A method for finding a minimal point of the lattice in cubic number fields

Kan Kaneko

February 2014

A method for finding a minimal point of the lattice in cubic number fields

Kan Kaneko
Doctoral Program in Mathematics

Submitted to the Graduate School of
Pure and Applied Sciences
in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy in
Science

at the
University of Tsukuba

Preface

I have studied Algebraic Number Theory. This dissertation is devoted to the study of a method for finding a minimal point of the reduced lattice in a cubic algebraic number field of negative discriminant and the study of two families of cubic number fields. One of the algorithms that find the fundamental units of the order of cubic number fields is Voronoi's algorithm. Voronoi's algorithm determines a chain of minimal point in such a lattice. It is known that to find all the minimal points of a reduced lattice, it is sufficient to know how to find a minimal point adjacent to 1 in any reduced lattice. Two versions are familiar as such a method: one version is by Delone (1940) and the other is by Williams, Cormack and Seah (1980). Williams, Cormack and Seah utilized the two-dimensional lattice obtained from a reduced lattice $\mathcal{R}$ to find a minimal point adjacent to 1 in $\mathcal{R}$. Subsequently, Adam(1995) utilized an isotropic vector of the quadratic form obtained from a basis of a reduced lattice $\mathcal{R}$. Later, Lahlou and Farhane(2005) generalise the Adam's method.

First, we give a method for finding a minimal point adjacent to 1 of the reduced lattice $\mathcal{R}$ in cubic number fields using an isotropic vector of the quadratic form and the two-dimensional lattice. Second, we consider a one-parameter family of cubic fields introduced by Ishida (1988). His family has a lot of interesting characteristics. For instance, each field of this family has an unramified cyclic extension of degree 9 under certain conditions. And finally, we consider a two-parameter family of cubic fields.

In chapter 1, firstly, we shall consider a $\mathbb{Q}$-linear map τ from reduced lattice $\mathcal{R}$ to $\mathbb{R}^2$ and investigate two-dimensional lattice $L = \mathcal{R}^\tau$. In that case we define terminology for F-point as a special element of $\mathcal{R}$. We also define terminology for normalized basis of $\mathcal{R}$. Secondly, we shall show the existence of a basis of $\mathcal{R}$ that contains an F-point and that satisfies some conditions. Next, we shall prove a theorem about the relationship between this basis and the normalized basis of $\mathcal{R}$. Then, we shall divide all the occurring cases for the basis of $\mathcal{R}$ into six cases. Then, we refine candidates of a minimal point adjacent to 1 in a reduced lattice $\mathcal{R}$. To narrow the candidates, we use three tools: the theorem of Williams, Cormack and Seah (1980) in which the candidates are described by the normalized basis , the relationship mentioned above and the isotropic vector of the quadratic form obtained from a basis of reduced lattice. Finally, we shall give several numerical examples. The result of this chapter is contained in my paper [21].

In chapter 2, we shall consider the cubic number fields $\mathbb{Q}(\theta)$ defined by $\theta^3 - 3\theta - b^3 = 0$ $(0, \pm 1 \neq b \in \mathbb{Z})$, which is introduced by Ishida as mentioned above.

In section 1, we shall find Voronoi-algorithm expansion of the order $\mathbb{Z}[\theta]$.

In section 2, we shall find an integral basis of $\mathbb{Q}(\theta)$ using Voronoi's theorem on integral basis.

In section 3, we shall prove a theorem which gives sufficient conditions so that ϵ (the fundamental unit of $\mathbb{Z}[\theta]$) is the fundamenatal unit of $\mathbb{Q}(\theta)$, using Artin's lemma on a unit of cubic number field. To prove the theorem, we also need a lemma about a diophantine system, which Lee and Spearman(2011) proved using theories including Algebraic Curves. Using the theorem, we shall show that there exist infinitely many cubic fields $\mathbb{Q}(\theta)$ such that ϵ is the fundamenatal unit of $\mathbb{Q}(\theta)$.

In section 4, we shall consider a family of biquadratic fields $F_b := \mathbb{Q}(\sqrt{-3}, \sqrt{b^6 - 4})$ $(0, \pm 3 \neq b \in \mathbb{Z}, b \equiv 0 \bmod 3)$. Given the property of ϵ, we can show that the length of the 3-class tower of F_b is greater than one using a theorem due to Yoshida(2003). The result of this chapter is contained in my paper [18,20,22].

In chapter 3, we shall consider a family of orders of complex cubic fields which depend on two parameters. Using the similar method in chaper 1, we shall obtain the Voronoi-algorithm expansions of orders and the fundamental units of orders. The result of this chapter is contained in my paper [19].

I would like to express my deep gratitude to Professor Shigeki Akiyama and Professor Shin-ichi Yasutomi for their helpful suggestions and constant encouragements.

Contents

1 A method for finding a minimal point of the lattice in cubic number fields

Let K be a cubic algebraic number field of negative discriminant. It is known that to find all the minimal points of a reduced lattice $\mathcal{R}$ of K, it is sufficient to know how to find a minimal point adjacent to 1 in any reduced lattice of K (refer to Definition 1.1 for a rigorous definition). Williams, Cormack and Seah [34] utilized the two-dimensional lattice obtained from a reduced lattice $\mathcal{R}$ to find a minimal point adjacent to 1 in $\mathcal{R}$ (the definition of such a two-dimensional lattice is forthcoming in Section 1). Moreover, Adam [1] utilized an isotropic vector of the quadratic form obtained from a basis of reduced lattice $\mathcal{R}$ (the definition of such a quadratic form is forthcoming in Section 3). Later, Lahlou and Farhane [24] generalise the Adam's method.

In this chapter, we shall prove six theorems which give candidates of a minimal point adjacent to 1 in a reduced lattice $\mathcal{R}$. In each case of the theorems, the maximum number of candidates $\varphi \in \mathcal{R}$ such that we must check whether $F(\varphi) < 1$ or not is at most four. Also, such six theorems contain all the occuring cases.

Definition 1.1. (1) Let $1, \beta, \gamma \in K$ be independent over $\mathbb{Q}$. We say that $\mathcal{R} = \langle 1, \beta, \gamma \rangle = \mathbb{Z} + \mathbb{Z}.\beta + \mathbb{Z}.\gamma$ is a *lattice* of K with basis $\{1, \beta, \gamma\}$.

(2) For $\alpha \in \mathcal{R}$ we define $F(\alpha) = \dfrac{N_K(\alpha)}{\alpha} = \alpha'\alpha''$, where N_K denotes the norm of K over $\mathbb{Q}$, and α' and α'' the conjugates of α.

(3) Let $\mathcal{R}$ be a lattice of K, and let $\varphi(> 0) \in \mathcal{R}$. We say that φ is a *minimal point* of $\mathcal{R}$ if for all α in $\mathcal{R}$ such that $0 < \alpha < \varphi$ we have $F(\alpha) > F(\varphi)$.

(4) Let $\mathcal{R}$ be a lattice of K and $\varphi,\ \psi \in \mathcal{R}$ be a minimal point. We say that ψ is a minimal point adjacent to φ in $\mathcal{R}$ if $\psi = \min\{\alpha \in \mathcal{R};\ \varphi < \alpha, F(\varphi) > F(\alpha)\}$.

(5) If $\mathcal{R}$ is a lattice of K in which 1 is a minimal point, we call $\mathcal{R}$ a *reduced lattice.*

1.1. Basis of reduced lattice (I)

Definition 1.2. Let $\alpha \in K$. We define $Y_\alpha := Re\ \alpha', Z_\alpha := Im\ \alpha', X_\alpha := \alpha - Y_\alpha$. Let $\lambda \in K, \mu \in K\backslash\mathbb{Q}$. We define $\omega_1(\lambda, \mu) := -(Z_\lambda/Z_\mu), \omega_2(\lambda, \mu) := -Y_\lambda - \omega_1(\lambda, \mu)Y_\mu$.

Remark. In [34] $Y_\alpha = Im\ \alpha',\ Z_\alpha = Re\ \alpha'$.

Proposition 1.1. Let $\alpha \in K, c \in \mathbb{Z}$. Then

(1) $F(\alpha) = Y_\alpha^2 + Z_\alpha^2$.

(2) $\alpha \notin \mathbb{Q} \;\Rightarrow\; Y_\alpha, X_\alpha \in K - \mathbb{Q}, Z_\alpha \notin \mathbb{Q}$.

(3) $K \ni 1, \lambda, \mu$ are independent over $\mathbb{Q} \;\Rightarrow\; \omega_1(\lambda, \mu) \notin \mathbb{Q}$.

(4) $K \ni 1, \lambda, \mu$ are independent over $\mathbb{Q}$

$\Rightarrow\; 1, X_\lambda, X_\mu$ are independent over $\mathbb{Q}$.

(5) $K \ni 1, \lambda, \mu$ are independent over $\mathbb{Q} \;\Rightarrow\; \det \begin{pmatrix} X_\lambda & X_\mu \\ Z_\lambda & Z_\mu \end{pmatrix} \neq 0$.

(6) Let $\alpha \notin \mathbb{Q}$. Then

(i) $-1 < Y_{\alpha+c} < 1 \;\Leftrightarrow\; c = [-Y_\alpha]$ or $[-Y_\alpha] + 1$,

(ii) $Y_{[-Y_\alpha]+\alpha} < 0, Y_{[-Y_\alpha]+1+\alpha} > 0$,

(iii) $|Y_{[-Y_\alpha]+\alpha}| < 1/2$ or $|Y_{[-Y_\alpha]+1+\alpha}| < 1/2$.

Proof. (3) Let $K = \mathbb{Q}(\theta)$ and $\lambda = a_0 + a_1\theta + a_2\theta^2 (a_i \in \mathbb{Q})$,
$\mu = b_0 + b_1\theta + b_2\theta^2 (b_i \in \mathbb{Q})$. Then we have
$Z_\lambda = \dfrac{1}{2i}(\lambda' - \lambda'') = \dfrac{1}{2i}\{a_1(\theta' - \theta'') + a_2(\theta'^2 - \theta''^2)\}$
$= \dfrac{1}{2i}(\theta' - \theta'')\{a_1 + a_2(\theta' + \theta'')\} = Z_\theta\{a_1 + (T_{K/\mathbb{Q}}\theta)a_2 - a_2\theta\}$ $(i^2 = -1)$.
Similarly we have $Z_\mu = Z_\theta\{b_1 + (T_{K/\mathbb{Q}}\theta)b_2 - b_2\theta\}$. Suppose that
$\omega_1(\lambda, \mu) = -\dfrac{Z_\lambda}{Z_\mu} = -\dfrac{a_1 + pa_2 - a_2\theta}{b_1 + pb_2 - b_2\theta} = r \in \mathbb{Q}$ $(p = T_{K/\mathbb{Q}}\theta)$. Then we have
$r(b_1 + pb_2 - b_2\theta) = -(a_1 + pa_2 - a_2\theta),\; rb_1 + rpb_2 + a_1 + pa_2 - (rb_2 + a_2)\theta = 0$.

Hence $rb_2 + a_2 = 0, rb_1 + a_1 = 0$, so $a_0 + rb_0 - \lambda - r\mu = 0$.

Since $1, \lambda, \mu$ are independent over $\mathbb{Q}$, we have reached a contradiction.

Therefore we have $\omega_1(\lambda, \mu) \notin \mathbb{Q}$.

(5) Since $1, \lambda, \mu$ are independent over $\mathbb{Q}$, by algebraic number theory
$\det \begin{pmatrix} 1 & \lambda & \mu \\ 1 & \lambda' & \mu' \\ 1 & \lambda'' & \mu'' \end{pmatrix} \neq 0$. Moreover, $\det \begin{pmatrix} 1 & \lambda & \mu \\ 1 & \lambda' & \mu' \\ 1 & \lambda'' & \mu'' \end{pmatrix} = 2i(X_\lambda Z_\mu - X_\mu Z_\lambda)$.
Therefore we have $X_\lambda Z_\mu - X_\mu Z_\lambda \neq 0$.

Otheres are easily deduced from definitions. $\square$

Definition 1.3. Let $\mathcal{R}$ be a reduced lattice of K. For $\mathcal{R} \ni \alpha$ we define
$\alpha_{(1)} := [-Y_\alpha] + \alpha,\; \alpha_{(2)} := [-Y_\alpha] + 1 + \alpha,\; \alpha_{(3)} := \begin{cases} \alpha_{(1)} \; if \; |Y_{\alpha_{(1)}}| < 1/2 \\ \alpha_{(2)} \; if \; |Y_{\alpha_{(2)}}| < 1/2 \end{cases}$,
$\alpha_{(0)} := \alpha - [\alpha]$, where [...] is the greatest integer function.

Note that $|Z_\alpha| < \sqrt{3}/2 \;\Rightarrow\; F(\alpha_{(3)}) < 1$.

Let $\mathcal{R} = \langle 1, \beta, \gamma \rangle$ be a reduced lattice of K. Let $\tau : K \to \mathbb{R}^2$ be the $\mathbb{Q}$-linear map defined by $\alpha^\tau = (X_\alpha, Z_\alpha)$. Note that for $\alpha_1, \alpha_2 \in \mathcal{R}$, $\alpha_1^\tau = \alpha_2^\tau \;\Leftrightarrow$ there exists some $c \in \mathbb{Z}$ such that $\alpha_2 = c + \alpha_1$. Let $L := \mathcal{R}^\tau = \langle \beta^\tau, \gamma^\tau \rangle$. By Proposition 1.1,(5) L is a two-dimensional lattice. Moreover, by Proposition 1.1,(3)(4) L has the following property (Δ):

$$(\Delta) \quad L \cap (\{0\} \times \mathbb{R}) = L \cap (\mathbb{R} \times \{0\}) = \{(0,0)\}.$$

Now we prepare two lemmas about the two-dimensional lattice which has property (Δ) from Delone's supplement I in [9].

Definition 1.4. Let $L(\subset \mathbb{R}^2)$ be a two-dimensional lattice which has property (Δ).

(1) For $\mathbb{R}^2 \ni S = (S_u, S_v) \neq (0,0)$ we define $C(S) := \{(u,v) \in \mathbb{R}^2;\ |u| < |S_u|, |v| < |S_v|\}$. Then we say that $S \in L$ is a minimal point of L if $L \cap C(S) = \{(0,0)\}$. The system of all the minimal points of L we denote by $M(L)$. We put $M(L)_{>0} := \{P \in M(L);\ P_u > 0\}$.

(2) Let $S(S_u > 0), Q(Q_u > 0) \in L$ be a minimal point of L. We say that Q is a minimal point adjacent to S in L if $Q_u = \min\{P_u;\ P \in L, S_u < P_u, |S_v| > |P_v|\}$.

Lemma 1.1. Let $L(\subset \mathbb{R}^2)$ be a two-dimensional lattice which has property (Δ). Let $L \ni S, Q$ $(S_u > 0, Q_u > 0)$. Then Q is a minimal point adjacent to S in L if and only if $L = \langle S, Q \rangle, S_u < Q_u, |S_v| > |Q_v|, S_v Q_v < 0$.

Proof. From Theorem XI,XII,XIII in [9,p.467-469]. (cf. Theorem 4.1 in [37]). □

Lemma 1.2. Let $L(\subset \mathbb{R}^2)$ be a two-dimensional lattice which has property (Δ) and let $E, G, H \in L$. We assume that G is a minimal point adjacent to E and that H is a minimal point adjacent to G. Then we have $H = E + [-E_v/G_v]G$.

Proof. From supplement I,Section 3,34 in [9,p.470]. □

Proposition 1.2. Let $\mathcal{R}$ be a reduced lattice of K, and let $L := \mathcal{R}^\tau$. Then there exists a basis $\{1, \lambda, \mu\}$ of $\mathcal{R}$ such that λ^τ is a minimal point adjacent to μ^τ in L, $0 < X_\lambda$, $F(\lambda_{(3)}) < 1$, $F(\mu_{(3)}) > 1$.

Proof. Let $\mathcal{R} = \langle 1, \beta, \gamma \rangle$. For $\varepsilon > 0$, we shall consider a rectangular neighbourhood of $(0,0)$, i.e. $W(\varepsilon, \sqrt{3}/2) = \{(u,v) \in \mathbb{R}^2;\ |u| < \varepsilon, |v| < \sqrt{3}/2\}$. By Minkowski's convex body theorem, there exists $\varepsilon > 0$ such that $L \cap W(\varepsilon, \sqrt{3}/2) \neq \{(0,0)\}$. We take such a $\varepsilon > 0$ and fix it. We put $W = W(\varepsilon, \sqrt{3}/2)$. Then there exists $Q = (Q_u, Q_v) \in L \cap W$ such that $Q_u = \min\{P_u;\ P \in L \cap W, 0 < P_u\}$. Note that such a $Q \in L$ is uniquely-determined. We have $L \cap C(Q) = \{(0,0)\}$. Hence Q is a minimal point of L. There exists $S \in L$ such that Q is a minimal point adjacent to S in L. By Lemma 1.1, $\{S, Q\}$ is a basis of L. Since both $\{S, Q\}$ and $\{\beta^\tau, \gamma^\tau\}$ are a basis of L, there exists $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in GL_2(\mathbb{Z})$ such that $(Q\ S) = (\beta^\tau\ \gamma^\tau) \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. We have $Q = p\beta^\tau + r\gamma^\tau = (p\beta + r\gamma)^\tau$. Similarly, we have $S = (q\beta + s\gamma)^\tau$. We define $\lambda, \mu \in K$ by $(\lambda\ \mu) = (\beta\ \gamma) \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Then we have $\mathcal{R} = \langle 1, \lambda, \mu \rangle, Q = \lambda^\tau, S = \mu^\tau$. Since $Q = (Q_u, Q_v) = \lambda^\tau = (X_\lambda, Z_\lambda)$, from $|Z_\lambda| < \sqrt{3}/2$, we have $F(\lambda_{(3)}) < 1$. From this, if we put $\mathcal{R}_F := \{\alpha \in \mathcal{R};\ \alpha^\tau \in M(L)_{>0}, F(\alpha_{(3)}) < 1\}$, then $\mathcal{R}_F \neq \emptyset$. Let $W(\varepsilon, 1) := \{(u, v) \in \mathbb{R}^2;\ |u| < \varepsilon, |v| < 1\}$. As $W(\varepsilon, \sqrt{3}/2) \subset W(\varepsilon, 1)$, we have $1 < |\mathcal{R}_F^\tau \cap W(\varepsilon, 1)| < \infty$. Hence there exists $\lambda^\tau \in \mathcal{R}_F^\tau \cap W(\varepsilon, 1)$ such that $X_\lambda = \min\{X_\alpha;\ \alpha^\tau \in \mathcal{R}_F^\tau \cap W(\varepsilon, 1)\}$. Since $F(\alpha_{(3)}) < 1 \ \Rightarrow\ |Z_\alpha| < 1$, it is easily seen that $X_\lambda = \min\{X_\alpha;\ \alpha^\tau \in \mathcal{R}_F^\tau \cap W(\varepsilon, 1)\} = \min\{X_\alpha;\ \alpha^\tau \in \mathcal{R}_F^\tau\} = \min\{X_\alpha;\ \alpha \in \mathcal{R}_F\}$. For this λ, there exists $\mu \in \mathcal{R}$ such that λ^τ is a minimal point adjacent to μ^τ in L. Moreover, for such a μ we have $F(\mu_{(3)}) > 1$. $\square$

Remark. Such a basis in Proposition 1.2 is easily found by modified version of Algorithm (A) in [34,p.581].

