3.2.40) \end{aligned}$$

Thus Bogoliubov transformation can be represented by $m_{f,r}$ and $n_{f,r}$ as

$$\begin{pmatrix} \gamma_r \\ \tilde{\gamma}_r \\ \tilde{\gamma}_r^{\dagger} \\ \gamma_r^{\dagger} \end{pmatrix} = \begin{pmatrix} 1+n_{f,r} & m_{f,r} & -n_{f,r} & -m_{f,r} \\ m_{f,r}^* & 1+n_{f,r} & -m_{f,r}^* & -n_{f,r} \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ \tilde{a} \\ \tilde{a}^\dagger \\ a^\dagger \end{pmatrix}. \tag{3.2.41}$$

3.2.4 Displaced Squeezed Vacuum

Here, we consider a displaced squeezed vacuum defined as

$$\begin{aligned} |0_{r,\alpha}\rangle &= \hat{D}(\alpha)|0_r\rangle \\ &= \hat{D}(\alpha)\hat{S}(r)|0\rangle, \end{aligned} \tag{3.2.42}$$

where

$$\hat{D}(\alpha) = D(\alpha)\tilde{D}(\alpha), \tag{3.2.43}$$

$$D(\alpha) = \mathrm{e}^{\alpha a^\dagger - \alpha^* a}. \tag{3.2.44}$$

The transformation of a and $a^\dagger$ by $\hat{D}(\alpha)$ are defined as

$$a_\alpha = \hat{D}(\alpha) a \hat{D}(\alpha)^{-1}, \tag{3.2.45}$$

$$a_\alpha^\dagger = \hat{D}(\alpha) a^\dagger \hat{D}(\alpha)^{-1}. \tag{3.2.46}$$

From (3.2.45) and (3.2.46), we have differential equations of a_α as

$$\begin{aligned} \frac{\partial}{\partial\alpha} a_\alpha &= \hat{D}(\alpha)[a^\dagger - \tilde{a}, a]\hat{D}(\alpha)^{-1} \\ &= -1, \end{aligned} \tag{3.2.47}$$

$$\begin{aligned} \frac{\partial}{\partial\alpha^*} a_\alpha &= \hat{D}(\alpha)[-a + \tilde{a}^\dagger, a]\hat{D}(\alpha)^{-1} \\ &= 0. \end{aligned} \tag{3.2.48}$$

The initial condition is

$$a_{\alpha=0} = a, \tag{3.2.49}$$

then we have the solution of the differential equations as

$$a_\alpha = a - \alpha. \tag{3.2.50}$$

We also have differential equations of a_α as

$$\begin{aligned} \frac{\partial}{\partial\alpha} a_\alpha^\dagger &= \hat{D}(\alpha)[a^\dagger - \tilde{a}, a^\dagger]\hat{D}(\alpha)^{-1} \\ &= 0, \end{aligned} \tag{3.2.51}$$

$$\begin{aligned} \frac{\partial}{\partial\alpha^*} a_\alpha^\dagger &= \hat{D}(\alpha)[-a + \tilde{a}^\dagger, a^\dagger]\hat{D}(\alpha)^{-1} \\ &= -1. \end{aligned} \tag{3.2.52}$$

The initial condition is

$$a^{\dagger}_{\alpha=0} = a^{\dagger}, \tag{3.2.53}$$

then we have the solution as

$$a^{\dagger}_{\alpha} = a^{\dagger} - \alpha^{*}. \tag{3.2.54}$$

Here, we consider an operator $\gamma_{r,\alpha}$ as

$$\begin{aligned}
\gamma_{r,\alpha} &= \hat{D}(\alpha)\gamma_r \hat{D}(\alpha)^{-1} \\
&= \hat{D}(\alpha)\left\{[1+n_{f,r}]\,a + m_{f,r}\tilde{a} - n_{f,r}\tilde{a}^{\dagger} - m_{f,r}a^{\dagger}\right\}\hat{D}(\alpha)^{-1} \\
&= [1+n_{f,r}]\,(a-\alpha) + m_{f,r}(\tilde{a}-\alpha^{*} - n_{f,r}(\tilde{a}^{\dagger}-\alpha) - m_{f,r}(a^{\dagger}-\alpha^{\dagger} \\
&= \gamma_r - \alpha,
\end{aligned} \tag{3.2.55}$$

which annihilates $|0_{r,\alpha}\rangle$,i.e.,

$$\begin{aligned}
\gamma_{r,\alpha}|0_{r,\alpha}\rangle &= \hat{D}(\alpha)\gamma_r\hat{D}(\alpha)^{-1}\hat{D}(\alpha)|0_r\rangle \\
&= \hat{D}(\alpha)\gamma_r 0_r\rangle \\
&= 0.
\end{aligned} \tag{3.2.56}$$

$\gamma_{r,\alpha}$ satisfies

$$[\gamma_{r,\alpha}, \gamma^{♀}] = [\gamma_r - \alpha, \gamma^{♀}] = 1, \tag{3.2.57}$$

$$[\gamma_{r,\alpha}, \tilde{\gamma}^{♀}] = [\gamma_r - \alpha, \tilde{\gamma}^{♀}] = 0, \tag{3.2.58}$$

where $\gamma^{♀}$ is the creation operator defined by (3.2.2). Then the Bogoliubov transformation is written as

$$\begin{aligned}
\begin{pmatrix} \gamma_{r,\alpha} \\ \tilde{\gamma}_{r,\alpha} \\ \tilde{\gamma}^{♀} \\ \gamma^{♀} \end{pmatrix}
&= \begin{pmatrix} 1+n_{f,r} & m_{f,r} & -n_{f,r} & -m_{f,r} \\ m^{*}_{f,r} & 1+n_{f,r} & -m^{*}_{f,r} & -n_{f,r} \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}
\begin{pmatrix} a \\ \tilde{a} \\ \tilde{a}^{\dagger} \\ a^{\dagger} \end{pmatrix}
+ \begin{pmatrix} -\alpha \\ -\alpha^{*} \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1+n_{f,r} & m_{f,r} & -n_{f,r} & -m_{f,r} \\ m^{*}_{f,r} & 1+n_{f,r} & -m^{*}_{f,r} & -n_{f,r} \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}
\begin{pmatrix} a-\alpha \\ \tilde{a}-\alpha^{*} \\ \tilde{a}^{\dagger}-\alpha \\ a^{\dagger}-\alpha^{*} \end{pmatrix},
\end{aligned} \tag{3.2.59}$$

and the inverse transformation is

$$\begin{pmatrix} a-\alpha \\ \tilde{a}-\alpha^{*} \\ \tilde{a}^{\dagger}-\alpha \\ a^{\dagger}-\alpha^{*} \end{pmatrix}
= \begin{pmatrix} 1 & 0 & n_{f,r} & m_{f,r} \\ 0 & 1 & m^{*}_{f,r} & n_{f,r} \\ 1 & 0 & 1+n_{f,r} & m_{f,r} \\ 0 & 1 & m^{*}_{f,r} & 1+n_{f,r} \end{pmatrix}
\begin{pmatrix} \gamma_{r,\alpha} \\ \tilde{\gamma}_{r,\alpha} \\ \tilde{\gamma}^{♀} \\ \gamma^{♀} \end{pmatrix}. \tag{3.2.60}$$

Calculating the expectation for a, we have

$$\begin{aligned}\langle 1|a|0_{r,\alpha}\rangle &= \langle 1|\left[\gamma_{r,\alpha}+\alpha+n_{f,r}\hat{\gamma}^{\ddagger}+m_{f,r}\gamma^{\ddagger}\right]|0_{r,\alpha}\rangle \\ &= \alpha. \end{aligned} \tag{3.2.61}$$

Then the expectation of $a^{\dagger}a$ is

$$\begin{aligned}\langle 1|a^{\dagger}a|0_{r,\alpha}\rangle &= \langle 1|\left\{\gamma_{r,\alpha}+\alpha^{*}+m^{*}_{f,r}\gamma^{\ddagger}+[1+n_{f,r}]\,\gamma^{\ddagger}\right\} \\ &\qquad \left\{\gamma_{r,\alpha}+\alpha+n_{f,r}\hat{\gamma}^{\ddagger}+m_{f,r}\gamma^{\ddagger}\right\}|0_{r,\alpha}\rangle \\ &= \langle 1|\left[n_{f,r}\tilde{\gamma}_{r,\alpha}\tilde{\gamma}^{\dagger}+|\alpha|^{2}\right]|0_{r,\alpha}\rangle \\ &= n_{f,r}+|\alpha|^{2}, \end{aligned} \tag{3.2.62}$$

and $n_{f,r}$ is represented as

$$\begin{aligned} n_{f,r} &= \langle 1|a^{\dagger}a|0_{r,\alpha}\rangle-\langle 1|a^{\dagger}|0_{r,\alpha}\rangle\langle 1|a|0_{r,\alpha}\rangle \\ &\equiv \quad n_{f,r,\alpha}. \end{aligned} \tag{3.2.63}$$

The expectation of a^{2} is

$$\begin{aligned}\langle 1|a^{2}|0_{r,\alpha}\rangle &= \langle 1|\left[\gamma_{r,\alpha}+\alpha+n_{f,r}\hat{\gamma}^{\ddagger}+m_{f,r}\gamma^{\ddagger}\right]^{2}|0_{r,\alpha}\rangle \\ &= \langle 1|\left[m_{f,r}\gamma_{r,\alpha}\gamma^{\ddagger}+\alpha^{2}\right]|0_{r,\alpha}\rangle \\ &= m_{f,r}+\alpha^{2}, \end{aligned} \tag{3.2.64}$$

and $m_{f,r}$ is represented as

$$\begin{aligned} m_{f,r} &= \langle 1|a^{2}|0_{r,\alpha}\rangle-\langle 1|a|0_{r,\alpha}\rangle^{2} \\ &\equiv \quad m_{f,r,\alpha} \end{aligned} \tag{3.2.65}$$

The bogoliubov transformation (3.2.59) can be rewritten by α, $n_{f,r,\alpha}$ and $m_{f,r,\alpha}$ as

$$\begin{pmatrix}\gamma_{r,\alpha}\\ \tilde{\gamma}_{r,\alpha}\\ \tilde{\gamma}^{\ddagger}\\ \gamma^{\ddagger}\end{pmatrix}=\begin{pmatrix}1+n_{f,r,\alpha} & m_{f,r,\alpha} & -n_{f,r,\alpha} & -m_{f,r,\alpha}\\ m^{*}_{f,r,\alpha} & 1+n_{f,r,\alpha} & -m^{*}_{f,r,\alpha} & -n_{f,r,\alpha}\\ -1 & 0 & 1 & 0\\ 0 & -1 & 0 & 1\end{pmatrix}\begin{pmatrix}a-\alpha\\ \tilde{a}-\alpha^{*}\\ \tilde{a}^{\dagger}-\alpha\\ a^{\dagger}-\alpha^{*}\end{pmatrix}. \tag{3.2.66}$$

3.3 Time-dependent Bogoliubov Transformation

3.3.1 Dissipative Bosonic System

The Hat-Hamiltonian within dissipative bosonic system is given by

$$\hat{H} = \hat{H}_0 + i\hat{\Pi}, \tag{3.3.1}$$

$$\hat{H}_0 = \omega(a^\dagger a - \tilde{a}^\dagger \tilde{a}), \tag{3.3.2}$$

$$\hat{\Pi} = -\kappa\left[(1+2\bar{n})(a^\dagger a + \tilde{a}^\dagger \tilde{a}) - 2(1+\bar{n})a\tilde{a} - 2\bar{n}a^\dagger \tilde{a}^\dagger\right] - 2\kappa\bar{n}. \tag{3.3.3}$$

The time-evolution of the ket vacuum is described by the dissipative Schrödinger equation

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}|0(t)\rangle, \tag{3.3.4}$$

and the initial thermal vacuum satisfies the thermal state condition

$$a|0\rangle = f a^\dagger |0\rangle. \tag{3.3.5}$$

From (3.2.14), we have the Bogoliubov transformation at the initial time $t = 0$ as

$$\begin{pmatrix} \gamma \\ \tilde{\gamma}^{\text{♀}} \end{pmatrix} = \begin{pmatrix} 1+n & -n \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ \tilde{a}^\dagger \end{pmatrix}, \tag{3.3.6}$$

$$\begin{pmatrix} a \\ \tilde{a}^\dagger \end{pmatrix} = \begin{pmatrix} 1 & n \\ 1 & 1+n \end{pmatrix} \begin{pmatrix} \gamma \\ \tilde{\gamma}^{\text{♀}} \end{pmatrix}. \tag{3.3.7}$$

$\hat{H}$ can be represented in the normal ordering for γ, $\tilde{\gamma}$, $\gamma^{\text{♀}}$ and $\tilde{\gamma}^{\text{♀}}$ as

$$\begin{aligned}
\hat{H} &= \omega\left[\left\{\tilde{\gamma} + (1+n)\gamma^{\text{♀}}\right\}\left(\gamma + n\theta\gamma^{\text{♀}}\right) - \left\{\gamma + (1+n)\tilde{\gamma}^{\text{♀}}\right\}\left(\tilde{\gamma} + n\gamma^{\text{♀}}\right)\right] \\
&\quad -i\kappa\left[(1+2\bar{n})\left\{\tilde{\gamma} + (1+n)\gamma^{\text{♀}}\right\}\left(\gamma + n\theta\gamma^{\text{♀}}\right) + (1+2\bar{n})\left\{\gamma + (1+n)\tilde{\gamma}^{\text{♀}}\right\}\left(\tilde{\gamma} + n\gamma^{\text{♀}}\right)\right. \\
&\quad \left. -2(1+\bar{n})\left(\gamma + n\tilde{\gamma}^{\text{♀}}\right)\left(\tilde{\gamma} + n\gamma^{\text{♀}}\right) - 2\bar{n}\left\{\tilde{\gamma} + (1+n)\gamma^{\text{♀}}\right\}\left\{\gamma + (1+n)\tilde{\gamma}^{\text{♀}}\right\}\right] \\
&\quad -i2\kappa\bar{n} \\
&= \omega\left[(1+n)\gamma^{\text{♀}}\gamma - n\gamma\gamma^{\text{♀}} + n\tilde{\gamma}\tilde{\gamma}^{\text{♀}} - (1+n)\tilde{\gamma}^{\text{♀}}\tilde{\gamma}\right] \\
&\quad -i\kappa\left[(n-2\bar{n})\tilde{\gamma}\tilde{\gamma}^{\text{♀}} + (2\bar{n}-n+1)\tilde{\gamma}^{\text{♀}}\tilde{\gamma} + (1+n)\gamma^{\text{♀}}\gamma - n\gamma\gamma^{\text{♀}} + 2(n-\bar{n})\gamma^{\text{♀}}\tilde{\gamma}^{\text{♀}}\right] \\
&\quad -i2\kappa\bar{n} \\
&= \omega\left(\gamma^{\text{♀}}\gamma - \tilde{\gamma}^{\text{♀}}\tilde{\gamma}\right) - i\kappa\left[\gamma^{\text{♀}}\gamma + \tilde{\gamma}^{\text{♀}}\tilde{\gamma} + 2(n-\bar{n})\gamma^{\text{♀}}\tilde{\gamma}^{\text{♀}}\right].
\end{aligned} \tag{3.3.8}$$

At time $t = 0$ γ satisfies

$$\gamma|0\rangle = 0, \tag{3.3.9}$$

however, γ does not annihilate the ket vacuum $|0(t)\rangle$ at general time t. It can be shown by seeing the time-evolution of the ket vector $\gamma|0(t)\rangle$, i.e,

$$\begin{aligned}\frac{\partial}{\partial t}\gamma|0(t)\rangle &= \gamma\frac{\partial}{\partial t}\mathrm{e}^{-i\hat{H}t}|0\rangle \\ &= \gamma\mathrm{e}^{-i\hat{H}t}\left(-i\hat{H}\right)|0\rangle \\ &= -2\kappa(n-\bar{n})\gamma\mathrm{e}^{-i\hat{H}t}\gamma^{\maltese}\tilde{\gamma}^{\maltese}|0\rangle \\ &\neq 0. \end{aligned} \tag{3.3.10}$$

On the other hand, $\gamma^{\maltese}$ is time-independent, and can be the creation operator at arbitrary time.

Here, We introduce the annihilation operator γ_t for $|0(t)\rangle$ at time t and the time-dependent Bogoliubov transformation as

$$\begin{pmatrix}\gamma_t \\ \tilde{\gamma}^{\maltese}\end{pmatrix} = \begin{pmatrix}1+n(t) & -n(t) \\ -1 & 1\end{pmatrix}\begin{pmatrix}a \\ \tilde{a}^{\dagger}\end{pmatrix}, \tag{3.3.11}$$

$$\begin{pmatrix}a \\ \tilde{a}^{\dagger}\end{pmatrix} = \begin{pmatrix}1 & n(t) \\ 1 & 1+n(t)\end{pmatrix}\begin{pmatrix}\gamma_t \\ \tilde{\gamma}^{\maltese}\end{pmatrix}. \tag{3.3.12}$$

Then γ_t, $\tilde{\gamma}_t$, $\gamma^{\maltese}$ and $\tilde{\gamma}^{\maltese}$ satisfy the canonical commutation relation

$$[\gamma_t, \gamma^{\maltese}] = 1, \tag{3.3.13}$$

$$[\gamma_t, \tilde{\gamma}^{\maltese}] = 0. \tag{3.3.14}$$

Assuming

$$\gamma_t|0(t)\rangle = 0, \tag{3.3.15}$$

the expectation of $a^{\dagger}a$ is given by

$$\begin{aligned}\langle 1|a^{\dagger}a|0(t)\rangle &= \langle 1|\left[\tilde{\gamma}_t + (1+n(t))\,\gamma^{\maltese}\right]\left[\gamma_t + n(t)\tilde{\gamma}^{\maltese}\right]|0(t)\rangle \\ &= \langle 1|\tilde{\gamma}_t\tilde{\gamma}^{\maltese}|0(t)\rangle n(t) \\ &= n(t). \end{aligned} \tag{3.3.16}$$

$\hat{H}$ can be represented by the normal ordering for γ_t, $\tilde{\gamma}_t$, $\gamma^{\maltese}$, and $\tilde{\gamma}^{\maltese}$ as

$$\hat{H} = \omega\left(\gamma^{\maltese}\gamma_t - \tilde{\gamma}^{\maltese}\tilde{\gamma}_t\right) - i\kappa\left[\gamma^{\maltese}\gamma_t + \tilde{\gamma}^{\maltese}\tilde{\gamma}_t + 2(n(t)-\bar{n})\gamma^{\maltese}\tilde{\gamma}^{\maltese}\right]. \tag{3.3.17}$$

Using (3.3.15), the dissipative Schrödinger equation (3.3.4) is rewritten as

$$\begin{aligned}\frac{\partial}{\partial t}|0(t)\rangle &= -i\hat{H}|0(t)\rangle \\ &= \omega\left(\gamma^{\maltese}\gamma_t - \tilde{\gamma}^{\maltese}\tilde{\gamma}_t\right) - i\kappa\left[\gamma^{\maltese}\gamma_t + \tilde{\gamma}^{\maltese}\tilde{\gamma}_t + 2(n(t)-\bar{n})\gamma^{\maltese}\tilde{\gamma}^{\maltese}\right]|0(t)\rangle \\ &= -2\kappa\left[n(t)-\bar{n}\right]\gamma^{\maltese}\tilde{\gamma}^{\maltese}|0(t)\rangle. \end{aligned} \tag{3.3.18}$$

Then the time-evolution of $n(t)$ is described as

$$\begin{aligned}\frac{d}{dt}n(t) &= \langle 1|a^{\dagger}a\frac{\partial}{\partial t}|0(t)\rangle \\ &= \langle 1|\left[\tilde{\gamma}_t + (1+n(t))\,\gamma^{♀}\right]\left[\gamma_t + n(t)\tilde{\gamma}^{♀}\right](-2\kappa)\left[n(t)-\bar{n}\right]\gamma^{♀}\tilde{\gamma}^{♀}|0(t)\rangle \\ &= -2\kappa\,(n(t)-\bar{n})\,. \end{aligned} \tag{3.3.19}$$

From (3.3.18) and (3.3.19), we have

$$\frac{\partial}{\partial t}|0(t)\rangle = \frac{dn(t)}{dt}\gamma^{♀}\tilde{\gamma}^{♀}|0(t)\rangle,$$

then the differential of $|0(t)\rangle$ is represented as

$$\delta|0(t)\rangle = \varDelta n(t)\gamma^{♀}\tilde{\gamma}^{♀}|0(t)\rangle. \tag{3.3.20}$$

(3.3.20) shows the time-evolution of $|0(t)\rangle$ is controlled by $n(t)$. Solving (3.3.20), we have

$$|0(t)\rangle = \hat{C}\mathrm{e}^{n(t)\gamma^{♀}\tilde{\gamma}^{♀}}|0\rangle. \tag{3.3.21}$$

From the initial condition $n(t=0)=n$, C is given by

$$|0\rangle = \hat{C}\mathrm{e}^{n\gamma^{♀}\tilde{\gamma}^{♀}}|0\rangle, \tag{3.3.22}$$

$$\therefore \hat{C} = \mathrm{e}^{-n\gamma^{♀}\tilde{\gamma}^{♀}} \tag{3.3.23}$$

Then we see

$$|0(t)\rangle = \mathrm{e}^{[n(t)-n]\gamma^{♀}\tilde{\gamma}^{♀}}|0\rangle. \tag{3.3.24}$$

(3.3.24) shows the time-evolution of the dissipative bosonic system is driven by condensation of $\gamma^{♀}\tilde{\gamma}^{♀}$ particle pairs. $n(t)-n$ is the order parameter of instability.

We confirm γ_t annihilates $|0(t)\rangle$ given by the solution (3.3.24). Applying γ_t on $|0(t)\rangle$, we have

$$\begin{aligned}\gamma_t|0(t)\rangle &= \gamma_t\mathrm{e}^{[n(t)-n]\gamma^{♀}\tilde{\gamma}^{♀}}|0\rangle \\ &= \mathrm{e}^{[n(t)-n]\gamma^{♀}\tilde{\gamma}^{♀}}\mathrm{e}^{-[n(t)-n]\gamma^{♀}\tilde{\gamma}^{♀}}\gamma_t\mathrm{e}^{[n(t)-n]\gamma^{♀}\tilde{\gamma}^{♀}}|0\rangle \\ &= \mathrm{e}^{[n(t)-n]\gamma^{♀}\tilde{\gamma}^{♀}}\hat{U}(t)|0\rangle, \end{aligned} \tag{3.3.25}$$

where

$$\hat{U}(t) = \mathrm{e}^{-[n(t)-n]\gamma^{♀}\tilde{\gamma}^{♀}}\gamma_t\mathrm{e}^{[n(t)-n]\gamma^{♀}\tilde{\gamma}^{♀}}. \tag{3.3.26}$$

The equation of the motion of $\hat{U}(t)$ is

$$\begin{aligned}\frac{d}{dt}\hat{U}(t) = &-\frac{dn(t)}{dt}\mathrm{e}^{-[n(t)-n]\gamma^{♀}\tilde{\gamma}^{♀}}\;[\gamma^{♀}\tilde{\gamma}^{♀},\gamma_t]\;\mathrm{e}^{[n(t)-n]\gamma^{♀}\tilde{\gamma}^{♀}} \\ &+\mathrm{e}^{-[n(t)-n]\gamma^{♀}\tilde{\gamma}^{♀}}\;\frac{d\gamma_t}{dt}\;\mathrm{e}^{[n(t)-n]\gamma^{♀}\tilde{\gamma}^{♀}}. \end{aligned} \tag{3.3.27}$$

Here, the commutation relation in (3.3.27) is

$$[\gamma^{♀}\tilde{\gamma}^{♀}, \gamma_t] = -\tilde{\gamma}^{♀}, \tag{3.3.28}$$

and the differential of γ_t is

$$\begin{aligned} \frac{d\gamma_t}{dt} &= \frac{dn(t)}{dt}a - \frac{dn(t)}{dt}\tilde{a}^\dagger \\ &= -\frac{dn(t)}{dt}\tilde{\gamma}^{♀}, \end{aligned} \tag{3.3.29}$$

then the right-hand side of (3.3.27) is canceled,i.e,

$$\frac{d}{dt}\hat{U}(t) = 0. \tag{3.3.30}$$

Thus we have

$$\hat{U}(t) = \hat{U}(0) = \gamma. \tag{3.3.31}$$

Putting (3.3.31) into (3.3.25), we see

$$\begin{aligned} \gamma_t|0(t)\rangle &= \mathrm{e}^{-[n(t)-n]\gamma^{♀}\tilde{\gamma}^{♀}}\gamma|0\rangle \\ &= 0, \end{aligned} \tag{3.3.32}$$

so (3.3.15) is satisfied.

3.3.2 Squeezing in Dissipative Environment

Interaction Representation

The Hat-Hamiltonian describing a squeezing process in dissipative environment is given by

$$\hat{H}_t = \hat{H}_0 + i\hat{\Pi} + \hat{H}_{sq,t}, \tag{3.3.33}$$

$$\hat{H}_0 = \omega(a^\dagger a - \tilde{a}^\dagger \tilde{a}), \tag{3.3.34}$$

$$\hat{\Pi} = -\kappa\left[(1+2\bar{n})(a^\dagger a + \tilde{a}^\dagger \tilde{a}) - 2(1+\bar{n})a\tilde{a} - 2\bar{n}a^\dagger \tilde{a}^\dagger\right] - 2\kappa\bar{n}, \tag{3.3.35}$$

$$\hat{H}_{sq,t} = \tfrac{i\chi}{2}\left(a^{\dagger 2}\mathrm{e}^{-2i\omega t} - a^2\mathrm{e}^{2i\omega t}\right) - t.c.. \tag{3.3.36}$$

The dissipative Schrödinger equation is given by

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}_t|0(t)\rangle. \tag{3.3.37}$$

Here, we describe the time-evolution of the system in interaction representation as

$$|0(t)\rangle_I = \mathrm{e}^{i\hat{H}_0 t}|0(t)\rangle. \tag{3.3.38}$$

The expectation of an operator A in interaction representation is written as

$$\begin{aligned}\langle A\rangle_t &= \langle 1|A|0(t)\rangle \\ &= \langle 1|A\mathrm{e}^{-i\hat{H}_0 t}|0\rangle_I \\ &= \langle 1|\mathrm{e}^{i\hat{H}_0 t}A\mathrm{e}^{-i\hat{H}_0 t}|0(t)\rangle_I \\ &= \langle 1|A^{(I)}(t)|0(t)\rangle_I, \end{aligned} \tag{3.3.39}$$

therefore an operator in interaction representation is defined as

$$A^{(I)}(t) = \mathrm{e}^{i\hat{H}_0 t}A\mathrm{e}^{-i\hat{H}_0 t}. \tag{3.3.40}$$

The equation of motion for $|0(t)\rangle_I$ can be derived as

$$\begin{aligned}\frac{\partial}{\partial t}|0(t)\rangle_I &= \frac{\partial}{\partial t}\left(\mathrm{e}^{i\hat{H}_0 t}|0(t)\rangle\right) \\ &= \mathrm{e}^{i\hat{H}_0 t}(i\hat{H}_0)|0(t)\rangle + \mathrm{e}^{i\hat{H}_0 t}(-i\hat{H}_t)|0(t)\rangle \\ &= -i\mathrm{e}^{i\hat{H}_0 t}\left(\hat{H}_{sq,t} + i\hat{\Pi}\right)|0(t)\rangle \\ &= -i\mathrm{e}^{i\hat{H}_0 t}\left(\hat{H}_{sq,t} + i\hat{\Pi}\right)\mathrm{e}^{-i\hat{H}_0 t}|0(t)\rangle_I \\ &= -i\left(\hat{H}^{(I)}_{sq,t}(t) + i\hat{\Pi}^{(I)}(t)\right)|0(t)\rangle_I. \end{aligned} \tag{3.3.41}$$

We can calculate

$$a^{(I)}(t) = \mathrm{e}^{i\hat{H}_0 t}a\mathrm{e}^{-i\hat{H}_0 t} = a\mathrm{e}^{-i\omega t}, \tag{3.3.42}$$

$$a^{\dagger(I)}(t) = \mathrm{e}^{i\hat{H}_0 t}a^\dagger\mathrm{e}^{-i\hat{H}_0 t} = a^\dagger\mathrm{e}^{i\omega t}, \tag{3.3.43}$$

and

$$\hat{H}^{(I)}_{sq,t}(t) = \frac{i\chi}{2}\left(a^{\dagger^2} - a^2\right) - t.c \tag{3.3.44}$$

$$\equiv \hat{H}^{(I)}_{sq}, \tag{3.3.45}$$

$$\hat{\Pi}^{(I)}(t) = -\kappa\left[(1+2\bar{n})(a^\dagger a + \tilde{a}^\dagger\tilde{a}) - 2(1+\bar{n})a\tilde{a} - 2\bar{n}a^\dagger\tilde{a}^\dagger\right] - 2\kappa\bar{n} \tag{3.3.46}$$

$$\equiv \hat{\Pi}^{(I)}, \tag{3.3.47}$$

then we see $\hat{H}^{(I)}_{sq,t}(t)$ and $\hat{\Pi}^{(I)}(t)$ are not dependent on t. Therefore, the equation of motion for $|0(t)\rangle_I$ is rewritten as

$$\frac{\partial}{\partial t}|0(t)\rangle_I = -i\left(\hat{H}^{(I)}_{sq} + i\hat{\Pi}^{(I)}\right)|0(t)\rangle_I. \tag{3.3.48}$$

Derivation for Time-Dependent Bogoliubov Transformation

We consider the system in which the Hat-Hamiltonian is given by

$$\hat{H} = \hat{H}_{sq} + i\hat{\Pi}, \tag{3.3.49}$$

$$\hat{H}_{sq} = \tfrac{i\chi}{2}\left(a^{\dagger^2} - a^2\right) - t.c, \tag{3.3.50}$$

$$\hat{\Pi} = -\kappa\left[(1+2\bar{n})(a^\dagger a + \tilde{a}^\dagger\tilde{a}) - 2(1+\bar{n})a\tilde{a} - 2\bar{n}a^\dagger\tilde{a}^\dagger\right] - 2\kappa\bar{n}. \tag{3.3.51}$$