Definition 1.5. Let $\mathcal{R}$ be a reduced lattice of K, and let $L := \mathcal{R}^\tau$. We say that $\lambda \in \mathcal{R}$ is a *F-point* of $M(L)_{>0}$ if $\lambda \in \mathcal{R}_F, X_\lambda = \min\{X_\alpha;\ \alpha \in \mathcal{R}_F\}$.

Lemma 1.3. Let $\mathcal{R}$ be a reduced lattice of K. If $0 < X_\lambda,\ F(\lambda_{(3)}) < 1$, then we have $0 < \lambda_{(1)}$.

Proof. We assume that $0 < X_\lambda,\ F(\lambda_{(3)}) < 1$. From $0 < X_\lambda = X_{\lambda_{(2)}} = \lambda_{(2)} - Y_{\lambda_{(2)}}$, we have $\lambda_{(2)} > Y_{\lambda_{(2)}} > 0$. Hence we have $\lambda_{(2)} > 0$. Suppose that $\lambda_{(1)} < 0$. We have $0 < \lambda_{(2)} = \lambda_{(1)} + 1 < 1$, so $-1 < \lambda_{(1)} < 0$. Since $\mathcal{R}$ is a reduced lattice of K, we have $F(\lambda_{(2)}) > 1$. Hence we have $\lambda_{(3)} = \lambda_{(1)}$, so $F(\lambda_{(1)}) < 1$. From this, $F(-\lambda_{(1)}) < 1$. Since $\mathcal{R}$ is a reduced lattice of K, we have reached a contradiction. Therefore, we have $\lambda_{(1)} > 0$. $\square$

Theorem 1.1. Let $\mathcal{R}$ be a reduced lattice of K. Then there exists a basis $\{1, \lambda, \mu\}$ of $\mathcal{R}$ such that

(a) $0 < \lambda < 1, -1/2 < \mu, F(\mu) > 1, 2|Y_\mu| < 1, 0 < X_\mu < X_\lambda, 0 < \omega_1(\lambda, \mu) < 1$,

(b) $\omega_2(\lambda, \mu) > 0$,

(c) $F([\omega_2] + \lambda) < 1$ or $F([\omega_2] + 1 + \lambda) < 1$.

Proof. By Proposition 1.2, we can take a basis $\{1, \lambda, \mu\}$ of $\mathcal{R}$ such that λ^τ is a minimal point adjacent to μ^τ in L, $0 < X_\lambda$, $F(\lambda_{(3)}) < 1$, $F(\mu_{(3)}) > 1$, λ is a F-point of $M(L)_{>0}$. Clearly, $\mathcal{R} = \langle 1, \lambda_{(0)}, \mu_{(3)} \rangle$.

(a) Clearly we have $0 < \lambda_{(0)} < 1, F(\mu_{(3)}) > 1, 2|Y_{\mu_{(3)}}| < 1, 0 < X_{\mu_{(3)}} = X_\mu < X_{\lambda_{(0)}} = X_\lambda$. From $0 < X_\mu = X_{\mu_{(3)}} = \mu_{(3)} - Y_{\mu_{(3)}}$, we have $-1/2 < \mu_{(3)}$. From Remark 1.1 bellow, we have $0 < \omega_1(\lambda, \mu) < 1$. Since $\omega_1(\lambda_{(0)}, \mu_{(3)}) = -(Z_{\lambda_{(0)}}/Z_{\mu_{(3)}}) = -(Z_\lambda/Z_\mu) = \omega_1(\lambda, \mu)$, we have $0 < \omega_1(\lambda_{(0)}, \mu_{(3)}) < 1$.

(b) Proof of "$\omega_2(\lambda_{(0)}, \mu_{(3)}) > 0$".

(i) The case $\lambda_{(1)} = [-Y_\lambda] + \lambda > 1$. $\lambda_{(1)} = [-Y_\lambda] + \lambda = [-Y_{\lambda_{(0)}}] + \lambda_{(0)} > 1$. Hence $-Y_{\lambda_{(0)}} > 1$. From this and from $0 < \omega_1 < 1, |Y_{\mu_{(3)}}| < 1/2$ we have $\omega_2(\lambda_{(0)}, \mu_{(3)}) = -Y_{\lambda_{(0)}} - \omega_1(\lambda_{(0)}, \mu_{(3)})Y_{\mu_{(3)}} > 0$.

(ii) The case $\lambda_{(1)} = [-Y_\lambda] + \lambda < 1$. By Lemma 1.3, we have $\lambda_{(1)} > 0$. From $0 < \lambda_{(1)} < 1$, we have $F(\lambda_{(1)}) > 1$ because $\mathcal{R}$ is a reduced lattice of K. Therefore we have $F(\lambda_{(2)}) < 1$. Since $F(\lambda_{(1)}) > 1$, we have $Y_{\lambda_{(1)}} < -1/2$. Note that $\lambda_{(1)} = \lambda_{(0)}$. Hence from $Y_{\lambda_{(0)}} = Y_{\lambda_{(1)}} < -1/2$ and from $0 < \omega_1 < 1, |Y_{\mu_{(3)}}| < 1/2$ we have $\omega_2(\lambda_{(0)}, \mu_{(3)}) = -Y_{\lambda_{(0)}} - \omega_1(\lambda_{(0)}, \mu_{(3)})Y_{\mu_{(3)}} > 0$.

(c) Proof of "$F([\omega_2] + \lambda_{(0)}) < 1$ or $F([\omega_2] + 1 + \lambda_{(0)}) < 1$".

(i) The case $Y_{\mu_{(3)}} < 0$. Since $\omega_2 - (-Y_{\lambda_{(0)}}) = -\omega_1 Y_{\mu_{(3)}} > 0$, we have $-Y_{\lambda_{(0)}} < \omega_2$. From this and $|-\omega_1 Y_{\mu_{(3)}}| < 1/2$, we have $[\omega_2] = [-Y_{\lambda_{(0)}}]$ or $[-Y_{\lambda_{(0)}}] + 1$. Note that $[\omega_2] = [-Y_{\lambda_{(0)}}] + 1 \Rightarrow 0 < [-Y_{\lambda_{(0)}}] + 1 - (-Y_{\lambda_{(0)}}) < 1/2 \Rightarrow 0 < Y_{\lambda_{(2)}} = [-Y_{\lambda_{(0)}}] + 1 + Y_{\lambda_{(0)}} < 1/2$. Hence if $[\omega_2] = [-Y_{\lambda_{(0)}}] + 1$, then we have $\lambda_{(3)} = \lambda_{(2)}$. Therefore, we have "$[\omega_2] + \lambda_{(0)} = [-Y_{\lambda_{(0)}}] + \lambda_{(0)} = \lambda_{(1)}, [\omega_2] + 1 + \lambda_{(0)} = \lambda_{(2)}$" or "$[\omega_2] + \lambda_{(0)} = [-Y_{\lambda_{(0)}}] + 1 + \lambda_{(0)} = \lambda_{(2)}, F(\lambda_{(2)}) < 1$".

(ii) The case $Y_{\mu_{(3)}} > 0$. Since $\omega_2 - (-Y_{\lambda_{(0)}}) = -\omega_1 Y_{\mu_{(3)}} < 0$, we have $-Y_{\lambda_{(0)}} > \omega_2$. From this and $|-\omega_1 Y_{\mu_{(3)}}| < 1/2$, we have $[\omega_2] = [-Y_{\lambda_{(0)}}]$ or $[-Y_{\lambda_{(0)}}] - 1$. Note that $[\omega_2] = [-Y_{\lambda_{(0)}}] - 1 \Rightarrow 0 < -Y_{\lambda_{(0)}} - [-Y_{\lambda_{(0)}}] < 1/2 \Rightarrow -1/2 < Y_{\lambda_{(1)}} = [-Y_{\lambda_{(0)}}] + Y_{\lambda_{(0)}} < 0$. Hence if $[\omega_2] = [-Y_{\lambda_{(0)}}] - 1$, then we have $\lambda_{(3)} = \lambda_{(1)}$. Therefore we have "$[\omega_2] + \lambda_{(0)} = [-Y_{\lambda_{(0)}}] + \lambda_{(0)} = \lambda_{(1)}, [\omega_2] + 1 + \lambda_{(0)} = \lambda_{(2)}$" or "$[\omega_2] + 1 + \lambda_{(0)} = \lambda_{(1)}, F(\lambda_{(1)}) < 1$". $\square$

Remark 1.1. Let $\mathcal{R} = \langle 1, \beta, \gamma \rangle,\ 0 < X_\gamma < X_\beta$. Then γ^τ is a minimal point adjacent to β^τ in $L \ \Leftrightarrow\ 0 < \omega_1(\beta, \gamma) < 1$.

1.2. Basis of reduced lattice (II)

Definition 1.6. Let $\mathcal{R}$ be a lattice of K, and let $\{1, N, M\}$ be a basis of $\mathcal{R}$. We say that $\{1, N, M\}$ is *normalized* provided that
$0 < X_M < X_N,\ |Z_M| > 1/2,\ |Z_N| < 1/2,\ Z_M \cdot Z_N < 0$.

We quote Williams [37],Theorem 8.1 as Theorem 1.2 for our convenience.

Theorem 1.2(Williams [37],Theorem 8.1)**.** Let $\mathcal{R}$ be a reduced lattice with the normalized basis $\{1, N, M\}$. If $\theta_g = x + yN + zM\ (x, y, z \in \mathbb{Z})$ is the minimal point adjacent to 1, then $(y, z) \in \{(1, 0), (0, 1), (1, 1), (1, -1), (2, 1)\}$.

In this paper, θ_g denotes the minimal point adjacent to 1 of any reduced lattice $\mathcal{R}$. We shall consider the relationship between F-point and the normalized basis.

Theorem 1.3. Let $\mathcal{R}$ be a reduced lattice with the normalized basis $\{1, N, M\}$. If $\mathcal{R} = \langle 1, \lambda, \mu \rangle,\ \lambda^\tau$ is adjacent to $\mu^\tau,\ \lambda$ is a F-point of $M(L)_{>0}$ $(L = \mathcal{R}^\tau)$, then λ^τ must be one of $N^\tau,\ (N - M)^\tau,\ M^\tau$. Moreover,

(1) The case $\lambda^\tau = (N - M)^\tau$: $N^\tau = (d + 1)\lambda^\tau + \mu^\tau,\ M^\tau = d\lambda^\tau + \mu^\tau$,

(2) The case $\lambda^\tau = M^\tau$: $N^\tau = d\lambda^\tau + \mu^\tau$,

where $d = d(\lambda, \mu) = [1/\omega_1(\lambda, \mu)]$.

Proof. Recall that $\mathcal{R}_F = \{\alpha \in \mathcal{R};\ \alpha^\tau \in M(L)_{>0}, F(\alpha_{(3)}) < 1\},\ X_\lambda = \min\{X_\alpha;\ \alpha \in \mathcal{R}_F\}$. By Lemma 1.1 and Definition 1.6, we have $N \in \mathcal{R}_F$. Hence, we have $X_\lambda \leq X_N$. Since $L = \langle N^\tau, M^\tau \rangle = \langle \lambda^\tau, \mu^\tau \rangle$, there exists $a, b \in \mathbb{Z}$ such that $\lambda^\tau = aN^\tau + bM^\tau$.

(i) The case $a < 0$. Since $X_\lambda > 0$, we have $b > 0$. Moreover, since $|Z_\lambda| = |aZ_N + bZ_M| = |a| \cdot |Z_N| + b \cdot |Z_M| < 1$ and $1/2 < |Z_M|$, we have $b \leq 1$. Therefore $b = 1$. Hence $X_\lambda = aX_N + bX_M = aX_N + X_M = X_M - |a| \cdot X_N < 0$. Therefore the case (i) is impossible.

(ii) The case $a = 0$. Since $X_\lambda = aX_N + bX_M = bX_M$, we have $b > 0$. Since $|Z_\lambda| = b|Z_M|$, we have $b = 1$. [i.e. $(a, b) = (0, 1)$]

(iii) The case $a \geq 1, b \leq 0$. Since $|Z_\lambda| = a|Z_N| + |b| \cdot |Z_M| < 1$, we have $|b| \leq 1$.

1) The case $b = -1$. Since $X_\lambda = aX_N - X_M = (a - 1)X_N + (X_N - X_M)$, if $a \geq 2$, then we

have $X_\lambda > X_N$, which is impossible. Therefore, we have $a = 1$. [i.e. $(a,b) = (1,-1)$]

2) The case $b = 0$. Since $X_\lambda = aX_N = (a-1)X_N + X_N$, if $a \geq 2$, then we have $X_\lambda > X_N$, which is impossible. Therefore, we have $a = 1$. [i.e. $(a,b) = (1,0)$]

(iv) The case $a \geq 1, b \geq 1$. We have $X_\lambda = aX_N + bX_M > X_N$, which is impossible. Therefore, the case (iv) is impossible.

By (i) to (iv), we conclude that $\lambda^\tau = aN^\tau + bM^\tau = M^\tau$ or $(N-M)^\tau$ or N^τ.

(a) The case $|Z_\lambda| < 1/2$. Since $|Z_\mu| > \sqrt{3}/2 > 1/2$, we have $\lambda^\tau = N^\tau, \mu^\tau = M^\tau$.

(b) The case $|Z_\lambda| > 1/2$. Since $\lambda^\tau \neq N^\tau$, we have $0 < X_\lambda < X_N$. Hence we have $\lambda^\tau = (N-M)^\tau$ or M^τ.

(b-1) The case $\lambda^\tau = (N-M)^\tau$. We have

(1.1) $X_\lambda = X_{N-M} < X_M < X_N$.

Because if $X_M < X_\lambda = X_{N-M} < X_N$, then from $X_M < X_{N-M}, |Z_M| < |Z_{N-M}|$, we have $L \cap C((N-M)^\tau) = L \cap \{(u,v) \in \mathbb{R}^2;\ |u| < X_{N-M}, |v| < |Z_{N-M}|\} \ni M^\tau \neq (0,0)$. Since $\lambda^\tau = (N-M)^\tau \in L$ is a minimal point, we have reached a contradiction. Therefore we have $X_\lambda = X_{N-M} < X_M < X_N$. By Remark 1.1 we have $0 < \omega_1(N,M) < 1$. Since $\omega_1(M, N-M) = \dfrac{1}{\omega_1(N,M)+1}$, we have $0 < \omega_1(M, N-M) < 1$. From this, if $X_{N-M} < X_M$, then M^τ is adjacent to $(N-M)^\tau$. Note that $\mathcal{R} = \langle 1, M, N-M \rangle$. Hence we have

(1.2) $X_{N-M} < X_M \ \Leftrightarrow\ M^\tau$ is adjacent to $(N-M)^\tau$.

Since M^τ is a minimal point adjacent to λ^τ, and λ^τ is a minimal point adjacent to μ^τ, by Lemma 1.2 we have $M^\tau = \mu^\tau + [-(Z_\mu/Z_\lambda)]\lambda^\tau$. We put $d = [-(Z_\mu/Z_\lambda)] = [1/\omega_1(\lambda,\mu)]$. We have $M^\tau = \mu^\tau + d\lambda^\tau$. From $\lambda^\tau = N^\tau - M^\tau$, we have $N^\tau = \mu^\tau + (d+1)\lambda^\tau$. Therefore we obtain formulas : $M^\tau = d\lambda^\tau + \mu^\tau,\ N^\tau = (d+1)\lambda^\tau + \mu^\tau$.

(b-2) The case $\lambda^\tau = M^\tau$.

Since N^τ is a minimal point adjacent to λ^τ, and λ^τ is a minimal point adjacent to μ^τ, by Lemma 1.2 we have $N^\tau = \mu^\tau + [-(Z_\mu/Z_\lambda)]\lambda^\tau = \mu^\tau + d\lambda^\tau$. Therefore we obtain formulas : $M^\tau = \lambda^\tau,\ N^\tau = d\lambda^\tau + \mu^\tau$. □

Corollary 1.1. Let $\mathcal{R}$ be a reduced lattice with basis $\{1, \lambda, \mu\}$ such that λ^τ is adjacent to μ^τ, λ is a F-point of $M(L)_{>0}$ $(L = \mathcal{R}^\tau)$. If $\theta_g = x + y\lambda + z\mu$ $(x, y, z \in \mathbb{Z})$, then

the case $\lambda^\tau = N^\tau$: $(y, z) \in \{(1, 0), (1, 1), (1, -1), (2, 1)\}$,

the case $\lambda^\tau = (N - M)^\tau$: $(y, z) \in \{(1, 0), (d, 1), (d + 1, 1), (2d + 1, 2), (3d + 2, 3)\}$,

the case $\lambda^\tau = M^\tau$: $(y, z) \in \{(1, 0), (d, 1), (d + 1, 1), (2d + 1, 2), (d - 1, 1)\}$,

where $d = [1/\omega_1(\lambda, \mu)] \geq 1$.