The dissipative Schrd̈inger equation is

$$\frac{\partial}{\partial t}|0_{r,\alpha}(t)\rangle = -i\hat{H}|0_{r,\alpha}(t)\rangle, \tag{3.3.52}$$

and we set the initial state by a displaced squeezed thermal vacuum

$$|0_{r,\alpha}\rangle = \hat{D}(\alpha)\hat{S}(r)|0\rangle, \tag{3.3.53}$$

where

$$\hat{D}(\alpha) = D(\alpha)\tilde{D}(\alpha), \tag{3.3.54}$$

$$D(\alpha) = \mathrm{e}^{-\alpha^* a + \alpha a^\dagger}, \tag{3.3.55}$$

$$\hat{S}(r) = S(r)\tilde{S}(r), \tag{3.3.56}$$

$$S(r) = \mathrm{e}^{\frac{r}{2}(-a^2 + a^{\dagger 2})}. \tag{3.3.57}$$

The Bogoliubov transform at time $t = 0$ is, from (3.2.59),

$$\begin{pmatrix} \gamma_{r,\alpha} \\ \tilde{\gamma}_{r,\alpha} \\ \tilde{\gamma}^{\text{♀}} \\ \gamma^{\text{♀}} \end{pmatrix} = \begin{pmatrix} 1+n_{f,r,\alpha} & m_{f,r,\alpha} & -n_{f,r,\alpha} & -m_{f,r,\alpha} \\ m^*_{f,r,\alpha} & 1+n_{f,r,\alpha} & -m^*_{f,r,\alpha} & -n_{f,r,\alpha} \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a-\alpha \\ \tilde{a}-\alpha^* \\ \tilde{a}^\dagger - \alpha \\ a^\dagger - \alpha^* \end{pmatrix}, \tag{3.3.58}$$

$$\begin{pmatrix} a-\alpha \\ \tilde{a}-\alpha^* \\ \tilde{a}^\dagger - \alpha \\ a^\dagger - \alpha^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & n_{f,r,\alpha} & m_{f,r,\alpha} \\ 0 & 1 & m^*_{f,r,\alpha} & n_{f,r,\alpha} \\ 1 & 0 & 1+n_{f,r,\alpha} & m_{f,r,\alpha} \\ 0 & 1 & m^*_{f,r,\alpha} & 1+n_{f,r,\alpha} \end{pmatrix} \begin{pmatrix} \gamma_{r,\alpha} \\ \tilde{\gamma}_{r,\alpha} \\ \tilde{\gamma}^{\text{♀}} \\ \gamma^{\text{♀}} \end{pmatrix}, \tag{3.3.59}$$

where α, $n_{f,r,\alpha}$ and $m_{f,r,\alpha}$ are

$$\alpha = \langle 1|a|0_{r,\alpha}\rangle, \tag{3.3.60}$$

$$n_{f,r,\alpha} = \langle 1|a^\dagger a|0_{r,\alpha}\rangle - \langle 1|a^\dagger 0_{r,\alpha}\rangle\langle 1|a|0_{r,\alpha}\rangle, \tag{3.3.61}$$

$$\tag{3.3.62}$$

and

$$m_{f,r,\alpha} = \langle 1|a^2|0_{r,\alpha}\rangle - \{\langle 1|a|0_{r,\alpha}\rangle\}^2. \tag{3.3.63}$$

In following, we describe

$$n_{f,r,\alpha} \longrightarrow \Delta n, \tag{3.3.64}$$

$$m_{f,r,\alpha} \longrightarrow \Delta m, \tag{3.3.65}$$

$$m^*_{f,r,\alpha} \longrightarrow \Delta m^*. \tag{3.3.66}$$

$\hat{H}$ can be represented by $\gamma_{r,\alpha}$, $\tilde{\gamma}_{r,\alpha}$, $\gamma^{♀}$ and $\tilde{\gamma}^{♀}$, i.e.,

$$\begin{aligned}
\hat{H}_{sq} &= \frac{i\chi}{2}\left(-a^2 + a^{\dagger^2} - \tilde{a}^2 + \tilde{a}^{\dagger 2}\right) \\
&= \frac{i\chi}{2}\Big[-\left(\gamma_{r,\alpha} + \alpha + \Delta m\gamma^{♀} + \Delta n\tilde{\gamma}^{♀}\right)^2 + \left(\tilde{\gamma}_{r,\alpha} + \alpha^* + (\Delta n+1)\gamma^{♀} + \Delta m^*\tilde{\gamma}^{♀}\right)^2 \\
&\qquad - \left(\tilde{\gamma}_{r,\alpha} + \alpha^* + \Delta n\gamma^{♀} + \Delta m^*\tilde{\gamma}^{♀}\right)^2 + \left(\gamma_{r,\alpha} + \alpha + \Delta m\gamma^{♀} + (\Delta n+1)\tilde{\gamma}^{♀}\right)^2\Big] \\
&= \frac{i\chi}{2}\Big[(2\Delta n+1)(\gamma^{♀^2} + \tilde{\gamma}^{♀^2}) + 2(\Delta m + \Delta m^*)\gamma^{♀}\tilde{\gamma}^{♀} + 2\gamma^{♀}(\tilde{\gamma}_{r,\alpha} + \alpha^*) + 2\tilde{\gamma}^{♀}(\gamma_{r,\alpha} + \alpha)\Big] \\
&= i\chi\Big[(\Delta n + \frac{1}{2})(\gamma^{♀^2} + \tilde{\gamma}^{♀^2}) + 2\Re\mathrm{e}\Delta m\gamma^{♀}\tilde{\gamma}^{♀} + \gamma^{♀}(\tilde{\gamma}_{r,\alpha} + \alpha^*) + \tilde{\gamma}^{♀}(\gamma_{r,\alpha} + \alpha)\Big],
\end{aligned} \tag{3.3.67}$$

where

$$\begin{aligned}
\hat{\Pi} &= -\kappa\Bigg[(1+2\bar{n})\left\{\tilde{\gamma}_{r,\alpha} + \alpha^* + (\Delta n+1)\gamma^{♀} + \Delta m^*\tilde{\gamma}^{♀}\right\}\left\{\gamma_{r,\alpha} + \alpha + \Delta m\gamma^{♀} + \Delta n\tilde{\gamma}^{♀}\right\} \\
&\quad + (1+2\bar{n})\left\{\gamma_{r,\alpha} + \alpha + \Delta m\gamma^{♀} + (\Delta n+1)\tilde{\gamma}^{♀}\right\}\left\{\tilde{\gamma}_{r,\alpha} + \alpha^* + \Delta n\gamma^{♀} + \Delta m^*\tilde{\gamma}^{♀}\right\} \\
&\quad - 2(1+\bar{n})\left\{\gamma_{r,\alpha} + \alpha + \Delta m\gamma^{♀} + \Delta n\tilde{\gamma}^{♀}\right\}\left\{\tilde{\gamma}_{r,\alpha} + \alpha^* + \Delta n\gamma^{♀} + \Delta m^*\tilde{\gamma}^{♀}\right\} \\
&\quad - 2\bar{n}\left\{\tilde{\gamma}_{r,\alpha} + \alpha^* + (\Delta n+1)\gamma^{♀} + \Delta m^*\tilde{\gamma}^{♀}\right\}\left\{\gamma_{r,\alpha} + \alpha + \Delta m\gamma^{♀} + (\Delta n+1)\tilde{\gamma}^{♀}\right\}\Bigg] \\
&\quad -2\kappa\bar{n}, \\
&= -\kappa\Bigg[\Delta m\gamma^{♀^2} + \Delta m^*\tilde{\gamma}^{♀^2} + \alpha\gamma^{♀} + \alpha^*\tilde{\gamma}^{♀} + (\Delta n+1)\gamma^{♀}\gamma_{r,\alpha} - \Delta n\gamma_{r,\alpha}\gamma^{♀} \\
&\quad + \{(1+2\bar{n})(\Delta n+1) - 2(\bar{n}+1)\Delta n\}\,\tilde{\gamma}^{♀}\tilde{\gamma}_{r,\alpha} + \{(1+2\bar{n})\Delta n - 2\bar{n}(\Delta n+1)\}\,\tilde{\gamma}_{r,\alpha}\tilde{\gamma}^{♀} \\
&\quad + \left\{2(2\bar{n}+1)(\Delta n+1)\Delta n - 2(1+\bar{n})\Delta n^2 - 2\bar{n}(\Delta n-1)^2\right\}\gamma^{♀}\tilde{\gamma}^{♀}\Bigg] \\
&\quad - 2\kappa\bar{n} \\
&= -\kappa\Big[\Delta m\gamma^{♀^2} + \Delta m^*\tilde{\gamma}^{♀^2} + 2(\Delta n + \bar{n})\gamma^{♀}\tilde{\gamma}^{♀} + \gamma^{♀}(\gamma_{r,\alpha} + \alpha) + \tilde{\gamma}^{♀}(\tilde{\gamma}_{r,\alpha} + \alpha^*)\Big].
\end{aligned} \tag{3.3.68}$$

Calculating the time-evolution of $\gamma_{r,\alpha}|0_{r,\alpha}(t)\rangle$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\gamma_{r,\alpha}|0_{r,\alpha}(t)\rangle &= \gamma_{r,\alpha}\frac{\partial}{\partial t}\left(\mathrm{e}^{-i(\hat{H}_{sq}+i\hat{\Pi})t}|0_{r,\alpha}\rangle\right) \\
&= \gamma_{r,\alpha}\mathrm{e}^{-i(\hat{H}_{sq}+i\hat{\Pi})t}(-i)(\hat{H}_{sq} + i\hat{\Pi})|0_{r,\alpha}\rangle \\
&= \gamma_{r,\alpha}\mathrm{e}^{-i(\hat{H}_{sq}+i\hat{\Pi})t}\Bigg[\left\{\chi(\Delta n + \frac{1}{2}) - \kappa\Delta m\right\}\gamma^{♀^2} + \left\{\chi(\Delta n + \frac{1}{2}) - \kappa\Delta m^*\right\}\tilde{\gamma}^{♀^2} \\
&\quad + 2\left\{\chi\Re\mathrm{e}\Delta m - \kappa(\Delta n - \bar{n})\right\}\gamma^{♀}\tilde{\gamma}^{♀} + \left\{\chi\alpha^* - \kappa\alpha\right\}\gamma^{♀} + \left\{\chi\alpha - \kappa\alpha^*\right\}\tilde{\gamma}^{♀}\Bigg]|0_{r,\alpha}(t)\rangle \\
&\neq 0.
\end{aligned} \tag{3.3.69}$$

It implies $\gamma_{r,\alpha}$ does not annihilate $|0_{r,\alpha}(t)\rangle$. Then, let us consider a Bogoliubov transformation at time

t as

$$\begin{pmatrix} \gamma_{r,\alpha;t} \\ \tilde{\gamma}_{\chi,\alpha;t} \\ \tilde{\gamma}^{\mathring{\dagger}} \\ \gamma^{\mathring{\dagger}} \end{pmatrix} = \begin{pmatrix} 1+\Delta n(t) & \Delta m(t) & -\Delta n(t) & -\Delta m(t) \\ \Delta m(t)^* & 1+\Delta n(t) & -\Delta m(t)^* & -\Delta n(t) \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a-\alpha(t) \\ \tilde{a}-\alpha(t)^* \\ \tilde{a}^\dagger-\alpha(t) \\ a^\dagger-\alpha(t)^* \end{pmatrix} . \tag{3.3.70}$$

This transformation satisfies

$$[\gamma_{r,\alpha;t}, \gamma^{\mathring{\dagger}}] = \left[(\Delta n(t)+1)a + \Delta m(t)\tilde{a} - \Delta m(t)a^\dagger - \Delta n(t)\tilde{a}^\dagger - \alpha, -\tilde{a}+a^\dagger\right]$$
$$= 1, \tag{3.3.71}$$
$$[\gamma_{r,\alpha;t}, \tilde{\gamma}^{\mathring{\dagger}}] = \left[(\Delta n(t)+1)a + \Delta m(t)\tilde{a} - \Delta m(t)a^\dagger - \Delta n(t)\tilde{a}^\dagger - \alpha, -a+\tilde{a}^\dagger\right]$$
$$= 0, \tag{3.3.72}$$

and the inverse transformation is

$$\begin{pmatrix} a-\alpha(t) \\ \tilde{a}-\alpha(t)^* \\ \tilde{a}^\dagger-\alpha(t) \\ a^\dagger-\alpha(t)^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & \Delta n(t) & \Delta m(t) \\ 0 & 1 & \Delta m(t)^* & \Delta n(t) \\ 1 & 0 & 1+\Delta n(t) & \Delta m(t) \\ 0 & 1 & \Delta m(t)^* & 1+\Delta n(t) \end{pmatrix} \begin{pmatrix} \gamma_{r,\alpha;t} \\ \tilde{\gamma}_{\chi,\alpha;t} \\ \tilde{\gamma}^{\mathring{\dagger}} \\ \gamma^{\mathring{\dagger}} \end{pmatrix} . \tag{3.3.73}$$

Assuming

$$\gamma_{r,\alpha;t}|0_{r,\alpha}(t)\rangle = 0, \tag{3.3.74}$$

we obtain the expectation for a and $a^\dagger a$ as

$$\langle 1|a|0_{r,\alpha}(t)\rangle = \langle 1| \left\{ \gamma_{r,\alpha} + \alpha(t) + \Delta m(t)\gamma^{\mathring{\dagger}} + \Delta n(t)\tilde{\gamma}^{\mathring{\dagger}} \right\} |0_{r,\alpha}(t)\rangle$$
$$= \alpha(t) \tag{3.3.75}$$

and

$$\begin{aligned} \langle 1|a^\dagger a|0_{r,\alpha}(t)\rangle &= \langle 1| \left\{ \tilde{\gamma}_{r,\alpha} + \alpha^* + (\Delta n+1)\gamma^{\mathring{\dagger}} + \Delta m^* \tilde{\gamma}^{\mathring{\dagger}} \right\} \\ &\quad \times \left\{ \gamma_{r,\alpha} + \alpha(t) + \Delta m(t)\gamma^{\mathring{\dagger}} + \Delta n(t)\tilde{\gamma}^{\mathring{\dagger}} \right\} |0_{r,\alpha}(t)\rangle \\ &= \langle 1| \left\{ \Delta n(t)\tilde{\gamma}_{\chi,\alpha;t}\tilde{\gamma}^{\mathring{\dagger}} + |\alpha(t)|^2 \right\} |0_{r,\alpha}(t)\rangle \\ &= \Delta n(t) + |\alpha(t)|^2. \end{aligned} \tag{3.3.76}$$

Then, $\Delta n(t)$ is written as

$$\Delta n(t) = \langle 1|a^\dagger a|0_{r,\alpha}(t)\rangle - \langle 1|a^\dagger|0_{r,\alpha}(t)\rangle\langle 1|a|0_{r,\alpha}(t)\rangle. \tag{3.3.77}$$

We also obtain the expectation for a^2 as

$$\begin{aligned}
\langle 1|a^2|0_{r,\alpha}(t)\rangle &= \langle 1| \left\{ \gamma_{r,\alpha} + \alpha(t) + \mathit{\Delta} m(t)\gamma^{♀} + \mathit{\Delta} n(t)\tilde{\gamma}^{♀} \right\}^2 |0_{r,\alpha}(t)\rangle \\
&= \langle 1| \left\{ \mathit{\Delta} m(t)\gamma_{r,\alpha;t}\gamma^{♀} + \alpha(t)^2 \right\} |0_{r,\alpha}(t)\rangle \\
&= \mathit{\Delta} m(t) + \alpha(t)^2,
\end{aligned} \tag{3.3.78}$$

and $\mathit{\Delta} m(t)$ is written as

$$\mathit{\Delta} m(t) = \langle 1|a^2|0_{r,\alpha}(t)\rangle - \left\{\langle 1|a|0_{r,\alpha}(t)\rangle\right\}^2. \tag{3.3.79}$$

$\hat{H}$ can be rewritten in normal order of $\gamma_{r,\alpha;t}$, $\tilde{\gamma}_{\chi,\alpha;t}$, $\gamma^{♀}$ and $\tilde{\gamma}^{♀}$ as

$$\begin{aligned}
\hat{H} &= \hat{H}_{sq} + i\hat{\mathit{\Pi}} \\
&= i\left\{ \chi(\mathit{\Delta} n(t) + \frac{1}{2}) - \kappa \mathit{\Delta} m(t) \right\} \gamma^{♀ 2} + i\left\{ \chi(\mathit{\Delta} n(t) + \frac{1}{2}) - \kappa \mathit{\Delta} m(t)^* \right\} \tilde{\gamma}^{♀ 2} \\
&\quad + i2\left\{ \chi \Re\mathrm{e} \mathit{\Delta} m(t) - \kappa(\mathit{\Delta} n(t) - \bar{n}) \right\} \gamma^{♀}\tilde{\gamma}^{♀} \\
&\quad + i\left\{ \chi\alpha(t)^* - \kappa\alpha(t) \right\} \gamma^{♀} + i\left\{ \chi\alpha(t) - \kappa\alpha(t)^* \right\} \tilde{\gamma}^{♀} \\
&\quad + i\chi\gamma^{♀}\tilde{\gamma}_{\chi,\alpha;t} + \chi\tilde{\gamma}^{♀}\gamma_{r,\alpha;t} - i\kappa\gamma^{♀}\gamma_{r,\alpha;t} - \kappa\tilde{\gamma}^{♀}\tilde{\gamma}_{\chi,\alpha;t}.
\end{aligned} \tag{3.3.80}$$

Therefore, the dissipative Schrödinger equation is represented as

$$\begin{aligned}
\frac{\partial}{\partial t}|0_{r,\alpha}(t)\rangle &= -i\left(\hat{H}_{sq} + i\hat{\mathit{\Pi}} \right) |0_{r,\alpha}(t)\rangle \\
&= \Bigg[\chi \Big\{ (\mathit{\Delta} n(t) + \frac{1}{2})(\gamma^{♀ 2} + \tilde{\gamma}^{♀ 2}) + 2\Re\mathrm{e}\mathit{\Delta} m(t)\gamma^{♀}\tilde{\gamma}^{♀} \\
&\qquad + \gamma^{♀}(\tilde{\gamma}_{\chi,\alpha;t} + \alpha(t)^*) + \tilde{\gamma}^{♀}(\gamma_{r,\alpha;t} + \alpha)(t) \Big\} \\
&\quad - \kappa \Big\{ \mathit{\Delta} m(t)\gamma^{♀ 2} + \mathit{\Delta} m(t)^*\tilde{\gamma}^{♀ 2} + 2(\mathit{\Delta} n(t) + \bar{n})\gamma^{♀}\tilde{\gamma}^{♀} \\
&\qquad + \gamma^{♀}(\gamma_{r,\alpha;t} + \alpha(t)) + \tilde{\gamma}^{♀}(\tilde{\gamma}_{\chi,\alpha;t} + \alpha(t)^*) \Big\} \Bigg] |0_{r,\alpha}(t)\rangle \\
&= \Bigg[\left\{ \chi(\mathit{\Delta} n(t) + \frac{1}{2}) - \kappa\mathit{\Delta} m(t) \right\} \gamma^{♀ 2} + \left\{ \chi(\mathit{\Delta} n(t) + \frac{1}{2}) - \kappa\mathit{\Delta} m(t)^* \right\} \tilde{\gamma}^{♀ 2} \\
&\quad + 2\left\{ \chi\Re\mathrm{e}\mathit{\Delta} m(t) - \kappa(\mathit{\Delta} n(t) - \bar{n}) \right\} \gamma^{♀}\tilde{\gamma}^{♀} \\
&\quad + \left\{ \chi\alpha(t)^* - \kappa\alpha(t) \right\} \gamma^{♀} + \left\{ \chi\alpha(t) - \kappa\alpha(t)^* \right\} \tilde{\gamma}^{♀} \Bigg] |0_{r,\alpha}(t)\rangle.
\end{aligned} \tag{3.3.81}$$

The equations of motion for $\alpha(t)$, $\mathit{\Delta} n(t)$ and $\mathit{\Delta} m(t)$ are derived as

$$
\begin{aligned}
\frac{d}{dt}\alpha(t) &= \langle 1|a\frac{\partial}{\partial t}|0_{r,\alpha}(t)\rangle \\
&= \langle 1|\left\{\gamma_{r,\alpha} + \alpha(t) + \mathit{\Delta} m(t)\gamma^{\overset{\circ}{+}} + \mathit{\Delta} n(t)\tilde{\gamma}^{\overset{\circ}{+}}\right\}\frac{\partial}{\partial t}|0_{r,\alpha}(t)\rangle \\
&= \langle 1|\left\{\gamma_{r,\alpha;t} + \alpha(t)\right\} \\
&\qquad \times \ \Big[\Big\{\chi(\mathit{\Delta} n(t) + \frac{1}{2}) - \kappa \mathit{\Delta} m(t)\Big\}\gamma^{\overset{\circ}{+}2} + \Big\{\chi(\mathit{\Delta} n(t) + \frac{1}{2}) - \kappa \mathit{\Delta} m(t)^*\Big\}\tilde{\gamma}^{\overset{\circ}{+}2} \\
&\qquad\quad + 2\left\{\chi \Re\mathrm{e}\mathit{\Delta} m(t) - \kappa(\mathit{\Delta} n(t) - \bar{n})\right\}\gamma^{\overset{\circ}{+}}\tilde{\gamma}^{\overset{\circ}{+}} \\
&\qquad\quad + \left\{\chi\alpha(t)^* - \kappa\alpha(t)\right\}\gamma^{\overset{\circ}{+}} + \left\{\chi\alpha(t) - \kappa\alpha(t)^*\right\}\tilde{\gamma}^{\overset{\circ}{+}}\Big]|0_{r,\alpha}(t)\rangle \\
&= \chi\alpha(t)^* - \kappa\alpha(t),
\end{aligned}
\tag{3.3.82}
$$

$$
\begin{aligned}
\frac{d}{dt}\mathit{\Delta} n(t) &= \langle 1|a^\dagger a\frac{\partial}{\partial t}|0_{r,\alpha}(t)\rangle - \frac{d}{dt}|\alpha(t)|^2 \\
&= \langle 1|\left\{\tilde{\gamma}_{r,\alpha} + \alpha^* + (\mathit{\Delta} n + 1)\gamma^{\overset{\circ}{+}} + \mathit{\Delta} m^*\tilde{\gamma}^{\overset{\circ}{+}}\right\} \\
&\qquad \times \left\{\gamma_{r,\alpha} + \alpha(t) + \mathit{\Delta} m(t)\gamma^{\overset{\circ}{+}} + \mathit{\Delta} n(t)\tilde{\gamma}^{\overset{\circ}{+}}\right\}\frac{\partial}{\partial t}|0_{r,\alpha}(t)\rangle \\
&\quad - \frac{d}{dt}\alpha(t)\alpha(t)^* \\
&= \langle 1|\left\{\tilde{\gamma}_{\chi,\alpha;t}\gamma_{r,\alpha;t} + \alpha(t)\gamma_{r,\alpha;t} + \alpha^*\tilde{\gamma}_{r,\alpha;t} + |\alpha(t)|^2 + \mathit{\Delta} n(t)\right\}\frac{\partial}{\partial t}|0_{r,\alpha}(t)\rangle \\
&\quad - \frac{d\alpha(t)}{dt}\alpha(t)^* - \alpha(t)\frac{d\alpha(t)^*}{dt} \\
&= \langle 1|\left\{\tilde{\gamma}_{\chi,\alpha;t}\gamma_{r,\alpha;t} + \alpha(t)\gamma_{r,\alpha;t} + \alpha^*\tilde{\gamma}_{r,\alpha;t} + |\alpha(t)|^2 + \mathit{\Delta} n(t)\right\} \\
&\qquad \times \ \Big[\Big\{\chi(\mathit{\Delta} n(t) + \frac{1}{2}) - \kappa \mathit{\Delta} m(t)\Big\}\gamma^{\overset{\circ}{+}2} + \Big\{\chi(\mathit{\Delta} n(t) + \frac{1}{2}) - \kappa \mathit{\Delta} m(t)^*\Big\}\tilde{\gamma}^{\overset{\circ}{+}2} \\
&\qquad\quad + 2\left\{\chi \Re\mathrm{e}\mathit{\Delta} m(t) - \kappa(\mathit{\Delta} n(t) - \bar{n})\right\}\gamma^{\overset{\circ}{+}}\tilde{\gamma}^{\overset{\circ}{+}} \\
&\qquad\quad + \left\{\chi\alpha(t)^* - \kappa\alpha(t)\right\}\gamma^{\overset{\circ}{+}} + \left\{\chi\alpha(t) - \kappa\alpha(t)^*\right\}\tilde{\gamma}^{\overset{\circ}{+}}\Big]|0_{r,\alpha}(t)\rangle \\
&\quad - \left[\chi\alpha(t)^* - \kappa\alpha(t)\right]\alpha(t)^* - \alpha(t)\left[\chi\alpha(t) - \kappa\alpha(t)^*\right] \\
&= 2\left[\chi\Re\mathrm{e}\mathit{\Delta} m(t) - \kappa\left(\mathit{\Delta} n(t) - \bar{n}\right)\right]
\end{aligned}
\tag{3.3.83}
$$

and

$$
\begin{aligned}
\frac{d}{dt}\mathit{\Delta} m(t) &= \langle 1|a^2\frac{\partial}{\partial t}|0_{r,\alpha}(t)\rangle - \frac{d}{dt}\alpha(t)^2 \\
&= \langle 1|\left\{\gamma_{r,\alpha} + \alpha(t) + \mathit{\Delta} m(t)\gamma^{\Venus} + \mathit{\Delta} n(t)\tilde{\gamma}^{\Venus}\right\}^2 \frac{\partial}{\partial t}|0_{r,\alpha}(t)\rangle \\
&\quad -2\alpha(t)\frac{d\alpha(t)}{dt} \\
&= \langle 1|\left\{\gamma_{r,\alpha;t}^2 + 2\alpha(t)\gamma_{r,\alpha;t} + \mathit{\Delta} m(t) + \alpha(t)^2\right\} \\
&\qquad \times \left[\left\{\chi(\mathit{\Delta} n(t)+\frac{1}{2}) - \kappa\mathit{\Delta} m(t)\right\}\gamma^{\Venus 2} + \left\{\chi(\mathit{\Delta} n(t)+\frac{1}{2}) - \kappa\mathit{\Delta} m(t)^*\right\}\tilde{\gamma}^{\Venus 2}\right. \\
&\qquad + 2\left\{\chi\Re\mathrm{e}\mathit{\Delta} m(t) - \kappa(\mathit{\Delta} n(t)-\bar{n})\right\}\gamma^{\Venus}\tilde{\gamma}^{\Venus} \\
&\qquad \left. + \left\{\chi\alpha(t)^* - \kappa\alpha(t)\right\}\gamma^{\Venus} + \left\{\chi\alpha(t) - \kappa\alpha(t)^*\right\}\tilde{\gamma}^{\Venus}\right]|0_{r,\alpha}(t)\rangle \\
&\quad -2\alpha(t)\left[\chi\alpha(t)^* - \kappa\alpha(t)\right] \\
&= 2\left[\chi\left(\mathit{\Delta} n(t)+\frac{1}{2}\right) - \kappa\mathit{\Delta} m(t)\right]
\end{aligned}
\tag{3.3.84}
$$

Using (3.3.82), (3.3.83) and (3.3.84), we can rewrite (3.3.81) to

$$
\begin{aligned}
&\frac{\partial}{\partial t}|0_{r,\alpha}(t)\rangle \\
&= \left\{\frac{1}{2}\frac{d\mathit{\Delta} m(t)}{dt}\gamma^{\Venus 2} + \frac{1}{2}\frac{d\mathit{\Delta} m(t)^*}{dt}\tilde{\gamma}^{\Venus 2} + \frac{d\mathit{\Delta} n(t)}{dt}\gamma^{\Venus}\tilde{\gamma}^{\Venus} + \frac{d\alpha(t)}{dt}\gamma^{\Venus} + \frac{d\alpha(t)^*}{dt}\tilde{\gamma}^{\Venus}\right\}|0_{r,\alpha}(t)\rangle
\end{aligned}
\tag{3.3.85}
$$

Therefore we obtain the differential equation for $\delta|0(t)\rangle$ as

$$
\begin{aligned}
&\delta|0_{r,\alpha}(t)\rangle \\
&= \left\{\delta\mathit{\Delta} m(t)\frac{1}{2}\gamma^{\Venus 2} + \delta\mathit{\Delta} m(t)^*\frac{1}{2}\tilde{\gamma}^{\Venus 2} + \delta\mathit{\Delta} n(t)\gamma^{\Venus}\tilde{\gamma}^{\Venus} + \delta\alpha(t)\gamma^{\Venus} + \delta\alpha(t)^*\tilde{\gamma}^{\Venus}\right\}|0_{r,\alpha}(t)\rangle,
\end{aligned}
\tag{3.3.86}
$$

and we see $\alpha(t)$, $\mathit{\Delta} n(t)$ and $\mathit{\Delta} m(t)$ control the time-evolution of the thermal vacuum. The formal solution of (3.3.86) can be written as

$$
\begin{aligned}
|0_{r,\alpha}(t)\rangle &= \hat{C}'\mathrm{e}^{\frac{\mathit{\Delta} m(t)}{2}\gamma^{\Venus 2} + \frac{\mathit{\Delta} m(t)^*}{2}\tilde{\gamma}^{\Venus 2} + \mathit{\Delta} n(t)\gamma^{\Venus}\tilde{\gamma}^{\Venus} + \alpha(t)\gamma^{\Venus} + \alpha(t)^*\tilde{\gamma}^{\Venus}}|0(\chi.\alpha)\rangle \\
&= \hat{C}'\mathrm{e}^{\hat{G}(t)}|0_{r,\alpha}\rangle,
\end{aligned}
\tag{3.3.87}
$$

where

$$
\hat{G}(t) = \frac{\mathit{\Delta} m(t)}{2}\gamma^{\Venus 2} + \frac{\mathit{\Delta} m(t)^*}{2}\tilde{\gamma}^{\Venus 2} + \mathit{\Delta} n(t)\gamma^{\Venus}\tilde{\gamma}^{\Venus} + \alpha(t)\gamma^{\Venus} + \alpha(t)^*\tilde{\gamma}^{\Venus}. \tag{3.3.88}
$$

From the initial condition (3.3.53), we have

$$
|0_{r,\alpha}\rangle = \hat{C}'\mathrm{e}^{\hat{G}(0)}|0_{r,\alpha}\rangle \tag{3.3.89}
$$

then $\hat{C}$ is determined as

$$
\begin{aligned}
\hat{C}' &= \mathrm{e}^{-\hat{G}(0)} \\
&= \mathrm{e}^{-\frac{\mathit{\Delta} m}{2}\gamma^{\Venus 2} - \frac{\mathit{\Delta} m^*}{2}\tilde{\gamma}^{\Venus 2} - \mathit{\Delta} n\gamma^{\Venus}\tilde{\gamma}^{\Venus} - \alpha\gamma^{\Venus} - \alpha^*\tilde{\gamma}^{\Venus}}.
\end{aligned}
\tag{3.3.90}
$$

Finally, we obtain

$$|0_{r,\alpha}(t)\rangle = \mathrm{e}^{\hat{G}(t)-\hat{G}(0)}|0_{r,\alpha}(0)\rangle \tag{3.3.91}$$

(3.3.88) and (3.3.91) imply that $\gamma^{♀}\tilde{\gamma}^{♀}$ pair, $\gamma^{♀^2}$ pair, $\tilde{\gamma}^{♀^2}$ pair, $\gamma^{♀}$ and $\tilde{\gamma}^{♀}$ condense for the thermal vacuum. The condensation of $\gamma^{♀^2}$ pair and $\tilde{\gamma}^{♀^2}$ pair cause symmetry breaking and $\Delta m(t) - m_{f,r,\alpha}$ is the order parameter.