Proof. From Theorem 1.2. $\square$

Remark 1.2. Since $1/(d + 1) < \omega_1 < 1/d$, we have

$[d\omega_1] = [(d - 1)\omega_1] = 0, [(d + 1)\omega_1] = 1, 1 \leq [(2d + 1)\omega_1] \leq 2, 2 \leq [(3d + 2)\omega_1] \leq 4.$

Theorem 1.4. Let $\mathcal{R}$ be a reduced lattice with basis $\{1, \lambda, \mu\}$ such that $F(\mu) > 1, 2|Y_\mu| < 1, 0 < X_\mu < X_\lambda, 0 < \omega_1(\lambda, \mu) < 1, F(\lambda_{(3)}) < 1$.

Then λ^τ must be one of N^τ, $(N - M)^\tau$, M^τ. Moreover, if $\lambda^\tau = (N - M)^\tau$ or M^τ, then λ is a F-point of $M(L)_{>0}$ $(L = \mathcal{R}^\tau)$.

Proof. At first, we note that λ^τ is adjacent to μ^τ. Also $\lambda \in \mathcal{R}_F$. From $2|Y_\mu| < 1$, $\mu = \mu_{(3)}$.

(a) The case $|Z_\lambda| < 1/2$. Since $F(\mu_{(3)}) = F(\mu) > 1$, we have $|Z_\mu| > \sqrt{3}/2 > 1/2$. Hence we have $\lambda^\tau = N^\tau, \mu^\tau = M^\tau$.

(b) The case $|Z_\lambda| > 1/2$. Let λ^* be a F-point of $M(L)_{>0}$. So we have $X_{\lambda^*} \leq X_\lambda$. We shall show that $\lambda^{*\tau} = \lambda^\tau$. Suppose that $\lambda^{*\tau} \neq \lambda^\tau$.

(i) The case $\lambda^\tau \neq M^\tau$. We have

(i-1) $X_{\lambda^*} < X_\mu < X_\lambda < X_M < X_N$.

Since $|Z_{\lambda^*}| > 1/2$, by Theorem 1.3, we have $\lambda^{*\tau} = M^\tau$ or $(N - M)^\tau$. Hence $\lambda^{*\tau} = (N - M)^\tau$. By (1.1) in the proof of Theorem 1.3, we have $X_{\lambda^*} = X_{N-M} < X_M$. From (i-1), we have $X_{\lambda^*} = X_{N-M} < X_\mu < X_\lambda < X_M < X_N$. Since M^τ is adjacent to $(N - M)^\tau$, we have reached a contradiction.

(ii) The case $\lambda^\tau = M^\tau$. Since $\lambda^{*\tau} \neq \lambda^\tau$, by Theorem 1.3, we have $\lambda^{*\tau} = (N - M)^\tau$. By (1.1) in the proof of Theorem 1.3, we have $X_{\lambda^*} = X_{N-M} < X_M$. Hence we have $X_{\lambda^*} = X_{N-M} < X_\mu < X_\lambda = X_M < X_N$. Since M^τ is adjacent to $(N - M)^\tau$, we have reached a contradiction.

By (i)(ii), an assumption $\lambda^{*\tau} \neq \lambda^\tau$ lead to a contradiction. Therefore we have $\lambda^{*\tau} = \lambda^\tau$.

Finally, if $\lambda^\tau = (N - M)^\tau$ or M^τ, then we must have only the case (b), so λ is a F-point of $M(L)_{>0}$. $\square$

Remark. $F(\lambda_{(3)}) < 1 \Leftrightarrow \exists c \in \mathbb{Z};\ F(c+\lambda) < 1$.

Corollary 1.2. Let $\mathcal{R}$ be a reduced lattice with basis $\{1, \lambda, \mu\}$ such that $F(\mu) > 1, 2|Y_\mu| < 1$, $0 < X_\mu < X_\lambda, 0 < \omega_1(\lambda, \mu) < 1, F(\lambda_{(3)}) < 1$. If $\theta_g = x + y\lambda + z\mu$ $(x, y, z \in \mathbb{Z})$, then (y, z) $\in \{(1,0),(1,1),(1,-1),(2,1),(d,1),(d+1,1),(2d+1,2),(d-1,1),(3d+2,3)\}$, where $d = [1/\omega_1(\lambda,\mu)] \geq 1$.

1.3. Preliminaries (I)

Definition 1.7. Let $\mathcal{R}$ be a lattice of K. For a basis $\{1, \lambda, \mu\}$ of $\mathcal{R}$, we define a mapping $F_{\lambda,\mu} : \mathbb{R}^3 \to \mathbb{R}$ by $F_{\lambda,\mu}(x, y, z) = x^2 + (\lambda' + \lambda'')xy + (\mu' + \mu'')xz + (\lambda'\mu'' + \lambda''\mu')yz + \lambda'\lambda''y^2 + \mu'\mu''z^2$. For any $(x, y, z) \in \mathbb{Z}^3$, we have $F_{\lambda,\mu}(x, y, z) = F(x + y\lambda + z\mu)$.

Remark. $F_{\lambda,\mu}$ is a positive quadratic form with real coefficients of rank 2. $(\omega_2, 1, \omega_1)$ is an isotropic vector of $F_{\lambda,\mu}$.

We quote Lahlou and Farhane [24],Lemma 2.2 as Lemma 1.4 for our convenience. (cf. [1],Lemma 2.2)

Lemma 1.4(Lahlou and Farhane [24],Lemma 2.2)**.** Let $\mathcal{R}$ be a lattice of K and let $\{1, \lambda, \mu\}$ be a basis of $\mathcal{R}$. Then we can write

(1) $F_{\lambda,\mu}(x, y, z) = a(z - \omega_1 y)^2 + 2b(z - \omega_1 y)(x - \omega_2 y) + (x - \omega_2 y)^2$

(2) $F_{\lambda,\mu}(x, y, z) = \dfrac{1}{2}(x - \omega_2 y)^2 + \dfrac{1}{2}(x - \omega_2 y + 2b(z - \omega_1 y))^2 + (a - 2b^2)(z - \omega_1 y)^2$

(3) $F_{\lambda,\mu}(x, y, z) = \dfrac{a}{2}(z - \omega_1 y)^2 + \dfrac{a}{2}(z - \omega_1 y + \dfrac{2b}{a}(x - \omega_2 y))^2 + (1 - \dfrac{2b^2}{a})(x - \omega_2 y)^2$

with $a = F(\mu),\ b = Y_\mu$.

Definition 1.8. Let $\mathcal{R}$ be a reduced lattice with basis $\{1, \lambda, \mu\}$
such that $\mu > -1/2, \omega_2(\lambda, \mu) > 0, 0 < \omega_1(\lambda, \mu) < 1$. Let $y \in \mathbb{Z}$. Then
we define

$$\begin{array}{ll}
\psi_{1,y} = [\omega_2 y] - 1 + y\lambda + [\omega_1 y]\mu & \psi_{7,y} = [\omega_2 y] + 1 + y\lambda + ([\omega_1 y] - 1)\mu \\
\psi_{2,y} = [\omega_2 y] - 1 + y\lambda + ([\omega_1 y] + 1)\mu & \psi_{8,y} = [\omega_2 y] + 1 + y\lambda + [\omega_1 y]\mu \\
\psi_{3,y} = [\omega_2 y] + y\lambda + ([\omega_1 y] - 1)\mu & \psi_{9,y} = [\omega_2 y] + 1 + y\lambda + ([\omega_1 y] + 1)\mu \\
\psi_{4,y} = [\omega_2 y] + y\lambda + [\omega_1 y]\mu & \psi_{10,y} = [\omega_2 y] + 1 + y\lambda + ([\omega_1 y] + 2)\mu \\
\psi_{5,y} = [\omega_2 y] + y\lambda + ([\omega_1 y] + 1)\mu & \psi_{11,y} = [\omega_2 y] + 2 + y\lambda + [\omega_1 y]\mu \\
\psi_{6,y} = [\omega_2 y] + y\lambda + ([\omega_1 y] + 2)\mu & \psi_{12,y} = [\omega_2 y] + 2 + y\lambda + ([\omega_1 y] + 1)\mu
\end{array}$$

$$\begin{array}{lll}
\phi_1 = \psi_{4,1} = [\omega_2] + \lambda & \phi_5 = \psi_{2,1} = [\omega_2] - 1 + \lambda + \mu & \phi_9 = 2\lambda + \mu \\
\phi_2 = \psi_{5,1} = [\omega_2] + \lambda + \mu & \phi_6 = \psi_{8,1} = [\omega_2] + 1 + \lambda & \phi_{10} = 3\lambda + 2\mu \\
\phi_3 = \psi_{3,1} = [\omega_2] + \lambda - \mu & \phi_7 = \psi_{7,1} = [\omega_2] + 1 + \lambda - \mu & \\
\phi_4 = \psi_{1,1} = [\omega_2] - 1 + \lambda & \phi_8 = \psi_{9,1} = [\omega_2] + 1 + \lambda + \mu &
\end{array}$$

Remark 1.3. (1) If $0 < \mu < 1$, then we have

$\psi_{1,y} < \psi_{2,y} < \psi_{4,y};\ \psi_{1,y} < \psi_{3,y} < \psi_{4,y};\ \psi_{4,y} < \psi_{5,y} < \psi_{6,y} < \psi_{9,y}$

$\psi_{4,y} < \psi_{5,y} < \psi_{8,y} < \psi_{9,y};\ \psi_{4,y} < \psi_{7,y} < \psi_{8,y} < \psi_{9,y}$

$\psi_{9,y} < \psi_{10,y} < \psi_{12,y};\ \psi_{9,y} < \psi_{11,y} < \psi_{12,y}$

(2) If $\mu > 1$, then we have

$\psi_{3,y} < \psi_{1,y} < \psi_{4,y};\ \psi_{3,y} < \psi_{7,y} < \psi_{4,y};\ \psi_{4,y} < \psi_{2,y} < \psi_{5,y} < \psi_{9,y}$

$\psi_{4,y} < \psi_{8,y} < \psi_{5,y} < \psi_{9,y};\ \psi_{4,y} < \psi_{8,y} < \psi_{11,y} < \psi_{9,y}$

$\psi_{9,y} < \psi_{6,y} < \psi_{10,y};\ \psi_{9,y} < \psi_{12,y} < \psi_{10,y}$

Lemma 1.5. Let $\mathcal{R}$ be a reduced lattice with basis $\{1, \lambda, \mu\}$ such that $\mu > -1/2, \omega_2(\lambda, \mu) > 0$ and $0 < \omega_1(\lambda, \mu) < 1$.
Let $a > \max(1, 2b^2, 2|b|)$, where $a = F(\mu), b = Y_\mu$. Then

(1) $\theta_g \in \{\psi_{i,y};\ y(\neq 0) \in \mathbb{Z}, 1 \leq i \leq 12\}$.

(2) $\lambda, \mu > 0 \ \Rightarrow\ \psi_{i,1} \leq \psi_{i,y}\ (y \geq 1)$.

(3) (i) $b < 0 \ \Rightarrow\ F(\psi_{2,y}) > 1, F(\psi_{6,y}) > 1, F(\psi_{7,y}) > 1, F(\psi_{11,y}) > 1$.

(ii) $b > 0 \ \Rightarrow\ F(\psi_{1,y}) > 1, F(\psi_{3,y}) > 1, F(\psi_{10,y}) > 1, F(\psi_{12,y}) > 1$.

(4) $F(\psi_{3,1}) > F(\psi_{4,1})$.

(5) $(0 <)b < 1/2 \ \Rightarrow\ F(\psi_{7,1}) > F(\psi_{4,1})$.

(6) $F(\psi_{5,1}) < F(\psi_{4,1}), 0 < b < 1 \ \Rightarrow\ F(\psi_{7,1}) > F(\psi_{4,1})$.

(7) $b > 1 \ \Rightarrow\ F(\psi_{7,1}) > 1$.

(8) $b > 0$ or $-1/2 < b < 0 \;\Rightarrow\; F(\psi_{1,1}) > F(\psi_{4,1})$.

(9) $F(\psi_{5,1}) > F(\psi_{8,1}), (0 <)b < 1 \;\Rightarrow\; F(\psi_{2,1}) > F(\psi_{4,1})$.

(10) $F(\psi_{4,1}) > F(\psi_{8,1}), b < 0 \;\Rightarrow\; c_2 = [\omega_2] - \omega_2 < -1/2$.

(11) $c_1 = [\omega_1] - \omega_1 < -1/2, b < 0 \;\Rightarrow\; F(\psi_{8,1}) > F(\psi_{9,1})$.

(12) $[2\alpha] = \begin{cases} 2\,[\alpha] & if \;\; 0 \le \alpha - [\alpha] < 1/2 \\ 2\,[\alpha] + 1 & if \;\; 1/2 \le \alpha - [\alpha] \end{cases}$.

Proof. We put $c_1 = [\omega_1] - \omega_1, c_2 = [\omega_2] - \omega_2$. Then $-1 < c_1, c_2 < 0$.

(1) was proved in Lahlou and Farhane [24],Theorem 2.1.

(2) obvious

(3) by Lemma 1.4,(1)

(4) By Lemma 1.4,(1), $F(\psi_{3,1}) - F(\psi_{4,1}) = -2ac_1 + a - 2bc_2 = -2ac_1 + a(1 - \frac{2b}{a}c_2) > 0$.

(5) By Lemma 1.4,(1), $F(\psi_{7,1}) - F(\psi_{4,1}) = -2ac_1 + a + 2bc_1 - 2bc_2 - 2b + 2c_2 + 1 = (1-2b)(1 + c_2) + a + c_2 - 2(a-b)c_1 > 0$.

(6) By Lemma 1.4,(1) since $F(\psi_{5,1}) < F(\psi_{4,1})$, $F(\psi_{4,1}) - F(\psi_{5,1}) = -2ac_1 - a - 2bc_2 > 0$. So $-2bc_2 > a(1 + 2c_1)$. From this and $a > 2b$, we have $-2bc_2 > 2b(1 + 2c_1), -c_2 > 1 + 2c_1$. Hence $-2c_1 > 1 + c_2$. By this, $F(\psi_{7,1}) - F(\psi_{4,1}) = -2ac_1 + a + 2bc_1 - 2bc_2 - 2b + 2c_2 + 1 = (1-2b)(1+c_2) + a + c_2 - 2c_1(a-b) > (1-2b)(1+c_2) + a + c_2 + (1+c_2)(a-b) = (1-2b)(1 + c_2) + a - 1 + 1 + c_2 + (1+c_2)(a-b) = (2-2b)(1+c_2) + a - 1 + (1+c_2)(a-b) > 0$.

(7) If $b > 1$, then we have $a > 2$ because $a > 2|b|$. From this and by Lemma 1.4,(3), we have $F(\psi_{7,1}) > 1$.

(8) By Lemma 1.4,(1), $F(\psi_{1,1}) - F(\psi_{4,1}) = -2bc_1 - 2c_2 + 1 > 0$.

(9) Since $F(\psi_{5,1}) > F(\psi_{8,1})$, we have $F(\psi_{5,1}) - F(\psi_{8,1}) = 2ac_1 + a + 2bc_2 - 2bc_1 - 2c_2 - 1 > 0$. From this, $F(\psi_{2,1}) - F(\psi_{4,1}) = 2ac_1 + a - 2bc_1 + 2bc_2 - 2b - 2c_2 + 1 = (2ac_1 + a + 2bc_2 - 2bc_1 - 2c_2 - 1) + 2 - 2b > 0$.

(10) Since $F(\psi_{4,1}) - F(\psi_{8,1}) > 0$, we have $bc_1 + c_2 < -1/2$. From this and $b < 0, c_1 < 0$, we have $c_2 < -1/2$.

(11) By Lemma 1.4,(1), $F(\psi_{9,1}) - F(\psi_{8,1}) = 2ac_1 + a + 2b(c_2 + 1) = a(2c_1 + 1) + 2b(c_2 + 1) < 0$.

(12) is easily deduced from the definitions. $\square$

Some of Lemma 1.5 were proved in Lahlou and Farhane [24],Theorem 2.1.

Remark. $a > 1, 2|b| < 1 \;\Rightarrow\; a > \max(1, 4b^2) \;\Rightarrow\; a > \max(1, 2b^2, 2|b|)$.

1.4. Preliminaries (II)

In this section, we make the following assumption;

Assumption 1.1. Let $\mathcal{R} = \langle 1, \lambda, \mu \rangle$ be a reduced lattice of K such that
(a) $0 < \lambda < 1, -1/2 < \mu, F(\mu) > 1, 2|Y_\mu| < 1, 0 < X_\mu < X_\lambda, 0 < \omega_1(\lambda, \mu) < 1$
(b) $\omega_2(\lambda, \mu) > 0$ (c) $F(\phi_1) < 1$ or $F(\phi_6) < 1$.

By Theorem 1.1, we can take such the basis. So in next section,
we shall consider six cases:

$(1A)$ $0 < \mu < 1, \phi_1 > 1$	$(2A)$ $\mu > 1, \phi_1 > 1$	$(3A)$ $\mu < 0, \phi_1 > 1$
$(1B)$ $0 < \mu < 1, \phi_1 < 1, F(\phi_6) < 1$	$(2B)$ $\mu > 1, \phi_1 < 1, F(\phi_6) < 1$	
$(3B)$ $\mu < 0, \phi_1 < 1, F(\phi_6) < 1$		

We note that
(A) $\phi_1 = [\omega_2] + \lambda > 1 \Leftrightarrow [\omega_2] \geq 1 \Leftrightarrow \omega_2 > 1$,
(B) $\phi_1 = [\omega_2] + \lambda < 1 \Leftrightarrow [\omega_2] = 0 \Leftrightarrow \omega_2 < 1$.

Lemma 1.6. If $\phi_1 < 1$, then (1) $Y_\lambda < -1/2$ (2) $\omega_2(\lambda, \mu) > 1/2 - \omega_1 Y_\mu$.