We confirm (3.3.74) is satisfied for the solution (3.3.91). Applying $\gamma_{r,\alpha;t}$ on $|0_{\theta,\alpha(t)}\rangle$, we obtain

$$\begin{aligned}
\gamma_{r,\alpha;t}|0_{r,\alpha}(t)\rangle &= \gamma_{r,\alpha;t}\mathrm{e}^{\hat{G}(t)-\hat{G}(0)}|0_{r,\alpha}\rangle \\
&= \mathrm{e}^{\hat{G}(t)-\hat{G}(0)}\mathrm{e}^{-\hat{G}(t)+\hat{G}(0)}\gamma_{r,\alpha;t}\mathrm{e}^{\hat{G}(t)-\hat{G}(0)}|0_{r,\alpha}\rangle \\
&= \mathrm{e}^{\hat{G}(t)-\hat{G}(0)}\hat{V}(t)|0_{r,\alpha}\rangle,
\end{aligned} \tag{3.3.92}$$

where

$$\hat{V}(t) = \mathrm{e}^{-\hat{G}(t)+\hat{G}(0)}\gamma_{r,\alpha;t}\mathrm{e}^{\hat{G}(t)-\hat{G}(0)}. \tag{3.3.93}$$

The equation of motion for $\hat{V}(t)$ is given by

$$\begin{aligned}
\frac{d}{dt}\hat{V}(t) = &-\mathrm{e}^{-\hat{G}(t)+\hat{G}(0)}\left[\frac{d\hat{G}(t)}{dt}, \gamma_{r,\alpha;t}\right]\mathrm{e}^{\hat{G}(t)-\hat{G}(0)} \\
&+\mathrm{e}^{-\hat{G}(t)+\hat{G}(0)}\frac{d\gamma_{r,\alpha;t}}{dt}\mathrm{e}^{\hat{G}(t)-\hat{G}(0)}.
\end{aligned} \tag{3.3.94}$$

The differentials in the right hand-side of (3.3.94) can be calculated as

$$\left[\frac{d\hat{G}(t)}{dt}, \gamma_{r,\alpha;t}\right] = -\frac{d\Delta m(t)}{dt}\gamma^{♀} - \frac{d\Delta n(t)}{dt}\tilde{\gamma}^{♀} - \frac{d\alpha}{dt} \tag{3.3.95}$$

and

$$\begin{aligned}
\frac{d\gamma_{r,\alpha;t}}{dt} &= \frac{d\Delta n(t)}{dt}a + \frac{d\Delta m(t)}{dt}\tilde{a} - \frac{d\Delta n(t)}{dt}\theta a^\dagger - \frac{d\Delta m(t)}{dt}a^\dagger - \frac{d\alpha(t)}{dt} \\
&= -\frac{d\Delta m(t)}{dt}\gamma^{♀} - \frac{d\Delta n(t)}{dt}\tilde{\gamma}^{♀} - \frac{d\alpha}{dt},
\end{aligned} \tag{3.3.96}$$

then, the right hand-side of (3.3.94) is canceled. Therefore, we see

$$\frac{d}{dt}\hat{V}(t) = 0, \tag{3.3.97}$$

and can solve the differential equation as

$$\hat{V}(t) = \hat{V}(0) = \gamma_{r,\alpha}. \tag{3.3.98}$$

(3.3.92) and (3.3.98) imply

$$\begin{aligned}
\gamma_{r,\alpha;t}|0_{r,\alpha}(t)\rangle &= \mathrm{e}^{\hat{G}(t)-\hat{G}(0)}\hat{V}(t)|0(\chi,\alpha)\rangle \\
&= \mathrm{e}^{\hat{G}(t)-\hat{G}(0)}\gamma_{r,\alpha}|0(\chi,\alpha)\rangle \\
&= 0,
\end{aligned} \tag{3.3.99}$$

thus, (3.3.74) is satisfied.

$$\alpha(t) = \mathrm{e}^{-\kappa t}\mathrm{ch}(\chi t)\alpha + \mathrm{e}^{-\kappa t}\mathrm{sh}(\chi t)\alpha^* \tag{3.3.100}$$

$$= \mathrm{e}^{-(\kappa-\chi)t}\alpha' + i\mathrm{e}^{-(\kappa+\chi)t}\alpha'' \tag{3.3.101}$$

where

$$\alpha' = \Re\mathrm{e}\ \alpha, \quad \alpha'' = \Im\mathrm{m}\ \alpha. \tag{3.3.102}$$

In followings, we represent $\varDelta m$ in the same way, i.e,

$$\varDelta m' = \Re\mathrm{e}\varDelta m, \quad \varDelta m'' = \Im\mathrm{m}\varDelta m. \tag{3.3.103}$$

Then we obtain

$$\begin{aligned}\varDelta n(t) = \varDelta n &- \left(1 - \mathrm{e}^{-2(\kappa-\chi)t}\right) \frac{\kappa(\varDelta n + \varDelta m' - \bar{n}) - \chi(\varDelta n + \varDelta m' + \frac{1}{2})}{2(\kappa-\chi)} \\ &- \left(1 - \mathrm{e}^{-2(\kappa+\chi)t}\right) \frac{\kappa(\varDelta n - \varDelta m' - \bar{n}) + \chi(\varDelta n - \varDelta m' + \frac{1}{2})}{2(\kappa+\chi)}\end{aligned} \tag{3.3.104}$$

,and

$$\begin{aligned}\varDelta m(t) = \varDelta m &- \left(1 - \mathrm{e}^{-2(\kappa-\chi)t}\right) \frac{\kappa(\varDelta n + \varDelta m' - \bar{n}) - \chi(\varDelta n + \varDelta m' + \frac{1}{2})}{2(\kappa-\chi)} \\ &+ \left(1 - \mathrm{e}^{-2(\kappa+\chi)t}\right) \frac{\kappa(\varDelta n - \varDelta m' - \bar{n}) + \chi(\varDelta n - \varDelta m' + \frac{1}{2})}{2(\kappa+\chi)} \\ &- \left(1 - \mathrm{e}^{-2\kappa t}\right) i\varDelta m''.\end{aligned} \tag{3.3.105}$$

The expectation values and the variances of X and P can be represented by the parameters $\alpha(t)$, $\varDelta m(t)$ and $\varDelta n(t)$, i.e,

$$\begin{aligned}\langle X\rangle_t &= \langle 1|\frac{1}{\sqrt{2}}(a + a^\dagger)|0(t)\rangle \\ &= \sqrt{2}\alpha'(t),\end{aligned} \tag{3.3.106}$$

$$\begin{aligned}\langle P\rangle_t &= \langle 1|\frac{1}{\sqrt{2}i}(a - a^\dagger)|0(t)\rangle \\ &= \sqrt{2}\alpha''(t),\end{aligned} \tag{3.3.107}$$

$$\begin{aligned}\langle(\varDelta X)^2)\rangle_t &= \frac{1}{2}\left\{\langle 1|(a + a^\dagger)^2|0(t)\rangle - \langle 1|(a + a^\dagger)|0(t)\rangle^2\right\} \\ &= \varDelta m'(t) + \varDelta n(t) + \frac{1}{2},\end{aligned} \tag{3.3.108}$$

$$\begin{aligned}\langle(\varDelta P)^2)\rangle_t &= -\frac{1}{2}\left\{\langle 1|(a - a^\dagger)^2|0(t)\rangle + \langle 1|(a - a^\dagger)|0(t)\rangle^2\right\} \\ &= -\varDelta m'(t) + \varDelta n(t) + \frac{1}{2}.\end{aligned} \tag{3.3.109}$$

We see that $|0(t)\rangle$ is squeezed when the parameters $\varDelta m'(t)$ and $\varDelta n(t)$ satisfy

$$-\varDelta m'(t) + \varDelta n(t) < 0. \tag{3.3.110}$$

At $t \to +\infty$, the variances achieve to

$$\langle(\Delta X)^2\rangle_t \to \begin{cases} +\infty & (\kappa < \chi) \\ +\infty & (\kappa = \chi) \\ \frac{\kappa(\bar{n}+\frac{1}{2})}{\kappa+\chi} & (\kappa > \chi) \end{cases} \tag{3.3.111}$$

$$\langle(\Delta P)^2\rangle_t \to \begin{cases} \frac{\kappa(\frac{1}{2}+\bar{n})}{\kappa+\chi} & (\kappa < \chi) \\ \frac{1}{4}+\frac{\bar{n}}{2} & (\kappa = \chi) \\ \frac{\kappa(\bar{n}+\frac{1}{2})}{\kappa+\chi} & (\kappa > \chi) \end{cases} \tag{3.3.112}$$

with the initial condition $\Delta n = \Delta m = 0$. In the case of $\kappa \le \chi$, $\langle(\Delta X)^2\rangle_t$ is enhanced by the effect of the antisqueezing, and otherwise, $\langle(\Delta X)^2\rangle_t$ comes to a constant by the effect of the dissipation.

3.4 Derivation of $\hat{\Pi}$ from Quantum Master Equation

In this section, we derivate the master equation in the squeezing process, in order to confirm the form of $\hat{\Pi}$. In the case of semi-free model, the master equation is derived in Appendix (A.3), and we use the same formalism in this section.

The Hamiltonian is given by

$$H_t = H_{0,t} + gH_1 \tag{3.4.1}$$

$$H_{0,t} = \omega a^\dagger a + \frac{i\chi}{2}\left(a^{\dagger 2}\mathrm{e}^{-i2\omega t} - a^2\mathrm{e}^{i2\omega t}\right) + \sum_j \omega_j b_j^\dagger b_j \tag{3.4.2}$$

$$H_1 = \sum_j (ab_j^\dagger + a^\dagger b_j). \tag{3.4.3}$$

The equation of motion of the density operator within interaction representation is written by the same form as (A.4.79):

$$\frac{\partial}{\partial t}\rho^{(I)}(t) = -g^2\int_{t_0}^{t} dt' \langle L_1^{(I)}(t)L_1^{(I)}(t')\rangle_B \rho^{(I)}(t). \tag{3.4.4}$$

The integrated terms expanded by

$$\begin{aligned} \langle L_1^{(I)}(t)L_1^{(I)}(t')\rangle_B &= \mathrm{tr}_B\left[H_{SB}^{(I)}(t), \left[H_{SB}^{(I)}(t'), \rho_B\rho^{(I)}(t)\right]\right] \\ &= \mathrm{tr}_B\left\{H_{SB}^{(I)}(t)\left(H_{SB}^{(I)}(t')\rho_B\rho^{(I)}(t) - \rho_B\rho^{(I)}(t)H_{SB}^{(I)}(t')\right)\right. \\ &\qquad \left. - \left(H_{SB}^{(I)}(t')\rho_B\rho^{(I)}(t) - \rho_B\rho^{(I)}(t)H_{SB}^{(I)}(t')\right)H_{SB}^{(I)}(t)\right\} \\ &= \mathrm{tr}_B H_{SB}^{(I)}(t)H_{SB}^{(I)}(t')\rho_B\rho^{(I)}(t) - \mathrm{tr}_B H_{SB}^{(I)}(t)\rho_B\rho^{(I)}(t)H_{SB}^{(I)}(t') \\ &\quad - \mathrm{tr}_B H_{SB}^{(I)}(t')\rho_B\rho^{(I)}(t)H_{SB}^{(I)}(t) + \mathrm{tr}_B\rho_B\rho^{(I)}(t)H_{SB}^{(I)}(t')H_{SB}^{(I)}(t) \end{aligned} \tag{3.4.5}$$

then, the first term is written by

$$\begin{aligned}
&\mathrm{tr}_B H_{SB}^{(I)}(t) H_{SB}^{(I)}(t')\rho_B \rho^{(I)}(t) \\
&= \sum_{\ell,m} \left\{ \langle b_\ell(t) b_m^\dagger(t')\rangle_B a^\dagger(t) a(t')\rho^{(I)}(t) + \langle b_\ell^\dagger(t) b_m(t')\rangle_B a(t) a^\dagger(t')\rho^{(I)}(t) \right\}
\end{aligned} \tag{3.4.6}$$

Here, $b(t)$ is the same as that in the case of semi-free, however, $a(t)$ is different. $a(t)$ in this case is estimated to

$$a(t) = U^\dagger(t) a U(t), \tag{3.4.7}$$

where $U(t)$ is given by

$$\frac{d}{dt} U(t) = -iH_{0,t} U(t). \tag{3.4.8}$$

Differentiating it with respect to t, we obtain

$$\begin{aligned}
\frac{d}{dt} a(t) &= U^\dagger(t) \left[iH_{0,t}, a \right] U(t) \\
&= -\omega a(t) + \mathrm{e}^{-i2\omega t} \chi a^\dagger(t),
\end{aligned} \tag{3.4.9}$$

and , in the same way, we also have

$$\begin{aligned}
\frac{d}{dt} a^\dagger(t) &= U^\dagger(t) [iH_{0,t}, a] U(t) \\
&= \omega a^\dagger(t) + \mathrm{e}^{i2\omega t} \chi a(t).
\end{aligned} \tag{3.4.10}$$

Solving these equations, we obtain

$$\begin{aligned}
a(t) &= \{\cosh(\chi t) a + \sinh(\chi t)\} \mathrm{e}^{-i\omega t} \\
&= A(t) \mathrm{e}^{-i\omega t},
\end{aligned} \tag{3.4.11}$$

where

$$A(t) = \cosh(\chi t) a + \sinh(\chi t). \tag{3.4.12}$$

Applying this solution to (A.4.87), we have

$$\begin{aligned}
\langle L_1^{(I)}(t) L_1^{(I)}(t')\rangle_B &= \phi_{-+}(t,t') \mathrm{e}^{i\omega(t-t')} A^\dagger(t) A(t') \rho^{(I)}(t) + \phi_{+-}(t,t') \mathrm{e}^{-i\omega(t-t')} A(t) A^\dagger(t') \rho^{(I)}(t) \\
&\quad -\phi_{-+}(t',t) \mathrm{e}^{-i\omega(t-t')} A \rho^{(I)} A^\dagger - \phi_{+-}(t',t) \mathrm{e}^{i\omega_S(t-t')} A^\dagger(t) \rho^{(I)}(t) A(t') \\
&\quad -\phi_{-+}(t,t') \mathrm{e}^{i\omega(t-t')} A \rho^{(I)}(t') A^\dagger(t) - \phi_{+-}(t,t') \mathrm{e}^{-i\omega(t-t')} A^\dagger(t') \rho^{(I)}(t) A(t) \\
&\quad +\phi_{-+}(t',t) \mathrm{e}^{-i\omega(t-t')} \rho^{(I)}(t) A^\dagger(t') A(t) + \phi_{+-}(t',t) \mathrm{e}^{i\omega(t-t')} \rho^{(I)}(t) A(t') A^\dagger(t) \\
&= \phi_{-+}(t,t') \mathrm{e}^{i\omega(t-t')} [A^\dagger(t), A(t') \rho^{(I)}(t)] + \phi_{+-}(t,t') \mathrm{e}^{-i\omega(t-t')} [A(t), A^\dagger(t') \rho^{(I)}(t)] \\
&\quad +\phi_{-+}(t',t) \mathrm{e}^{-i\omega(t-t')} [\rho^{(I)}(t) A^\dagger(t'), A(t)] + \phi_{+-}(t',t) \mathrm{e}^{i\omega(t-t')} [\rho^{(I)}(t) A(t'), A^\dagger(t)].
\end{aligned} \tag{3.4.13}$$

The term $A(t')$ contents $\mathrm{e}^{\pm\chi t'}$, so we should quantify this integration with this exponential term. Here, we consider following integral:

$$\int_{-\infty}^{\infty} d\omega \int_{0}^{\infty} dt g(\omega) f(t) \mathrm{e}^{i\omega t-\epsilon t-\delta\omega^2}, \tag{3.4.14}$$

where $g(\omega)$ and $f(t)$ are general functions. Using Taylor expansion, this integral is written to

$$\begin{aligned}
\int_{-\infty}^{\infty} d\omega \int_{0}^{\infty} dt g(\omega) f(t) \mathrm{e}^{i\omega t-\epsilon t-\delta\omega^2} &= \int_{\infty}^{\infty} d\omega \int_{0}^{\infty} dt \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n g(\omega) \mathrm{e}^{i\omega t-\epsilon t-\delta\omega^2} \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \int_{\infty}^{\infty} d\omega \int_{0}^{\infty} dt g(\omega) \mathrm{e}^{-\delta\omega^2} (-i)^n \frac{d^n}{d\omega^n} \mathrm{e}^{i\omega t-\epsilon t} \\
&= \sum_{n=0}^{\infty} i^n \frac{f^{(n)}(0)}{n!} \int_{\infty}^{\infty} d\omega \int_{0}^{\infty} dt \mathrm{e}^{i\omega t-\epsilon t} \frac{d^n}{d\omega^n} \left[g(\omega) \mathrm{e}^{-\delta\omega^2} \right] \\
&= \sum_{n=0}^{\infty} i^n \frac{f^{(n)}(0)}{n!} \int_{\infty}^{\infty} d\omega \frac{1}{i\omega-\epsilon} \frac{d^n}{d\omega^n} \left[g(\omega) \mathrm{e}^{-\delta\omega^2} \right] \\
&= \sum_{n=0}^{\infty} i^n \frac{f^{(n)}(0)}{n!} \left[\int_{\infty}^{\infty} d\omega \pi \delta(\omega) g^{(n)}(\omega) + iP \int_{\infty}^{\infty} d\omega \frac{g^{(n)}(\omega)}{\omega} \right] \\
&= \sum_{n=0}^{\infty} i^n \frac{1}{n!} \pi f^{(n)}(0) g^{(n)}(0) + \sum_{n=0}^{\infty} i^{n+1} \frac{f^{(n)}(0)}{n!} P \int_{-\infty}^{\infty} d\omega \frac{g^{(n)}(\omega)}{\omega}.
\end{aligned} \tag{3.4.15}$$

At the third equality, we use partial integration, and P is the principle value. In order to apply this expansion to our problem, we put

$$\begin{aligned}
g(\omega) &= \mathcal{D}(\omega+\omega_0)\bar{n}(\omega) \\
&= \mathcal{D}_d(\omega+\omega_0)^d \frac{1}{\mathrm{e}^{\omega/k_BT}-1} \qquad (3.4.16) \\
f(t) &= \mathrm{e}^{\pm\chi}. \qquad (3.4.17)
\end{aligned}$$

where $\mathcal{D}(\omega)$ is the density of states, $\bar{n}(\omega)$ is particle number at stationary state, and d is dimension of the space. Here, we suppose the temperature of bath is low enough corresponding to the energy ω, and approximate

$$g(\omega) = \mathcal{D}_d(\omega+\omega_0)^d \mathrm{e}^{-\omega/k_BT}. \tag{3.4.18}$$

In this case, the nth order differentials are estimated to

$$\begin{aligned}
g^{(n)}(\omega) &= \sum_{k=1}^{n} \mathcal{D}_{d\,n}C_k \left[\frac{d^k}{d\omega^k} (\omega+\omega_0)^{d-1} \right] \left[\frac{d^{n-k}}{d\omega^{n-k}} \mathrm{e}^{-\beta(\omega+\omega_0)} \right] \\
&= \sum_{k=1}^{n} {}_nC_k \frac{(d-1)!}{(d-1-k)!} \frac{\beta^{n-k}}{(\omega+\omega_0)^k} \mathcal{D}(\omega+\omega_0)\bar{n}(\omega+\omega_0), \qquad (3.4.19) \\
f^{(n)}(t) &= (\pm\chi)^n \mathrm{e}^{\pm\chi t}, \qquad (3.4.20)
\end{aligned}$$

then, the nth term in (3.4.15) is written by

$$f^{(n)}(0) g^{(n)}(0) = \sum_{k=1}^{n} (\pm 1)^n {}_nC_k \frac{(d-1)!}{(d-1-k)!} \frac{\chi^n \beta^{n-k}}{\omega_0^k} \mathcal{D}(\omega_0)\bar{n}(\omega_0). \tag{3.4.21}$$

Therefore the effective order of the integral (3.4.15) is depends on the ratio χ to ω_0 and $\beta = 1/(k_B T)$. In practice, the order of these parameter in usual experiments are $\chi \sim 10^8$[Hz], $\omega_0 \sim 10^{15}$[Hz], and $k_B T \sim 10^{11}$[Hz][4, 33]. Quantifying the ratio, $\chi^n \beta^{n-k}/\omega_0^k \sim 10^{-2} \ll 1$, we can approximate the leading order of the integral (3.4.15) is zero. Therefore, $A(t)$ in (3.4.13) can be approximated to a, then, the equation of motion (3.4.4) comes to the same form to that of semi-free model (A.4.107), i.e,

$$\begin{aligned}\frac{\partial}{\partial t}\rho^{(I)}(t) &= -ig^2(\phi''_{-+} - \phi''_{+-})[a^\dagger a, \rho^{(I)}(t)] - \pi g^2 \mathcal{D}(\omega_0)\left\{[a^\dagger, a\rho^{(I)}(t)] - [a, \rho^{(I)}(t)a^\dagger]\right\} \\ &\quad +2g^2\phi'_{+-}[a, [\rho^{(I)}(t), a^\dagger]] \\ &= -i\Delta\omega[a^\dagger a, \rho^{(I)}(t)] + \kappa\left\{[a\rho^{(I)}(t), a^\dagger] + [a, \rho^{(I)}(t)a^\dagger]\right\} \\ &\quad +2\kappa\bar{n}[a, [\rho^{(I)}(t), a^\dagger]], \end{aligned} \tag{3.4.22}$$

where

$$\kappa = \pi g^2 \mathcal{D}(\omega_0). \tag{3.4.23}$$

Here, we cancel the interaction representation by

$$\rho(t) = U(t)\rho^{(I)}(t)U^\dagger(t) \tag{3.4.24}$$

then we obtain

$$\frac{\partial}{\partial t}\rho(t) = -i[H_{0,t}, \rho(t)] + \kappa\left\{[a\rho(t), a^\dagger] + [a, \rho(t)a^\dagger]\right\} + 2\kappa\bar{n}\left[a, [\rho(t), a^\dagger]\right] \\ . \tag{3.4.25}$$

It is the master equation in the squeezing process, and the dissipation term is the same to that of semi-free model.

Chapter 4

Quantum Teleportation with Nonorthogonal Measurement

4.1 Generation of Entangled Suqueezed thermal vacuum

In this chapter, we consider quantum teleportation using squeezed vacuum generated in dissipative environment.

The Hat-Hamiltonian $\hat{H}$ in the squeezing process is given by (3.3.49). The state $|0\rangle$ at the initial time ($t = 0$) satisfies

$$a|0\rangle = \hat{a}|0\rangle = 0. \tag{4.1.1}$$

At $t = t_0$, we have squeezed vacuum given by

$$|0(r)\rangle = |0(t_0)\rangle \tag{4.1.2}$$

$$= \mathrm{e}^{\hat{G}(t_0) - \hat{G}(0)}|0\rangle, \tag{4.1.3}$$

where

$$\hat{G}(t_0) = \frac{m_r}{2}\gamma^{♀2} + \frac{m_r^*}{2}\tilde{\gamma}^{♀2} + n_r\gamma^{♀}\tilde{\gamma}^{♀} \tag{4.1.4}$$

$$n_r = \frac{2\bar{n}\kappa/\chi + 1}{4((\kappa/\chi - 1))}\left\{ 1 - \mathrm{e}^{-2(\kappa/\chi - 1)r}\right\} - \frac{-2\bar{n}\kappa/\chi + 1}{4((\kappa/\chi + 1))}\left\{ 1 - \mathrm{e}^{-2(\kappa/\chi + 1)r}\right\} \tag{4.1.5}$$

$$= \frac{1}{2}\left(s_1(t_0) - s_2(t_0)\right), \tag{4.1.6}$$

$$m_r = \frac{2\bar{n}\kappa/\chi + 1}{4(\kappa/\chi - 1)}\left\{ 1 - \mathrm{e}^{-2(\kappa/\chi - 1)r}\right\} + \frac{-2\bar{n}(\kappa/\chi + 1)}{4(\kappa/\chi + 1)}\left\{ 1 - \mathrm{e}^{-2(\kappa/\chi + 1)r}\right\} \tag{4.1.7}$$

$$= \frac{1}{2}\left(s_1(t_0) + s_2(t_0)\right), \tag{4.1.8}$$

$$s_1 = s_1(t_0) = \frac{2\bar{n}\kappa/\chi + 1}{2(\kappa/\chi - 1)} \left\{ 1 - \mathrm{e}^{-2(\kappa/\chi - 1)r} \right\}, \tag{4.1.9}$$

$$s_2 = s_2(t_0) = \frac{-2\bar{n}\kappa/\chi + 1}{2(\kappa/\chi + 1)} \left\{ 1 - \mathrm{e}^{-2(\kappa/\chi + 1)r} \right\}, \tag{4.1.10}$$

and, we used $\alpha(t) = 0$ in (3.3.88), which is derived from $\alpha(0) = \langle 1|\alpha|0\rangle = 0$ and (A.5.11). We defined $r = \chi t_0$ as the squeezing parameter.

From (3.3.108) and (3.3.109), we have

$$\langle X \rangle = 0, \tag{4.1.11}$$

$$\langle P \rangle = 0, \tag{4.1.12}$$

$$\langle (\varDelta X)^2 \rangle_r = m_r + n_r + \frac{1}{2} = s_1 + \frac{1}{2}, \tag{4.1.13}$$

$$\langle (\varDelta P)^2 \rangle_r = -m_r + n_r + \frac{1}{2} = -s_2 + \frac{1}{2}. \tag{4.1.14}$$

Passing a pair of squeezed vacua $|0(r)\rangle_A$ and $|0(-r)\rangle_B$ through a Half Beam Spriter(HBS), the entangled state between Alice and Bob

$$|\{r, -r\}\rangle_{AB} = \hat{V}_{AB}|0(r)\rangle_A \otimes |0(-r)\rangle_B \tag{4.1.15}$$

is generated, where $\hat{V}_{AB}$ is the HBS operator defined as

$$\hat{V}_{AB} = \mathrm{e}^{-\frac{\pi}{4}(a_A^{\dagger} a_B - a_A a_B^{\dagger})}. \tag{4.1.16}$$

and the entangled vacuum under environment as

$$\begin{aligned}
|\{r, -r\}\rangle_{AB} &= \hat{V}_{AB}|0(r)\rangle_A|0(-r)\rangle_B \\
&= F(-\pi/4)\tilde{F}(-\pi/4)\mathrm{e}^{\hat{G}_A(r)+\hat{G}_B(r)}|0\rangle_A|0\rangle_B \\
&= \exp\Big[-\frac{1}{2}m_r(\gamma^{\dagger 2}_A - \gamma^{\dagger 2}_B) + m_r\gamma^{\dagger}_A\gamma^{\dagger}_B - \frac{1}{2}m_r(\tilde{\gamma}^{\dagger 2}_A - \tilde{\gamma}^{\dagger 2}_B) + m_r\tilde{\gamma}^{\dagger}_A\tilde{\gamma}^{\dagger}_B \\
&\quad + \frac{m_r}{2}\gamma^{\dagger 2}_A + \frac{m_r^*}{2}\tilde{\gamma}^{\dagger 2}_A + n_r\gamma^{\dagger}_A\tilde{\gamma}^{\dagger}_A - \frac{m_r}{2}\gamma^{\dagger 2}_B - \frac{m_r^*}{2}\tilde{\gamma}^{\dagger 2}_B + n_r\gamma^{\dagger}_B\tilde{\gamma}^{\dagger}_B\Big]|0\rangle_A|0\rangle_B \\
&= \exp\Big[m_r(\gamma^{\dagger}_A\gamma^{\dagger}_B + \tilde{\gamma}^{\dagger}_A\tilde{\gamma}^{\dagger}_B) + n_r(\gamma^{\dagger}_A\tilde{\gamma}^{\dagger}_A + \gamma^{\dagger}_B\tilde{\gamma}^{\dagger}_B)\Big]|0\rangle_A|0\rangle_B \\
&= \exp\Big[m_r(b^{\dagger 2}_A + \tilde{b}^{\dagger 2}_A - b^{\dagger 2}_B - \tilde{b}^{\dagger 2}_B) + 2n_r(b^{\dagger}_A\tilde{b}^{\dagger}_A + b^{\dagger}_B\tilde{b}^{\dagger}_B)\Big]|0\rangle_A|0\rangle_B,
\end{aligned} \tag{4.1.17}$$

where $b^{\dagger}_J$ is given by

$$\begin{pmatrix} b_A \\ b_B \\ b^{\dagger}_A \\ b^{\dagger}_B \end{pmatrix} = \hat{V}_{AB} \begin{pmatrix} \gamma_A \\ \gamma_B \\ \gamma^{\dagger}_A \\ \gamma^{\dagger}_B \end{pmatrix} \hat{V}^{-1}_{AB} \tag{4.1.18}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & -2m_r \\ -1 & 1 & -2m_r & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \gamma_A \\ \gamma_B \\ \gamma^{\dagger}_A \\ \gamma^{\dagger}_B \end{pmatrix}. \tag{4.1.19}$$

$b_J^{♀}$ and b_J are the creation and annihilation operators for the entangled state $|0(r,-r)\rangle$, i.e.,

$$b_J|\{r,-r\}\rangle_{AB} = \tilde{b}_J|\{r,-r\}\rangle_{AB} = 0, \tag{4.1.20}$$

$$_{AB}\langle 1|b_J^{♀} = {}_{AB}\langle 1|\tilde{b}_J^{♀} = 0, \tag{4.1.21}$$

$$\left[b_J, b_L^{♀}\right] = \left[\tilde{b}_J, \tilde{b}_L^{♀}\right] = \delta_{JL}. \tag{4.1.22}$$

The inverse transformation is

$$\begin{pmatrix} \gamma_A \\ \gamma_B \\ \gamma_A^{♀} \\ \gamma_B^{♀} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 2m_r \\ 1 & 1 & 2m_r & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} b_A \\ b_B \\ b_A^{♀} \\ b_B^{♀} \end{pmatrix}. \tag{4.1.23}$$

4.2 Generalized Measurement in NETFD

In NETFD, a measurement with a continuous value is described by a set of operators $\{\hat{M}(x)\}$ which is represented by the product of tilde an operator and a non-tilde operator, i.e.,

$$\hat{M}(x) = M(x)\tilde{M}(x), \tag{4.2.1}$$

and satisfies the completeness

$$\int dx \langle 1|\hat{M}(x) = \langle 1|. \tag{4.2.2}$$

If the initial thermal vacuum is $|0\rangle$, the measurement gives outcome x with probability density

$$w(x) = \langle 1|\hat{M}(x)|0\rangle, \tag{4.2.3}$$

and the thermal vacuum after the measurement is

$$|0(x)\rangle = \frac{\hat{M}(x)|0\rangle}{\langle 1|\hat{M}(x)|0\rangle}. \tag{4.2.4}$$

For example, let us consider a measurement given by

$$M(\bar{x}) = \int dx m(\bar{x}|x)|x)(x|, \tag{4.2.5}$$

where $\{|x)\}$ is an orthogonal complete set, i.e.,

$$(x'|x) = \Delta(X - x'), \tag{4.2.6}$$

$$\int dx|x)(x| = I, \tag{4.2.7}$$

and $m(\bar{x}|x)$ satisfies

$$\int dx|m(\bar{x}|x)|^2 = 1. \tag{4.2.8}$$

$|m(\bar{x}|x)|^2$ is the probability density to detect value $\bar{x}$ when the correct value is x.