Proof. (1) From $\phi_1 = [\omega_2] + \lambda < 1$, we have $[\omega_2] = 0$.
By definition $\lambda_{(1)} = [-Y_\lambda] + \lambda, \lambda_{(2)} = [-Y_\lambda] + 1 + \lambda$.
Since $\mathcal{R}$ is a reduced lattice, from $\phi_1 < 1$, we have $F(\phi_1) > 1$.
Hence, by Assumpsion 1.1,(c), we have $F(\phi_6) < 1$.
From $F(\phi_6) = F([\omega_2] + 1 + \lambda) = F(1 + \lambda) < 1$, we have $1 + \lambda = \lambda_{(1)}$ or $\lambda_{(2)}$.
(i) The case $1 + \lambda = \lambda_{(1)}$. Since $-1 < Y_\lambda + 1 = Y_{\lambda_{(1)}} < 0$,
we have $-2 < Y_\lambda < -1$.
(ii) The case $1 + \lambda = \lambda_{(2)}$. We have $\lambda = \lambda_{(1)}$. Since $F(\lambda_{(2)}) < 1$,
we have $0 < Y_{\lambda_{(2)}} < 1/2$. From this, $0 < Y_\lambda + 1 = Y_{\lambda_{(2)}} < 1/2$, so $-1 < Y_\lambda < -1/2$.
Finally, from (i)(ii), we have $Y_\lambda < -1/2$.
(2) From (1), we have $-Y_\lambda > 1/2$. Hence, $\omega_2(\lambda, \mu) = -Y_\lambda - \omega_1 Y_\mu > 1/2 - \omega_1 Y_\mu$. □

Corollary 1.3. $Y_\mu < 0 \Rightarrow \omega_2(\lambda, \mu) > 1/2$.

By Corollary 1.2 if $\theta_g = x + y\lambda + z\mu$ $(x, y, z \in \mathbb{Z})$, then $(y, z) \in \{(1, 0), (1, 1), (1, -1), (2, 1), (d, 1),$ $(d + 1, 1), (2d + 1, 2), (d - 1, 1), (3d + 2, 3)\}$, where $d = [1/\omega_1(\lambda, \mu)] \geq 1$.

From Remark 1.2 and Corollary 1.3, we make the following tables in which we deside whether the possibility that $\theta_g = \psi_{i,y}$ $(1 \le i \le 10, i = 12)$ exists. Note that $y \ge 1 \;\Rightarrow\; [y\omega_2] \ge y[\omega_2]$.

Table 1

(y,z)	$\psi_{1,y} = [\omega_2 y] - 1 + y\lambda + [\omega_1 y]\mu$	$\mu > 0$ $\omega_2 > 1$	$\mu < 0$ $\omega_2 > 1$	$\mu > 0$ $\omega_2 < 1$	$\mu < 0$ $\omega_2 < 1$	No.
$(1,0)$	$[\omega_2] - 1 + \lambda$			< 1	< 1	(1-1)
$(1,1)$	$[\omega_2] - 1 + \lambda$	impossible	impossible	impossible	impossible	
$(1,-1)$	$[\omega_2] - 1 + \lambda$	impossible	impossible	impossible	impossible	
$(2,1)$	$[2\omega_2] - 1 + 2\lambda + [2\omega_1]\mu$					(1-2)
$(d,1)$	$[d\omega_2] - 1 + d\lambda$	impossible	impossible	impossible	impossible	
$(d+1,1)$	$[(d+1)\omega_2] - 1 + (d+1)\lambda + \mu$					(1-3)
$(2d+1,2)$	$[(2d+1)\omega_2] - 1 + (2d+1)\lambda + [(2d+1)\omega_1]\mu$	$> \phi_6$				(1-4)
$(d-1,1)$	$[(d-1)\omega_2] - 1 + (d-1)\lambda$	impossible	impossible	impossible	impossible	
$(3d+2,3)$	$[(3d+2)\omega_2] - 1 + (3d+2)\lambda + [(3d+2)\omega_1]\mu$	$> \phi_6$	$> \phi_6$			(1-5)

Table 2 $(\mu > 0)$

(y,z)	$\psi_{2,y} = [\omega_2 y] - 1 + y\lambda + ([\omega_1 y] + 1)\mu$	$\mu > 0$ $\omega_2 > 1$	$\mu > 0$ $\omega_2 < 1$	No.
$(1,0)$	$[\omega_2] - 1 + \lambda + \mu$	impossible	impossible	
$(1,1)$	$[\omega_2] - 1 + \lambda + \mu$			(2-1)
$(1,-1)$	$[\omega_2] - 1 + \lambda + \mu$	impossible	impossible	
$(2,1)$	$[2\omega_2] - 1 + 2\lambda + ([2\omega_1] + 1)\mu$			(2-2)
$(d,1)$	$[d\omega_2] - 1 + d\lambda + \mu$			(2-3)
$(d+1,1)$	$[(d+1)\omega_2] - 1 + (d+1)\lambda + 2\mu$	impossible	impossible	
$(2d+1,2)$	$[(2d+1)\omega_2] - 1 + (2d+1)\lambda + ([(2d+1)\omega_1] + 1)\mu$	$> \phi_6$		(2-4)
$(d-1,1)$	$[(d-1)\omega_2] - 1 + (d-1)\lambda + \mu$			(2-5)
$(3d+2,3)$	$[(3d+2)\omega_2] - 1 + (3d+2)\lambda + ([(3d+2)\omega_1] + 1)\mu$	$> \phi_6$		(2-6)

Table 3

(y,z)	$\psi_{3,y} = [\omega_2 y] + y\lambda + ([\omega_1 y] - 1)\mu$	$\mu > 0$ $\omega_2 > 1$	$\mu < 0$ $\omega_2 > 1$	$\mu > 0$ $\omega_2 < 1$	$\mu < 0$ $\omega_2 < 1$	No.
$(1,0)$	$[\omega_2] + \lambda - \mu$	impossible	impossible	impossible	impossible	
$(1,1)$	$[\omega_2] + \lambda - \mu$	impossible	impossible	impossible	impossible	
$(1,-1)$	$[\omega_2] + \lambda - \mu$			< 1		(3-1)
$(2,1)$	$[2\omega_2] + 2\lambda + ([2\omega_1] - 1)\mu$	impossible	impossible	impossible	impossible	
$(d,1)$	$[d\omega_2] + d\lambda - \mu$	impossible	impossible	impossible	impossible	
$(d+1,1)$	$[(d+1)\omega_2] + (d+1)\lambda$	impossible	impossible	impossible	impossible	
$(2d+1,2)$	$[(2d+1)\omega_2] + (2d+1)\lambda + ([(2d+1)\omega_1] - 1)\mu$	impossible	impossible	impossible	impossible	
$(d-1,1)$	$[(d-1)\omega_2] + (d-1)\lambda - \mu$	impossible	impossible	impossible	impossible	
$(3d+2,3)$	$[(3d+2)\omega_2] + (3d+2)\lambda + ([(3d+2)\omega_1] - 1)\mu$	$> \phi_6$	$> \phi_6$			(3-2)

Table 4

(y,z)	$\psi_{4,y} = [\omega_2 y] + y\lambda + [\omega_1 y]\mu$	$\mu > 0$ $\omega_2 > 1$	$\mu < 0$ $\omega_2 > 1$	$\mu > 0$ $\omega_2 < 1$	$\mu < 0$ $\omega_2 < 1$	No.
$(1,0)$	$[\omega_2] + \lambda$			< 1	< 1	(4-1)
$(1,1)$	$[\omega_2] + \lambda$	impossible	impossible	impossible	impossible	
$(1,-1)$	$[\omega_2] + \lambda$	impossible	impossible	impossible	impossible	
$(2,1)$	$[2\omega_2] + 2\lambda + [2\omega_1]\mu$	$> \phi_6$				(4-2)
$(d,1)$	$[d\omega_2] + d\lambda$	impossible	impossible	impossible	impossible	
$(d+1,1)$	$[(d+1)\omega_2] + (d+1)\lambda + \mu$	$> \phi_6$				(4-3)
$(2d+1,2)$	$[(2d+1)\omega_2] + (2d+1)\lambda + [(2d+1)\omega_1]\mu$	$> \phi_6$	$> \phi_6$			(4-4)
$(d-1,1)$	$[(d-1)\omega_2] + (d-1)\lambda$	impossible	impossible	impossible	impossible	
$(3d+2,3)$	$[(3d+2)\omega_2] + (3d+2)\lambda + [(3d+2)\omega_1]\mu$	$> \phi_6$	$> \phi_6$			(4-5)

Table 5

(y,z)	$\psi_{5,y} = [\omega_2 y] + y\lambda + ([\omega_1 y] + 1)\mu$	$\mu > 0$ $\omega_2 > 1$	$\mu < 0$ $\omega_2 > 1$	$\mu > 0$ $\omega_2 < 1$	$\mu < 0$ $\omega_2 < 1$	No.
$(1,0)$	$[\omega_2] + \lambda + \mu$	impossible	impossible	impossible	impossible	
$(1,1)$	$[\omega_2] + \lambda + \mu$				< 1	(5-1)
$(1,-1)$	$[\omega_2] + \lambda + \mu$	impossible	impossible	impossible	impossible	
$(2,1)$	$[2\omega_2] + 2\lambda + ([2\omega_1] + 1)\mu$	$> \phi_6$				(5-2)
$(d,1)$	$[d\omega_2] + d\lambda + \mu$	$> \phi_6 (d \geq 2)$				(5-3)
$(d+1,1)$	$[(d+1)\omega_2] + (d+1)\lambda + 2\mu$	impossible	impossible	impossible	impossible	
$(2d+1,2)$	$[(2d+1)\omega_2] + (2d+1)\lambda + ([(2d+1)\omega_1] + 1)\mu$	$> \phi_6$	$> \phi_6$			(5-4)
$(d-1,1)$	$[(d-1)\omega_2] + (d-1)\lambda + \mu$	$> \phi_6 (d \geq 3)$				(5-5)
$(3d+2,3)$	$[(3d+2)\omega_2] + (3d+2)\lambda + ([(3d+2)\omega_1] + 1)\mu$	$> \phi_6$	$> \phi_6$			(5-6)

Table 6 ($\mu > 0$)

(y,z)	$\psi_{6,y} = [\omega_2 y] + y\lambda + ([\omega_1 y] + 2)\mu$	$\mu > 0$ $\omega_2 \lesseqgtr 1$	No.
$(1,0)$	$[\omega_2] + \lambda + 2\mu$	impossible	
$(1,1)$	$[\omega_2] + \lambda + 2\mu$	impossible	
$(1,-1)$	$[\omega_2] + \lambda + 2\mu$	impossible	
$(2,1)$	$[2\omega_2] + 2\lambda + ([2\omega_1] + 2)\mu$	impossible	
$(d,1)$	$[d\omega_2] + d\lambda + 2\mu$	impossible	
$(d+1,1)$	$[(d+1)\omega_2] + (d+1)\lambda + 3\mu$	impossible	
$(2d+1,2)$	$[(2d+1)\omega_2] + (2d+1)\lambda + ([(2d+1)\omega_1] + 2)\mu$	impossible	
$(d-1,1)$	$[(d-1)\omega_2] + (d-1)\lambda + 2\mu$	impossible	
$(3d+2,3)$	$[(3d+2)\omega_2] + (3d+2)\lambda + ([(3d+2)\omega_1] + 2)\mu$	impossible	

Table 7 ($\mu > 0$)

(y,z)	$\psi_{7,y} = [\omega_2 y] + 1 + y\lambda + ([\omega_1 y] - 1)\mu$	$\mu > 0$ $\omega_2 \lesseqgtr 1$	No.
$(1,0)$	$[\omega_2] + 1 + \lambda - \mu$	impossible	
$(1,1)$	$[\omega_2] + 1 + \lambda - \mu$	impossible	
$(1,-1)$	$[\omega_2] + 1 + \lambda - \mu$		(7-1)
$(2,1)$	$[2\omega_2] + 1 + 2\lambda + ([2\omega_1] - 1)\mu$	impossible	
$(d,1)$	$[d\omega_2] + 1 + d\lambda - \mu$	impossible	
$(d+1,1)$	$[(d+1)\omega_2] + 1 + (d+1)\lambda$	impossible	
$(2d+1,2)$	$[(2d+1)\omega_2] + 1 + (2d+1)\lambda + ([(2d+1)\omega_1] - 1)\mu$	impossible	
$(d-1,1)$	$[(d-1)\omega_2] + 1 + (d-1)\lambda - \mu$	impossible	
$(3d+2,3)$	$[(3d+2)\omega_2] + 1 + (3d+2)\lambda + ([(3d+2)\omega_1] - 1)\mu$	$> \phi_6$	

Table 8

(y,z)	$\psi_{8,y} = [\omega_2 y] + 1 + y\lambda + [\omega_1 y]\mu$	$\mu \lesseqgtr 0$ $\omega_2 \lesseqgtr 1$	No.
$(1,0)$	$[\omega_2] + 1 + \lambda$		(8-1)
$(1,1)$	$[\omega_2] + 1 + \lambda$	impossible	
$(1,-1)$	$[\omega_2] + 1 + \lambda$	impossible	
$(2,1)$	$[2\omega_2] + 1 + 2\lambda + [2\omega_1]\mu$	$> \phi_6$	
$(d,1)$	$[d\omega_2] + 1 + d\lambda$	impossible	
$(d+1,1)$	$[(d+1)\omega_2] + 1 + (d+1)\lambda + \mu$	$> \phi_6$	
$(2d+1,2)$	$[(2d+1)\omega_2] + 1 + (2d+1)\lambda + [(2d+1)\omega_1]\mu$	$> \phi_6$	
$(d-1,1)$	$[(d-1)\omega_2] + 1 + (d-1)\lambda$	impossible	
$(3d+2,3)$	$[(3d+2)\omega_2] + 1 + (3d+2)\lambda + [(3d+2)\omega_1]\mu$	$> \phi_6$	

Table 9 ($\mu < 0$)

(y,z)	$\psi_{9,y} = [\omega_2 y] + 1 + y\lambda + ([\omega_1 y] + 1)\mu$	$\mu < 0$ $\omega_2 \lesseqgtr 1$	No.
$(1,0)$	$[\omega_2] + 1 + \lambda + \mu$	impossible	
$(1,1)$	$[\omega_2] + 1 + \lambda + \mu$		(9-1)
$(1,-1)$	$[\omega_2] + 1 + \lambda + \mu$	impossible	
$(2,1)$	$[2\omega_2] + 1 + 2\lambda + ([2\omega_1] + 1)\mu$	$> \phi_6$	
$(d,1)$	$[d\omega_2] + 1 + d\lambda + \mu$	$> \phi_6(d \geq 2)$	
$(d+1,1)$	$[(d+1)\omega_2] + 1 + (d+1)\lambda + 2\mu$	impossible	
$(2d+1,2)$	$[(2d+1)\omega_2] + 1 + (2d+1)\lambda + ([(2d+1)\omega_1] + 1)\mu$	$> \phi_6$	
$(d-1,1)$	$[(d-1)\omega_2] + 1 + (d-1)\lambda + \mu$	$> \phi_6(d \geq 3)$	
$(3d+2,3)$	$[(3d+2)\omega_2] + 1 + (3d+2)\lambda + ([(3d+2)\omega_1] + 1)\mu$	$> \phi_6$	

Table 10 ($\mu < 0$)

(y,z)	$\psi_{10,y} = [\omega_2 y] + 1 + y\lambda + ([\omega_1 y] + 2)\mu$	$\mu < 0$ $\omega_2 \lesseqgtr 1$	No.
$(1,0)$	$[\omega_2] + 1 + \lambda + 2\mu$	impossible	
$(1,1)$	$[\omega_2] + 1 + \lambda + 2\mu$	impossible	
$(1,-1)$	$[\omega_2] + 1 + \lambda + 2\mu$	impossible	
$(2,1)$	$[2\omega_2] + 1 + 2\lambda + ([2\omega_1] + 2)\mu$	impossible	
$(d,1)$	$[d\omega_2] + 1 + d\lambda + 2\mu$	impossible	
$(d+1,1)$	$[(d+1)\omega_2] + 1 + (d+1)\lambda + 3\mu$	impossible	
$(2d+1,2)$	$[(2d+1)\omega_2] + 1 + (2d+1)\lambda + ([(2d+1)\omega_1] + 2)\mu$	impossible	
$(d-1,1)$	$[(d-1)\omega_2] + 1 + (d-1)\lambda + 2\mu$	impossible	
$(3d+2,3)$	$[(3d+2)\omega_2] + 1 + (3d+2)\lambda + ([(3d+2)\omega_1] + 2)\mu$	impossible	

(y,z)	$\psi_{12,y} = [\omega_2 y] + 2 + y\lambda + ([\omega_1 y] + 1)\mu$	$\mu < 0$ $\omega_2 \lesseqgtr 1$	No.
$(1,0)$	$[\omega_2] + 2 + \lambda + \mu$	impossible	
$(1,1)$	$[\omega_2] + 2 + \lambda + \mu$	$> \phi_6$	
$(1,-1)$	$[\omega_2] + 2 + \lambda + \mu$	impossible	
$(2,1)$	$[2\omega_2] + 2 + 2\lambda + ([2\omega_1] + 1)\mu$	$> \phi_6$	
$(d,1)$	$[d\omega_2] + 2 + d\lambda + \mu$	$> \phi_6$	
$(d+1,1)$	$[(d+1)\omega_2] + 2 + (d+1)\lambda + 2\mu$	impossible	
$(2d+1,2)$	$[(2d+1)\omega_2] + 2 + (2d+1)\lambda + ([(2d+1)\omega_1] + 1)\mu$	$> \phi_6$	
$(d-1,1)$	$[(d-1)\omega_2] + 2 + (d-1)\lambda + \mu$	$> \phi_6(d \geq 2)$	
$(3d+2,3)$	$[(3d+2)\omega_2] + 2 + (3d+2)\lambda + ([(3d+2)\omega_1] + 1)\mu$	$> \phi_6$	

1.5. Main theorems

Theorem 1.5A. Let $\mathcal{R} = \langle 1, \lambda, \mu \rangle$ be a reduced lattice of K such that

$0 < \lambda < 1, 0 < X_\mu < X_\lambda, 0 < \omega_1(\lambda, \mu) < 1, \omega_2(\lambda, \mu) > 0, a > 1, 2|b| < 1,$

$0 < \mu < 1, \phi_1 > 1$, where $a = F(\mu), b = Y_\mu$. Then

(1) If $F(\phi_1) < 1$:

(i) if $b < 0$, then the minimal point adjacent to 1 is ϕ_1, ϕ_3 or ϕ_4;

(ii) if $b > 0$, then the minimal point adjacent to 1 is ϕ_1 or ϕ_5.

(2) If $F(\phi_1) > 1, F(\phi_2) < 1$:

(i) if $b < 0$, then the minimal point adjacent to 1 is ϕ_2;

(ii) if $b > 0$, then the minimal point adjacent to 1 is ϕ_2 or ϕ_5.