$\hat{M}(x)$ can be written by

$$\begin{aligned} \hat{M}(\bar{x}) &= M(\bar{x})\tilde{M}(\bar{x}) \\ &= \int dx \int dx' m(\bar{x}|x)|x)(x|\dot{m}^*(\bar{x}|x')|\tilde{x'})(\tilde{x'}| \\ &= \int dx \int dx' m(\bar{x}|x)|x)m^*(\bar{x}|x')|x,\tilde{x}'\rangle\langle x,\tilde{x'}|, \end{aligned} \tag{4.2.9}$$

where

$$|x,\tilde{x}'\rangle = |x)|\tilde{x'}). \tag{4.2.10}$$

The bra vector $\langle 1|\hat{M}(\bar{x})$ is estimated as

$$\begin{aligned} \langle 1|\hat{M}(\bar{x}) &= \int dx \int dx' m(\bar{x}|x)|x)m^*(\bar{x}|x')\langle 1|x,\tilde{x}'\rangle\langle x,\tilde{x'}| \\ &= \int dx \int dx' m(\bar{x}|x)|x)m^*(\bar{x}|x')\mathit{\Delta}(X-x')\langle x,\tilde{x'}| \\ &= \int dx|m(\bar{x}|x)|^2\langle x|, \end{aligned} \tag{4.2.11}$$

then $\hat{M}(\bar{x})$ satisfies the completeness (4.2.2), i.e.,

$$\begin{aligned} \int d\bar{x}\langle 1|\hat{M}(\bar{x}) &= \int d\bar{x}\int dx|m(\bar{x}|x)|^2\langle x| \\ &= \int dx\langle x| \\ &= \langle 1|. \end{aligned} \tag{4.2.12}$$

The probability density function (PDF) to detect $\bar{x}$ is given by

$$w(\bar{x}) = \int dx|m(\bar{x}|x))|^2\langle x|0\rangle \tag{4.2.13}$$

$$= \langle 1|\hat{M}(\bar{x})|0\rangle, \tag{4.2.14}$$

and the thermal vacuum after the measurement is

$$|0(\bar{x})\rangle = \frac{1}{w(\bar{x})}\hat{M}(\bar{x})|0\rangle. \tag{4.2.15}$$

4.3 Teleportation

We concider the case that Victor is sending a message $|\psi_{in}\rangle_V$ to Bob by quantum teleportation with the entangled state (4.1.15) between Alice and Bob. The initial state is described by

$$|in\rangle = |\psi_{in}\rangle_V \otimes |\{r,-r\}\rangle_{AB} \tag{4.3.1}$$

Alice, the secretary of Victor, conducts a simultaneous measurement of position and momentum in AV space for the state $|in\rangle$. In this case, the measurement basis is represented by

$$|\beta)_{AV} = \frac{1}{\sqrt{2\pi}}\int dx|x)_A|x+x_-)_V e^{ip_+x}, \tag{4.3.2}$$

where the parameter β is a complex number $\beta = (x_- + ip_+)/\sqrt{2}$. $|\beta)_{AV}$ is the simultaneous eigenstate of $X_V - X_A$ and $P_V + P_A$ given by

$$\begin{pmatrix} X_J \\ P_J \end{pmatrix} = \Lambda \begin{pmatrix} a_J \\ a_J^\dagger \end{pmatrix}, \tag{4.3.3}$$

$$\Lambda = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \tag{4.3.4}$$

i.e.,

$$(X_V - X_A)|\beta)_{AV} = x_-|\beta)_{AV}, \tag{4.3.5}$$

$$(P_V + P_A)|\beta)_{AV} = p_+|\beta)_{AV}. \tag{4.3.6}$$

and the set $\{|\beta)_{AV}\}$ is satisfies the orthogonality

$$_{AV}(\beta'|\beta)_{AV} = \delta^2(\beta - \beta') \tag{4.3.7}$$

and the completeness

$$\int d^2\beta|\beta)_{AV}(\beta| = I_{AV}. \tag{4.3.8}$$

Then we can describe the measurement operator by

$$\hat{M}_{AV}(\bar{\beta}) = M_{AV}(\bar{\beta})\tilde{M}_{AV}(\bar{\beta}), \tag{4.3.9}$$

$$M_{AV}(\bar{\beta}) = \int d^2\beta\, m(\bar{\beta}|\beta)|\beta)(\beta|, \tag{4.3.10}$$

where $m(\bar{\beta}|\beta)$ satisfies

$$\int d^2\bar{\beta}|m(\bar{\beta}|\beta)|^2 = 1. \tag{4.3.11}$$

Here, we rewrite $\hat{M}_{AV}(\bar{\beta})$ as

$$\hat{M}_{AV}(\bar{\beta}) = \int d^2\beta \int d^2\beta' m(\bar{\beta}|\beta)m^*(\bar{\beta}|\beta')|\beta, \tilde{\beta}'\rangle_{AV}\langle\beta, \tilde{\beta}'|, \tag{4.3.12}$$

where

$$|\beta, \tilde{\beta}'\rangle_{AV} = |\beta)_{AV}|\tilde{\beta}')_{AV}. \tag{4.3.13}$$

The operator $\hat{M}_{AV}(\bar{\beta})$ satisfies the completeness(4.2.2)

$$\begin{aligned} \int d^2\bar{\beta}_{AV}\langle 1|\hat{M}_{AV}(\bar{\beta}) &= \int d^2\bar{\beta} \int d^2\beta' \int d^2\beta m(\bar{\beta}|\beta)m^*(\bar{\beta}|\beta')_{AV}\langle 1|\beta, \tilde{\beta}'\rangle_{AV}\langle\beta, \tilde{\beta}'| \\ &= \int d^2\bar{\beta} \int d^2\beta' \int d^2\beta m(\bar{\beta}|\beta)m^*(\bar{\beta}|\beta')\delta^2(\beta - \beta')_{AV}\langle\beta, \tilde{\beta}'| \\ &= \int d^2\bar{\beta} \int d^2\beta|m(\bar{\beta}|\beta)|^2_{AV}\langle\{\beta\}| \\ &= \int d^2\beta_{AV}\langle\{\beta\}| \\ &= \langle 1|, \end{aligned} \tag{4.3.14}$$

where

$$|\{\beta\}\rangle_{AV} = |\beta, \tilde{\beta}\rangle_{AV}. \tag{4.3.15}$$

The PDF that Alice gains outcome $\bar{\beta}$ is

$$w(\bar{\beta}, r, \kappa, \bar{n}) = \langle 1|\hat{M}(\bar{\beta})|in\rangle, \tag{4.3.16}$$

and Bob's state after the measurement reduces to

$$|\{\bar{\beta}\}; r, \psi_{in}\rangle_B = \frac{1}{w(\bar{\beta}, r, \kappa, \bar{n})}{}_{AV}\langle 1|\hat{M}(\bar{\beta})|in\rangle \tag{4.3.17}$$

Note that this measurement is conducted with a degree of error described by $m(\bar{\beta}|\beta)$ and the outcome $\bar{\beta}$ is not correct parameter of measured system. $|m(\bar{\beta}|\beta)|^2$ is the PDF that Alice gains outcome $\bar{\beta}$ when the correct parameter is β.

Alice informs Bob about the measured value $\bar{\beta}$ by a classical way, then Bob operate displacement and gains the state

$$|\psi_{out}(\bar{\beta}, r)\rangle_B = \hat{D}(\bar{\beta})|\{\bar{\beta}\}; r, \psi_{in}\rangle_B \tag{4.3.18}$$

Bob consider $|\psi_{out}(\bar{\beta}, r)\rangle$ is the message from Victor.

4.4 Quantification

In order to quantfy the teleportation, we introduce one-shot fidelity (OSF), fidelity density and fidelity. OSF is given by

$$F(\bar{\beta}, r, \kappa, \bar{n}) = {}_B\langle\psi_{in}|\psi_{out}(\bar{\beta}, r)\rangle_B, \tag{4.4.1}$$

and the fidelity density is

$$\mathcal{F}(\bar{\beta}, r, \kappa, \bar{n}) = F(\bar{\beta}, r, \kappa, \bar{n})w(\bar{\beta}, r, \kappa, \bar{n}) \tag{4.4.2}$$

$$= {}_B\langle\psi_{in}|_{AV}\langle 1|\hat{M}(\beta)|in\rangle. \tag{4.4.3}$$

The fidelity of the teleportation is given by the average of OSF with respected to the PDF $w(\bar{\beta}, r, \kappa, \bar{n})$, i.e.,

$$F(r, \kappa, \bar{n}) = \int d^2\bar{\beta} w(\bar{\beta}, r, \kappa, \bar{n})F(\bar{\beta}, r, \kappa, \bar{n}), nn \tag{4.4.4}$$

$$= \int d^2\bar{\beta}\mathcal{F}(\bar{\beta}, r, \kappa, \bar{n}). \tag{4.4.5}$$

We estimate the teleportation by these quantities.

Here, we set the coherent state $|\alpha\rangle_V$ as Victor's original state, i.e.,

$$|\psi_{in}\rangle_V = |\alpha\rangle_V, \tag{4.4.6}$$

and the error is given by

$$m(\bar{\beta}|\beta) = \frac{1}{\sqrt{\pi}\Delta} \mathrm{e}^{-|\bar{\beta}-\beta|^2/(2\Delta)}. \tag{4.4.7}$$

In this case, from (A.7.24), the state Bob gains right after the measurement is written by

$$\begin{aligned} |\{\bar{\beta}\}; r, \psi_{in}\rangle_B &= \frac{1}{w(\bar{\beta}, r, \kappa, \bar{n})} {}_{AV}\langle 1|\hat{M}(\bar{\beta})|in\rangle \\ &= \frac{1}{w(\bar{\beta}, r, \kappa, \bar{n})} \frac{1}{(2+s_-)\pi\Delta} \int d^2\beta \mathrm{e}^{-|\bar{\beta}-\beta|^2/\Delta} \mathrm{e}^{-\frac{|\mu|^2}{2+s_-}} \hat{D}_B(s_+\mu^*) \mathrm{e}^{g\gamma_B^\dagger \tilde{\gamma}_B^\dagger} |0\rangle_B, \end{aligned} \tag{4.4.8}$$

where

$$\mu = \frac{1}{2+s_-}(\beta - \alpha), \tag{4.4.9}$$

$$g = \frac{s_-^2; s_- - s_+^2}{2+s_-} \tag{4.4.10}$$

The PDF to observe the values $\bar{\beta}$ is derived to

$$\begin{aligned} w(\bar{\beta}, r, \kappa, \bar{n}, \Delta) &= \langle 1|\hat{M}(\beta)|in\rangle \\ &= \frac{1}{(2+s_-)\pi\Delta} \int d^2\beta \mathrm{e}^{-|\bar{\beta}-\beta|^2/\Delta} \mathrm{e}^{-\frac{|\mu|^2}{2+s_-}} {}_B\langle 1|\hat{D}_B(s_+\mu^*) \mathrm{e}^{g\gamma_B^\dagger \tilde{\gamma}_B^\dagger}|0\rangle_B \\ &= \frac{2}{\pi(2+s_-)} \int d^2\beta \mathrm{e}^{-\frac{1}{\Delta}|\bar{\beta}-\beta|^2 - \frac{2}{2+s_-}|\mu|^2} \\ &= \frac{1}{1+s_-/2+\Delta} \exp\left[-\frac{1}{1+s_-/2+2\Delta}|\bar{\beta}-\alpha|^2\right], \end{aligned} \tag{4.4.11}$$

where, $s_\pm$ are given by

$$s_\pm = s_1 \pm s_2, \tag{4.4.12}$$

and, then we see the variance of the PDF is

$$\sigma_w(r, \kappa, \bar{n}, \Delta) = 1 + \frac{s_-}{2} + \Delta. \tag{4.4.13}$$

In the case of $\kappa/\chi \ll 1 < r$, we obtain

$$\sigma_w(r, \kappa, \bar{n}, \Delta) \simeq \mathrm{e}^{2r} \left\{ \frac{1}{2} + \frac{\kappa}{\chi}(\bar{n} + \frac{1}{2} - r) \right\} + \Delta. \tag{4.4.14}$$

The OSF with observed values $\bar{\beta}$ is given by

$$\begin{aligned} F(\bar{\beta}, r, \kappa, \bar{n}, \Delta) &= {}_B\langle \alpha|\psi_{out}(\beta, r)\rangle_B \\ &= \frac{1+s_-/2+\Delta}{(1+s_1)(1-s_2)+\Delta(1+s_-/2)} \\ &\quad \times \exp\left[-\frac{1-s_2+2\Delta(3+s_1-\frac{1+s_-/2}{1+s_1})+2\Delta^2\frac{1+s_1}{1+s_-/2}}{(1+s_1+\Delta\frac{1+s_-/2}{1-s_2})(1+s_-/2+\Delta)}|\bar{\beta}-\alpha|^2\right], \end{aligned} \tag{4.4.15}$$

$$= \left(1 + \frac{\sigma_\omega}{\sigma_F}\right) \frac{1}{(1-s_2)(2+\Delta\frac{1+s_1}{1+s_-/2})} \mathrm{e}^{-\frac{1}{\sigma_F}|\bar{\beta}-\alpha|^2}, \tag{4.4.16}$$

then the variance of the OSF is

$$\sigma_F(r,\kappa,\bar{n},\Delta) = \frac{\left(1+s_1+\Delta^2\frac{1+s_-/2}{1-s_2}\right)(1+s_-/2+\Delta)}{1-s_2+\Delta\left(3+s_1-\frac{1+s_-/2}{1+s_1}\right)+2\Delta\frac{1+s_1}{1+s_-/2}}, \tag{4.4.17}$$

and the fidelity density is given by

$$\begin{aligned}\mathcal{F}(\bar{\beta},r,\kappa,\bar{n},\Delta) &= F(\bar{\beta},r,\kappa,\bar{n},\Delta)w(\bar{\beta},r,\kappa,\bar{n},\Delta)\\ &= \frac{1}{(1+s_1)(1-s_2)+2\Delta(1+s_-/2)}\exp\left[-\frac{2+2\Delta\frac{1+s_1}{1+s_-/2}}{1+s_1+2\Delta\frac{1+s_-/2}{1-s_2}}|\bar{\beta}-\beta|^2\right].\end{aligned} \tag{4.4.18}$$

Integrating (4.4.18), we obtain the fidelity as

$$\begin{aligned}F(r,\kappa,\bar{n},\Delta) &= \int\frac{d^2\bar{\beta}}{\pi}\mathcal{F}(\bar{\beta},r,\kappa,\bar{n},\Delta)\\ &= \frac{1}{2\left(1-s_2+\Delta\frac{(1+s_1)(1-s_2)}{1+s_-/2}\right)}.\end{aligned} \tag{4.4.19}$$

In the case of $\kappa/\chi \ll 1 < r$ and $\Delta \ll 1$, we obtain

$$F(r,\kappa,\bar{n},\Delta) \simeq \frac{1}{1+\mathrm{e}^{-2r}}\left[1-\frac{\kappa}{\chi}\left\{(2\bar{n}+1)\tanh r-\frac{2r}{1+\mathrm{e}^{2r}}\right\}-\Delta\right] \tag{4.4.20}$$

$$\simeq 1-\frac{\kappa}{\chi}(2\bar{n}+1)-\Delta. \tag{4.4.21}$$

From fig.4.1 and fig.4.2, we find the fidelity is close to 1 with large r, and to 0 with large $\bar{n}$ and Δ. In Fig.(4.3) and Fig.(4.4), the variance of the PDF gets large along r. It means that Alice hardly can guess the original state and guarantee the secrecy of the information.

The variance is decreasing as κ increasing. This and (4.1.9) implies κ reduces the degree of squeezing.

$\bar{n}$ involves the dependence of κ for the PDF. In fig. 4.3(d), the variance of the PDF has peak.

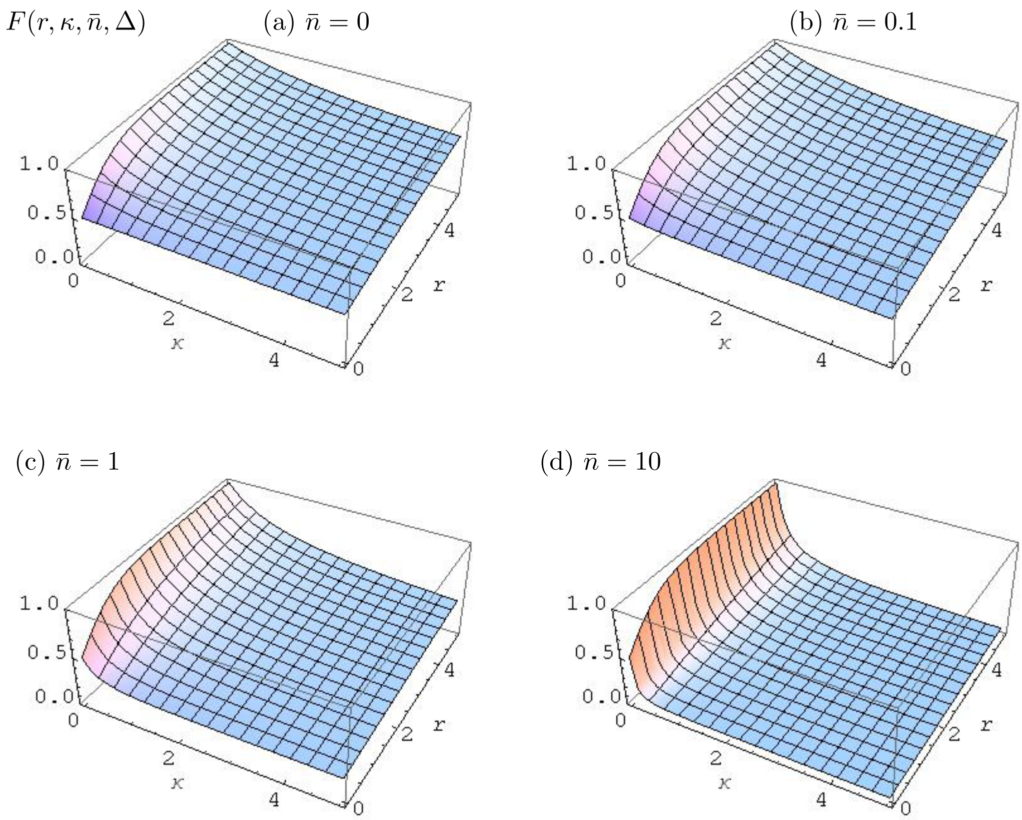

Figure 4.1: Bird's-eye view of the fidelity $F(r,\kappa,\bar{n},\Delta)$ with $\Delta=0$ for the cases (a) $\bar{n}=0$, (b) $\bar{n}=0.1$, (c) $\bar{n}=1$ and (d) $\bar{n}=10$.

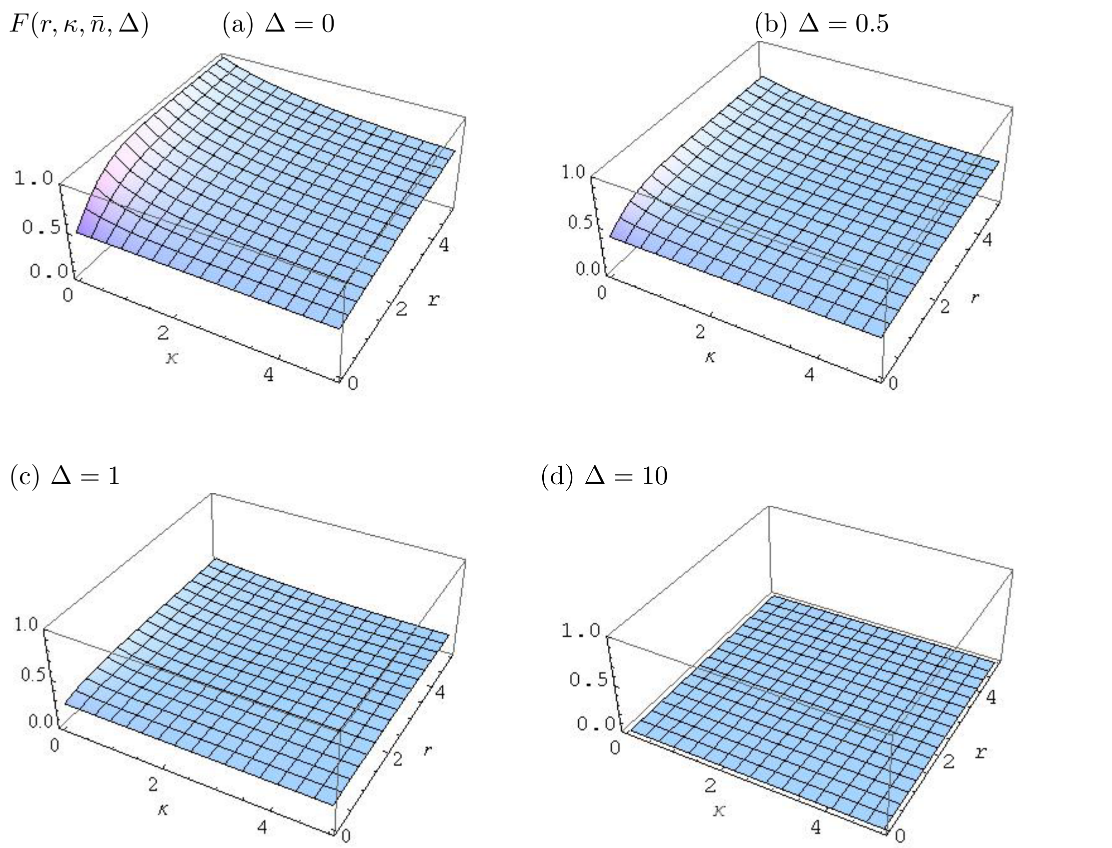

Figure 4.2: Bird's-eye view of the fidelity $F(r, \kappa, \bar{n}, \Delta)$ with $\Delta = 0$ for the cases (a) $\Delta = 0$, (b) $\Delta = 0.5$, (c) $\Delta = 1$ and (d) $\Delta = 10$.

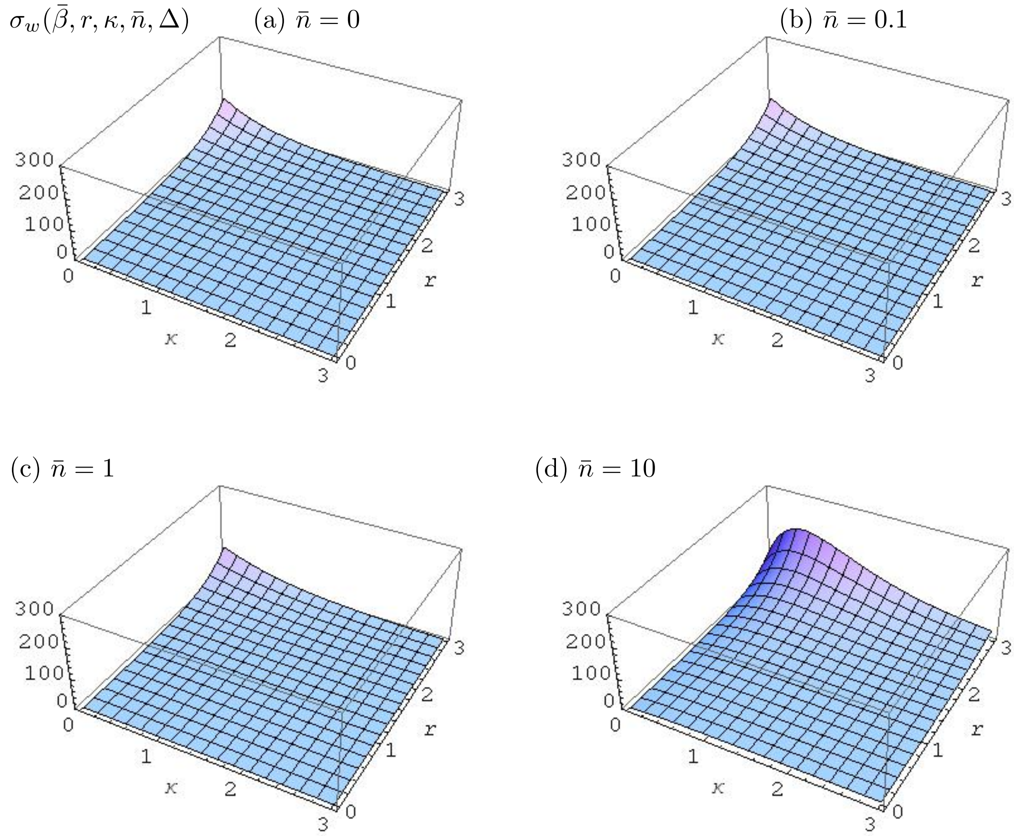

Figure 4.3: Bird's-eye view of $\sigma_w(\bar{\beta}, r, \kappa, \bar{n}, \Delta)$ with $\Delta = 0$ for the cases (a) $\bar{n} = 0$, (b) $\bar{n} = 0.1$, (c) $\bar{n} = 1$ and (d) $\bar{n} = 10$.

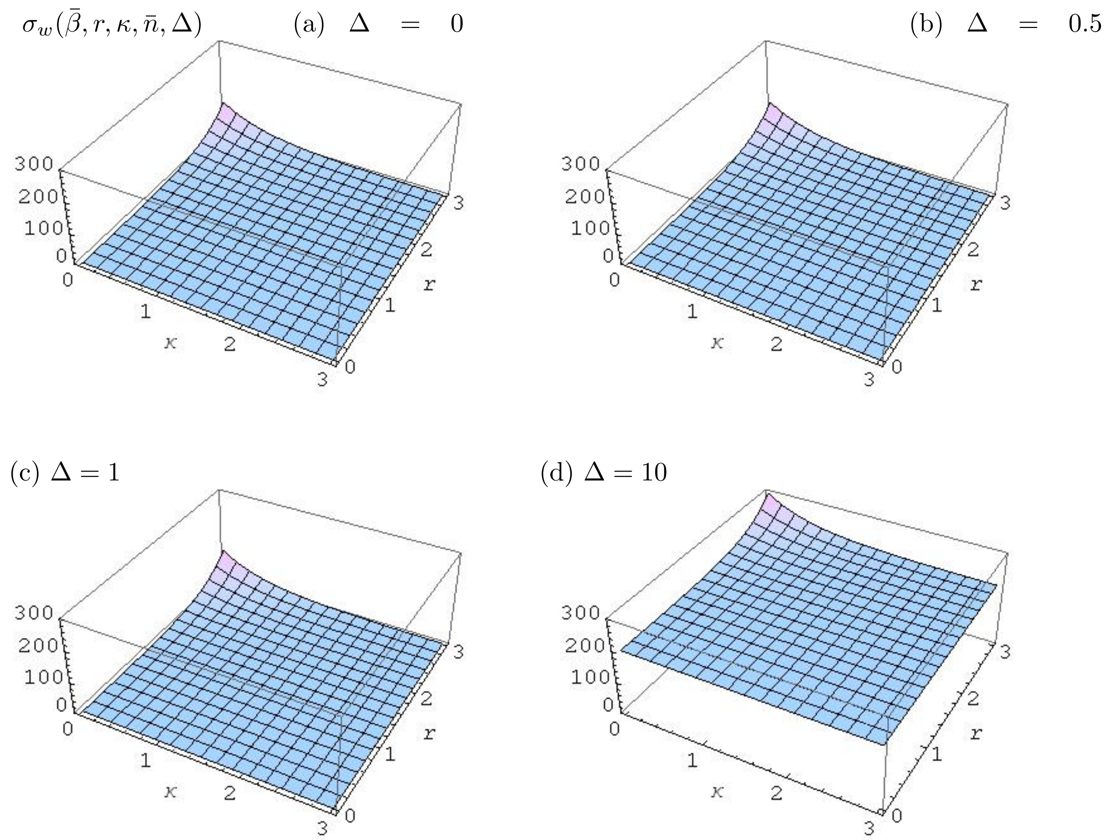

Figure 4.4: Bird's-eye view of $\sigma_w(\bar{\beta}, r, \kappa, \bar{n}, \Delta)$ with $\bar{n} = 0$ for the cases (a) $\Delta = 0$, (b) $\Delta = 0.5$, (c) $\Delta = 1$ and (d) $\Delta = 10$.

4.5 Estimation of Parameters

In experiments, the squeezing levels and the antisqueezing levels are determined by observing $\langle(\mathit{\Delta} X)^2\rangle$ and $\langle(\mathit{\Delta} P)^2\rangle$. For the conditions in this thesis, the squeezing level Ω_{sq} and antisqueezing level Ω_{anti} are defined by

$$\Omega_{sq} = 10\log_{10} 2\langle(\mathit{\Delta} P)^2\rangle_r = 10\log_{10}(1-2s_2), \tag{4.5.1}$$

$$\Omega_{anti} = 10\log_{10} 2\langle(\mathit{\Delta} X)^2\rangle_r = 10\log_{10}(1+2s_1). \tag{4.5.2}$$

In practice, the squeezing levels are achieved to $\Omega_{sq} = -9.01$ dB and $\Omega_{anti} = 15.12$ dB by Takeno and et al in the University of Tokyo in 2007[34], and they also demonstrated the quantum teleportation by using this squeezed state in 2008 and gained fidelity $F_{gain} = 0.83$[35].

Here, we try to estimate the parameters κ/χ, r and Δ in the case of these experiments. From (4.1.10), (4.5.1) and $\Omega_{sq} = -9.01$, we have

$$\frac{1}{2(1+\kappa/\chi)}\left(1-\mathrm{e}^{-2(1+\kappa/\chi)r}\right) \approx 0.44, \tag{4.5.3}$$

and from (4.1.9), (4.5.2) and $\Omega_{anti} = -15.12$, we have

$$-\frac{1}{2(1-\kappa/\chi)}\left(1-\mathrm{e}^{2(1-\kappa/\chi)r}\right) \approx 16.7, \tag{4.5.4}$$

where we approximate $\bar{n} = 0$. Solving these simultaneous equations, we obtain

$$\kappa/\chi \approx 0.13, \tag{4.5.5}$$

$$r \approx 1.9, \tag{4.5.6}$$

where κ/χ is the effect of the environment. From this result, the loss of the squeezing levels caused by thermal bath can be estimated by

$$\Delta\Omega_{sq} = \Omega_{sq} - 10\log_{10}\mathrm{e}^{-2r} \approx 7.5. \tag{4.5.7}$$

If there is no dissipation in the process of the teleportation, i.e. $s_2 = (1-\mathrm{e}^{-2r})/2$, we expect the fidelity is

$$F_0 = \frac{1}{1+\mathrm{e}^{-2r}} \approx 0.98, \tag{4.5.8}$$

and if there are the dissipation only in the squeezing process, we expect the fidelity is

$$F_1 = \frac{1}{2(1-s_2)} \approx 0.89. \tag{4.5.9}$$

In the report of [35], the obtained fidelity was $F_{gain} = 0.83$, therefore, we conclude there must be other sources of dissipations. By supposing the loss occurred in the process of the measurement, from (4.4.19), we obtain

$$F_{gain} = \frac{1}{\frac{1}{F_1} + \Delta\frac{(1+s_1)(1-s_2)}{1+s_-/2}}, \tag{4.5.10}$$

therefore, we can estimate the parameter of the measurement error Δ;

$$\begin{aligned} \Delta &= \left(\frac{1}{F_{gain}} - \frac{1}{F_{ex}}\right) \frac{1+s_-/2}{2(1+s_1)(1-s_2)} \\ &\approx 0.042. \end{aligned} \tag{4.5.11}$$

Then, we see $\kappa/\chi > \Delta$, i.e., the thermal noises in the squeezing process are more effective than the measurement error Δ to decrease the fidelity in this case.

Chapter 5

Conclusion and Discussion

5.1 Dissipation in Squeezing Process

We estimated squeezed vacuum generated in thermal bath, within NETFD. This derivation had been established in [32], and built on the assumption that the property of the thermal bath is the same to that of semi-free model. In this thesis, we show the conditions in which this assumption is proper. In the case of $\chi \ll k_B T \ll \omega_0$, the dissipative term in the master equation, or $\hat{\Pi}$ within NETFD, is the same to that of semi-free model. In practical experiment, these parameters are $\chi \sim 10^8$, $\omega_0 \sim 10^{15}$, and $k_B T \sim 10^{11}$ [4, 12, 33], then this condition is satisfied.

5.2 Measurement within NETFD

The formulation of generalized measurement in NETFD is constructed in this thesis firstly.

A measurement operator$\hat{M}(\beta)$ is separable to tilde part and non-tilde part;

$$\hat{M}(\beta) = M(\beta)\tilde{M}(\beta) \tag{5.2.1}$$

and satisfies completeness

$$\int d^2\beta \langle 1|\hat{M}(\beta) = \langle 1|. \tag{5.2.2}$$

These requirement correspond to the completeness of generalized measurement within quantum mechanics(A.8.6).

In this formulation, we describe the Bell measurement with a Gaussian error by

$$\hat{M}_{AV}(\bar{\beta}) = \int d^2\beta \int d^2\beta' m(\bar{\beta}|\beta) m^*(\bar{\beta}|\beta')|\beta, \tilde{\beta}'\rangle_{AV}\langle\beta, \tilde{\beta}'|, \tag{5.2.3}$$

$$m(\bar{\beta}|\beta) = \frac{1}{\sqrt{\pi}\Delta} \mathrm{e}^{-|\bar{\beta}-\beta|^2/(2\Delta)}, \tag{5.2.4}$$

where $m(\bar{\beta}|\beta)$ is the probability density to detect $\bar{\beta}$ when the correct value is β.

5.3 Quantum Teleportation

We estimate quantum teleportation for continuous variables via squeezed vacua generated in dissipative environment and via non-orthogonal measurement, within NETFD. We obtain the average of fidelity by

$$F(\bar{\beta}, r, \kappa, \bar{n}, \Delta) = \frac{1}{2\left(1 - s_2 + \Delta \frac{(1+s_1)(1-s_2)}{1+s_-/2}\right)}. \tag{5.3.1}$$

In the case of $\Delta = 0$, the fidelity comes to

$$F(\bar{\beta}, r, \kappa, \bar{n}, 0) = \frac{1}{2(1 - s_2)}. \tag{5.3.2}$$

In the case of $\kappa/\chi \ll 1 < r$ and $\Delta \ll 1$, we obtain

$$F(r, \kappa, \bar{n}, \Delta) \simeq \frac{1}{1 + \mathrm{e}^{-2r}} \left[1 - \frac{\kappa}{\chi}\left\{(2\bar{n}+1)\tanh r - \frac{2r}{1+\mathrm{e}^{2r}}\right\} - \Delta\right] \tag{5.3.3}$$

$$\simeq 1 - \frac{\kappa}{\chi}(2\bar{n}+1) - \Delta. \tag{5.3.4}$$

We also obtain the variance of PDF with respect to the result of the Bell measurement $\bar{\beta}$ by

$$\sigma_w(r, \kappa, \bar{n}, \Delta) = 1 + \frac{s_-}{2} + \Delta, \tag{5.3.5}$$

and in the case of $\kappa/\chi \ll 1 < r$,

$$\sigma_w(r, \kappa, \bar{n}, \Delta) \simeq \mathrm{e}^{2r}\left\{\frac{1}{2} + \frac{\kappa}{\chi}(\bar{n} + \frac{1}{2} - r)\right\} + \Delta. \tag{5.3.6}$$

When PDF is spread, it means Alice can hardly gain information of Victor's state from the Bell measurement.

By comparing the experimental reports, we can estimate the parameters κ/χ, r and Δ using (4.1.13), (4.1.14) and (4.4.19). Then we can obtain the upper bound of the measurement error Δ. For example, in the case of the experiment of Takeno et al.[35], we obtain $\chi/\kappa \approx 0.13$, $r \approx 1.9$ and $\Delta \approx 0.042$.

In theoretical analysis of the quantum teleportation with continuous quadrature states in thermal environments, the interaction of the squeezed states and the environment have been introduced in the transmission path[18, 19] or by an arbitrary intertaction parameter[20]. In this thesis, we considered the dissipative interaction in the squeezing process. The form of estimated fidelity quantitatively corresponds to that in above reports, and we could represent it by the more detailed parameters χ, κ, $\bar{n}$ and the measurement error Δ.

Appendix

A.1 Derivation of Eq.(2.2.5)

At first,we consider a transformation of

$$a_J(\theta) = V_{AB}^\dagger(\theta) a_J V_{AB}(\theta) \tag{A.1.1}$$

where

$$V_{AB}(\theta) = \mathrm{e}^{\theta(a_A a_B^\dagger - a_A^\dagger a_B)}, \tag{A.1.2}$$

and the HBS operator is $V_{AB} = V_{AB}(\theta = -\pi/4)$ Differentiating (A.1.1) with respect to θ, we obtain

$$\frac{d}{d\theta} a_A(\theta) = V_{AB}^\dagger(\theta) i[a_A, a_A a_B^\dagger - a_A^\dagger a_B] V_{AB}(\theta) = -a_B^(\theta), \tag{A.1.3}$$

$$\frac{d}{d\theta} a_B(\theta) = V_{AB}^\dagger(\theta)[a_B, a_A a_B^\dagger - a_A^\dagger a_B] V_{AB}(\theta) = a_A(\theta). \tag{A.1.4}$$

Solving these equations with the initial condition $a_J(0) = aJ$, we have

$$a_A(\theta) = \cos\theta a_A - \sin\theta a_B, \tag{A.1.5}$$

$$a_B(\theta) = \cos\theta a_B + \sin\theta a_A. \tag{A.1.6}$$

Then, the equation (2.2.5) can be rewritten by

$$\begin{aligned}
V_{AB} S_A(r) S_B(-r) |0)_A |0)_B &= V_{AB} S_A(r) V_{AB}^\dagger V_{AB} S_B(-r) V_{AB}^\dagger V_{AB} |0)_A |0)_B \\
&= \exp\left[\frac{r}{2}\{a_A^\dagger\left(-\frac{\pi}{4}\right)^2 - a_A\left(-\frac{\pi}{4}\right)^2\}\right] \exp\left[-\frac{r}{2}\{a_B^\dagger\left(-\frac{\pi}{4}\right)^2 - a_B\left(-\frac{\pi}{4}\right)^2\}\right] |0)_A |0)_B \\
&= \exp\left[\frac{r}{4}(a_A^\dagger + a_B^\dagger)^2 - \frac{r}{4}(a_A + a_B)^2\right] \\
&\quad \times \exp\left[-\frac{r}{4}(-a_A^\dagger + a_B^\dagger)^2 + \frac{r}{4}(-a_A + a_B)^2\right] |0)_A |0)_B \\
&= \exp\left[\frac{r}{4}(-a_A^2 + a_A^{\dagger^2} - a_B^2 + a_B^{\dagger 2} - 2a_A a_B + 2a_A^\dagger a_B^\dagger)\right] \\
&\quad \times \exp\left[\frac{r}{4}(a_A^2 - a_A^{\dagger^2} + a_B^2 - a_B^{\dagger 2} - 2a_A a_B + 2a_A^\dagger a_B^\dagger)\right] |0)_A |0)_B \\
&= \exp\left[-r a_A a_B + r a_A^\dagger a_B^\dagger\right] |0)_A |0)_B.
\end{aligned} \tag{A.1.7}$$

At the third equality, we used (A.1.5), (A.1.6) and $a|0) = 0$,. At the final equality, we used

$$[-a_A^2 + a_A^{\dagger^2} - a_B^2 + a_B^{\dagger 2},\ a_A a_B + a_A^\dagger a_B^\dagger] = 0. \tag{A.1.8}$$

Here, we put

$$\mathrm{e}^{r(a_A^\dagger a_B^\dagger - r a_A a_B)} = \mathrm{e}^{q(r) a_A^\dagger a_B^\dagger} Q(r), \tag{A.1.9}$$

where $Q(r)$ is an operator and $q(r)$ is a c-number function satisfying $Q(0) = 1$, $q(0) = 0$. Differentiating (A.1.9) with respect to r, we obtain

$$(a_A^\dagger a_B^\dagger - a_A a_B)\mathrm{e}^{q(r) a_A^\dagger a_B^\dagger} Q(r) = \frac{dq(r)}{dr} a_A^\dagger a_B^\dagger \mathrm{e}^{q(r) a_A^\dagger a_B^\dagger} Q(r) + \mathrm{e}^{q(r) a_A^\dagger a_B^\dagger} \frac{dQ(r)}{dr}, \tag{A.1.10}$$

therefore,

$$\frac{dQ(r)}{dr} = \left[\left\{ 1 - \frac{dq(r)}{dr} \right\} a_A^\dagger a_B^\dagger - \mathrm{e}^{-q(r) a_A^\dagger a_B^\dagger} a_A a_B \mathrm{e}^{q(r) a_A^\dagger a_B^\dagger} \right] Q(r). \tag{A.1.11}$$

Here, we consider the transformation

$$a_J(r) = \mathrm{e}^{-q(r) a_A^\dagger a_B^\dagger} a_J \mathrm{e}^{q(r) a_A^\dagger a_B^\dagger}. \tag{A.1.12}$$

Differentiating (A.1.12) with respect to r, we have

$$\frac{d}{dr} a_A(r) = \frac{dq(r)}{dr} \mathrm{e}^{-q(r) a_A^\dagger a_B^\dagger} [a_A, a_A^\dagger a_B^\dagger] \mathrm{e}^{q(r) a_A^\dagger a_B^\dagger} = \frac{dq(r)}{dr} a_B^\dagger, \tag{A.1.13}$$

$$\frac{d}{dr} a_B(r) = \frac{dq(r)}{dr} \mathrm{e}^{-q(r) a_A^\dagger a_B^\dagger} [a_B, a_A^\dagger a_B^\dagger] \mathrm{e}^{q(r) a_A^\dagger a_B^\dagger} = \frac{dq(r)}{dr} a_A^\dagger. \tag{A.1.14}$$

Solving these equations with $a_J((r = 0) = a_J$, we obtain

$$a_A(r) = a_A + q(r) a_B^\dagger \tag{A.1.15}$$

$$a_B(r) = a_B + q(r) a_A^\dagger. \tag{A.1.16}$$

Then the equation (A.1.11) is rewritten by

$$\begin{aligned} \frac{dQ(r)}{dr} &= \left[\left\{ 1 - \frac{dq(r)}{dr} \right\} a_A^\dagger a_B^\dagger - \left\{ a_A + q(r) a_B^\dagger \right\} \left\{ a_B + q(r) a_A^\dagger \right\} \right] Q(r) \\ &= \left[\left\{ 1 - \frac{dq(r)}{dr} - q(r)^2 \right\} a_A^\dagger a_B^\dagger - a_A a_B - q(r) a_A^\dagger a_A - q(r) a_B^\dagger a_B + q(r) \right] Q(r). \end{aligned} \tag{A.1.17}$$

We require $q(r)$ to satisfy

$$1 - \frac{dq(r)}{dr} - q(r)^2 = 0, \tag{A.1.18}$$

and by solving it, $q(r)$ is decided by

$$q(r) = \tanh r. \tag{A.1.19}$$

Then we obtain

$$\frac{dQ(r)}{dr} = \left[-a_A a_B - \tanh r a_A^\dagger a_A - \tanh r a_B^\dagger a_B - \tanh r \right] Q(r). \tag{A.1.20}$$

Solving this differential equation with $Q(0) = 1$, we have

$$Q(r) = \frac{1}{\cosh r} \mathrm{e}{-r a_A a_B - \ln \cosh r (a_A^\dagger a_A - \tanh r a_B^\dagger a_B)}. \tag{A.1.21}$$

Therefore, (A.1.7) is rewritten by

$$
\begin{aligned}
|r,-r)_{AB} &= V_{AB}S_A(r)S_B(-r)|0)_A|0)_B \\
&= \frac{1}{\cosh r}\mathrm{e}^{\tanh r a_A^\dagger a_B^\dagger}\mathrm{e}^{-r a_A a_B - \ln\cosh r(a_A^\dagger a_A - \tanh r a_B^\dagger a_B)|0)_A|0)_B} \\
&= \frac{1}{\cosh r}\mathrm{e}^{\tanh r a_A^\dagger a_B^\dagger |0)_A|0)_B}. \qquad \text{(A.1.22)} \\
&= \frac{1}{\cosh r}\sum_{n=0}^{\infty}(\tanh r)^n|n)_A|n)_B, \qquad \text{(A.1.23)}
\end{aligned}
$$

where $|n)_J$ is number state which satisfies

$$a_J^\dagger a_J|n)_J = n|n)_J. \qquad \text{(A.1.24)}$$

Here, $|n)_J$ is written by

$$|n)_J = \int dx H_n(x)|x)_J, \qquad \text{(A.1.25)}$$

where $H_n(x)$ is Hermite polynomial. Therefore, (A.1.22) can be rewritten by

$$|r,-r)_{AB} = \frac{1}{\cosh r}\sum_{n=0}^{\infty}\int dx \int dx'(\tanh r)^n H_n(x)H_n(x')|x)_A|x')_B. \qquad \text{(A.1.26)}$$

Since Hermite polynomial satisfies the orthogonality $\sum_{n=0}^{\infty} H_n(x)H_n(x') = \Delta(X-x')$, we obtain

$$\lim_{r\to\infty}|r,-r)_{AB} \propto \int dx|x)_A|x)_B. \qquad \text{(A.1.27)}$$

A.2 Inseparability Criterion

By supposing the composite system is separable, i.e. the density operator can be written by

$$\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i, \tag{A.2.1}$$

the variances of $XA - XB$ is estimated to

$$\begin{aligned}
\langle\{\mathit{\Delta}(X_A - X_B)\}^2\rangle &= \mathrm{tr}\rho(X_A - X_B)^2 - [\mathrm{tr}\rho(X_A - X_B)]^2 \\
&= \sum_i p_i \langle X_A - X_B\rangle_i^2 - \left[\sum_i p_i \langle X_A - X_B\rangle_i\right]^2 \\
&= \sum_i p_i \left[\langle(\mathit{\Delta}X_A)^2\rangle_i + \langle(\mathit{\Delta}X_B)^2\rangle_i - 2\langle X_A\rangle_i \langle X_B\rangle_i + \langle X_A\rangle_i^2 + \langle X_B\rangle_i^2\right] \\
&\quad - \left[\sum_i p_i \langle X_A - X_B\rangle_i\right]^2 \\
&= \sum_i p_i \left[\langle(\mathit{\Delta}X_A)^2\rangle_i + \langle(\mathit{\Delta}X_B)^2\rangle_i\right] + \sum_i p_i \langle X_A - X_B\rangle_i^2 \\
&\quad - \left[\sum_i p_i \langle X_A - X_B\rangle_i\right]^2,
\end{aligned} \tag{A.2.2}$$

where $\langle\cdots\rangle_i$ means the expectation with respect to $\rho_A^{(i)} \otimes \rho_B^{(i)}$. In the same way, the variance of $P_A + P_B$ is estimated to

$$\begin{aligned}
\langle\{\mathit{\Delta}(P_A + P_B)\}^2\rangle &= \sum_i p_i \left[\langle(\mathit{\Delta}P_A)^2\rangle_i + \langle(\mathit{\Delta}P_B)^2\rangle_i\right] + \sum_i p_i \langle P_A + P_B\rangle_i^2 \\
&\quad - \left[\sum_i p_i \langle P_A + P_B\rangle_i\right]^2.
\end{aligned} \tag{A.2.3}$$

Then we obtain

$$\begin{aligned}
\langle\{\mathit{\Delta}(X_A - X_B)\}^2\rangle + \langle\{\mathit{\Delta}(P_A + P_B)\}^2\rangle &= \sum_i p_i \left[\langle(\mathit{\Delta}X_A)^2\rangle_i + \langle(\mathit{\Delta}X_B)^2\rangle_i + \langle(\mathit{\Delta}P_A)^2\rangle_i + \langle(\mathit{\Delta}P_B)^2\rangle_i\right] \\
&\quad + \sum_i p_i \langle X_A - X_B\rangle_i^2 - \left[\sum_i p_i \langle X_A - X_B\rangle_i\right]^2 \\
&\quad + \sum_i p_i \langle P_A + P_B\rangle_i^2 - \left[\sum_i p_i \langle P_A + P_B\rangle_i\right]^2.
\end{aligned} \tag{A.2.4}$$

From the uncertainty relation we have

$$\langle(\mathit{\Delta}X_J)^2\rangle\rangle_i + \langle(\mathit{\Delta}P_J)^2\rangle\rangle_i \quad \geq \quad 2\sqrt{\langle(\mathit{\Delta}X_J)^2\rangle\rangle_i \langle(\mathit{\Delta}P_J)^2\rangle\rangle_i} \quad = \quad 1, \tag{A.2.5}$$

then the first line in (A.2.4) is bounded from below by 2. From Cauchy-Schwarz inequality, we have

$$\sum_i p_i \langle\cdot\rangle_i^2 = \left(\sum_i p_i\right)\left(\sum_i p_i \langle\cdot\rangle_i^2\right) \geq \left(\sum_i p_i \langle\cdot\rangle_i\right)^2, \tag{A.2.6}$$

then the second and the third lines in (A.2.4) are bounded from below by zero. Therefore, we obtain

$$\langle\{\mathit{\Delta}(X_A - X_B)\}^2\rangle + \langle\{\mathit{\Delta}(P_A + P_B)\}^2\rangle \geq 2 \tag{A.2.7}$$

for any separable state.

If this inequality is broken, the system is inseparable.

A.3 Representation of $\langle 1|$

Thermal bra vacuum $\langle 1|$ can be represented by

$$\langle 1| = \sum_{n=0}^{\infty} (n|(\tilde{n}|, \tag{A.3.1}$$

where $|n)|\tilde{m})$ is the eigenvector of $a^\dagger a$ and $\tilde{a}^\dagger \tilde{a}$, i.e.,

$$a^\dagger a|n)|\tilde{m}) = n|n)|\tilde{m}) \tag{A.3.2}$$

$$\tilde{a}^\dagger \tilde{a}|\tilde{n})|\tilde{m}) = m|\tilde{n})|\tilde{m}) \tag{A.3.3}$$

$$(n, m = 0, 1, 2, \cdots).$$

Proof.

Putting a and $a^\dagger$ on $\sum_{n=0}^{\infty}(n|(\tilde{n}|$ from right, we have

$$\begin{aligned} \sum_{n=0}^{\infty}(n|(\tilde{n}|a &= \sum_{n=0}^{\infty}(n+1|(\tilde{n}|\sqrt{n+1} \\ &= \sum_{m=1}^{\infty}(m|(\widetilde{m-1}|\sqrt{m} \\ &= \sum_{m=1}^{\infty}(m|(\tilde{m}|\tilde{a}^\dagger \\ &= \sum_{n=0}^{\infty}(n|(\tilde{n}|\tilde{a}^\dagger, \end{aligned} \tag{A.3.4}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty}(n|(\tilde{n}|a^\dagger &= \sum_{n=1}^{\infty}(n-1|(\tilde{n}|\sqrt{n} \\ &= \sum_{\ell=0}^{\infty}(\ell|(\widetilde{\ell+1}|\sqrt{\ell} \\ &= \sum_{\ell=0}^{\infty}(\ell|(\tilde{\ell}|\tilde{a}. \end{aligned} \tag{A.3.5}$$

Then, for normally ordered product $a^{\dagger k}a^{\ell}$ $(k = 0, 1, 2, \cdots,\ \ell = 0, 1, 2,)$, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty}(n|(\tilde{n}|a^{\dagger k}a^{\ell} &= \sum_{n=0}^{\infty}(n|(\tilde{n}|\tilde{a}^{k}a^{\ell} \\ &= \sum_{n=0}^{\infty}(n|(\tilde{n}|\tilde{a}^{\dagger \ell}\tilde{a}^{k} \\ &= \sum_{n=0}^{\infty}(n|(\tilde{n}|\left(a^{\dagger k}a^{\ell}\right)^{\dagger \sim}. \end{aligned} \tag{A.3.6}$$

Thus, for arbitrary non-tilde operator $A(a, a^\dagger)$, $\sum_{n=0}^{\infty}(n|(\tilde{n}|$ satisfies

$$\sum_{n=0}^{\infty}(n|(\tilde{n}|A = \sum_{n=0}^{\infty}(n|(\tilde{n}|\tilde{A}, \tag{A.3.7}$$

which is the thermal condition (3.1.6). Therefore, we see

$$\langle 1| = \sum_{n=0}^{\infty} (n|(\tilde{n}|. \tag{A.3.8}$$

We can also represent ⟨1| by continuous basis $\{(x|(\tilde{x}|\}$;

$$\begin{aligned}\langle 1| &= \sum_{n=0}^{\infty} (n|(\tilde{n}| \\ &= \sum_{n=0}^{\infty} \int dx \int dx' (x|(\tilde{x}'|H_n(x)H_n(x') \\ &= \int dx \int dx' (x|(\tilde{x}'|\Delta(X - x') \\ &= \int dx (x|(\tilde{x}|, \end{aligned} \tag{A.3.9}$$

.

A.4 Derivation of Quantum Master Equation

A.4.1 Damping Theory

We consider a system of which Hamiltonian is given by

$$H = H_0 + gH_1. \tag{A.4.1}$$

The equation of motion of the density operator is written by Liouville von-Neumann equation:

$$\frac{\partial}{\partial t} W(t) = -iH^{\times} W(t) = -iLW(t), \tag{A.4.2}$$

where

$$H^{\times} \bullet = [H, \bullet] \equiv L \bullet . \tag{A.4.3}$$

In followings, we describe

$$L_0 \bullet = [H_0, \bullet], \qquad gL_1 \bullet = g[H_1, \bullet]. \tag{A.4.4}$$

The density operator in interaction representation is

$$W^{(I)}(t) = \mathrm{e}^{iL_0 t} W(t), \tag{A.4.5}$$

and the equation of motion of it comes to

$$\begin{aligned} \frac{\partial}{\partial t} W^{(I)}(t) &= \mathrm{e}^{iL_0 t} iL_0 W(t) + \mathrm{e}^{iL_0 t}(-iL)W(t) \\ &= -ig\mathrm{e}^{iL_0 t} L_1 W(t) \\ &= -ig\mathrm{e}^{iL_0 t} L_1 \mathrm{e}^{-iL_0 t} W^{(I)}(t) \\ &= -igL_1^{(I)}(t) W^{(I)}(t), \end{aligned} \tag{A.4.6}$$

where

$$L_1^{(I)}(t) = \mathrm{e}^{iL_0 t} L_1 \mathrm{e}^{-iL_0 t}. \tag{A.4.7}$$

Here, we introduce a projection operator P, satisfying

$$P^2 = P, \tag{A.4.8}$$

$$P^{\dagger} = P. \tag{A.4.9}$$

Then an operator Q defined

$$Q = 1 - P \tag{A.4.10}$$

$$\tag{A.4.11}$$

also satisfies

$$Q^2 = (1 - 2P + P^2) = 1 - P = Q, \tag{A.4.12}$$

$$Q^\dagger = Q. \tag{A.4.13}$$

Putting P on (A.4.6), we obtain the equation of motion of $PW(t)^{(I)}(t)$:

$$\begin{aligned}
\frac{\partial}{\partial t} PW^{(I)}(t) &= -igPL_1^{(I)}(t)W^{(I)}(t) \\
&= -igPL_1^{(I)}(t)(P+Q)W^{(I)}(t) \\
&= -igPL_1^{(I)}(t)PW^{(I)}(t) - igPL_1^{(I)}(t)QW^{(I)}(t) \\
&= -igL_{PP}(t)PW^{(I)}(t) - igL_{PQ}(t)QW^{(I)}(t),
\end{aligned} \tag{A.4.14}$$

where

$$L_{PP}(t) = PL_1^{(I)}(t)P, \tag{A.4.15}$$

$$L_{PQ}(t) = PL_1^{(I)}(t)Q. \tag{A.4.16}$$

In the same way, putting Q on (A.4.6), we obtain the equation of motion of $QW^{(I)}(t)$:

$$\begin{aligned}
\frac{\partial}{\partial t} QW^{(I)}(t) &= -igQL_1^{(I)}(t)W^{(I)}(t) \\
&= -igQL_1^{(I)}(t)(P+Q)W^{(I)}(t) \\
&= -igQL_1^{(I)}(t)PW^{(I)}(t) - igPL_1^{(I)}(t)QW^{(I)}(t) \\
&= -igL_{QP}(t)PW^{(I)}(t) - igL_{QQ}(t)QW^{(I)}(t),
\end{aligned} \tag{A.4.17}$$

where

$$L_{QP}(t) = QL_1^{(I)}(t)P, \tag{A.4.18}$$

$$L_{QQ}(t) = QL_1^{(I)}(t)Q. \tag{A.4.19}$$

Here, we consider a general differential equation

$$\frac{\partial}{\partial t} X(t) = -igL(t)X(t) + Y(t), \tag{A.4.20}$$

where $L(t)$ is Hermitian:

$$L^\dagger(t) = L(t) \tag{A.4.21}$$

In the case of$Y(t) = 0$, (A.4.20) can be solved to

$$\begin{aligned}
X(t) &= X(t_0) + (-ig)\int_{t_0}^{t} dt_1 L(t_1)X(t_1) \\
&= X(t_0) + (-ig)\int_{t_0}^{t} dt_1 L(t_1)X(t_0) + (-ig)^2 \int_{t_0}^{t}\int_{t_0}^{t_1} dt_1 L(t_1)L(t_2)X(t_2) \\
&= \left[1 + \sum_{n=1}^{\infty}(-ig)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n L(t_1)L(t_2)\cdots L(t_n)\right] X(t_0).
\end{aligned} \tag{A.4.22}$$