(3) If $F(\phi_1) > 1, F(\phi_2) > 1, F(\phi_6) < 1$,

then the minimal point adjacent to 1 is ϕ_6.

Proof. Since $\phi_1 = [\omega_2] + \lambda > 1$, we have $[\omega_2] \geq 1$.

(1) was proved in [24],Theorem 2.1.

(2) We assume that $F(\psi_{4,1}) > 1,\ F(\psi_{5,1}) < 1$.

(i) the case $b < 0$,by Lemma 1.5,(4), we have $\phi_3 = \psi_{3,1} \neq \theta_g$. By Lemma 1.5,(8), we have $\phi_4 = \psi_{1,1} \neq \theta_g$. The others were proved in [24],Theorem 2.1;

(ii) The case $b > 0$. The case were all proved in [24],Theorem 2.1.

(3) We assume that $F(\psi_{4,1}) > 1, F(\psi_{5,1}) > 1, F(\psi_{8,1}) < 1$.

By Lemma 1.5,(1)(2) and Remark 1.3,(1), we have $\theta_g \in \{\psi_{1,y}, \psi_{2,y}, \psi_{3,y}, \psi_{4,y}, \psi_{5,y}, \psi_{6,y}, \psi_{7,y}, \psi_{8,1}\}$.

(i) The case $b < 0$. By Lemma 1.5,(3), we have $\theta_g \in \{\psi_{1,y}, \psi_{3,y}, \psi_{4,y}, \psi_{5,y}, \psi_{8,1}\}$. Also by Lemma 1.5,(10) we have $c_2 = [\omega_2] - \omega_2 < -1/2$.

(a) In the case of $\psi_{1,y}$, based on Table 1,

(1-1) from $\psi_{1,1} = \psi_{8,1} - 2$ and $F(\psi_{8,1}) < 1$, we have $F(\psi_{1,1}) > 1$.

(1-2) by Lemma 1.5,(12), $\psi_{1,2} = [2\omega_2] - 1 + 2\lambda + \mu = 2[\omega_2] + 2\lambda + \mu$ or $2[\omega_2] - 1 + 2\lambda + \mu$. Since $c_2 < -1/2,\ \psi_{1,2} \neq 2[\omega_2] - 1 + 2\lambda + \mu$. Hence $\psi_{1,2} = 2[\omega_2] + 2\lambda + \mu > \psi_{8,1}$.

(1-3) $d \geq 2\ \Rightarrow\ \psi_{1,d+1} > \psi_{8,1}$. If $d = 1$, then $\psi_{1,d+1} = \psi_{1,2} = [2\omega_2] - 1 + 2\lambda + \mu$. This case is just the same as (1-2).

(b) In the case of $\psi_{3,y}$, based on Table 3,

(3-1) by Lemma 1.5,(4) $\phi_3 = \psi_{3,1} \neq \theta_g$.

(c) In the case of $\psi_{4,y}$, based on Table 4, (4-1) by the assumption $\psi_{4,1} \neq \theta_g$.

(d) In the case of $\psi_{5,y}$, based on Table 5,

(5-1) by the assumption $\psi_{5,1} \neq \theta_g$.

As a result, $\psi_{8,1}$ remains.

(ii) The case $b > 0$. By Lemma 1.5,(3), we have $\theta_g \in \{\psi_{2,y}, \psi_{4,y}, \psi_{5,y}, \psi_{6,y}, \psi_{7,y}, \psi_{8,1}\}$.

(a) In the case of $\psi_{2,y}$, based on Table 2,

(2-1) by Lemma 1.5,(9), $\psi_{2,1} \neq \theta_g$.

(2-2) by Lemma 1.5,(12), $\psi_{2,2} = [2\omega_2] - 1 + 2\lambda + \mu = 2[\omega_2] + 2\lambda + \mu (> \psi_{8,1})$ or $2[\omega_2] - 1 + 2\lambda + \mu$. The case $\psi_{2,2} = 2[\omega_2] - 1 + 2\lambda + \mu$. If $[\omega_2] \geq 2$, then we have $2[\omega_2] - 1 + 2\lambda + \mu > \psi_{8,1}$. If $[\omega_2] = 1$, then $\psi_{2,2} = 1 + 2\lambda + \mu$. We shall show that $F(1 + 2\lambda + \mu) > 1$. Since $F(\phi_6) = F(2 + \lambda) < 1$, we have $-1 < Y_{2+\lambda} < 1$, so $-3 < Y_\lambda < -1$. Suppose that $Y_\lambda > -3/2$. Then $Y_{2+\lambda} = 2 + Y_\lambda > 1/2$. From this, we have $1/4 + Z_{2+\lambda}^2 < Y_{2+\lambda}^2 + Z_{2+\lambda}^2 < 1$. Hence, $|Z_{2+\lambda}| < \sqrt{3}/2$. Since $Y_\lambda > -3/2$ and $Y_\lambda < -1$, we have $-1/2 < Y_{1+\lambda} < 0$. Hence, $F(1+\lambda) = Y_{1+\lambda}^2 + Z_{1+\lambda}^2 = Y_{1+\lambda}^2 + Z_{2+\lambda}^2 < 1/4 + 3/4 = 1$. Since $F(\phi_1) = F(1+\lambda) > 1$, we have reached a contradiction. Therefore, we have $Y_\lambda < -3/2$. From this, we have $Y_{1+2\lambda+\mu} = 1 + 2Y_\lambda + Y_\mu < 1 - 3 + Y_\mu < -3/2$. Hence, $F(1 + 2\lambda + \mu) > 1$.

(2-3) $d \geq 3 \;\Rightarrow\; \psi_{2,d} = [d\omega_2] - 1 + d\lambda + \mu > \psi_{8,1}$.

The case $d = 1, 2$ are just the same as (2-1) or (2-2).

(2-5) Similar to (2-3).

(b) In the case of $\psi_{4,y}$, based on Table 4,

(4-1) by the assumption, $\psi_{4,1} \neq \theta_g$.

(c) In the case of $\psi_{5,y}$, based on Table 5,

(5-1) by the assumption $\psi_{5,1} \neq \theta_g$.

(d) In the case of $\psi_{6,y}$, based on Table 6,

no case is included

(e) In the case of $\psi_{7,y}$, based on Table 7,

(7-1) by Lemma 1.5,(5), $\psi_{7,1} \neq \theta_g$.

As a result, $\psi_{8,1}$ remains. □

Remark 1.4. From the proof in [24,Theorem 2.1], (1) and (2) don't require the assumption $0 < X_\mu < X_\lambda$. Moreover, in (1) and (2) (except for the part of ϕ_4), we can weaken the condition from $a > 1, 2|b| < 1$ to $a > \max(1, 2b^2, 2|b|)$.

Theorem 1.6A. Let $\mathcal{R} = \langle 1, \lambda, \mu \rangle$ be a reduced lattice of K such that $0 < \lambda < 1, 0 < X_\mu < X_\lambda, 0 < \omega_1(\lambda, \mu) < 1, \omega_2(\lambda, \mu) > 0, a > 1, 2|b| < 1,$ $\mu > 1, \phi_1 > 1$, where $a = F(\mu), b = Y_\mu$. Then

(1) If $F(\phi_1) < 1$:

(i) if $b < 0$, then the minimal point adjacent to 1 is ϕ_1, ϕ_3 or ϕ_4;

(ii) if $b > 0$, then the minimal point adjacent to 1 is ϕ_1 or ϕ_7.

(2) If $F(\phi_1) > 1, F(\phi_6) < 1$:

(i) if $b < 0$, then the minimal point adjacent to 1 is ϕ_6;

(ii) if $b > 0$, then the minimal point adjacent to 1 is ϕ_5 or ϕ_6.

Proof. Since $\phi_1 = \psi_{4,1} = [\omega_2] + \lambda > 1$, we have $[\omega_2] \geq 1$.

(1) We assume that $F(\psi_{4,1}) < 1$.

By Lemma 1.5,(1)(2) and Remark 1.3,(2), we have $\theta_g \in \{\psi_{1,y}, \psi_{3,y}, \psi_{7,y}, \psi_{4,1}\}$.

(i) The case $b < 0$. By Lemma 1.5,(3), we have $\theta_g \in \{\psi_{1,y}, \psi_{3,y}, \psi_{4,1}\}$.

(a) In the case of $\psi_{1,y}$, based on Table 1,

(1-1) $\psi_{1,1}$.

(1-2) $\psi_{1,2} = [2\omega_2] - 1 + 2\lambda + \mu > \psi_{8,1}$.

(1-3) $\psi_{1,d+1} > \psi_{8,1} > \psi_{4,1}$.

(b) In the case of $\psi_{3,y}$, based on Table 3,

(3-1) $\psi_{3,1}$.

As a result, $\psi_{4,1}, \psi_{3,1}$ and $\psi_{1,1}$ remain.

(ii) The case $b > 0$. By Lemma 1.5,(3), we have $\theta_g \in \{\psi_{7,y}, \psi_{4,1}\}$.

(a) In the case of $\psi_{7,y}$, based on Table 7,

(7-1) $\psi_{7,1}$.

As a result, $\psi_{4,1}$ and $\psi_{7,1}$ remain.

(2) We assume that $F(\psi_{4,1}) > 1, F(\psi_{8,1}) < 1$.

By Lemma 1.5,(1)(2) and Remark 1.3,(2), we have $\theta_g \in \{\psi_{1,y}, \psi_{2,y}, \psi_{3,y}, \psi_{4,y}, \psi_{7,y}, \psi_{8,1}\}$.

(i) The case $b < 0$. By Lemma 1.5,(3), we have $\theta_g \in \{\psi_{1,y}, \psi_{3,y}, \psi_{4,y}, \psi_{8,1}\}$.

(a) In the case of $\psi_{1,y}$, based on Table 1,

(1-1) from $\psi_{1,1} = \psi_{8,1} - 2$ and $F(\psi_{8,1}) < 1$, we have $F(\psi_{1,1}) > 1$.

(1-2) $\psi_{1,2} = [2\omega_2] - 1 + 2\lambda + \mu > \psi_{8,1}$.

(1-3) $\psi_{1,d+1} > \psi_{8,1}$.

(b) In the case of $\psi_{3,y}$, based on Table 3, (3-1) by Lemma 1.5,(4) $\phi_3 = \psi_{3,1} \neq \theta_g$.

(c) In the case of $\psi_{4,y}$, based on Table 4,

(4-1) by the assumption $\psi_{4,1} \neq \theta_g$.

As a result, $\psi_{8,1}$ remains.

(ii) The case $b > 0$. By Lemma 1.5,(3), we have $\theta_g \in \{\psi_{2,y}, \psi_{4,y}, \psi_{7,y}, \psi_{8,1}\}$.

(a) In the case of $\psi_{2,y}$, based on Table 2,

(2-1) $\psi_{2,1} = [\omega_2] - 1 + \lambda + \mu (> \psi_{4,1})$.

(2-2) $\psi_{2,2} = [2\omega_2] - 1 + 2\lambda + \mu > \psi_{8,1}$.

(2-3) $d \geq 3 \;\Rightarrow\; \psi_{2,d} = [d\omega_2] - 1 + d\lambda + \mu > \psi_{8,1}$.

The cases $d = 1, 2$ are just the same as (2-1) or (2-2).

(2-5) Similar to (2-3).

(b) In the case of $\psi_{4,y}$, based on Table 4,

(4-1) by the assumption $\psi_{4,1} \neq \theta_g$.

(c) In the case of $\psi_{7,y}$, based on Table 7,

(7-1) by Lemma 4.5,(5) $\psi_{7,1} \neq \theta_g$.

As a result, $\psi_{8,1}$ and $\psi_{2,1}$ remain. $\square$

Remark 1.5. By [24,Theorem 2.1], if $\phi_1 > \mu$, then in (1) we can weaken the condition from $0 < \lambda < 1, 0 < X_\mu < X_\lambda, 0 < \omega_1(\lambda,\mu) < 1, \omega_2(\lambda,\mu) > 0, a > 1, 2|b| < 1$ to $0 < \omega_1(\lambda,\mu) < 1, \omega_2(\lambda,\mu) > 0, a > \max(1, 2b^2, 2|b|)$.

Theorem 1.7A. Let $\mathcal{R} = \langle 1, \lambda, \mu \rangle$ be a reduced lattice of K such that
$0 < \lambda < 1, 0 < X_\mu < X_\lambda, 0 < \omega_1(\lambda,\mu) < 1, \omega_2(\lambda,\mu) > 0, a > 1, 2|b| < 1$,
$\mu < 0, \phi_1 > 1$, where $a = F(\mu), b = Y_\mu$. Then

(1) If $F(\phi_1) < 1$:

(i) if $[\omega_2] \geq 2$, then the minimal point adjacent to 1 is ϕ_1, ϕ_2 or ϕ_4;

(ii-a) if $[\omega_2] = 1, \lambda + \mu < 0$, then the minimal point adjacent to 1 is ϕ_1 or $1 + \phi_9$,

(ii-b) if $[\omega_2] = 1, \lambda + \mu > 0$, then the minimal point adjacent to 1 is ϕ_1 or ϕ_2.

(2) If $F(\phi_1) > 1, F(\phi_6) < 1$, then the minimal point adjacent to 1 is ϕ_2, ϕ_6 or ϕ_8.

Proof. Since $\mu < 0$ and $0 < X_\mu$, we have $b < 0$ and $-1/2 < \mu$.

From Table 10 and Lemma 1.5,(3), we have $\theta_g \in \{\psi_{1,y}, \psi_{3,y}, \psi_{4,y}, \psi_{5,y}, \psi_{8,y}, \psi_{9,y}\}$.

(1) We assume that $F(\psi_{4,1}) < 1$.

(a) In the case of $\psi_{1,y}$, based on Table 1,

(1-1) $\psi_{1,1}$.

(1-2) by Lemma 1.5,(12) $\psi_{1,2} = [2\omega_2] - 1 + 2\lambda + \mu = 2[\omega_2] + 2\lambda + \mu (> \psi_{4,1})$ or $2[\omega_2] - 1 + 2\lambda + \mu$. The case $\psi_{1,2} = 2[\omega_2] - 1 + 2\lambda + \mu$. If $[\omega_2] \geq 2$, then we have $\psi_{1,2} > \psi_{4,1}$. If $[\omega_2] = 1$, $\psi_{1,2} = 1 + 2\lambda + \mu$.

(1-3) $d \geq 2 \ \Rightarrow \ \psi_{1,d+1} \geq [3\omega_2] - 1 + 3\lambda + \mu > \psi_{4,1}$. The case $d = 1$ is just the same as (1-2).

(1-4) $\psi_{1,2d+1} > \psi_{4,1}$.

(b) In the case of $\psi_{3,y}$, based on Table 3,

(3-1) $\psi_{3,1} = [\omega_2] + \lambda - \mu > [\omega_2] + \lambda = \psi_{4,1}$.

(c) In the case of $\psi_{4,y}$, based on Table 4,

(4-1) $\psi_{4,1}$.

(4-2) $\psi_{4,2} = [2\omega_2] + 2\lambda + \mu > \psi_{4,1}$.

(4-3) $\psi_{4,d+1} > \psi_{4,1}$.

(d) In the case of $\psi_{5,y}$, based on Table 5,

(5-1) $\psi_{5,1} = [\omega_2] + \lambda + \mu$.

(5-2) $\psi_{5,2} = [2\omega_2] + 2\lambda + \mu > \psi_{4,1}$.

(5-3) $d \geq 2 \ \Rightarrow \ \psi_{5,d} \geq [2\omega_2] + 2\lambda + \mu > \psi_{4,1}$.

The case $d = 1$ is just the same as (5-1).

(5-5) Similar to (5-3).

(e) In the case of $\psi_{8,y}$, based on Table 8,

(8-1) $\psi_{8,1} > \psi_{4,1}$.

(f) In the case of $\psi_{9,y}$, based on Table 9,

(9-1) $\psi_{9,1} = [\omega_2] + 1 + \lambda + \mu > \psi_{4,1}$.

As a result, $\psi_{4,1}, \psi_{5,1}, \psi_{1,1}$ and $1 + 2\lambda + \mu$ remain. Moreover, If $[\omega_2] \geq 2$, then we have $\theta_g \neq 1 + 2\lambda + \mu$. The case $[\omega_2] = 1$. Since $\phi_4 = \psi_{1,1} = [\omega_2] - 1 + \lambda = \lambda < 1$, we have $\theta_g \neq \psi_{1,1}$. If $\lambda + \mu < 0$, then we have $\phi_2 = 1 + \lambda + \mu < 1$. If $\lambda + \mu > 0$, then we have $1 + 2\lambda + \mu \neq \theta_g$, because $1 + 2\lambda + \mu = 1 + \lambda + (\lambda + \mu) > 1 + \lambda = \psi_{4,1}$.

(2) We assume that $F(\phi_1) > 1, F(\phi_6) < 1$.

We note that by Lemma 1.5,(10), we have $c_2 = [\omega_2] - \omega_2 < -1/2$. So by Lemma 1.5,(12), we have $[2\omega_2] = 2[\omega_2] + 1$.

(a) In the case of $\psi_{1,y}$, based on Table 1,

(1-1) from $\psi_{1,1} = \psi_{8,1} - 2$ and $F(\psi_{8,1}) < 1$, we have $F(\psi_{1,1}) > 1$.

(1-2) $\psi_{1,2} = [2\omega_2] - 1 + 2\lambda + \mu = 2[\omega_2] + 2\lambda + \mu$. If such a $\psi_{1,2}$ exist, then by $[2\omega_1] = 1$, we have $c_1 < -1/2$ ($\Leftrightarrow$ $[2\omega_1] = 1$).

(i) The case $[\omega_2] \geq 2$. We have $\psi_{1,2} > \psi_{8,1}$.

(ii) The case $[\omega_2] = 1$. $\psi_{1,2} = 2 + 2\lambda + \mu > 2 + \lambda + \mu = \psi_{9,1}$.

From Lemma 1.5,(11), we have $F(\psi_{9,1}) < F(\psi_{8,1})$. So we have $F(\psi_{9,1}) < 1$. Therefore, $\psi_{1,2} = 2 + 2\lambda + \mu \neq \theta_g$.

(1-3) (i) The case $d \geq 2$. We have $\psi_{1,d+1} \geq [3\omega_2] - 1 + 3\lambda + \mu \geq [2\omega_2] + [\omega_2] - 1 + 3\lambda + \mu$ $= 3[\omega_2] + 3\lambda + \mu > \psi_{8,1}$.

(ii) The case $d = 1$. Since $d = 1 \;\Leftrightarrow\; [2\omega_1] = 1$, this case is just the same as (1-2).

(1-4) $\psi_{1,2d+1} \geq [3\omega_2] - 1 + 3\lambda + 2\mu \geq [2\omega_2] + [\omega_2] - 1 + 3\lambda + 2\mu = 3[\omega_2] + 3\lambda + 2\mu > \psi_{8,1}$.

(b) In the case of $\psi_{3,y}$, based on Table 3,

(3-1) by Lemma 1.5,(4) $\phi_3 = \psi_{3,1} \neq \theta_g$.

(c) In the case of $\psi_{4,y}$, based on Table 4,

(4-1) $F(\psi_{4,1}) > 1$.

(4-2) $\psi_{4,2} = [2\omega_2] + 2\lambda + \mu = 2[\omega_2] + 1 + 2\lambda + \mu > \psi_{8,1}$.

(4-3) $\psi_{4,d+1} \geq [2\omega_2] + 2\lambda + \mu > \psi_{8,1}$.

(d) In the case of $\psi_{5,y}$, based on Table 5,

(5-1) $\psi_{5,1} = [\omega_2] + \lambda + \mu$.

(5-2) $\psi_{5,2} = [2\omega_2] + 2\lambda + \mu = 2[\omega_2] + 1 + 2\lambda + \mu > \psi_{8,1}$.

(5-3) $d \geq 2 \;\Rightarrow\; \psi_{5,d} \geq [2\omega_2] + 2\lambda + \mu > \psi_{8,1}$.

The case $d = 1$ is just the same as (5-1).

(5-5) Similar to (5-3).

(e) In the case of $\psi_{8,y}$, based on Table 8,

(8-1) $F(\psi_{8,1}) < 1$.

(f) In the case of $\psi_{9,y}$, based on Table 9,

(9-1) $\psi_{9,1} = [\omega_2] + 1 + \lambda + \mu$.

As a result, $\psi_{8,1}, \psi_{5,1}$ and $\psi_{9,1}$ remain. $\square$

Theorem 1.5B. Let $\mathcal{R} = \langle 1, \lambda, \mu \rangle$ be a reduced lattice of K such that $0 < \lambda < 1, 0 < X_\mu < X_\lambda, 0 < \omega_1(\lambda, \mu) < 1, \omega_2(\lambda, \mu) > 0, a > 1, 2|b| < 1,$ $0 < \mu < 1, \phi_1 < 1, F(\phi_6) < 1,$ where $a = F(\mu), b = Y_\mu$. Then

(1) If $F(\phi_2) < 1$, then the minimal point adjacent to 1 is ϕ_2.

(2) If $\phi_2 > 1, F(\phi_2) > 1$, then the minimal point adjacent to 1 is ϕ_6.

(3) If $\phi_2 < 1$:

(i) if $b < 0$, then the minimal point adjacent to 1 is ϕ_6;

(ii-a) if $b > 0,\ 2\lambda + \mu < 1$, then the minimal point adjacent to 1 is ϕ_6 or ϕ_{10},

(ii-b) if $b > 0,\ 2\lambda + \mu > 1$, then the minimal point adjacent to 1 is ϕ_6 or ϕ_9.

Proof. From the assumption $\phi_1 < 1$, by Lemma 1.6,(1), we have $Y_\lambda < -1/2$. By Corollary 1.3, if $b < 0$, then we have $1 > \omega_2 > 1/2$.

(1) We assume that $F(\psi_{5,1}) < 1$. Since $\mathcal{R}$ is a reduced lattice, we have $\psi_{5,1} = [\omega_2] + \lambda + ([\omega_1] + 1)\mu = \lambda + \mu > 1$.

By Lemma 1.5,(1)(2) and Remark 1.3,(1) we have $\theta_g \in \{\psi_{1,y}, \psi_{2,y}, \psi_{3,y}, \psi_{4,y}, \psi_{7,y}, \psi_{5,1}\}$.

(i) The case $b < 0$. By Lemma 1.5,(3), we have $\theta_g \in \{\psi_{1,y}, \psi_{3,y}, \psi_{4,y}, \psi_{5,1}\}$.

(a) In the case of $\psi_{1,y}$, based on Table 1,

(1-2) since $[2\omega_2] = 1$, we have $\psi_{1,2} = 2\lambda + \mu > \psi_{8,1} > \psi_{5,1}$.

(1-3) $[(d+1)\omega_2] \geq 2 \ \Rightarrow\ \psi_{1,d+1} > \psi_{8,1} > \psi_{5,1}$. $[(d+1)\omega_2] = 1 \ \Rightarrow\ \psi_{1,d+1} = (d+1)\lambda + \mu$ $\Rightarrow\ Y_{\psi_{1,d+1}} = (d+1)Y_\lambda + Y_\mu < -1$.

(1-4) $[(2d+1)\omega_2] \geq 2 \ \Rightarrow\ \psi_{1,2d+1} > \psi_{8,1} > \psi_{5,1}$. $[(2d+1)\omega_2] = 1 \ \Rightarrow\ \psi_{1,2d+1} = (2d+1)\lambda$ $+2\mu > \psi_{8,1} > \psi_{5,1}$.

(1-5) from $[(3d+2)\omega_2] \geq 2$, we have $\psi_{1,3d+2} \geq 1 + (3d+2)\lambda + 3\mu > \psi_{8,1} > \psi_{5,1}$.

(b) In the case of $\psi_{3,y}$, based on Table 3,

(3-2) $\psi_{3,3d+2} > \psi_{8,1} > \psi_{5,1}$.

(c) In the case of $\psi_{4,y}$, based on Table 4,

(4-2) since $[2\omega_2] = 1$, we have $\psi_{4,2} = 1 + 2\lambda + \mu > \psi_{8,1} > \psi_{5,1}$.

(4-3) $\psi_{4,d+1} > \psi_{8,1} > \psi_{5,1}$.

(4-4) $\psi_{4,2d+1} > \psi_{8,1} > \psi_{5,1}$.

(4-5) $\psi_{4,3d+2} > \psi_{8,1} > \psi_{5,1}$.

(ii) The case $b > 0$. By Lemma 1.5,(3), we have $\theta_g \in \{\psi_{2,y}, \psi_{4,y}, \psi_{7,y}, \psi_{5,1}\}$.

(a) In the case of $\psi_{2,y}$, based on Table 2,

(2-1) $\psi_{2,1} = -1 + \lambda + \mu < 1$.

(2-2) $[2\omega_2]=0 \Rightarrow \psi_{2,2}=-1+2\lambda+\mu \Rightarrow Y_{\psi_{2,2}}=-1+2Y_\lambda+Y_\mu<-1$. $[2\omega_2]=1$ $\Rightarrow \psi_{2,2}=2\lambda+\mu>\psi_{8,1}>\psi_{5,1}$.

(2-3) $[d\omega_2]\geq 2 \Rightarrow \psi_{2,d}>\psi_{8,1}>\psi_{5,1}$. $[d\omega_2]=1 \Rightarrow$ Since $d\geq 2$, $\psi_{2,d}=d\lambda+\mu>\psi_{8,1}$ $>\psi_{5,1}$. $[d\omega_2]=0 \Rightarrow \psi_{2,d}=-1+d\lambda+\mu \Rightarrow Y_{\psi_{2,d}}=-1+dY_\lambda+Y_\mu<-1$.

(2-4) $[(2d+1)\omega_2]\geq 2 \Rightarrow \psi_{2,2d+1}>\psi_{8,1}>\psi_{5,1}$. $[(2d+1)\omega_2]=1 \Rightarrow \psi_{2,2d+1}$ $=(2d+1)\lambda+2\mu>\psi_{8,1}>\psi_{5,1}$. $[(2d+1)\omega_2]=0 \Rightarrow \psi_{2,2d+1}=-1+(2d+1)\lambda+2\mu \Rightarrow Y_{\psi_{2,2d+1}}$ $=-1+(2d+1)Y_\lambda+2Y_\mu<-1$.

(2-5) Similar to (2-3).

(2-6) $[(3d+2)\omega_2]\geq 2 \Rightarrow \psi_{2,3d+2}>\psi_{8,1}>\psi_{5,1}$. $[(3d+2)\omega_2]=1 \Rightarrow \psi_{2,3d+2}$ $=(3d+2)\lambda+3\mu>\psi_{8,1}>\psi_{5,1}$. $[(3d+2)\omega_2]=0 \Rightarrow \psi_{2,3d+2}=-1+(3d+2)\lambda+3\mu \Rightarrow Y_{\psi_{2,3d+2}}=$ $-1+(3d+2)Y_\lambda+3Y_\mu<-1$.

(b) In the case of $\psi_{4,y}$, based on Table 4,

(4-2) $[2\omega_2]=0 \Rightarrow \psi_{4,2}=2\lambda+\mu>\psi_{8,1}>\psi_{5,1}$. $[2\omega_2]=1 \Rightarrow \psi_{4,2}=1+2\lambda+\mu>\psi_{8,1}$ $>\psi_{5,1}$.

(4-3) $\psi_{4,d+1}>\psi_{8,1}>\psi_{5,1}$.

(4-4) $\psi_{4,2d+1}>\psi_{8,1}>\psi_{5,1}$.

(4-5) $\psi_{4,3d+2}>\psi_{8,1}>\psi_{5,1}$.

(c) In the case of $\psi_{7,y}$, based on Table 7,

(7-1) by Lemma 1.5,(5) $\psi_{7,1}\neq\theta_g$.

As a result, $\psi_{5,1}$ remains.

(2) We assume that $\psi_{5,1}=\lambda+\mu>1, F(\psi_{5,1})>1$.

By Lemma 1.5,(1)(2) and Remark 1.3,(1) we have $\theta_g\in\{\psi_{1,y},\psi_{2,y},\psi_{3,y},\psi_{4,y},\psi_{5,y},\psi_{6,y},\psi_{7,y},\psi_{8,1}\}$.

(i) The case $b<0$. By Lemma 1.5,(3), we have $\theta_g\in\{\psi_{1,y},\psi_{3,y},\psi_{4,y},\psi_{5,y},\psi_{8,1}\}$.

(a) In the case of $\psi_{1,y}$, based on Table 1,
similar to (1).

(b) In the case of $\psi_{3,y}$, based on Table 3,
similar to (1).

(c) In the case of $\psi_{4,y}$, based on Table 4,
similar to (1).

(d) In the case of $\psi_{5,y}$, based on Table 5,

(5-1) from the assumption, $F(\psi_{5,1})>1$.

(5-2) $\psi_{5,2}>\phi_6$. (5-3) $\psi_{5,d}>\phi_6(d\geq 2)$.

(5-4) $\psi_{5,2d+1} > \phi_6$. (5-5) $\psi_{5,d-1} > \phi_6 (d \geq 3)$.

As a result, $\psi_{8,1}$ remains.

(ii) The case $b > 0$. By Lemma 1.5,(3), we have $\theta_g \in \{\psi_{2,y}, \psi_{4,y}, \psi_{5,y}, \psi_{6,y}, \psi_{7,y}, \psi_{8,1}\}$.

(a) In the case of $\psi_{2,y}$, based on Table 2,

similar to (1).

(b) In the case of $\psi_{4,y}$, based on Table 4,

similar to (1).

(c) In the case of $\psi_{5,y}$, based on Table 5,

(5-1) from the assumption, $F(\psi_{5,1}) > 1$.

(5-2) $\psi_{5,2} > \phi_6$. (5-3) $\psi_{5,d} > \phi_6 (d \geq 2)$.

(5-4) $\psi_{5,2d+1} > \phi_6$. (5-5) $\psi_{5,d-1} > \phi_6 (d \geq 3)$.

(d) In the case of $\psi_{6,y}$, based on Table 6,

no case included

(e) In the case of $\psi_{7,y}$. based on Table 7,

similar to (1).

As a result, $\psi_{8,1}$ remains.

(3) We assume that $\psi_{5,1} < 1$.

By Lemma 1.5,(1)(2) and Remark 1.3,(1) we have $\theta_g \in \{\psi_{1,y}, \psi_{2,y}, \psi_{3,y}, \psi_{4,y}, \psi_{5,y}, \psi_{6,y}, \psi_{7,y}, \psi_{8,1}\}$.

(i) The case $b < 0$. By Lemma 1.5,(3), we have $\theta_g \in \{\psi_{1,y}, \psi_{3,y}, \psi_{4,y}, \psi_{5,y}, \psi_{8,1}\}$.

(a) In the case of $\psi_{1,y}$, based on Table 1,

(1-2) $\psi_{1,2} = 2\lambda + \mu$, $Y_{\psi_{1,2}} = 2Y_\lambda + Y_\mu < -1$.

(1-3) The case $d \geq 3$. $\psi_{1,d+1} > 1 + 4\lambda + \mu > \phi_6$. The case $d = 2$. $\psi_{1,d+1} = [3\omega_2] - 1 + 3\lambda + \mu$. $[3\omega_2] = 2 \Rightarrow \psi_{1,d+1} = 1+3\lambda+\mu > \phi_6$. $[3\omega_2] = 1 \Rightarrow \psi_{1,d+1} = 3\lambda+\mu$. $Y_{\psi_{1,d+1}} = 3Y_\lambda+Y_\mu < -1$.

(1-4) The case $d \geq 2$. $\psi_{1,2d+1} > \phi_6$.

The case $d = 1$. $\psi_{1,2d+1} = [3\omega_2] - 1 + 3\lambda + 2\mu$.$[3\omega_2] = 2 \Rightarrow \psi_{1,2d+1} = 1 + 3\lambda + 2\mu > \phi_6$. $[3\omega_2] = 1 \Rightarrow \psi_{1,2d+1} = 3\lambda + 2\mu$. $Y_{\psi_{1,2d+1}} = 3Y_\lambda + 2Y_\mu < -1$.

(1-5) $\psi_{1,3d+2} > \phi_6$.

(b) In the case of $\psi_{3,y}$, based on Table 3,

(3-2) $\psi_{3,3d+2} > \phi_6$.

(c) In the case of $\psi_{4,y}$, based on Table 4,

(4-2) $\psi_{4,2} > \phi_6$. (4-3) $\psi_{4,d+1} > \phi_6$. (4-4) $\psi_{4,2d+1} > \phi_6$. (4-5) $\psi_{4,3d+2} > \phi_6$.

(d) In the case of $\psi_{5,y}$, based on Table 5, (5-1) from the assumption, $\psi_{5,1} < 1$. (5-2) $\psi_{5,2} > \phi_6$.

(5-3) $\psi_{5,d} > \phi_6 (d \geq 2)$. (5-4) $\psi_{5,2d+1} > \phi_6$. (5-5) $\psi_{5,d-1} > \phi_6 (d \geq 3)$.

As a result, $\psi_{8,1}$ remains.

(ii) The case $b > 0$. by Lemma 1.5,(3), we have $\theta_g \in \{\psi_{2,y}, \psi_{4,y}, \psi_{5,y}, \psi_{6,y}, \psi_{7,y}, \psi_{8,1}\}$.

(a) In the case of $\psi_{2,y}$, based on Table 2,

(2-1) $\psi_{2,1} = -1 + \lambda + \mu < 1$.

(2-2) $[2\omega_2] = 0 \Rightarrow \psi_{2,2} = -1 + 2\lambda + \mu < \lambda < 1.[2\omega_2] = 1 \Rightarrow \psi_{2,2} = 2\lambda + \mu$.

(2-3) The case $[d\omega_2] \geq 2$. $\psi_{2,d} > \psi_{8,1} > \psi_{5,1}$.

The case $[d\omega_2] = 1$. We have $d \geq 2 \Rightarrow \psi_{2,d} = d\lambda + \mu$. If $d \geq 3$, then we have $Y_{\psi_{2,d}} = dY_\lambda + Y_\mu < -1$. Hence, only when $d = 2$, it is possible to have $\theta_g = \psi_{2,d} = \psi_{2,2} = 2\lambda + \mu$. The case $[d\omega_2] = 0$. $\psi_{2,d} = -1 + d\lambda + \mu$. $Y_{\psi_{2,d}} = -1 + dY_\lambda + Y_\mu < -1$.

(2-4) The case $[(2d+1)\omega_2] \geq 2$. $\psi_{2,2d+1} > \psi_{8,1}$. The case $[(2d+1)\omega_2] = 1$. $\psi_{2,2d+1} = (2d+1)\lambda + 2\mu$. If $d \geq 2$, then we have $Y_{\psi_{2,2d+1}} = (2d+1)Y_\lambda + 2Y_\mu < -1$. Hence, only when $d = 1$, it is possible to have $\theta_g = \psi_{2,3} = 3\lambda + 2\mu$. The case $[(2d+1)\omega_2] = 0$. $\psi_{2,2d+1} = -1 + (2d+1)\lambda + 2\mu$. $Y_{\psi_{2,2d+1}} = -1 + (2d+1)Y_\lambda + 2Y_\mu < -1$.

(2-5) Similar to (2-3).

(2-6) The case $[(3d+2)\omega_2] \geq 2$. $\psi_{2,3d+2} > \psi_{8,1}$. The case $[(3d+2)\omega_2] = 1$. $\psi_{2,3d+2} = (3d+2)\lambda + 3\mu$. $Y_{\psi_{2,3d+2}} = (3d+2)Y_\lambda + 3Y_\mu < -1$. The case $[(3d+2)\omega_2] = 0$. $\psi_{2,3d+2} = -1 + (3d+2)\lambda + 3\mu$. $Y_{\psi_{2,3d+2}} = -1 + (3d+2)Y_\lambda + 3Y_\mu < -1$.

(b) In the case of $\psi_{4,y}$, based on Table 4,

(4-2) $[2\omega_2] = 0 \Rightarrow \psi_{4,2} = 2\lambda + \mu.[2\omega_2] = 1 \Rightarrow \psi_{4,2} = 1 + 2\lambda + \mu > \psi_{8,1}$.