Here, we define

$$
\begin{aligned}
V(t) &= 1 + \sum_{n=1}^{\infty}(-ig)^n \int^t dt_1 \int^{t_1} dt_2 \cdots \int^{t_{n-1}} dt_n L(t_1)L(t_2)\cdots L(t_n) \\
&= T\exp\left[-ig\int^t dt' L(t')\right], \qquad \text{(A.4.23)}
\end{aligned}
$$

where T is the time-ordering operator. The Hermite conjugate of $V(t)$ is estimated by

$$
\begin{aligned}
V^\dagger(t) &= 1 + \sum_{n=1}^{\infty}(ig)^n \int^t dt_1 \int^{t_1} dt_2 \cdots \int^{t_{n-1}} dt_n L^\dagger(t_n)L^\dagger(t_{n-1})\cdots L^\dagger(t_1) \\
&= 1 + \sum_{n=1}^{\infty}(ig)^n \int^t dt_1 \int^{t_1} dt_2 \cdots \int^{t_{n-1}} dt_n L(t_n)L(t_{n-1})\cdots L(t_1) \\
&= \tilde{T}\exp\left[ig\int^t dt' L(t')\right], \qquad \text{(A.4.24)}
\end{aligned}
$$

where $\tilde{T}$ is the anti-time-ordering operator. Since the integral is estimated by

$$
\int^t dt' f(t) = -\int_t dt' f(t'), \qquad \text{(A.4.25)}
$$

(A.4.24) can be rewritten by time-ordered:

$$
\begin{aligned}
V^\dagger(t) &= 1 + \sum_{n=1}^{\infty}(-ig)^n \int_t dt_1 \int_{t_1} dt_2 \cdots \int_{t_{n-1}} dt_n L(t_n)L(t_{n-1})\cdots L(t_1) \\
&= T\exp\left[-ig\int_t dt' L(t')\right]. \qquad \text{(A.4.26)}
\end{aligned}
$$

For $t \geq t_1$ we define

$$
V(t,t_1) = V(t)V^\dagger(t_1) = V^\dagger(t_1)V(t) = T\exp\left[-ig\int_{t_1}^t dt' L(t')\right], \qquad \text{(A.4.27)}
$$

and, in the case of $t = t_1$, we see

$$
V(t,t) = V(t)V^\dagger(t) = V^\dagger(t)V(t) = 1. \qquad \text{(A.4.28)}
$$

Then, we see $V(t)$ is unitary:

$$
V^\dagger(t) = V^{-1}(t) \qquad \text{(A.4.29)}
$$

In the geveral case of $Y(t) \neq 0$, by putting

$$
X(t) = V(t,t_0)C(t), \qquad X(t_0) = C(t_0), \qquad \text{(A.4.30)}
$$

and by differentiating with respect to t, (A.4.20) is rewritten by

$$
\begin{aligned}
\frac{\partial}{\partial t}X(t) &= \frac{\partial V(t,t_0)}{\partial t}C(t) + V(t,t_0)\frac{\partial C(t)}{\partial t} \\
&= -igL(t)V(t,t_0)C(t) + V(t,t_0)\frac{\partial C(t)}{\partial t} \\
&= -igL(t)X(t) + V(t,t_0)\frac{\partial C(t)}{\partial t}. \qquad \text{(A.4.31)}
\end{aligned}
$$

Comparing (A.4.20) and (A.4.31), we estimate

$$V(t,t_0)\frac{\partial C(t)}{\partial t} = Y(t), \tag{A.4.32}$$

$$\therefore \quad \frac{\partial C(t)}{\partial t} = V^\dagger(t)Y(t), \tag{A.4.33}$$

$$\therefore \quad C(t) = X(t_0) + \int_{t_0}^{t} dt' V(t)^\dagger(t',t_0)Y(t'). \tag{A.4.34}$$

Applying it to (A.4.30), we have

$$\begin{aligned} X(t) &= V(t,t_0)X(t_0) + \int_{t_0}^{t} dt' V(t,t_0)V^\dagger(t',t_0)Y(t') \\ &= V(t,t_0)X(t_0) + \int_{t_0}^{t} dt' V(t,t')Y(t'). \end{aligned} \tag{A.4.35}$$

L_0 and L_1 satisfy Hermit

$$L_0^\dagger \bullet = [H_0^\dagger, \bullet] = [H_0, \bullet] = L_0 \bullet, \tag{A.4.36}$$

$$L_1^\dagger \bullet = [H_1^\dagger, \bullet] = [H_1, \bullet] = L_1 \bullet . \tag{A.4.37}$$

Then, $L_{QQ}(t)$ also satisfies Hermitian:

$$L_1^{(I)\dagger}(t) = \left\{ \mathrm{e}^{iL_0 t} L_1 \mathrm{e}^{-iL_0 t} \right\}^\dagger = \mathrm{e}^{iL_0 t} L_1 \mathrm{e}^{-iL_0 t} = L_1^{(I)}(t), \tag{A.4.38}$$

$$L_{QQ}^\dagger(t) = \left\{ QL_1^{(I)}(t)Q \right\}^\dagger = QL_1^{(I)}(t)Q = L_{QQ}(t). \tag{A.4.39}$$

Applying (A.4.35) to (A.4.17), we obtain

$$QW^{(I)}(t) = V_{QQ}(t,t_0)QW^{(I)}(t_0) - ig \int_{t_0}^{t} dt' V_{QQ}(t,t')L_{QP}(t')PW^{(I)}(t'), \tag{A.4.40}$$

where

$$\frac{\partial}{\partial t} V_{QQ}(t,t') = -igL_{QQ}(t)V(t,t'), \tag{A.4.41}$$

$$V_{QQ}(t,t') = T\mathrm{exp}\left[-ig \int_{t'}^{t} dt'' L_{QQ}(t'') \right]. \tag{A.4.42}$$

Putting (A.4.40) into (A.4.14), we obtain the Time Convolution (TC) equation of motion[25]:

$$\begin{aligned} \frac{\partial}{\partial t} PW^{(I)}(t) = & -igL_{PP}(t)PW^{(I)}(t) \\ & +(-ig)^2 \int_{t_0}^{t} dt' L_{PQ}(t)V_{QQ}(t,t')L_{QP}(t')PW^{(I)}(t') \\ & -igL_{PQ}(t)V_{QQ}(t,t_0)QW^{(I)}(t_0). \end{aligned} \tag{A.4.43}$$

(A.4.6) can also be solved by

$$W^{(I)}(t') = V(t',t)W^{(I)}(t), \tag{A.4.44}$$

where

$$\frac{\partial}{\partial t'} V(t',t) = -igL_1^{(I)}(t')V(t',t), \tag{A.4.45}$$

$$V(t',t) = T\exp\left[-ig \int_{t}^{t'} dt'' L_1^{(I)}(t'') \right] \tag{A.4.46}$$

Putting (A.4.44) into (A.4.40), we have

$$
\begin{aligned}
QW^{(I)}(t) &= V_{QQ}(t,t_0)QW^{(I)}(t_0) \\
&\quad -ig\int_{t_0}^{t} dt' V_{QQ}(t,t')L_{QP}(t')PV(t',t)W^{(I)}(t) \\
&= V_{QQ}(t,t_0)QW^{(I)}(t_0) \\
&\quad -ig\int_{t_0}^{t} dt' V_{QQ}(t,t')L_{QP}(t')PV(t',t)(P+Q)W^{(I)}(t) \\
&= V_{QQ}(t,t_0)QW^{(I)}(t_0) \\
&\quad -ig\int_{t_0}^{t} dt' V_{QQ}(t,t')L_{QP}(t')PV(t',t)PW^{(I)}(t) \\
&\quad -ig\int_{t_0}^{t} dt' V_{QQ}(t,t')L_{QP}(t')PV(t',t)QW^{(I)}(t) \qquad \text{(A.4.47)} \\
&= V_{QQ}(t,t_0)QW^{(I)}(t_0) \\
&\quad -ig\int_{t_0}^{t} dt' V_{QQ}(t,t')L_{QP}(t')PV(t',t)PW^{(I)}(t) \\
&\quad -\Sigma(t,t_0)PV(t',t)QW^{(I)}(t), \qquad \text{(A.4.48)}
\end{aligned}
$$

(A.4.49)

where

$$\Sigma(t,t_0) = ig\int_{t_0}^{t} dt' V_{QQ}(t,t')L_{QP}(t')PV(t',t). \qquad \text{(A.4.50)}$$

Solving it with respect to $QW^{(I)}(t)$, we obtain

$$
\begin{aligned}
\left[1+\Sigma(t,t_0)\right]QW^{(I)}(t) &= V_{QQ}(t,t_0)QW^{(I)}(t_0) \\
&\quad -ig\int_{t_0}^{t} dt' V_{QQ}(t,t')L_{QP}(t')PV(t',t)PW^{(I)}(t) \qquad \text{(A.4.51)}
\end{aligned}
$$

$$
\begin{aligned}
\therefore QW^{(I)}(t) &= \left[1+\Sigma(t,t_0)\right]^{-1} V_{QQ}(t,t_0)QW^{(I)}(t_0) \\
&\quad -ig\int_{t_0}^{t} dt' \left[1+\Sigma(t,t_0)\right]^{-1} V_{QQ}(t,t')L_{QP}(t')PV(t',t)PW^{(I)}(t). \qquad \text{(A.4.52)}
\end{aligned}
$$

Putting (A.4.52) into (A.4.14), we obtain time Tim-Convolution-less (TCL) equation of motion.[25]

$$
\begin{aligned}
\frac{\partial}{\partial t}PW^{(I)}(t) &= -igL_{PP}(t)PW^{(I)}(t) \\
&\quad +(-ig)^2\int_{t_0}^{t} dt' L_{PQ}(t)\left[1+\Sigma(t,t_0)\right]^{-1} V_{QQ}(t,t')L_{QP}(t')PV(t',t)PW^{(I)}(t) \\
&\quad -igL_{PQ}(t)\left[1+\Sigma(t,t_0)\right]^{-1} V_{QQ}(t,t_0)QW^{(I)}(t_0) \qquad \text{(A.4.53)}
\end{aligned}
$$

In followings, we neglect $\mathcal{O}(g^3)$, and require

$$L_{PP}(t) = 0, \qquad \text{(A.4.54)}$$

$$QW^{(I)}(t) = 0. \qquad \text{(A.4.55)}$$

In this case, the TCL equation of motion (A.4.53) is written by

$$\frac{\partial}{\partial t}PW^{(I)}(t) = (-ig)^2 \int_{t_0}^{t} dt' L_{PQ}(t) L_{QP}(t') PW^{(I)}(t). \tag{A.4.56}$$

A.4.2 Derivation of Master Equation from Microscopic Interaction

We consider the system interacted with thermal bath, of which the Hamiltonian is given by

$$H = H_S + H_B + gH_{SB}, \tag{A.4.57}$$

$$H_S = \omega_S a^\dagger a, \tag{A.4.58}$$

$$H_B = \sum_\ell \omega_\ell b_\ell^\dagger b_\ell, \tag{A.4.59}$$

$$H_{SB} = \sum_\ell \left(a^\dagger b_\ell + a b_\ell^\dagger \right), \tag{A.4.60}$$

where $a^\dagger$, a are operators of the system, and b, $b^\dagger$ are operators of bath, satisfying

$$[a, a^\dagger] = 1, \qquad [b_\ell, b_k^\dagger] = \delta_{\ell k}, \tag{A.4.61}$$

$$[a, b_\ell] = [a, b_\ell^\dagger] = 0. \tag{A.4.62}$$

Using (A.4.4), we describe

$$L_0 = H_S^\times + H_B^\times, \quad gL_1 = gH_{SB}^\times. \tag{A.4.63}$$

In the interaction representation, these equations come to

$$L_1^{(I)}(t) = \mathrm{e}^{iL_0 t} H_{SB}^\times \mathrm{e}^{-iL_0 t} = H_{SB}^{(I)}(t)^\times \tag{A.4.64}$$

$$\begin{aligned} H_{SB}^{(I)}(t) &= \mathrm{e}^{iL_0 t} H_{SB} \mathrm{e}^{-iL_0 t} \\ &= \mathrm{e}^{i(H_S+H_B)t} H_{SB} \mathrm{e}^{-i(H_S+H_B)t} \\ &= \sum_\ell \left(a^\dagger(t) b_\ell(t) + a(t) b_\ell^\dagger(t) \right), \end{aligned} \tag{A.4.65}$$

where

$$a(t) = \mathrm{e}^{iH_S t} a \mathrm{e}^{-iH_S t} = \mathrm{e}^{-i\omega_S t} a, \tag{A.4.66}$$

$$b_\ell(t) = \mathrm{e}^{iH_B t} b_\ell \mathrm{e}^{-iH_B t} = \mathrm{e}^{-i\omega_\ell t} b_\ell. \tag{A.4.67}$$

We define a projection operator P by

$$P = \rho_B \mathrm{tr}_B, \tag{A.4.68}$$

where ρ_B is the density operator of bath defined by

$$\rho_B = \frac{1}{Z_B} \mathrm{e}^{-\beta H_B}, \qquad Z_B = \mathrm{tr}_B \mathrm{e}^{-\beta H_B}. \tag{A.4.69}$$

Then we obtain

$$\begin{aligned}\langle H_{SB}^{(I)}(t)\rangle_B &= \sum_\ell \langle a^\dagger(t)b_\ell(t) + a(t)b_\ell^\dagger(t)\rangle_B \\ &= \sum_\ell \left[a^\dagger(t)\langle b_\ell(t)\rangle_B + a(t)\langle b_\ell^\dagger(t)\rangle_B\right] \\ &= 0, \end{aligned} \tag{A.4.70}$$

where $\langle\cdots\rangle_B$ is expectation value for bath:

$$\langle\cdots\rangle_B = \mathrm{tr}_B \cdots \rho_B, \tag{A.4.71}$$

and the final equality is given from

$$\langle b_\ell(t)\rangle_B = \mathrm{e}^{-i\omega_\ell t}\langle b_\ell\rangle_B = 0. \tag{A.4.72}$$

Furthermore, we obtain

$$\begin{aligned}L_{PP}(t)X &= PH_{SB}^{(I)}(t)^\times PX \\ &= P[H_{SB}^{(I)}(t), PX] \\ &= PH_{SB}^{(I)}(t)(PX) - P(PX)H_{SB}^{(I)}(t) \\ &= \rho_B \mathrm{tr}_B H_{SB}^{(I)}(t)\rho_B \mathrm{tr}_B X - \rho_B \mathrm{tr}_B \rho_B(\mathrm{tr}_B X)H_{SB}^{(I)}(t) \\ &= \rho_B\langle H_{SB}^{(I)}(t)\rangle_B \mathrm{tr}_B X - \rho_B \mathrm{tr}_B X\langle H_{SB}^{(I)}(t)\rangle_B \\ &= 0. \end{aligned} \tag{A.4.73}$$

this $L_{PP}(t)$ satisfies (A.4.54).

We suppose the system and bath are separable at time $t = t_0$, i.e.,

$$W(t_0) = \rho_B\rho(t_0) = P\rho_B\rho(t_0). \tag{A.4.74}$$

In this case, we have

$$\begin{aligned}QW(t_0) &= QP\rho_B\rho(t_0) \\ &= (1-P)P\rho_B\rho(t_0) \\ &= 0 \end{aligned} \tag{A.4.75}$$

This initial condition satisfies (A.4.55), then in the case of $g \ll 1$, we can apply (A.4.56) to this model.

From (A.4.68), $PW^{(I)}(t)$ is estimated to

$$PW^{(I)}(t) = \rho_B \mathrm{tr}_B W^{(I)}(t) = \rho_B \rho^{(I)}(t), \tag{A.4.76}$$

where $\rho^{(I)}(t)$ is the density operator of the system defined by

$$\rho^{(I)}(t) = \mathrm{tr}_B W^{(I)}(t). \tag{A.4.77}$$

Putting it into (A.4.56), we have

$$\begin{aligned}\frac{\partial}{\partial t}\rho_B\rho^{(I)}(t) &= (-ig)^2\int_{t_0}^{t} dt' L_{PQ}(t)L_{QP}(t')\rho_B\rho^{(I)}(t)\\ &= (-ig)^2\int_{t_0}^{t} dt' \rho_B \mathrm{tr}_B L_1^{(I)}(t)QL_1^{(I)}(t')\rho_B\rho^{(I)}(t).\end{aligned} \tag{A.4.78}$$

By taking ρ_B^{-1}, it comes to

$$\begin{aligned}\frac{\partial}{\partial t}\rho^{(I)}(t) &= (-ig)^2\int_{t_0}^{t} dt' \mathrm{tr}_B L_1^{(I)}(t)QL_1^{(I)}(t')\rho_B\rho^{(I)}(t)\\ &= (-ig)^2\int_{t_0}^{t} dt' \mathrm{tr}_B L_1^{(I)}(t)(1-\rho_B\mathrm{tr}_B)L_1^{(I)}(t')\rho_B\rho^{(I)}(t)\\ &= (-ig)^2\int_{t_0}^{t} dt' \langle L_1^{(I)}(t)L_1^{(I)}(t')\rangle_B\rho^{(I)}(t)\\ &\quad -(-ig)^2\int_{t_0}^{t} dt' \langle L_1^{(I)}(t)\rangle_B\langle L_1^{(I)}(t')\rangle_B\rho^{(I)}(t)\\ &= -g^2\int_{t_0}^{t} dt' \langle L_1^{(I)}(t)L_1^{(I)}(t')\rangle_B\rho^{(I)}(t).\end{aligned} \tag{A.4.79}$$

the integrated functions are estimated to

$$\begin{aligned}\langle L_1^{(I)}(t)L_1^{(I)}(t')\rangle_B &= \mathrm{tr}_B\left[H_{SB}^{(I)}(t),\left[H_{SB}^{(I)}(t'),\rho_B\rho^{(I)}(t)\right]\right]\\ &= \mathrm{tr}_B\left\{H_{SB}^{(I)}(t)\left(H_{SB}^{(I)}(t')\rho_B\rho^{(I)}(t)-\rho_B\rho^{(I)}(t)H_{SB}^{(I)}(t')\right)\right.\\ &\qquad\left.-\left(H_{SB}^{(I)}(t')\rho_B\rho^{(I)}(t)-\rho_B\rho^{(I)}(t)H_{SB}^{(I)}(t')\right)H_{SB}^{(I)}(t)\right\}\\ &= \mathrm{tr}_B H_{SB}^{(I)}(t)H_{SB}^{(I)}(t')\rho_B\rho^{(I)}(t)-\mathrm{tr}_B H_{SB}^{(I)}(t)\rho_B\rho^{(I)}(t)H_{SB}^{(I)}(t')\\ &\quad -\mathrm{tr}_B H_{SB}^{(I)}(t')\rho_B\rho^{(I)}(t)H_{SB}^{(I)}(t)+\mathrm{tr}_B\rho_B\rho^{(I)}(t)H_{SB}^{(I)}(t')H_{SB}^{(I)}(t).\end{aligned} \tag{A.4.80}$$

The first term of (A.4.80) is estimated to

$$\begin{aligned}&\mathrm{tr}_B H_{SB}^{(I)}(t)H_{SB}^{(I)}(t')\rho_B\rho^{(I)}(t)\\ &= \sum_{\ell,m}\mathrm{tr}_B\left\{a^\dagger(t)b_m(t)+a(t)b_\ell^\dagger(t)\right\}\left\{a^\dagger(t')b_m(t')+a(t')b_m^\dagger(t')\right\}\rho_B\rho^{(I)}(t)\\ &= \sum_{\ell,m}\left\{\langle b_\ell(t)b_m^\dagger(t')\rangle_B a^\dagger(t)a(t')\rho^{(I)}(t)+\langle b_\ell^\dagger(t)b_m(t')\rangle_B a(t)a^\dagger(t')\rho^{(I)}(t)\right\}\\ &= \phi_{-+}(t,t')\mathrm{e}^{i\omega_S(t-t')}a^\dagger a\rho^{(I)}(t)+\phi_{+-}(t,t')\mathrm{e}^{-i\omega_S(t-t')}aa^\dagger\rho^{(I)}(t).\end{aligned} \tag{A.4.81}$$

where

$$\phi_{-+}(t,t') = \sum_{\ell,m}\langle b_\ell(t)b_m^\dagger(t')\rangle_B = \phi^*(t',t), \tag{A.4.82}$$

$$\phi_{+-}(t,t') = \sum_{\ell,m}\langle b_\ell^\dagger(t)b_m(t')\rangle_B = \phi_{+-}(t',t). \tag{A.4.83}$$

In the same way, each term of (A.4.80) come to

$$\begin{aligned}
&\mathrm{tr}_B H_{SB}^{(I)}(t)\rho_B\rho^{(I)}(t)H_{SB}^{(I)}(t') \\
&=\sum_{\ell,m}\mathrm{tr}_B\left\{a^\dagger(t)b_\ell(t)+a(t)b_\ell^\dagger(t)\right\}\rho_B\rho^{(I)}(t)\left\{a^\dagger(t')b_m(t')+a(t')b_m^\dagger(t')\right\} \\
&=\sum_{\ell,m}\left\{\langle b_m^\dagger(t')b_\ell(t)\rangle_B a(t)\rho^{(I)}(t)a^\dagger(t')+\langle b_m^\dagger(t')b_\ell(t)\rangle_B a^\dagger(t)\rho^{(I)}(t)a(t')\right\} \\
&=\phi_{-+}(t',t)\mathrm{e}^{-i\omega_S(t-t')}a\rho^{(I)}(t)a^\dagger+\phi_{+-}(t',t)\mathrm{e}^{i\omega_S(t-t')}a^\dagger\rho^{(I)}(t)a,, \qquad \text{(A.4.84)}
\end{aligned}$$

$$\begin{aligned}
&\mathrm{tr}_B H_{SB}^{(I)}(t')\rho_B\rho^{(I)}(t)H_{SB}^{(I)}(t) \\
&=\sum_{\ell,m}\mathrm{tr}_B\left\{a^\dagger(t')b_\ell(t')+a(t')b_\ell^\dagger(t')\right\}\rho_B\rho^{(I)}(t)\left\{a^\dagger(t)b_m(t)+a(t)b_m^\dagger(t)\right\} \\
&=\sum_{\ell,m}\left\{\langle b_m(t)b_\ell^\dagger(t')\rangle_B a(t')\rho^{(I)}(t)a^\dagger(t)+\langle b_m^\dagger(t)b_\ell(t')\rangle_B a^\dagger(t')\rho^{(I)}(t)a(t)\right\} \\
&=\phi_{-+}(t,t')\mathrm{e}^{i\omega_S(t-t')}a\rho^{(I)}(t)a^\dagger+\phi_{+-}(t,t')\mathrm{e}^{-i\omega_S(t-t')}a^\dagger\rho^{(I)}(t)a,, \qquad \text{(A.4.85)}
\end{aligned}$$

$$\begin{aligned}
&\mathrm{tr}_B\rho_B\rho^{(I)}(t)H_{SB}^{(I)}(t')H_{SB}^{(I)}(t) \\
&=\sum_{\ell,m}\mathrm{tr}_B\rho_B\rho^{(I)}(t)\left\{a^\dagger(t')b_\ell(t')+a(t')b_\ell^\dagger(t')\right\}\left\{a^\dagger(t)b_m(t)+a(t)b_m^\dagger(t)\right\} \\
&=\sum_{\ell,m}\left\{\langle b_\ell(t')b_m^\dagger(t)\rangle_B\rho^{(I)}(t)a^\dagger(t')a(t)+\langle b_\ell^\dagger(t')b_m(t)\rangle_B\rho^{(I)}(t)a(t')a^\dagger(t)\right\} \\
&=\phi_{-+}(t',t)\mathrm{e}^{-i\omega_S(t-t')}\rho^{(I)}(t)a^\dagger a+\phi_{+-}(t',t)\mathrm{e}^{i\omega_S(t-t')}\rho^{(I)}(t)aa^\dagger. \qquad \text{(A.4.86)}
\end{aligned}$$

Then we rewrite the integrated terms by

$$\begin{aligned}
\langle L_1^{(I)}(t)L_1^{(I)}(t')\rangle_B &= \phi_{-+}(t,t')\mathrm{e}^{i\omega_S(t-t')}a^\dagger a\rho^{(I)}(t)+\phi_{+-}(t,t')\mathrm{e}^{-i\omega_S(t-t')}aa^\dagger\rho^{(I)}(t) \\
&\quad -\phi_{-+}(t',t)\mathrm{e}^{-i\omega_S(t-t')}a\rho^{(I)}(t)a^\dagger-\phi_{+-}(t',t)\mathrm{e}^{i\omega_S(t-t')}a^\dagger\rho^{(I)}(t)a \\
&\quad -\phi_{-+}(t,t')\mathrm{e}^{i\omega_S(t-t')}a\rho^{(I)}(t)a^\dagger-\phi_{+-}(t,t')\mathrm{e}^{-i\omega_S(t-t')}a^\dagger\rho^{(I)}(t)a \\
&\quad +\phi_{-+}(t',t)\mathrm{e}^{-i\omega_S(t-t')}\rho^{(I)}(t)a^\dagger a+\phi_{+-}(t',t)\mathrm{e}^{i\omega_S(t-t')}\rho^{(I)}(t)aa^\dagger \\
&= \phi_{-+}(t,t')\mathrm{e}^{i\omega_S(t-t')}[a^\dagger,a\rho^{(I)}(t)]+\phi_{+-}(t,t')\mathrm{e}^{-i\omega_S(t-t')}[a,a^\dagger\rho^{(I)}(t)] \\
&\quad +\phi_{-+}(t',t)\mathrm{e}^{-i\omega_S(t-t')}[\rho^{(I)}(t)a^\dagger,a]+\phi_{+-}(t',t)\mathrm{e}^{i\omega_S(t-t')}[\rho^{(I)}(t)a,a^\dagger].
\end{aligned}$$

(A.4.87)

Putting (A.4.87) into (A.4.79), we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\rho^{(I)}(t) &= -g^2\int_{t_0}^{t} dt' \Big\{\phi_{-+}(t,t')\mathrm{e}^{i\omega_S(t-t')}[a^\dagger, a\rho^{(I)}(t)] \\
&\quad + \phi_{+-}(t,t')\mathrm{e}^{-i\omega_S(t-t')}[a, a^\dagger\rho^{(I)}(t)] \\
&\quad + \phi_{-+}(t',t)\mathrm{e}^{-i\omega_S(t-t')}[\rho^{(I)}(t)a^\dagger, a] \\
&\quad + \phi_{+-}(t',t)\mathrm{e}^{i\omega_S(t-t')}[\rho^{(I)}(t)a, a^\dagger]\Big\} \\
&= -g^2\int_{0}^{t-t_0} dt'' \Big\{\phi_{-+}(t,t-t'')\mathrm{e}^{i\omega_S t''}[a^\dagger, a\rho^{(I)}(t)] \\
&\quad + \phi_{+-}(t,t-t'')\mathrm{e}^{-i\omega_S t''}[a, a^\dagger\rho^{(I)}(t)] \\
&\quad + \phi_{-+}(t-t'',t)\mathrm{e}^{-i\omega_S t''}[\rho^{(I)}(t)a^\dagger, a] \\
&\quad + \phi_{+-}(t-t'',t)\mathrm{e}^{i\omega_S t''}[\rho^{(I)}(t)a, a^\dagger]\Big\} \\
&= -g^2\int_{0}^{\infty} dt'' \Big\{\phi_{-+}(t,t-t'')\mathrm{e}^{i\omega_S t''}[a^\dagger, a\rho^{(I)}(t)] \\
&\quad + \phi_{+-}(t,t-t'')\mathrm{e}^{-i\omega_S t''}[a, a^\dagger\rho^{(I)}(t)] \\
&\quad + \phi_{-+}(t-t'',t)\mathrm{e}^{-i\omega_S t''}[\rho^{(I)}(t)a^\dagger, a] \\
&\quad + \phi_{+-}(t-t'',t)\mathrm{e}^{i\omega_S t''}[\rho^{(I)}(t)a, a^\dagger]\Big\}. \qquad \text{(A.4.88)}
\end{aligned}$$

At the third equality, supposing the specific time scale of $\rho^{(I)}(t)$ is much larger than that of bath, we took the limit $t_0 \longrightarrow \infty$.