(4-3) The case $[(d+1)\omega_2] \geq 1$. $\psi_{4,d+1} > \psi_{8,1}$. The case $[(d+1)\omega_2] = 0$. $\psi_{4,d+1} = (d+1)\lambda + \mu$. If $d \geq 2$, then we have $Y_{\psi_{4,d+1}} = (d+1)Y_\lambda + Y_\mu < -1$. Hence, only when $d = 1$, it is possible to have $\theta_g = \psi_{4,2} = 2\lambda + \mu$.

(4-4) The case $[(2d+1)\omega_2] \geq 1$. $\psi_{4,2d+1} > \psi_{8,1}$. The case $[(2d+1)\omega_2] = 0$. $\psi_{4,2d+1} = (2d+1)\lambda + 2\mu$. If $d \geq 2$, then we have $Y_{\psi_{4,2d+1}} = (2d+1)Y_\lambda + 2Y_\mu < -1$. Hence, only when $d = 1$, it is possible to have $\theta_g = \psi_{4,3} = 3\lambda + 2\mu$.

(4-5) $[(3d+2)\omega_2] \geq 1 \Rightarrow \psi_{4,3d+2} > \psi_{8,1}.[(3d+2)\omega_2] = 0 \Rightarrow \psi_{4,3d+2} = (3d+2)\lambda + 3\mu$. $Y_{\psi_{4,3d+2}} = (3d+2)Y_\lambda + 3Y_\mu < -1$.

(c) In the case of $\psi_{5,y}$, based on Table 5,

(5-1) from the assumption, $F(\psi_{5,1}) > 1$.

(5-2) $[2\omega_2] = 0 \Rightarrow \psi_{5,2} = 2\lambda + \mu.[2\omega_2] = 1 \Rightarrow \psi_{5,2} = 1 + 2\lambda + \mu > \psi_{8,1}$.

(5-3) The case $[d\omega_2] \geq 1$. $\psi_{5,d} > \psi_{8,1}$.

The case $[d\omega_2] = 0$. $\psi_{5,d} = d\lambda + \mu$. If $d \geq 3$, then we have $Y_{\psi_{5,d}} = dY_\lambda + Y_\mu < -1$. Hence, only when $d = 2$, it is possible to have $\theta_g = \psi_{5,2} = 2\lambda + \mu$.

(5-4) The case $[(2d+1)\omega_2] \geq 1$. $\psi_{5,2d+1} > \psi_{8,1}$. The case $[(2d+1)\omega_2] = 0$. $\psi_{5,2d+1}$ $= (2d+1)\lambda + 2\mu$. If $d \geq 2$, then we have $Y_{\psi_{5,2d+1}} = (2d+1)Y_\lambda + 2Y_\mu < -1$. Hence, only when $d = 1$, it is possible to have $\theta_g = \psi_{5,3} = 3\lambda + 2\mu$.

(5-5) The case $[(d-1)\omega_2] \geq 1$. $\psi_{5,d-1} > \psi_{8,1}$. The case $[(d-1)\omega_2] = 0$. $\psi_{5,d-1} = (d-1)\lambda$ $+\mu$. If $d \geq 4$, then we have $Y_{\psi_{5,d-1}} = (d-1)Y_\lambda + Y_\mu < -1$. Hence, only when $d = 3$, it is possible to have $\theta_g = \psi_{5,2} = 2\lambda + \mu$.

(d) In the case of $\psi_{6,y}$. based on Table 6,

no case included

(e) In the case of $\psi_{7,y}$, based on Table 7,

(7-1) By Lemma 1.5,(5) $\psi_{7,1} \neq \theta_g$.

As a result, $2\lambda + \mu, 3\lambda + 2\mu$ and $\psi_{8,1}$ remain. If $2\lambda + \mu < 1$, then we have $2\lambda + \mu \neq \theta_g$. If $2\lambda + \mu > 1$, then we have $3\lambda + 2\mu \neq \theta_g$, because $3\lambda + 2\mu = (2\lambda + \mu) + \lambda + \mu > 1 + \lambda = \psi_{8,1}$. $\square$

Theorem 1.6B. Let $\mathcal{R} = \langle 1, \lambda, \mu \rangle$ be a reduced lattice of K such that
$0 < \lambda < 1, 0 < X_\mu < X_\lambda, 0 < \omega_1(\lambda, \mu) < 1, \omega_2(\lambda, \mu) > 0, a > 1, 2|b| < 1,$
$\mu > 1, \phi_1 < 1, F(\phi_6) < 1$, where $a = F(\mu), b = Y_\mu$. Then
the minimal point adjacent to 1 is ϕ_6.

Proof. From the assumption $\phi_1 < 1$, by Lemma 1.6,(1), we have $Y_\lambda < -1/2$. By Corollary 1.3, if $b < 0$, then we have $\omega_2 > 1/2$.

By Lemma 1.5,(1)(2) and Remark 1.3,(2) we have $\theta_g \in \{\psi_{1,y}, \psi_{2,y}, \psi_{3,y}, \psi_{4,y}, \psi_{7,y}, \psi_{8,1}\}$.

(i) The case $b < 0$. By Lemma 1.5,(3), we have $\theta_g \in \{\psi_{1,y}, \psi_{3,y}, \psi_{4,y}, \psi_{8,1}\}$.

(a) In the case of $\psi_{1,y}$, based on Table 1,

(1-1) from $\psi_{1,1} = \psi_{8,1} - 2$ and $F(\psi_{8,1}) < 1$, we have $F(\psi_{1,1}) > 1$.

(1-2) $\psi_{1,2} = 2\lambda + \mu > \psi_{8,1}$. (1-3) $\psi_{1,d+1} > \psi_{8,1}$.

(1-4) $\psi_{1,2d+1} > \psi_{8,1}$. (1-5) $\psi_{1,3d+2} > \psi_{8,1}$.

(b) In the case of $\psi_{3,y}$, based on Table 3,

(3-2) $\psi_{3,3d+2} > \psi_{8,1}$.

(c) In the case of $\psi_{4,y}$, based on Table 4,

(4-2) $\psi_{4,2} > \psi_{8,1}$. (4-3) $\psi_{4,3} > \psi_{8,1}$.

(4-4) $\psi_{4,2d+1} > \psi_{8,1}$. (4-5) $\psi_{4,3d+2} > \psi_{8,1}$.

As a result $\psi_{8,1}$ remains.

(ii) The case $b > 0$. By Lemma 1.5,(3), we have $\theta_g \in \{\psi_{2,y}, \psi_{4,y}, \psi_{7,y}, \psi_{8,1}\}$.

(a) In the case of $\psi_{2,y}$, based on Table 2,

(2-1) $\psi_{2,1} = -1 + \lambda + \mu$. $Y_{\psi_{2,1}} = -1 + Y_\lambda + Y_\mu < -1$.

(2-2) $\psi_{2,2} = [2\omega_2] - 1 + 2\lambda + \mu$. $[2\omega_2] = 0 \;\Rightarrow\; \psi_{2,2} = -1 + 2\lambda + \mu$. $Y_{\psi_{2,2}} = -1 + 2Y_\lambda + Y_\mu$ < -1. $[2\omega_2] = 1 \;\Rightarrow\; \psi_{2,2} = 2\lambda + \mu > \psi_{8,1}$.

(2-3) $[d\omega_2] \geq 1 \;\Rightarrow\; \psi_{2,d} = [d\omega_2] - 1 + d\lambda + \mu > \psi_{8,1}$. $[d\omega_2] = 0 \;\Rightarrow\; \psi_{2,d} = -1 + d\lambda + \mu$. $Y_{\psi_{2,d}} = -1 + dY_\lambda + Y_\mu < -1$.

(2-4) $\psi_{2,2d+1} > \psi_{8,1}$. (2-5) Similar to (2-3).

(2-6) $\psi_{2,3d+2} > \psi_{8,1}$.

(b) In the case of $\psi_{4,y}$, based on Table 4,

(4-2) $\psi_{4,2} > \psi_{8,1}$. (4-3) $\psi_{4,d+1} > \psi_{8,1}$.

(4-4) $\psi_{4,2d+1} > \psi_{8,1}$. (4-5) $\psi_{4,3d+2} > \psi_{8,1}$.

(c) In the case of $\psi_{7,y}$, based on Table 7,

$\psi_{7,1} = 1 + \lambda - \mu < \lambda < 1$.

As a result, $\psi_{8,1}$ remains. $\square$

Theorem 1.7B. Let $\mathcal{R} = \langle 1, \lambda, \mu \rangle$ be a reduced lattice of K such that $0 < \lambda < 1, 0 < X_\mu < X_\lambda, 0 < \omega_1(\lambda, \mu) < 1, \omega_2(\lambda, \mu) > 0, a > 1, 2|b| < 1,$ $\mu < 0, \phi_1 < 1, F(\phi_6) < 1$, where $a = F(\mu), b = Y_\mu$. Then

(1) If $F(\phi_8) < 1$, then the minimal point adjacent to 1 is ϕ_8.

(2) If $F(\phi_8) > 1$:

(i) if $2\lambda + \mu < 0$, then the minimal point adjacent to 1 is ϕ_6 or $\phi_6 + \phi_9$;

(ii) if $2\lambda + \mu > 0$, then the minimal point adjacent to 1 is ϕ_6 or $1 + \phi_9$.

Proof. From the assumption $\phi_1 < 1$, by Lemma 1.6,(1), we have $Y_\lambda < -1/2$. Since $\mu < 0$ and $0 < X_\mu$, we have $b < 0$. By Corollary 1.3, we have $\omega_2 > 1/2$. From Table 10 and Lemma 1.5,(3), we have $\theta_g \in \{\psi_{1,y}, \psi_{3,y}, \psi_{4,y}, \psi_{5,y}, \psi_{8,y}, \psi_{9,y}\}$.

(a) In the case of $\psi_{1,y}$, based on Table 1,

(1-2) $\psi_{1,2} = 2\lambda + \mu$. $Y_{\psi_{1,2}} = 2Y_\lambda + Y_\mu < -1$.

*(1-3) $d \geq 5 \;\Rightarrow\; \psi_{1,d+1} \geq [6\omega_2] - 1 + 6\lambda + \mu \geq 2 + 6\lambda + \mu > \psi_{8,1}$.

$d = 1 \;\Rightarrow\; \psi_{1,d+1} = 2\lambda + \mu$. $Y_{\psi_{1,d+1}} = 2Y_\lambda + Y_\mu < -1$.

Hence, only when $2 \le d \le 4$, it is possible to have $\theta_g = \psi_{1,d+1}$.

*(1-4) $d \ge 3 \;\Rightarrow\; \psi_{1,2d+1} \ge [7\omega_2] - 1 + 7\lambda + 2\mu \ge 2 + 7\lambda + 2\mu > \psi_{8,1}$.

Hence, only when $1 \le d \le 2$, it is possible to have $\theta_g = \psi_{1,2d+1}$.

*(1-5) $d \ge 2 \;\Rightarrow\; \psi_{1,3d+2} \ge [8\omega_2] - 1 + 8\lambda + 3\mu \ge 3 + 8\lambda + 3\mu > \psi_{8,1}$.

Hence, only when $d = 1$, it is possible to have $\theta_g = \psi_{1,2d+1} = \psi_{1,5}$.

(b) In the case of $\psi_{3,y}$, based on Table 3,

(3-1) By Lemma 1.5,(4), $\phi_3 = \psi_{3,1} \ne \theta_g$.

*(3-2) $d \ge 2 \;\Rightarrow\; \psi_{3,3d+2} > \psi_{8,1}$. Hence, only when $d = 1$, it is possible to have $\theta_g = \psi_{3,3d+2}$ $= \psi_{3,5}$.

(c) In the case of $\psi_{4,y}$, based on Table 4,

*(4-2) $\psi_{4,2} = 1 + 2\lambda + \mu$.

*(4-3) $d \ge 3 \;\Rightarrow\; \psi_{4,d+1} \ge [4\omega_2] + 4\lambda + \mu \ge 2 + 4\lambda + \mu > \psi_{8,1}$.

Hence, only when $1 \le d \le 2$, it is possible to have $\theta_g = \psi_{4,d+1}$.

*(4-4) $d \ge 2 \;\Rightarrow\; \psi_{4,2d+1} \ge [5\omega_2] + 5\lambda + 2\mu \ge 2 + 5\lambda + 2\mu > \psi_{8,1}$.

Hence, only when $d = 1$, it is possible to have $\theta_g = \psi_{4,2d+1}$.

*(4-5) $d \ge 2 \;\Rightarrow\; \psi_{4,3d+2} \ge [8\omega_2] + 8\lambda + 3\mu \ge 4 + 8\lambda + 3\mu > \psi_{8,1}$.

Hence, only when $d = 1$, it is possible to have $\theta_g = \psi_{4,3d+2}$.

(d) In the case of $\psi_{5,y}$, based on Table 5,

*(5-2) $\psi_{5,2} = 1 + 2\lambda + \mu$.

*(5-3) $d \ge 4 \;\Rightarrow\; \psi_{5,d} \ge [4\omega_2] + 4\lambda + \mu \ge 2 + 4\lambda + \mu > \psi_{8,1}$.

$d = 1 \;\Rightarrow\; \psi_{5,d} = \lambda + \mu < 1$.

Hence, only when $2 \le d \le 3$, it is possible to have $\theta_g = \psi_{5,d}$.

*(5-4) $d \ge 2 \;\Rightarrow\; \psi_{5,2d+1} \ge [5\omega_2] + 5\lambda + 2\mu \ge 2 + 5\lambda + 2\mu > \psi_{8,1}$.

Hence, only when $d = 1$, it is possible to have $\theta_g = \psi_{5,2d+1}$.

*(5-5) $d \ge 5 \;\Rightarrow\; \psi_{5,d-1} \ge [4\omega_2] + 4\lambda + \mu \ge 2 + 4\lambda + \mu > \psi_{8,1}$.

$d = 2 \;\Rightarrow\; \psi_{5,d} = \lambda + \mu < 1$.

Hence, only when $3 \le d \le 4$, it is possible to have $\theta_g = \psi_{5,d-1}$.

*(5-6) $d \ge 2 \;\Rightarrow\; \psi_{5,3d+2} \ge [8\omega_2] + 8\lambda + 3\mu \ge 4 + 8\lambda + 3\mu > \psi_{8,1}$.

Hence, only when $d = 1$, it is possible to have $\theta_g = \psi_{5,3d+2}$.

(e) In the case of $\psi_{8,y}$, based on Table 8,

*(8-1) From the assumption, $F(\psi_{8,1}) < 1$.

(f) In the case of $\psi_{9,y}$, based on Table 9,

*(9-1) $\psi_{9,1} = [\omega_2] + 1 + \lambda + \mu$.

From described above, we shall select all the elements in each part with asterisk (*), using $1 \le [3\omega_2] \le 2, 2 \le [4\omega_2] \le 3, 2 \le [5\omega_2] \le 4$.

Then we have the following set

$\{1+\lambda, 1+\lambda+\mu, 1+2\lambda+\mu, j+3\lambda+\mu (0 \le j \le 2),$

$j+3\lambda+2\mu (0 \le j \le 2), j+4\lambda+\mu (1 \le j \le 2), j+5\lambda+\mu (1 \le j \le 3),$

$j+5\lambda+2\mu (1 \le j \le 3), j+5\lambda+3\mu (1 \le j \le 4)\} = \Sigma.$

Here, we eliminate elements $\psi \in \Sigma$ such that $\psi > \phi_6$ or $Y_\psi < -1$.

Then we have

$\Sigma' = \{1+\lambda, 1+\lambda+\mu, 1+2\lambda+\mu, 1+3\lambda+\mu, 1+3\lambda+2\mu, 2+5\lambda+3\mu\}$.

(1) We assume that $F(\phi_8) < 1$. Since $\mathcal{R}$ is a reduced lattice, we have $\phi_8 = \psi_{9,1} = 1+\lambda+\mu > 1$. Hence, we have $\lambda + \mu > 0$.

From this, we have $1+\lambda+\mu < 1+2\lambda+\mu, 1+3\lambda+\mu, 1+3\lambda+2\mu, 2+5\lambda+3\mu$.

Therefore we conclude that $\theta_g = \phi_8 = 1+\lambda+\mu$ because $\phi_8 < \phi_6 = 1+\lambda$.

(2) We assume that $F(\phi_8) > 1$. We note that $d(\lambda,\mu) = 1 \;\Leftrightarrow\; 1/2 < \omega_1$.

Hence, if $d = 1$, then by Lemma 1.5,(11), we have $F(\phi_8) < 1$. Therefore we have $d \ge 2$. So we have $\theta_g \ne 1+3\lambda+2\mu, 2+5\lambda+3\mu$.

(i) The case $2\lambda + \mu < 0$. We have $\theta_g = 1+\lambda$ or $1+3\lambda+\mu$.

(ii) The case $2\lambda + \mu > 0$. We have $\theta_g = 1+\lambda$ or $1+2\lambda+\mu$. □

1.6. Examples

Voronoi-algorithm :

Let K be a cubic algebraic number field of negative discriminant and let $\mathcal{R}$ be a reduced lattice of K. We define the increasing chain of the minimal points of $\mathcal{R}$ by :

$\theta_0 = 1, \quad \theta_{k+1} = \min\{\gamma \in \mathcal{R}; \theta_k < \gamma, F(\theta_k) > F(\gamma)\}$ if $k \ge 0$.

Then θ_{k+1} is the minimal point adjacent to θ_k in $\mathcal{R}$.

Let $\mathcal{O}_K$ be the ring of integers in K and $\mathcal{R} = \mathcal{O}_K$. By Voronoi we know that the previous chain is of purely periodic form :

$1 = \theta_0,\ \theta_1, \ldots,\ \theta_{\ell-1},\ \epsilon,\ \epsilon\theta_1, \ldots,\ \epsilon\theta_{\ell-1}, \ldots,$

where ℓ denotes the period length and $\epsilon(>1)$ is the fundamental unit of $\mathcal{O}_K$.
To calculate such a sequence, it is sufficient to know how to find the minimal point adjacent to 1 in a lattice $\mathcal{R}$.
Indeed, let $\theta_g^{(1)}$ be the minimal point adjacent to 1 in $\mathcal{R}_1 = \mathcal{O}_K = \langle 1, \beta, \gamma\rangle$ and $\theta_1 = \theta_g^{(1)}$.
(i) We choose an appropriate point $\theta_h^{(1)}$ so that $\{1, \theta_g^{(1)}, \theta_h^{(1)}\}$ is a basis of $\mathcal{R}_1$.
(ii) Let $\mathcal{R}_2 = \dfrac{1}{\theta_g^{(1)}}\mathcal{R}_1$, then $\mathcal{R}_2$ is a reduced lattice. $\theta_g^{(2)}$ is the minimal point adjacent to 1 in $\mathcal{R}_2 = \dfrac{1}{\theta_g^{(1)}}\mathcal{R}_1 = \langle 1, 1/\theta_g^{(1)}, \theta_h^{(1)}/\theta_g^{(1)}\rangle$, is equivalent to $\theta_2 = \theta_1\theta_g^{(2)} = \theta_g^{(1)}\theta_g^{(2)}$ being the minimal point adjacent to θ_1 in $\mathcal{R}_1$.
This process can be continued by induction.