Here, we define

$$\begin{aligned}
\phi_{-+}(t,t-t'') &= \sum_{\ell,m}\langle b_\ell(t)b_m^\dagger(t-t'')\rangle_B \\
&= \sum_{\ell,m}\langle b_\ell b_m^\dagger\rangle_B \mathrm{e}^{-i\omega_\ell t}\mathrm{e}^{i\omega_m(t-t'')} \\
&= \sum_{\ell}\langle b_\ell b_\ell^\dagger\rangle_B \mathrm{e}^{-i\omega_\ell t''} \\
&= \sum_{\ell}(\bar{n}_\ell + 1)\mathrm{e}^{-i\omega_\ell t''} \\
&\equiv \phi_{-+}(t''), \qquad \text{(A.4.89)}
\end{aligned}$$

where

$$\langle b_\ell^\dagger b_\ell\rangle_B = \bar{n}_\ell. \qquad \text{(A.4.90)}$$

In the same way, we define

$$
\begin{aligned}
\phi_{+-}(t, t-t'') &= \sum_{\ell,m} \langle b_\ell^\dagger(t) b_m(t-t'')\rangle_B \\
&= \sum_{\ell,m} \langle b_\ell^\dagger b_m\rangle_B \mathrm{e}^{i\omega_\ell t}\mathrm{e}^{-i\omega_m(t-t'')} \\
&= \sum_{\ell} \langle b_\ell^\dagger b_\ell\rangle_B \mathrm{e}^{i\omega_\ell t''} \\
&= \sum_{\ell} \bar{n}_\ell \mathrm{e}^{i\omega_\ell t''} \\
&\equiv \;\; \phi_{+-}(t''),
\end{aligned} \tag{A.4.91}
$$

$$
\begin{aligned}
\phi_{-+}(t-t'', t) &= \phi^*_{-+}(t, t-t'') \\
&= \phi^*_{-+}(t''),
\end{aligned} \tag{A.4.92}
$$

$$
\begin{aligned}
\phi_{+-}(t-t'', t) &= \phi^*_{+-}(t, t-t'') \\
&= \phi^*_{+-}(t'').
\end{aligned} \tag{A.4.93}
$$

We estimate Fourier transform for $\phi_{-+}(t)$

$$
\int_0^\infty dt\phi_{-+}(t)\mathrm{e}^{i\omega t} = \sum_\ell (\bar{n}_\ell + 1)\int_0^\infty dt \mathrm{e}^{i(\omega-\omega_\ell)}. \tag{A.4.94}
$$

Here, we take continues mode limit:

$$
\sum_\ell \longrightarrow \int_{-\infty}^{\infty} d\omega \mathcal{D}(\omega), \tag{A.4.95}
$$

where $\mathcal{D}(\omega)$ is the density of states. Then (A.4.94) is estimated to

$$
\begin{aligned}
\int_0^\infty dt\phi_{-+}(t)\mathrm{e}^{i\omega t} &= \int_{-\infty}^{\infty} d\omega' \mathcal{D}(\omega')[\bar{n}(\omega')+1]\int_0^\infty dt \mathrm{e}^{i(\omega-\omega')-\delta t} \\
&= \int_{-\infty}^{\infty} d\omega' \mathcal{D}(\omega')[\bar{n}(\omega')+1]\frac{-1}{i[(\omega-\omega')+i\delta]} \\
&= \pi\int_{-\infty}^{\infty} d\omega' \mathcal{D}(\omega')[\bar{n}(\omega')+1]\delta(\omega-\omega') + iP\int_{-\infty}^{\infty} d\omega' \frac{\mathcal{D}(\omega')[\bar{n}(\omega')+1]}{\omega-\omega'} \\
&= \pi\mathcal{D}(\omega)[\bar{n}(\omega)+1] + iP\int_{-\infty}^{\infty} d\omega' \frac{\mathcal{D}(\omega')[\bar{n}(\omega')+1]}{\omega-\omega'} \\
&= \phi'_{-+}(\omega) + i\phi''_{-+}(\omega),
\end{aligned} \tag{A.4.96}
$$

where

$$
\phi'_{-+}(\omega) = \pi\mathcal{D}(\omega)[\bar{n}(\omega)+1], \tag{A.4.97}
$$

$$
\phi''_{-+}(\omega) = P\int_{-\infty}^{\infty} d\omega' \frac{\mathcal{D}(\omega')[\bar{n}(\omega')+1]}{\omega-\omega'}. \tag{A.4.98}
$$

Fourier transform for $\phi_{+-}(t)$ is also estimated to

$$\begin{aligned}
\int_0^\infty dt\phi_{+-}(t)\mathrm{e}^{-i\omega t} &= \sum_\ell \bar{n}_\ell \int_0^\infty dt\mathrm{e}^{-i(\omega-\omega_\ell)} \\
&= \int_{-\infty}^\infty d\omega' \mathcal{D}(\omega')\bar{n}(\omega') \int_0^\infty dt\mathrm{e}^{-i(\omega-\omega')-\delta t} \\
&= \int_{-\infty}^\infty d\omega' \mathcal{D}(\omega')\bar{n}(\omega') \frac{-1}{-i[(\omega-\omega')-i\delta]} \\
&= \pi\int_{-\infty}^\infty d\omega' \mathcal{D}(\omega')\bar{n}(\omega')\delta(\omega-\omega') - iP\int_{-\infty}^\infty d\omega' \frac{\mathcal{D}(\omega')\bar{n}(\omega')}{\omega-\omega'} \\
&= \pi\mathcal{D}(\omega)[\bar{n}(\omega)+1] - iP\int_{-\infty}^\infty d\omega' \frac{\mathcal{D}(\omega')\bar{n}(\omega')}{\omega-\omega'} \\
&= \phi'_{+-}(\omega) - i\phi''_{+-}(\omega),
\end{aligned} \tag{A.4.99}$$

where

$$\phi'_{+-}(\omega) = \pi\mathcal{D}(\omega)\bar{n}(\omega) = \phi'_{-+}(\omega) - \pi\mathcal{D}(\omega), \tag{A.4.100}$$

$$\phi''_{+-}(\omega) = P\int_{-\infty}^\infty d\omega' \frac{\mathcal{D}(\omega')\bar{n}(\omega')}{\omega-\omega'}. \tag{A.4.101}$$

From (A.4.96) and (A.4.99), (A.4.88) is rewritten by

$$\begin{aligned}
\frac{\partial}{\partial t}\rho^{(I)}(t) &= -g^2\left(\phi'_{-+}+\phi''_{-+}\right)[a^\dagger, a\rho^{(I)}(t)] - g^2\left(\phi'_{+-}-\phi''_{+-}\right)[a, a^\dagger\rho^{(I)}(t)] \\
&\quad +g^2\left(\phi'_{-+}-\phi''_{-+}\right)[a, \rho^{(I)}(t)a^\dagger] + g^2\left(\phi'_{+-}+\phi''_{+-}\right)[a^\dagger, \rho^{(I)}(t)a] \\
&= -ig^2\phi''_{-+}\left\{[a^\dagger, a\rho^{(I)}(t)] + [a, \rho^{(I)}(t)a^\dagger]\right\} \\
&\quad +ig^2\phi''_{+-}\left\{[a, a^\dagger\rho^{(I)}(t)] + [a^\dagger, \rho^{(I)}(t)a]\right\} \\
&\quad -g^2\phi'_{-+}\left\{[a^\dagger, a\rho^{(I)}(t)] - [a, \rho^{(I)}(t)a^\dagger]\right\} \\
&\quad -g^2\phi'_{+-}\left\{[a, a^\dagger\rho^{(I)}(t)] - [a^\dagger, \rho^{(I)}(t)a]\right\},
\end{aligned} \tag{A.4.102}$$

where

$$\phi'_{\pm\mp} = \phi'_{\pm\mp}(\omega_S), \qquad \phi''_{\pm\mp} = \phi''_{\pm\mp}(\omega_S). \tag{A.4.103}$$

By calculating commutation relation, we obtain

$$\begin{aligned}
[a^\dagger, a\rho^{(I)}(t)] + [a, \rho^{(I)}(t)a^\dagger] &= a^\dagger a\rho^{(I)}(t) - a^\dagger\rho^{(I)}(t)a + a^\dagger\rho^{(I)}(t)a - \rho^{(I)}(t)a^\dagger a \\
&= a^\dagger a\rho^{(I)}(t) - \rho^{(I)}(t)a^\dagger a \\
&= [a^\dagger a, \rho^{(I)}(t)],
\end{aligned} \tag{A.4.104}$$

$$\begin{aligned}
[a, a^\dagger\rho^{(I)}(t)] + [a^\dagger, \rho^{(I)}(t)a] &= aa^\dagger\rho^{(I)}(t) - a^\dagger\rho^{(I)}(t)a + a^\dagger\rho^{(I)}(t)a - \rho^{(I)}(t)aa^\dagger \\
&= a^\dagger a\rho^{(I)}(t) - \rho^{(I)}(t)a^\dagger a \\
&= [a^\dagger a, \rho^{(I)}(t)],
\end{aligned} \tag{A.4.105}$$

$$\begin{aligned}
&[a^\dagger, a\rho^{(I)}(t)] - [a, \rho^{(I)}(t)a^\dagger] + [a, a^\dagger\rho^{(I)}(t)] - [a^\dagger, \rho^{(I)}(t)a] \\
&\quad = a^\dagger a\rho^{(I)}(t) - a\rho^{(I)}(t)a^\dagger - a\rho^{(I)}(t)a^\dagger + \rho^{(I)}(t)a^\dagger a \\
&\qquad + aa^\dagger\rho^{(I)}(t) - a^\dagger\rho^{(I)}(t)a - a^\dagger\rho^{(I)}(t)a + \rho^{(I)}(t)aa^\dagger \\
&\quad = 2\left\{aa^\dagger\rho^{(I)}(t) - a\rho^{(I)}(t)a^\dagger - a^\dagger\rho^{(I)}(t)a + \rho^{(I)}(t)a^\dagger a\right\} \\
&\quad = 2\left\{a[a^\dagger, \rho^{(I)}(t)] - [a^\dagger, \rho^{(I)}(t)]a\right\} \\
&\quad = 2[a, [a^\dagger, \rho^{(I)}(t)]] \\
&\quad = -2[a, [\rho^{(I)}(t), a^\dagger]], \qquad \text{(A.4.106)}
\end{aligned}$$

then, (A.4.102) comes to

$$\begin{aligned}
\frac{\partial}{\partial t}\rho^{(I)}(t) &= -ig^2(\phi''_{-+} - \phi''_{+-})[a^\dagger a, \rho^{(I)}(t)] - \pi g^2\mathcal{D}(\omega_S)\left\{[a^\dagger, a\rho^{(I)}(t)] - [a, \rho^{(I)}(t)a^\dagger]\right\} \\
&\quad + 2g^2\phi'_{+-}[a, [\rho^{(I)}(t), a^\dagger]] \\
&= -i\Delta\omega[a^\dagger a, \rho^{(I)}(t)] + \kappa\left\{[a\rho^{(I)}(t), a^\dagger] + [a, \rho^{(I)}(t)a^\dagger]\right\} \\
&\quad + 2\kappa\bar{n}[a, [\rho^{(I)}(t), a^\dagger]], \qquad \text{(A.4.107)}
\end{aligned}$$

where

$$\Delta\omega = g^2\left(\phi''_{+-} - \phi''_{-+}\right) = P\int d\omega' \frac{g^2\mathcal{D}(\omega')}{\omega - \omega'}, \qquad \text{(A.4.108)}$$

$$\kappa = \pi g^2\mathcal{D}(\omega_S). \qquad \text{(A.4.109)}$$

Here, we cancel the interaction representation by

$$\begin{aligned}
\rho(t) &= \mathrm{e}^{-iL_0 t}\rho^{(I)}(t) \\
&= \mathrm{e}^{-i(H_S+H_B)t}\rho^{(I)}(t)\mathrm{e}^{i(H_S+H_B)t}, \qquad \text{(A.4.110)}
\end{aligned}$$

then we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\rho(t) &= \frac{\partial}{\partial t}\left\{\mathrm{e}^{-iL_0 t}\rho^{(I)}(t)\right\} \\
&= -iL_0\mathrm{e}^{-iL_0 t}\rho^{(I)}(t) + \mathrm{e}^{-iL_0 t}\frac{\partial}{\partial t}\rho^{(I)}(t) \\
&= -i[H_S + H_B, \rho(t)] + \mathrm{e}^{-i(H_S+H_B)t}\frac{\partial}{\partial t}\rho^{(I)}(t)\mathrm{e}^{i(H_S+H_B)t} \\
&= -i\omega_S[a^\dagger a, \rho(t)] - i\Delta\omega_S[a^\dagger(t)a(t), \rho(t)] \\
&\quad + \kappa\left\{[a(t)\rho(t), a^\dagger(t)] + [a(t), \rho(t)a^\dagger(t)]\right\} + 2\kappa\bar{n}\left[a(t), [\rho(t), a^\dagger(t)]\right] \\
&= -i(\omega_S + \Delta\omega_S)[a^\dagger a, \rho(t)] + \kappa\left\{[a\rho(t), a^\dagger] + [a, \rho(t)a^\dagger]\right\} + 2\kappa\bar{n}\left[a, [\rho(t), a^\dagger]\right] \\
&. \qquad \text{(A.4.111)}
\end{aligned}$$

It is the master equation.

A.5 Derivation of Parameters $\Delta m(t)$ and $\Delta n(t)$

We will derivate the order parameters $\Delta n(t)$, $\Delta m(t), \Delta m(t)^*, \alpha(t) and \alpha(t)^*$. From (3.3.82), we obtain

$$\frac{d}{dt}\begin{pmatrix}\alpha(t)\\ \alpha(t)^*\end{pmatrix} = M\begin{pmatrix}\alpha(t)\\ \alpha(t)^*\end{pmatrix}, \tag{A.5.1}$$

where

$$M = \begin{pmatrix}-\kappa & \chi\\ \chi & -\kappa\end{pmatrix}. \tag{A.5.2}$$

Defining λ as the eigenvalue of M, we have

$$|M - \lambda I| = (\lambda + \kappa)^2 - \chi^2 = 0, \tag{A.5.3}$$

then it can be solved as

$$\lambda_{\pm} = -\kappa \pm \chi. \tag{A.5.4}$$

The eigenvector for the eigenvalue $\boldsymbol{v}_{\pm}$ satisfies

$$\begin{pmatrix}0 & \chi\\ \chi & 0\end{pmatrix}\boldsymbol{v}_{\pm} = (-\kappa \pm \chi)\boldsymbol{v}_{\pm}, \tag{A.5.5}$$

then we obtain

$$\boldsymbol{v}_{\pm} = \begin{pmatrix}\pm 1\\ 1\end{pmatrix}. \tag{A.5.6}$$

Therefore, M can be orthogonalized as

$$M = \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}-\kappa+\chi & 0\\ 0 & -\kappa-\chi\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}^{-1}. \tag{A.5.7}$$

Thus (A.5.1) is written as

$$\frac{d}{dt}\begin{pmatrix}\alpha(t)\\ \alpha(t)^*\end{pmatrix} = \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}-\kappa+\chi & 0\\ 0 & -\kappa-\chi\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}^{-1}\begin{pmatrix}\alpha(t)\\ \alpha(t)^*\end{pmatrix}, \tag{A.5.8}$$

and it can be solved as

$$\begin{aligned}\begin{pmatrix}\alpha(t)\\ \alpha(t)^*\end{pmatrix} &= \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\mathrm{e}^{-(\kappa-\chi)t} & 0\\ 0 & \mathrm{e}^{-(\kappa+\chi)t}\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}^{-1}\begin{pmatrix}\alpha\\ \alpha^*\end{pmatrix}\\ &= \mathrm{e}^{-\kappa t}\begin{pmatrix}\mathrm{e}^{\chi t} & -\mathrm{e}^{-\chi t}\\ \mathrm{e}^{\chi t} & \mathrm{e}^{-\chi t}\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}\alpha\\ \alpha^*\end{pmatrix}\\ &= \mathrm{e}^{-\kappa t}\begin{pmatrix}\mathrm{ch}(\chi t) & \mathrm{sh}(\chi t)\\ \mathrm{sh}(\chi t) & \mathrm{ch}(\chi t)\end{pmatrix}\begin{pmatrix}\alpha\\ \alpha^*\end{pmatrix}.\end{aligned} \tag{A.5.9}$$

Therefore we have

$$\alpha(t) = \mathrm{e}^{-\kappa t}\mathrm{ch}(\chi t)\alpha + \mathrm{e}^{-\kappa t}\mathrm{sh}(\chi t)\alpha^* \tag{A.5.10}$$

$$= \mathrm{e}^{-(\kappa-\chi)t}\alpha' + i\mathrm{e}^{-(\kappa+\chi)t}\alpha'' \tag{A.5.11}$$

where

$$\alpha' = \Re\mathrm{e}\ \alpha, \quad \alpha'' = \Im\mathrm{m}\ \alpha. \tag{A.5.12}$$

In followings, we represent Δm in the same way,i.e,

$$\Delta m' = \Re\mathrm{e}\Delta m, \quad \Delta m'' = \Im\mathrm{m}\Delta m. \tag{A.5.13}$$

From (3.3.83) and (3.3.84), we obtain

$$\frac{d}{dt}\begin{pmatrix} \Delta n(t) \\ \Delta m(t) \\ \Delta m(t)^* \end{pmatrix} = \begin{pmatrix} -2\kappa & \chi & \chi \\ 2\chi & -2\kappa & 0 \\ 2\chi & 0 & -2\kappa \end{pmatrix}\begin{pmatrix} \Delta n(t) \\ \Delta m(t) \\ \Delta m(t)^* \end{pmatrix} + \begin{pmatrix} 2\kappa\bar{n} \\ \chi \\ \chi \end{pmatrix} \tag{A.5.14}$$

$$= L\left[\begin{pmatrix} \Delta n(t) \\ \Delta m(t) \\ \Delta m(t)^* \end{pmatrix} + L^{-1}\begin{pmatrix} 2\chi \\ \chi \\ \chi \end{pmatrix}\right], \tag{A.5.15}$$

where

$$L = \begin{pmatrix} -2\kappa & \chi & \chi \\ 2\chi & -2\kappa & 0 \\ 2\chi & 0 & -2\kappa \end{pmatrix}. \tag{A.5.16}$$

We can write (A.5.15) to

$$\frac{d}{dt}\left[\begin{pmatrix} \Delta n(t) \\ \Delta m(t) \\ \Delta m(t)^* \end{pmatrix} + L^{-1}\begin{pmatrix} 2\kappa\bar{n} \\ \chi \\ \chi \end{pmatrix}\right] = L\left[\begin{pmatrix} \Delta n(t) \\ \Delta m(t) \\ \Delta m(t)^* \end{pmatrix} + L^{-1}\begin{pmatrix} 2\kappa\bar{n} \\ \chi \\ \chi \end{pmatrix}\right]. \tag{A.5.17}$$

Defining λ as the eigenvalue of L, we have

$$\begin{aligned} |L - \lambda I| &= -(\lambda + 2\kappa)^3 + 4\chi^2(\lambda + 2\chi) \\ &= -(\lambda + 2\kappa)\{(\lambda + 2\kappa)^2 - 4\chi\} \\ &= 0, \end{aligned} \tag{A.5.18}$$

and it can be solved as

$$\lambda = -2\kappa, -2(\kappa + \chi), -2(\kappa - \chi). \tag{A.5.19}$$

The eigenvector $\boldsymbol{v}_1$ for the eigenvalue $\lambda_1 = -2\kappa$ satisfies

$$\begin{pmatrix} -2\kappa & \chi & \chi \\ 2\chi & -2\kappa & 0 \\ 2\chi & 0 & -2\kappa \end{pmatrix} \boldsymbol{v}_1 = -2\kappa \boldsymbol{v}_1, \tag{A.5.20}$$

therefore, we obtain

$$\boldsymbol{v}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \tag{A.5.21}$$

The eigenvector $\boldsymbol{v}_2$ for the eigenvalue $\lambda_2 = -2(\kappa + \chi)$ satisfies

$$\begin{pmatrix} -2\kappa & \chi & \chi \\ 2\chi & -2\kappa & 0 \\ 2\chi & 0 & -2\kappa \end{pmatrix} \boldsymbol{v}_2 = -2(\kappa + \chi) \boldsymbol{v}_2, \tag{A.5.22}$$

therefore, we obtain

$$\boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}. \tag{A.5.23}$$

The eigenvector $\boldsymbol{v}_3$ for the eigenvalue $\lambda_3 = -2(\kappa - \chi)$ satisfies

$$\begin{pmatrix} -2\kappa & \chi & \chi \\ 2\chi & -2\kappa & 0 \\ 2\chi & 0 & -2\kappa \end{pmatrix} \boldsymbol{v}_3 = -2(\kappa - \chi) \boldsymbol{v}_3, \tag{A.5.24}$$

therefore, we obtain

$$\boldsymbol{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{A.5.25}$$

Thus L can be orthogonalized as

$$L = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2\kappa & 0 & 0 \\ 0 & -2(\kappa + \chi) & 0 \\ 0 & 0 & -2(\kappa - \chi) \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}^{-1}, \tag{A.5.26}$$

where

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}^{-1} = \frac{1}{4} \begin{pmatrix} 0 & 2 & -2 \\ -2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}. \tag{A.5.27}$$

Then (A.5.15) can be solved as

$$
\begin{aligned}
&\begin{pmatrix} \Delta n(t) \\ \Delta m(t) \\ \Delta m(t)^* \end{pmatrix} + L^{-1} \begin{pmatrix} 2\kappa\bar{n} \\ \chi \\ \chi \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-2\kappa t} & 0 & 0 \\ 0 & \mathrm{e}^{-2(\kappa+\chi)t} & 0 \\ 0 & & \mathrm{e}^{-2(\kappa-\chi)t} \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}^{-1} \\
&\quad \times \left[\begin{pmatrix} \Delta n \\ \Delta m \\ \Delta m^* \end{pmatrix} + L^{-1} \begin{pmatrix} 2\kappa\bar{n} \\ \chi \\ \chi \end{pmatrix} \right] \\
&= \frac{\mathrm{e}^{-2\kappa t}}{4} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{-2\chi t} & 0 \\ 0 & & \mathrm{e}^{2\chi t} \end{pmatrix} \begin{pmatrix} 0 & 2 & -2 \\ -2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \\
&\quad \times \left[\begin{pmatrix} \Delta n \\ \Delta m \\ \Delta m^* \end{pmatrix} + L^{-1} \begin{pmatrix} 2\kappa\bar{n} \\ \chi \\ \chi \end{pmatrix} \right] \\
&= \mathrm{e}^{-2\kappa t} \begin{pmatrix} \mathrm{ch}(\kappa t) & \frac{1}{2}\mathrm{sh}(\kappa t) & \frac{1}{2}\mathrm{sh}(\kappa t) \\ \mathrm{sh}(\kappa t) & \frac{1}{2} + \frac{1}{2}\mathrm{ch}(\kappa t) & -\frac{1}{2} + \frac{1}{2}\mathrm{ch}(\kappa t) \\ \mathrm{sh}(\kappa t) & -\frac{1}{2} + \frac{1}{2}\mathrm{ch}(\kappa t) & \frac{1}{2} + \frac{1}{2}\mathrm{ch}(\kappa t) \end{pmatrix} \left[\begin{pmatrix} \Delta n \\ \Delta m \\ \Delta m^* \end{pmatrix} + L^{-1} \begin{pmatrix} 2\kappa\bar{n} \\ \chi \\ \chi \end{pmatrix} \right],
\end{aligned}
\tag{A.5.28}
$$

therefore, we obtain

$$
\begin{aligned}
\begin{pmatrix} \Delta n(t) \\ \Delta m(t) \\ \Delta m(t)^* \end{pmatrix} &= \mathrm{e}^{-2\kappa t} \begin{pmatrix} \mathrm{ch}(\kappa t) & \frac{1}{2}\mathrm{sh}(\kappa t) & \frac{1}{2}\mathrm{sh}(\kappa t) \\ \mathrm{sh}(\kappa t) & \frac{1}{2} + \frac{1}{2}\mathrm{ch}(\kappa t) & -\frac{1}{2} + \frac{1}{2}\mathrm{ch}(\kappa t) \\ \mathrm{sh}(\kappa t) & -\frac{1}{2} + \frac{1}{2}\mathrm{ch}(\kappa t) & \frac{1}{2} + \frac{1}{2}\mathrm{ch}(\kappa t) \end{pmatrix} \left[\begin{pmatrix} \Delta n \\ \Delta m \\ \Delta m^* \end{pmatrix} + L^{-1} \begin{pmatrix} 2\kappa\bar{n} \\ \chi \\ \chi \end{pmatrix} \right] \\
&\quad - L^{-1} \begin{pmatrix} 2\kappa\bar{n} \\ \chi \\ \chi \end{pmatrix} ..
\end{aligned}
\tag{A.5.29}
$$

The inverse of L is

$$
L^{-1} = -\frac{1}{4\kappa(\kappa-\chi)(\kappa+\chi)} \begin{pmatrix} 2\kappa^2 & \kappa\chi & \kappa\chi \\ 2\kappa\chi & 2\kappa^2 - \chi^2 & \chi^2 \\ 2\kappa\chi & \chi^2 & 2\kappa^2 - \chi^2 \end{pmatrix},
\tag{A.5.30}
$$

then, the second term in the right hand-side of (A.5.29) is

$$L^{-1}\begin{pmatrix} 2\kappa\bar{n} \\ \chi \\ \chi \end{pmatrix} = -\frac{1}{4\kappa(\kappa-\chi)(\kappa+\chi)}\begin{pmatrix} 2\kappa(2\kappa^2\bar{n}+\chi^2) \\ 2\kappa^2\chi(2\bar{n}+1) \\ 2\kappa^2\chi(2\bar{n}+1) \end{pmatrix}$$
$$= -\frac{1}{2(\kappa-\chi)(\kappa+\chi)}\begin{pmatrix} 2\kappa^2\bar{n}+\chi^2 \\ \kappa\chi(2\bar{n}+1) \\ \kappa\chi(2\bar{n}+1) \end{pmatrix}. \tag{A.5.31}$$

From (A.5.29), we can calculate $\varDelta n(t)$ as

$$\begin{aligned}
\varDelta n(t) &= \mathrm{e}^{-2\kappa t}\left\{\mathrm{ch}(\chi t)\varDelta n + \frac{1}{2}\mathrm{sh}(\chi t)\varDelta m + \frac{1}{2}\mathrm{sh}(\chi t)\varDelta m^*\right\} \\
&\quad -\frac{\mathrm{e}^{-2\kappa t}}{2(\kappa-\chi)(\kappa+\chi)}\left\{\mathrm{ch}(\chi t)(2\kappa^2\bar{n}+\chi) + \frac{1}{2}\mathrm{sh}(\chi t)(2\bar{n}+1)\kappa\chi + \frac{1}{2}\mathrm{sh}(\chi t)(2\bar{n}+1)\kappa\chi\right\} \\
&\quad +\frac{2\kappa^2\bar{n}+\chi^2}{2(\kappa-\chi)(\kappa+\chi)} \\
&= \frac{\mathrm{e}^{-2(\kappa-\chi)t}}{2}\left\{\varDelta n + \frac{\varDelta m+\varDelta m^*}{2} - \frac{2\kappa^2\bar{n}+\chi^2+(2\bar{n}+1)\kappa\chi}{2(\kappa-\chi)(\kappa+\chi)}\right\} \\
&\quad +\frac{\mathrm{e}^{-2(\kappa+\chi)t}}{2}\left\{\varDelta n - \frac{\varDelta m+\varDelta m^*}{2} - \frac{2\kappa^2\bar{n}+\chi^2-(2\bar{n}+1)\kappa\chi}{2(\kappa-\chi)(\kappa+\chi)}\right\} \\
&\quad +\frac{2\kappa^2\bar{n}+\chi^2}{2(\kappa-\chi)(\kappa+\chi)} \\
&= \frac{\mathrm{e}^{-2(\kappa-\chi)t}}{2}\left\{\varDelta n + \varDelta m' - \frac{\kappa\bar{n}+\frac{1}{2}\chi}{(\kappa-\chi)}\right\} + \frac{\mathrm{e}^{-2(\kappa+\chi)t}}{2}\left\{\varDelta n - \varDelta m' - \frac{\kappa\bar{n}-\frac{1}{2}\chi}{(\kappa+\chi)}\right\} \\
&\quad +\frac{\kappa\bar{n}+\frac{1}{2}\chi}{2(\kappa-\chi)} + \frac{\kappa\bar{n}-\frac{1}{2}\chi}{2(\kappa+\chi)} \\
&= \varDelta n + \left(1-\mathrm{e}^{-2(\kappa-\chi)t}\right)\left\{\frac{\kappa\bar{n}+\frac{1}{2}\chi}{2(\kappa-\chi)} - \frac{\varDelta n}{2} - \frac{\varDelta m'}{2}\right\} \\
&\quad +\left(1-\mathrm{e}^{-2(\kappa+\chi)t}\right)\left\{\frac{\kappa\bar{n}-\frac{1}{2}\chi}{2(\kappa+\chi)} - \frac{\varDelta n}{2} + \frac{\varDelta m'}{2}\right\} \\
&= \varDelta n - \left(1-\mathrm{e}^{-2(\kappa-\chi)t}\right)\frac{\kappa(\varDelta n+\varDelta m'-\bar{n}) - \chi(\varDelta n+\varDelta m'+\frac{1}{2})}{2(\kappa-\chi)} \\
&\quad -\left(1-\mathrm{e}^{-2(\kappa+\chi)t}\right)\frac{\kappa(\varDelta n-\varDelta m'-\bar{n}) + \chi(\varDelta n-\varDelta m'+\frac{1}{2})}{2(\kappa+\chi)}
\end{aligned} \tag{A.5.32}$$

and $\Delta m(t)$ as

$$\begin{aligned}
\Delta m(t) &= \mathrm{e}^{-2\kappa t}\left\{\mathrm{sh}(\chi t)\Delta n + \left(\frac{1}{2}+\frac{1}{2}\mathrm{ch}(\chi t)\right)\Delta m + \left(-\frac{1}{2}+\frac{1}{2}\mathrm{ch}(\chi t)\Delta m^*\right)\right\} \\
&\quad -\frac{\mathrm{e}^{-2\kappa t}}{2(\kappa-\chi)(\kappa+\chi)}\left\{\mathrm{sh}(\chi t)(2\kappa^2\bar{n}+\chi^2) + \left(\frac{1}{2}+\frac{1}{2}\mathrm{ch}(\chi t)\right)(2\bar{n}+1)\kappa\chi \right.\\
&\quad\qquad \left. + \left(-\frac{1}{2}+\frac{1}{2}\mathrm{ch}(\chi t)\right)(2\bar{n}+1)\kappa\chi\right\} \\
&\quad -\frac{(2\bar{n}+1)\kappa\chi}{2(\kappa-\chi)(\kappa+\chi)} \\
&= \frac{\mathrm{e}^{-2(\kappa-\chi)t}}{2}\left\{\Delta n + \frac{\Delta m+\Delta m^*}{2} - \frac{2\kappa^2\bar{n}+\chi^2+(2\bar{n}+1)\kappa\chi}{2(\kappa-\chi)(\kappa+\chi)}\right\} \\
&\quad +\frac{\mathrm{e}^{-2(\kappa+\chi)t}}{2}\left\{-\Delta n + \frac{\Delta m+\Delta m^*}{2} - \frac{-(2\kappa^2\bar{n}+\chi^2)-(2\bar{n}+1)\kappa\chi}{2(\kappa-\chi)(\kappa+\chi)}\right\} \\
&\quad +\mathrm{e}^{-2\kappa t}\frac{\Delta m-\Delta m^*}{2} + \frac{2\kappa^2\bar{n}+\chi^2}{2(\kappa-\chi)(\kappa+\chi)} \\
&= \mathrm{e}^{-2(\kappa-\chi)t}\left\{\frac{\Delta n}{2}+\frac{\Delta m'}{2}-\frac{\kappa\bar{n}+\frac{1}{2}\chi}{(\kappa-\chi)}\right\} \\
&\quad +\mathrm{e}^{-2(\kappa+\chi)t}\left\{-\frac{\Delta n}{2}+\frac{\Delta m'}{2}+\frac{\kappa\bar{n}-\frac{1}{2}\chi}{(\kappa+\chi)}\right\} \\
&\quad +\mathrm{e}^{-2\kappa t} i\Delta m'' - \frac{\kappa\bar{n}+\frac{1}{2}\chi}{2(\kappa-\chi)} + \frac{\kappa\bar{n}-\frac{1}{2}\chi}{2(\kappa+\chi)} \\
&= \Delta m' + \left(1-\mathrm{e}^{-2(\kappa-\chi)t}\right)\left\{\frac{\kappa\bar{n}+\frac{1}{2}\chi}{2(\kappa-\chi)} - \frac{\Delta n}{2} - \frac{\Delta m'}{2}\right\} \\
&\quad + \left(1-\mathrm{e}^{-2(\kappa+\chi)t}\right)\left\{-\frac{\kappa\bar{n}-\frac{1}{2}\chi}{2(\kappa+\chi)} + \frac{\Delta n}{2} - \frac{\Delta m'}{2}\right\} \\
&\quad - \left(1-\mathrm{e}^{-2\kappa t}\right) i\Delta m'' + i\Delta m'' \\
&= \Delta m - \left(1-\mathrm{e}^{-2(\kappa-\chi)t}\right)\frac{\kappa(\Delta n+\Delta m'-\bar{n})-\chi(\Delta n+\Delta m'+\frac{1}{2})}{2(\kappa-\chi)} \\
&\quad + \left(1-\mathrm{e}^{-2(\kappa+\chi)t}\right)\frac{\kappa(\Delta n-\Delta m'-\bar{n})+\chi(\Delta n-\Delta m'+\frac{1}{2})}{2(\kappa+\chi)} \\
&\quad - \left(1-\mathrm{e}^{-2\kappa t}\right) i\Delta m''. \qquad \text{(A.5.33)}
\end{aligned}$$

A.6 Derivation of (4.1.17)

A general spriter operator is described as

$$
\begin{aligned}
\hat{V}_{AB}(\theta) &= \mathrm{e}^{\theta(a_A^\dagger a_B - a_A a_B^\dagger)+\theta(\tilde{a}_A^\dagger \tilde{a}_B - \tilde{a}_A \tilde{a}_B^\dagger)} \\
&= \exp\left[-2\theta m_r(\gamma_A^{\venus}\gamma_B^{\venus} + \tilde{\gamma}_A^{\venus} + \tilde{\gamma}_B^{\venus}) + \theta(\gamma_A^{\venus}\gamma_B - \gamma_A\gamma_B^{\venus} + \tilde{\gamma}_A^{\venus}\tilde{\gamma}_B - \tilde{\gamma}_A\tilde{\gamma}_B^{\venus})\right] \\
&= \mathrm{e}^{-2\theta m_r \gamma_A^{\venus}\gamma_B^{\venus} + \theta(\gamma_A^{\venus}\gamma_B - \gamma_A\gamma_B^{\venus}) + t.c.}. \qquad \text{(A.6.1)}
\end{aligned}
$$

Here, we rewrite the non-tilde term as

$$
\mathrm{e}^{-2\theta m_r \gamma_A^{\venus}\gamma_B^{\venus} + \theta(\gamma_A^{\venus}\gamma_B - \gamma_A\gamma_B^{\venus})} = F(\theta)\mathrm{e}^{f(\theta)(\gamma_A^{\venus}\gamma_B - \gamma_A\gamma_B^{\venus})}, \qquad \text{(A.6.2)}
$$

where $F(\theta)$ is an operator satisfying $F(0) = 1$ and $f(\theta)$ is a c-number function satisfying $f(0) = 0$. Differentiating (A.6.2) with respected to θ, we have

$$
\begin{aligned}
& F(\theta)\mathrm{e}^{f(\theta)(\gamma_A^{\venus}\gamma_B - \gamma_A\gamma_B^{\venus})}\left\{-2m_\lambda \gamma_A^{\venus}\gamma_B^{\venus} + \gamma_A^{\venus}\gamma_B - \gamma_A\gamma_B^{\venus}\right\} \\
= \; & \frac{dF(\theta)}{d\theta}\mathrm{e}^{f(\theta)(\gamma_A^{\venus}\gamma_B - \gamma_A\gamma_B^{\venus})} + F(\theta)\mathrm{e}^{f(\theta)(\gamma_A^{\venus}\gamma_B - \gamma_A\gamma_B^{\venus})}\frac{df(\theta)}{d\theta}(\gamma_A^{\venus}\gamma_B - \gamma_A\gamma_B^{\venus}), \qquad \text{(A.6.3)}
\end{aligned}
$$

and therefore the differential equation for $F(\theta)$ is given by

$$
\begin{aligned}
\frac{dF(\theta)}{d\theta} = F(\theta)\Bigg[&\left\{1 - \frac{df(\theta)}{d\theta}\right\}(\gamma_A^{\venus}\gamma_B - \gamma_A\gamma_B^{\venus}) \\
&-2m_r \mathrm{e}^{f(\theta)(\gamma_A^{\venus}\gamma_B - \gamma_A\gamma_B^{\venus})}\gamma_A^{\venus}\gamma_B^{\venus}\mathrm{e}^{-f(\theta)(\gamma_A^{\venus}\gamma_B - \gamma_A\gamma_B^{\venus})}\Bigg]. \qquad \text{(A.6.4)}
\end{aligned}
$$

Here, we define

$$
\gamma_J^{\venus}(\theta) = \mathrm{e}^{f(\theta)(\gamma_A^{\venus}\gamma_B - \gamma_A\gamma_B^{\venus})}\gamma_J^{\venus}\mathrm{e}^{-f(\theta)(\gamma_A^{\venus}\gamma_B - \gamma_A\gamma_B^{\venus})}, \qquad \text{(A.6.5)}
$$

where J is A or B. The differential equations for $\gamma_J^{\venus}(\theta)$ are given by

$$
\begin{aligned}
\frac{d}{d\theta}\gamma_A^{\venus}(\theta) &= -\frac{df(\theta)}{d\theta}\mathrm{e}^{f(\theta)(\gamma_A^{\venus}\gamma_B - \gamma_A\gamma_B^{\venus})}\left[\gamma_A^{\venus}, \gamma_A^{\venus}\gamma_B - \gamma_A\gamma_B^{\venus}\right]\mathrm{e}^{-f(\theta)(\gamma_A^{\venus}\gamma_B - \gamma_A\gamma_B^{\venus})} \\
&= -\frac{df(\theta)}{d\theta}\gamma_B^{(r_0)}, \qquad \text{(A.6.6)} \\
\frac{d}{d\theta}\gamma_B^{\venus}(\theta) &= -\frac{df(\theta)}{d\theta}\mathrm{e}^{f(\theta)(\gamma_A^{\venus}\gamma_B - \gamma_A\gamma_B^{\venus})}\left[\gamma_B^{\venus}, \gamma_A^{\venus}\gamma_B - \gamma_A\gamma_B^{\venus}\right]\mathrm{e}^{-f(\theta)(\gamma_A^{\venus}\gamma_B - \gamma_A\gamma_B^{\venus})} \\
&= \frac{df(\theta)}{d\theta}\gamma_A^{(r_0)}. \qquad \text{(A.6.7)}
\end{aligned}
$$

Solving this equation with the initial condition $\gamma_A^{\venus}(\theta = 0) = \gamma_A^{\venus}, \quad \gamma_B^{\venus}(\theta = 0) = \gamma_B^{\venus}$, we obtain

$$
\gamma_A^{\venus}(\theta) = \gamma_A^{\venus}\cos f(\theta) - \gamma_B^{\venus}\sin f(\theta), \qquad \text{(A.6.8)}
$$

$$
\gamma_B^{\venus}(\theta) = \gamma_A^{\venus}\sin f(\theta) + \gamma_B^{\venus}\cos f(\theta). \qquad \text{(A.6.9)}
$$

Then (A.6.4) is rewritten as

$$
\frac{dF(\theta)}{d\theta} = F(\theta)\left[\left\{1 - \frac{df(\theta)}{d\theta}\right\}(\gamma_A^{\venus}\gamma_B - \gamma_A\gamma_B^{\venus}) - 2m_\lambda\left(\gamma_A^{\venus}\cos f(\theta) - \gamma_B^{\venus}\sin f(\theta)\right)\left(\gamma_A^{\venus}\sin f(\theta) + \gamma_B^{\venus}\cos f(\theta)\right)\right]. \qquad \text{(A.6.10)}
$$

Here we set

$$f(\theta) = \theta. \tag{A.6.11}$$

Then (A.6.10) is rewritten as

$$\frac{dF(\theta)}{d\theta} = F(\theta)\left[-m_r \sin(2\theta)\left(\gamma^{\dagger 2}_{A} - \gamma^{\dagger 2}_{B}\right) - 2m_r \cos(2\theta)\gamma^{\dagger}_{A}\gamma^{\dagger}_{B}\right]. \tag{A.6.12}$$

Solving this equation with the initial condition $F(0) = 1$, we obtain

$$F(\theta) = \exp\left[-\frac{1}{2}m_r 1(1-\cos(2\theta))\left(\gamma^{\dagger 2}_{A} - \gamma^{\dagger 2}_{B}\right) - m_\lambda \sin(2\theta)\gamma^{\dagger}_{A}\gamma^{\dagger}_{B}\right]. \tag{A.6.13}$$

The entangled state after BS is described as

$$\begin{aligned} |0(r,-r)\rangle_{AB} &= \hat{V}_{AB}(\theta)|0(r)\rangle_A|0(-r)\rangle_B \\ &= F(\theta)\tilde{F}(\theta)\mathrm{e}^{f(\theta)(\gamma^{\dagger}_{A}\gamma_B - \gamma_A\gamma^{\dagger}_{B})}\mathrm{e}^{f(\theta)(\tilde{\gamma}^{\dagger}_{A}\tilde{\gamma}_B - \tilde{\gamma}_A\tilde{\gamma}^{\dagger}_{B})}|0(r)\rangle_A|0(-r)\rangle_B \\ &= F(\theta)\tilde{F}(\theta)|0(r)\rangle_A|0(-r)\rangle_B \end{aligned} \tag{A.6.14}$$

where, we use $\gamma_J|0(r)\rangle_J = 0$ in the third equality. Putting $\theta = -\pi/4$ to (A.6.14), we obtain the entangled vacua after passing HBS.

A.7 Derivation of $|\{\bar{\beta}\}; r, \psi_{in}\rangle_B$

A.7.1 Derivation of ${}_{VA}\langle\beta|\alpha\rangle_V$

We consider the bra vector ${}_A\langle v|$ defined by

$$ {}_A\langle v| = {}_{VA}\{\beta\}|v\alpha\rangle_V = {}_{VA}\{\beta\}\hat{D}_V(v\alpha)|0\rangle_V. \tag{A.7.1} $$

Differentiating $A.7.1$ with respect to v,we obtain

$$ \begin{aligned} \frac{d}{dv}{}_A\langle v| &= {}_{VA}\langle\{\beta\}|(\alpha a_V^\dagger - \alpha^* a_V + \alpha^*\tilde{a}_V^\dagger - \alpha\tilde{a}_V)|v\alpha\rangle_V \\ &= i\sqrt{2}{}_{VA}\langle\{\beta\}|(\alpha'' X_V - \alpha' P_V - \alpha''\tilde{X}_V + \alpha'\tilde{P}_V)|v\alpha\rangle_V \\ &= i\sqrt{2}{}_{VA}\langle\{\beta\}|\{\alpha''(x_- + X_A) - \alpha'(p_+ - P_A) - \alpha''(x_- + \tilde{X}_A) + \alpha'(p_+ - \tilde{P}_A)\}|v\alpha\rangle_V \\ &= {}_{VA}\langle\{\beta\}|(a_A - \alpha^* a_A^\dagger + \alpha^*\tilde{a}_A - \alpha\tilde{a}_A^\dagger)|v\alpha\rangle_V. \end{aligned} \tag{A.7.2} $$

Solving this equation, we have

$$ {}_A\langle v| = {}_A\langle v=0|\hat{D}_A(-\alpha^*). \tag{A.7.3} $$

The bra vector ${}_A\langle v=0|$ can be derived by

$$ \begin{aligned} {}_A\langle v=0| &= {}_{VA}\langle\{\beta\}|0\rangle_V \\ &= \frac{1}{2\pi}\int dx \int dx' \mathrm{e}^{ip_+(x-x')}{}_V(x+x_-|{}_V(\tilde{x}'+\tilde{x}|{}_A(x|{}_A(\tilde{x}'|0)_V|\tilde{0})_V \\ &= \frac{1}{2\pi\sqrt{\pi}}\int dx \int dx' \mathrm{e}^{-\frac{1}{2}x^2-(x_- - ip_+)x-\frac{1}{2}x'^2-(x_- + ip_+)x'-x_-^2}{}_A(x|{}_A(\tilde{x}'| \\ &= \frac{1}{2\pi\sqrt{\pi}}\mathrm{e}\int dx \mathrm{e}^{-\frac{1}{2}|\beta|^2-\frac{1}{2}\beta^{*2}-\sqrt{2}\beta^* x-\frac{1}{2}x^2}\int dx' \mathrm{e}^{-\frac{1}{2}|\beta|^2-\frac{1}{2}\beta^2-\sqrt{2}\beta x'-\frac{1}{2}x^2}{}_A(x|(\tilde{x'}| \\ &= \frac{1}{2\pi}{}_A(-\beta^*|{}_A(-\tilde{\beta}^*| \\ &= \frac{1}{2\pi}{}_A\langle 0|\hat{D}_A(\beta^*). \end{aligned} \tag{A.7.4} $$

At the second and fifth equality, we used the property

$$ {}_J(x|\alpha\rangle_J = \frac{1}{\pi^{\frac{4}{4}}}\mathrm{e}^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}\alpha^2+\sqrt{2}\alpha x-\frac{1}{2}x^2}. \tag{A.7.5} $$

Finally we obtain

$$ {}_A\langle v| = \frac{1}{2\pi}{}_A\langle 0|\hat{D}(\beta^* - \alpha^*). \tag{A.7.6} $$

A.7.2 Derivation of$|\{\bar{\beta}\}; r, \psi_{in}\rangle_B$

We consider a ket vector $|u\rangle_B$defined by

$$ \begin{aligned} |u\rangle_B &= {}_A\langle u\mu|\{r,-r\}\rangle_{AB} \\ &= {}_A\langle 0|\hat{D}_A^\dagger(u\mu)|\{r,-r\}\rangle_{AB}. \end{aligned} \tag{A.7.7} $$

Differentiating (A.7.7) with respect to u, we obtain

$$\begin{aligned}\frac{d}{du}|u\rangle_B &= {}_A\langle u\mu|(-\mu a_A^\dagger + \mu^* a_A - \mu^*\tilde{a}_A^\dagger + \mu\tilde{a}_A)|\{r,-r\}\rangle_{AB} \\ &= -2u|\mu|^2|u\rangle_B + {}_A\langle u\mu|(\mu^* a_A + \mu\tilde{a}_A)|\{r,-r\}\rangle_{AB}. \end{aligned} \tag{A.7.8}$$

To solve (A.7.8), at first, we calculate the third term in the left-hand-side of (A.7.8).

$$\begin{aligned} a_A|\{r,-r\}\rangle_{AB} &= (\gamma_A + n_\lambda \gamma_A^{♀} + m_\lambda \gamma_A^{♀})|\{r,-r\}\rangle_{AB} \\ &= \frac{1}{2}\Big\{ b_{A\xi} - b_{B\xi} + 2m_\lambda b_{B\xi}^{♀} + i(b_{A\eta} - b_{B\eta} - 2m_{B\eta}^{♀}) \\ &\qquad + (n_\lambda + m_\lambda)(b_{A\xi}^{♀} - b_{B\xi}^{♀}) + i(n_\lambda - m_\lambda)(b_{A\eta}^{♀} - b_{B\eta}^{♀})\Big\}|\{r,-r\}\rangle_{AB} \\ &= \frac{1}{2}\Big\{(n_\lambda + m_\lambda)b_{A\xi}^{♀} - (n_\lambda - m_\lambda)b_{B\xi}^{♀} + i(n_\lambda - m_\lambda)b_{A\eta}^{♀} - i(n_\lambda + m_\lambda)b_{B\eta}^{♀}\Big\}|\{r,-r\}\rangle_{AB} \\ &= \Big(-n_\lambda a_A + n_\lambda \tilde{a}_A^\dagger - m_\lambda \tilde{a}_B + m_\lambda a_B^\dagger\Big)|\{r,-r\}\rangle_{AB}, \end{aligned} \tag{A.7.9}$$

where, we used (4.1.23) at the second equality and (4.1.20) at the third equality. From (A.7.9) , we have

$$\therefore a_A|\{r,-r\}\rangle_{AB} = \frac{1}{1+n_\lambda}\Big(n_\lambda \tilde{a}_A^\dagger - m_\lambda \tilde{a}_B + m_\lambda a_B^\dagger\Big)|\{r,-r\}\rangle_{AB}, \tag{A.7.10}$$

then we can rewrite (A.7.8) to

$$\begin{aligned}\frac{d}{du}|u\rangle_B &= -2u|\mu|^2|u\rangle_B + {}_A\langle u\mu|\frac{\mu^*}{1+n_\lambda}(n_\lambda u\mu - m_\lambda\tilde{a}_B + m_\lambda a_B^\dagger)|\{r,-r\}\rangle_{AB} \\ &\quad + {}_A\langle u\mu|\frac{\mu}{1+n_\lambda}(n_\lambda u\mu^* - m_\lambda a_B + m_\lambda \tilde{a}_B^\dagger)|\{r,-r\}\rangle_{AB} \\ &= -\frac{1}{1+n_\lambda}\Big\{2|\mu|^2 u + m_\lambda(\mu^* a_B - \mu a_B^\dagger + \mu\tilde{a}_B - \mu^*\tilde{a}_B^\dagger)\Big\}|u\rangle_B. \end{aligned} \tag{A.7.11}$$

. Here, we estimate ket vector $|u=0\rangle_B$,

$$\begin{aligned}|u=0\rangle_B &= {}_A\langle 0|\{r,-r\}\rangle_{AB} \\ &= {}_A\langle 0|\exp\Big[m_r(\gamma_A^{♀}\gamma_B^{♀} + \tilde{\gamma}_A^{♀}\tilde{\gamma}_B^{♀}) + n_r(\gamma_A^{♀}\tilde{\gamma}_A^{♀} + \gamma_B^{♀}\tilde{\gamma}_B^{♀})\Big]|0\rangle_A|0\rangle_B. \end{aligned} \tag{A.7.12}$$

$$\begin{aligned} e^{n_r\gamma_J^{♀}\tilde{\gamma}_J^{♀}} &= e^{n_r(a_J\tilde{a}_J - a_J^\dagger a_J - \tilde{a}_J\tilde{a}_J^\dagger + a_J^\dagger\tilde{a}_J^\dagger)} \\ &= \hat{G}_J(n_r)e^{g(n_r)a_J^\dagger\tilde{a}_J^\dagger} \end{aligned} \tag{A.7.13}$$

where, $\hat{G}_J(n_r)$ is an operator and $g(v_r)$ is c-number function satisfying

$$G(0) = 1 \tag{A.7.14}$$

$$g(0) = 0. \tag{A.7.15}$$

Differentiating (A.7.13) with respect to n_r, we obtain

$$\begin{aligned}\hat{G}_J(n_r)e^{g(n_r)a_J^\dagger\tilde{a}_J^\dagger}(a_J\tilde{a}_J &- a_J^\dagger a_J - \tilde{a}_J\tilde{a}_J^\dagger + a_J^\dagger\tilde{a}_J^\dagger) \\ &= \frac{d\hat{G}_J(n_r)}{dn_r}e^{g(n_r)a_J^\dagger\tilde{a}_J^\dagger} + \frac{dg(n_r)}{dn_r}\hat{G}_J(n_r)e^{g(n_r)a_J^\dagger\tilde{a}_J^\dagger}a_J^\dagger\tilde{a}_J^\dagger, \end{aligned} \tag{A.7.16}$$

therefore,

$$
\begin{aligned}
\frac{d}{dn_r}\hat{G}_J(n_r) &= \hat{G}_J(n_r)\left[\left\{1-\frac{dg(n_r)}{dn_r}\right\}a_J^{\dagger}\tilde{a}_J^{\dagger}+\mathrm{e}^{g(n_r)a_J^{\dagger}\tilde{a}_J^{\dagger}}(a_J\tilde{a}_J-a_Ja_J^{\dagger}-\tilde{a}_J\tilde{a}_J^{\dagger})\mathrm{e}^{-g(n_r)a_J^{\dagger}\tilde{a}_J^{\dagger}}\right]\\
&=\hat{G}_J(n_r)\Big[\left\{1-\frac{dg(n_r)}{dn_r}\right\}a_J^{\dagger}\tilde{a}_J^{\dagger}\\
&\qquad+(a_J+g(n_r)\tilde{a}_J^{\dagger})(\tilde{a}_J+g(n_r)a_J^{\dagger})-a_J^{\dagger}(a_J+g(n_r)\tilde{a}_J^{\dagger})-(\tilde{a}_J+g(n_r)a_J^{\dagger})\tilde{a}_J^{\dagger}\Big]\\
&=\hat{G}_J(n_r)\Big[\left\{1-\frac{dg(n_r)}{dn_r}-2g(n_r)+g(n_r)^2\right\}a_J^{\dagger}\tilde{a}_J^{\dagger}\\
&\qquad+a_J\tilde{a}_J+(g(n_r)-1)a_J^{\dagger}a_J+(g(n_r)-1)\tilde{a}_J^{\dagger}\tilde{a}_J+g(n_r)-1\Big].
\end{aligned}
\tag{A.7.17}
$$

Here, we require $g(n_r)$ satisfying

$$
1-\frac{dg(n_r)}{dn_r}-2g(n_r)+g(n_r)^2=0,
\tag{A.7.18}
$$

and solving this differential equation, $g(n_r)$ is decided by

$$
g(n_r)=\frac{n_r}{1+n_r.}
\tag{A.7.19}
$$

Then the equation (A.7.17) comes to

$$
\frac{d}{dn_r}\hat{G}_J(n_r)=\hat{G}_J(n_r)\left[a_J\tilde{a}_J-\frac{1}{1+n_r}a_J^{\dagger}a_J-\frac{1}{1+n_r}\tilde{a}_J^{\dagger}\tilde{a}_J-\frac{1}{1+n_r}\right].
\tag{A.7.20}
$$

Solving this equation, we obtain

$$
\begin{aligned}
\hat{G}_J(n_r) &= \mathrm{e}^{-\ln(1+n_r)+n_ra_J\tilde{a}_J-\ln(1+n_r)(a_J^{\dagger}a_J+\tilde{a}_J^{\dagger}\tilde{a}_J)}\\
&=\frac{1}{1+n_r}\mathrm{e}^{+n_ra_J\tilde{a}_J-\ln(1+n_r)(a_J^{\dagger}a_J+\tilde{a}_J^{\dagger}\tilde{a}_J)}
\end{aligned}
\tag{A.7.21}
$$

therefore, from (A.7.13), we see

$$
\begin{aligned}
\mathrm{e}^{n_r\gamma_J^{\mathring{\dagger}}\tilde{\gamma}_J^{\mathring{\dagger}}}|0\rangle_J &= \mathrm{e}^{\frac{n_r}{1+n_r}a_J^{\dagger}\tilde{a}_J^{\dagger}}\hat{G}_J(n_r)|0\rangle_J\\
&=\frac{1}{1+n_r}\mathrm{e}^{\frac{n_r}{1+n_r}a_J^{\dagger}\tilde{a}_J^{\dagger}}|0\rangle_J.
\end{aligned}
\tag{A.7.22}
$$

Then the equation (A.7.12) is rewritten by

$$
\begin{aligned}
|u=0\rangle_B &= \frac{1}{1+n_r}{}_A\langle 0|\mathrm{e}^{m_r(\gamma_A^{\mathring{\dagger}}\gamma_B^{\mathring{\dagger}}+\tilde{\gamma}_A^{\mathring{\dagger}}\tilde{\gamma}_B^{\mathring{\dagger}})}\mathrm{e}^{\frac{n_r}{1+n_r}a_A^{\dagger}\tilde{a}_A^{\dagger}}\mathrm{e}^{n_r\gamma_B^{\mathring{\dagger}}\tilde{\gamma}_B^{\mathring{\dagger}}}|0\rangle_A|0\rangle_B\\
&=\frac{1}{1+n_r}{}_A\langle 0|\mathrm{e}^{m_r(-a_A\tilde{\gamma}_B^{\mathring{\dagger}}-\tilde{a}_A\gamma_B^{\mathring{\dagger}}+a_A^{\dagger}\gamma_B{}^{\mathring{\dagger}}+\tilde{a}_A^{\dagger}\tilde{\gamma}^{\mathring{\dagger}})}\mathrm{e}^{\frac{n_r}{1+n_r}a_A^{\dagger}\tilde{a}_A^{\dagger}}\mathrm{e}^{n_r\gamma_B^{\mathring{\dagger}}\tilde{\gamma}_B^{\mathring{\dagger}}}|0\rangle_A|0\rangle_B\\
&=\frac{1}{1+n_r}{}_A\langle 0|\mathrm{e}^{m_r(a_A^{\dagger}\gamma_B^{\mathring{\dagger}}+\tilde{a}_A^{\dagger}\tilde{\gamma}_B^{\mathring{\dagger}})}\mathrm{e}^{-m_r(a_A\tilde{\gamma}_B^{\mathring{\dagger}}+\tilde{a}_A\gamma_B^{\mathring{\dagger}})}\mathrm{e}^{-m_r^2\gamma_B^{\mathring{\dagger}}\tilde{\gamma}_B^{\mathring{\dagger}}}\mathrm{e}^{\frac{n_r}{1+n_r}a_A^{\dagger}\tilde{a}_A^{\dagger}}\mathrm{e}^{n_r\gamma_B^{\mathring{\dagger}}\tilde{\gamma}_B^{\mathring{\dagger}}}|0\rangle_A|0\rangle_B\\
&=\frac{1}{1+n_r}{}_A\langle 0|\mathrm{e}^{-m_r(a_A\tilde{\gamma}_B^{\mathring{\dagger}}+\tilde{a}_A\gamma_B^{\mathring{\dagger}})}\mathrm{e}^{-m_r^2\gamma_B^{\mathring{\dagger}}\tilde{\gamma}_B^{\mathring{\dagger}}}\mathrm{e}^{\frac{n_r}{1+n_r}a_A^{\dagger}\tilde{a}_A^{\dagger}}\mathrm{e}^{n_r\gamma_B^{\mathring{\dagger}}\tilde{\gamma}_B^{\mathring{\dagger}}}|0\rangle_A|0\rangle_B\\
&=\frac{1}{1+n_r}{}_A\langle 0|\mathrm{e}^{\frac{n_r}{1+n_r}(a_A^{\dagger}-m_r\gamma_B^{\mathring{\dagger}})(a_A^{\dagger}-m_r\tilde{\gamma}_B^{\mathring{\dagger}})}\mathrm{e}^{-m_r(a_A\tilde{\gamma}_B^{\mathring{\dagger}}+\tilde{a}_A\gamma_B^{\mathring{\dagger}})}\mathrm{e}^{(n_r-m_r^2)\gamma_B^{\mathring{\dagger}}\tilde{\gamma}_B^{\mathring{\dagger}}}|0\rangle_A|0\rangle_B\\
&=\frac{1}{1+n_r}\mathrm{e}^{\frac{n_r+n_r^2-m_r^2}{1+n_r}\gamma_B^{\mathring{\dagger}}\tilde{\gamma}_B^{\mathring{\dagger}}}|0\rangle_B.
\end{aligned}
\tag{A.7.23}
$$

Then we solve the equation (A.7.7) to

$$|u\rangle_B = \frac{1}{1+n_r}\mathrm{e}^{-\frac{u^2|\mu|^2}{1+n_r}}\hat{D}_B\left(\frac{m_r}{1+n_r}u\mu\right)\mathrm{e}^{\frac{n_r+n_r^2-m_r^2}{1+n_r}\gamma_B^{\female}\tilde{\gamma}_B^{\female}}|0\rangle_B \tag{A.7.24}$$

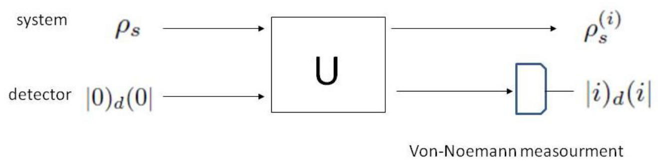

Figure A.1: Schematic drawings of a quantum measurement.

A.8 Generalized Measurement

A measurement for quantum system is illustrated by fig. A.1[22]. We describe the initial density operator of the system by ρ_s and that of the detector by $|0)_d(0|$. The system and the detector are entangled by an unitary operator U, then a projective measurement with orthogonal complete basis $\{|i)_d\}$ (von-Noemann measurement) is conducted on the detector. When we gain outcome i by this measurement, the state of the system comes to

$$\begin{aligned}\rho_s^{(i)} &= \frac{1}{w_i}\mathrm{tr}_d\left[|i)_d(i|U\rho_s|0)_d(0|U^\dagger\right] \\ &= \frac{1}{w_i}{}_d(i|U|0)_d\rho_s{}_d(0|U^\dagger|i)_d \\ &= \frac{1}{w_i}M_s^{(i)}\rho_s M_s^{(i)\dagger}\end{aligned} \tag{A.8.1}$$

where

$$M_s^{(i)} = {}_d(i|U|0)_d \tag{A.8.2}$$

is called measurement operator, and

$$\begin{aligned}w_i &= \mathrm{tr}_s\left[M_s^{(i)}\rho_s M_s^{(i)\dagger}\right] \\ &= \mathrm{tr}_s\left[M_s^{(i)\dagger}M_s^{(i)}\rho_s\right]\end{aligned} \tag{A.8.3}$$

is the probability to gain outcome i.

The measurement operator $M_s^{(i)}$ satisfies the completeness

$$\sum_i M_s^{\dagger(i)}M_s^{(i)} = \sum_i {}_d(0|U^\dagger|i)_{dd}(i|U|0)_d \tag{A.8.4}$$

$$= \sum_i {}_d(0|I|0)_d \tag{A.8.5}$$

$$= I_s. \tag{A.8.6}$$

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