Example 1.1. Let $K = \mathbb{Q}(\theta)$ be a cubic number field defined by $\theta^3 - 7\theta - 12 = 0$ $(\theta = 3.2669)$. Then $\mathcal{R}_8 = \langle 1, -2 + \frac{1}{6}\theta + \frac{1}{6}\theta^2, 2 + \frac{2}{3}\theta - \frac{1}{3}\theta^2\rangle = \langle 1, \lambda, \mu\rangle$.
It is easily seen that $0 < \lambda < 1, 0 < \mu < 1$.
Since $\mathcal{R}_8$ is a reduced lattice, we have $a = F(\mu) > 1$.
$Y_\theta = \frac{1}{2}(T_{K/\mathbb{Q}}\theta - \theta) = -\frac{1}{2}\theta,\ Y_{\theta^2} = \frac{1}{2}(T_{K/\mathbb{Q}}\theta^2 - \theta^2) = \frac{1}{2}(14 - \theta^2)$.
$X_\theta = \frac{1}{2}(3\theta - T_{K/\mathbb{Q}}\theta) = \frac{3}{2}\theta,\ X_{\theta^2} = \frac{1}{2}(3\theta^2 - T_{K/\mathbb{Q}}\theta^2) = \frac{1}{2}(3\theta^2 - 14)$.
$X_\mu = X_{2+\frac{2}{3}\theta-\frac{1}{3}\theta^2} = \frac{2}{3}X_\theta - \frac{1}{3}X_{\theta^2} = \frac{7}{3} + \theta - \frac{1}{2}\theta^2 > 0$,
$X_\lambda - X_\mu = -\frac{7}{2} - \frac{3}{4}\theta + \frac{3}{4}\theta^2 > 0$.
$Y_\mu = Y_{2+\frac{2}{3}\theta-\frac{1}{3}\theta^2} = 2 + \frac{2}{3}Y_\theta - \frac{1}{3}Y_{\theta^2} = \frac{1}{6}(-2 - 2\theta + \theta^2),\ 0 < Y_\mu < \frac{1}{2}$.
$Y_\lambda = \frac{1}{12}(-10 - \theta - \theta^2)$. $\omega_1(\lambda, \mu) = \dfrac{\theta - 1}{2(\theta + 2)},\ 0 < \omega_1 < 1$.
$\omega_2(\lambda, \mu) = -\frac{1}{12}(-10 - \theta - \theta^2) - \dfrac{\theta - 1}{2(\theta + 2)} \times \frac{1}{6}(-2 - 2\theta + \theta^2) = \frac{1}{4}(\theta^2 - 3),\ [\omega_2] = 1$.
$F([\omega_2] + \lambda) = F(1 + \lambda) = 1 + \frac{1}{2}(\theta - 3) > 1$.
$F([\omega_2] + \lambda + \mu) = F(1 + \lambda + \mu) = 2 - 5\theta + \theta^2 + \dfrac{50}{\theta} > 1$.
$F([\omega_2] + 1 + \lambda) = F(2 + \lambda) = F(\frac{1}{6}\theta + \frac{1}{6}\theta^2) = \frac{1}{3\theta^2}(12 + \theta - \theta^2) < 1$.
Therefore, by Theorem 1.5A,(3), we have $\theta_g = [\omega_2] + 1 + \lambda = 2 + \lambda$.

Example 1.2. Let $K = \mathbb{Q}(\theta)$ be a cubic number field defined by $\theta^3 - 2\theta - 111 = 0$ $(\theta = 4.9445)$. Then

$\mathcal{R}_7 = \langle 1, (-71 + 15\theta + \theta^2)/98, (-61 - 23\theta + 5\theta^2)/196\rangle = \langle 1, \lambda, \mu\rangle$.

It is easily seen that $0 < \lambda < 1, \mu < 0$.

Since $\mathcal{R}_7$ is a reduced lattice, we have $a = F(\mu) > 1$.

$X_\theta = \frac{3}{2}\theta,\ X_{\theta^2} = \frac{1}{2}(3\theta^2 - 4)$.

$X_\mu = \frac{1}{2c}(15\theta^2 - 69\theta - 20) = 0.0141 > 0\ (c = 196)$.

$X_\lambda - X_\mu = \frac{1}{2c}(-9\theta^2 + 159\theta + 12) = 1.4748 > 0$.

$Y_\mu = \frac{1}{2c}(-5\theta^2 + 23\theta - 102) = -0.2819,\ 0 < |Y_\mu| < \frac{1}{2}$.

$Y_\lambda = \frac{1}{2\times 98}(-\theta^2 - 15\theta - 138) = \frac{1}{c}(-\theta^2 - 15\theta - 138) = -1.2072$. $\omega_1(\lambda,\mu) = \frac{-2\theta + 30}{5\theta + 23} = 0.4214$,

$0 < \omega_1 < 1$. $\omega_2(\lambda,\mu) = -Y_\lambda - \omega_1 Y_\mu = 1.2072 - 0.4214 \times -0.2819,\ [\omega_2] = 1$.

(1) $N_{K/\mathbb{Q}}(x + y\theta + z\theta^2) =$

$x^3 + 2\times 2x^2z - 2xy^2 - 3\times 111xyz + 2^2xz^2 + 111y^3 - 2\times 111yz^2 + 111^2z^3$.

(a) By (1), $F(\phi_1) = F([\omega_2] + \lambda) = F(\frac{1}{98}(27 + 15\theta + \theta^2))$

$= \frac{1}{98^2}F(27 + 15\theta + \theta^2) = \frac{1}{98^2}\frac{N_{K/\mathbb{Q}}(27 + 15\theta + \theta^2)}{27 + 15\theta + \theta^2}$

$= \frac{1}{98^2}\frac{259308}{27 + 15\theta + \theta^2} = 0.2149 < 1$.

(b) $\lambda + \mu = \frac{1}{c}(7\theta^2 + 7\theta - 203) = \frac{1}{c} \times 2.7480 > 0$.

(c) By (1), $F(\phi_2) = F([\omega_2] + \lambda + \mu) = F(\frac{1}{c}(-7 + 7\theta + 7\theta^2))$

$= \frac{1}{c^2}F(-7 + 7\theta + 7\theta^2) = \frac{1}{c^2}\frac{N_{K/\mathbb{Q}}(-7 + 7\theta + 7\theta^2)}{-7 + 7\theta + 7\theta^2}$

$= \frac{1}{c^2}\frac{4302592}{-7 + 7\theta + 7\theta^2} = 0.5635 < 1$.

Therefore, by Theorem 1.7A,(1),(ii-b), we have $\theta_g = \phi_2$.

Example 1.3. Let $K = \mathbb{Q}(\theta)$ be a cubic number field defined by

$\theta^3 - 77\theta - 513 = 0\ (\theta = 11.1002)$. Then

$\mathcal{R}_{39} = \langle 1, (-674 - 28\theta + 9\theta^2)/613, (1205 + 121\theta - 17\theta^2)/613\rangle = \langle 1, \lambda, \mu\rangle$.

It is easily seen that $0 < \lambda < 1, 0 < \mu < 1$.

Since $\mathcal{R}_{39}$ is a reduced lattice, we have $a = F(\mu) > 1$.

$X_\theta = \frac{3}{2}\theta,\ X_{\theta^2} = \frac{1}{2}(3\theta^2 - 154)$.

$X_\mu = \frac{1}{2c}(-51\theta^2 + 363\theta + 2618) = \frac{1}{2c} \times 363.4361 > 0\ (c = 613)$.

$X_\lambda - X_\mu = \frac{1}{2c}(78\theta^2 - 457\theta - 4004) = \frac{1}{2c} \times 533.9349 > 0$.

$Y_\mu = \frac{1}{2c}(17\theta^2 - 121\theta - 208) = 0.4433,\ 0 < Y_\mu < \frac{1}{2}$.

$Y_\lambda = \frac{1}{2c}(-9\theta^2 + 28\theta + 38) = -0.6200$. $\omega_1(\lambda,\mu) = \frac{9\theta + 28}{17\theta + 121} = 0.4129$,

$0 < \omega_1 < 1$. $\omega_2(\lambda,\mu) = -Y_\lambda - \omega_1 Y_\mu = 0.6200 - 0.4129 \times 0.4433$, $[\omega_2] = 0$.

(a) $\phi_2 = \lambda + \mu = \dfrac{1}{c}(-8\theta^2 + 93\theta + 521) = 0.9259 < 1$.

(b) $2\lambda + \mu = \dfrac{1}{c}(\theta^2 + 65\theta - 143) = 1.1447 > 1$.

(1) $N_{K/\mathbb{Q}}(x + y\theta + z\theta^2) =$

$x^3 + 2 \times 77x^2z - 77xy^2 - 3 \times 513xyz + 77^2xz^2 + 513y^3 - 77 \times 513yz^2 + 513^2z^3$.

(c) By (1), $F(\phi_6) = F([\omega_2] + 1 + \lambda) = F(\dfrac{1}{c}(-61 - 28\theta + 9\theta^2))$

$= \dfrac{1}{c^2}F(-61 - 28\theta + 9\theta^2) = \dfrac{1}{c^2}\dfrac{N_{K/\mathbb{Q}}(-61 - 28\theta + 9\theta^2)}{-61 - 28\theta + 9\theta^2}$

$= \dfrac{1}{c^2}\dfrac{225837169}{-61 - 28\theta + 9\theta^2} = 0.8153 < 1$.

(d) By (1), $F(2\lambda + \mu) = \dfrac{1}{c^2}F(\theta^2 + 65\theta - 143)$

$= \dfrac{1}{c^2}\dfrac{N_{K/\mathbb{Q}}(\theta^2 + 65\theta - 143)}{\theta^2 + 65\theta - 143} = \dfrac{1}{c^2}\dfrac{198781801}{\theta^2 + 65\theta - 143} = 0.7538 < 1$.

Therefore, by Theorem 1.5B,(3),(ii-b), we have $\theta_g = 2\lambda + \mu$.

Example 1.4. (Williams and Dueck [35,p.690])

Let $K = \mathbb{Q}(\theta)$ be a cubic number field defined by $\theta^3 - 68781 = 0$

($\theta = 40.97221992$). Then $\mathcal{R}_{2307} = \langle 1, \phi, \psi \rangle$

$= \langle 1, (-72036 + 1809\theta + 2\theta^2)/126539, (117574 - 2668\theta + 67\theta^2)/126539 \rangle$

$= \langle 1, \phi, \psi - 1 \rangle = \langle 1, (-72036 + 1809\theta + 2\theta^2)/126539, (-8965 - 2668\theta + 67\theta^2)/126539 \rangle$

$= \langle 1, \lambda, \mu \rangle$. $0 < \lambda < 1$, $\mu < 0$. $0 < X_\mu < X_\lambda$.

Since $\mathcal{R}_{2307}$ is a reduced lattice, we have $a = F(\mu) > 1$.

$\omega_1(\lambda,\mu) = \dfrac{-2\theta + 1809}{67\theta + 2668}$. $Y_\lambda = -\dfrac{1}{2c}(2\theta^2 + 1809\theta + 144072)$ $(c = 126539)$.

$Y_\mu = -\dfrac{1}{2c}(67\theta^2 - 2668\theta + 17930)$.

$\omega_1 = 0.31904891$. $Y_\lambda = -0.87541450$. $Y_\mu = -0.08333592$.

$\omega_2 = 0.90200274$. Hence $[\omega_2] = 0$, $\phi_1 = [\omega_2] + \lambda = \lambda < 1$.

(1) $N_{K/\mathbb{Q}}(x + y\theta + z\theta^2) = x^3 - 3 \times 68781xyz + 68781y^3 + 68781^2z^3$.

(a) By (1), $F(\phi_6) = F([\omega_2] + 1 + \lambda) = F(1 + \lambda) = F(\dfrac{1}{c}(54503 + 1809\theta + 2\theta^2))$

$= \dfrac{1}{c^2}F(54503 + 1809\theta + 2\theta^2) = \dfrac{1}{c^2}\dfrac{N_{K/\mathbb{Q}}(54503 + 1809\theta + 2\theta^2)}{54503 + 1809\theta + 2\theta^2}$

$= \dfrac{1}{c^2}\dfrac{528431935430042}{54503 + 1809\theta + 2\theta^2} = 0.25005464 < 1$.

(b) By (1), $F(1 + 2\lambda + \mu) = F(\dfrac{-26498 + 950\theta + 71\theta^2}{c})$

$= \dfrac{1}{c^2}F(-26498 + 950\theta + 71\theta^2) = \dfrac{1}{c^2}\dfrac{N_{K/\mathbb{Q}}(-26498 + 950\theta + 71\theta^2)}{-26498 + 950\theta + 71\theta^2}$

$= \dfrac{1}{c^2}\dfrac{2102375149688779}{-26498 + 950\theta + 71\theta^2} = 0.99760062 < 1$.

(c) By (1), $F(\phi_8) = F(1+\lambda+\mu) = F(\dfrac{45538-859\theta+69\theta^2}{c})$
$= \dfrac{1}{c^2}F(45538-859\theta+69\theta^2) = \dfrac{1}{c^2}\dfrac{N_{K/\mathbb{Q}}(45538-859\theta+69\theta^2)}{45538-859\theta+69\theta^2}$
$= \dfrac{1}{c^2}\dfrac{2161892194231336}{45538-859\theta+69\theta^2} = 1.07007239 > 1.$

(d) Since $-153037+950\theta+71\theta^2 > 0,\ 2\lambda+\mu = \dfrac{-153037+950\theta+71\theta^2}{c} > 0.$

(e) Since $\lambda+\mu = \dfrac{-81001-859\theta+69\theta^2}{c} < 0$, we have $1+2\lambda+\mu < 1+\lambda$.

Therefore, by Theorem 1.7B,(2),(ii), we have $\theta_g = 1+2\lambda+\mu$.

2 A one-parameter family of cubic fields

Let $\mathbb{Z}$ be the set of rational integers,and let θ be the real root of the irreducible cubic polynomial

$$f(X) = X^3 - 3X - b^3\ ,\ b(\neq 0) \in \mathbb{Z}.$$

The discriminant of $f(X)$ is $D_f = -3^3(b^3-2)(b^3+2)$ and $D_f < 0$ provided $b \neq \pm 1$.
Let $K = \mathbb{Q}(\theta)$ be the cubic field formed by adjoining θ to the rationals $\mathbb{Q}$, and let $\mathbb{Z}_K$ be the ring of algebraic integers in K. The family of cubic fields were introduced by Ishida [17]. Ishida constructed an unramified cyclic extension, of degree 3^2, of K provided $b \equiv -1 (\mathrm{mod}\ 3^2)$.

In this chapter we shall consider this family from various points of view.

Remark 2.1. "$f(X) = X^3 - 3X + b^3$" in [17] is replaced by "$f(X) = X^3 - 3X - b^3$".

Remark 2.2. If $b \equiv \pm 1 (\mathrm{mod}\ 3)$, then K is of Eisenstein type with respect to 3 (cf. [17]).

2.1. Voronoi-algorithm expansion of the order $\mathbb{Z}[\theta]$

In this section we shall find Voronoi-algorithm expansion of the order $\mathbb{Z}[\theta]$ using the method in chapter 1. We need the following easily proved facts:

Proposition 2.1. For $\alpha = x + y\theta + z\theta^2 (x, y, z \in \mathbb{Q})$, we have
(i) $N_K(x + y\theta) = x^3 - 3xy^2 + b^3y^3$,
(ii) $x + y\theta + z\theta^2 = \dfrac{-xy + b^3z^2 + (-y^2 + xz + 3z^2)\theta}{-y + z\theta}\quad (z \neq 0)$,
(iii) $Y_\alpha = \dfrac{1}{2}(2x + 6z - y\theta - z\theta^2),\ X_\alpha = \dfrac{3}{2}(-2z + y\theta + z\theta^2)$.
(iv) For $\lambda = a_1 + a_2\theta + a_3\theta^2 (a_i \in \mathbb{Q}),\ \mu = b_1 + b_2\theta + b_3\theta^2 (b_i \in \mathbb{Q})$, we obtain
$\omega_1(\lambda, \mu) = -\dfrac{Z_\lambda}{Z_\mu} = -\dfrac{a_2 - a_3\theta}{b_2 - b_3\theta},\quad \omega_2(\lambda, \mu) = -a_1 - 3a_3 + a_2\theta + \omega_1(-b_1 - 3b_3 + b_2\theta)$.

Since $\theta^3 - 3\theta - b^3 = 0 \Rightarrow (-\theta)^3 - 3(-\theta) - (-b)^3 = 0$, we may assume that $b > 1$.
Since $f(b + \frac{1}{b^2}) = 3(1-b) + \dfrac{3(1-b)}{b^3} + \dfrac{1}{b^6} < 0$ and $f(b + \frac{1}{b}) = \dfrac{1}{b^3} > 0$, we get

$$b + \frac{1}{b^2} < \theta < b + \frac{1}{b}. \tag{1.1}$$

We consider the defining polynomial $g(X) = X^3 - 6X^2 + 9X - b^6$ of θ^2.
Since $g(b^2 + 2) = -3b^2 + 2 < 0$ and $g(b^2 + 3) = 3b^4 > 0$, we have

$$b^2 + 2 < \theta^2 < b^2 + 3. \tag{1.2}$$

(V-1) Let $\mathcal{R}_1 := \mathbb{Z}[\theta] = \langle 1, -3 + b\theta + \theta^2, \theta\rangle = \langle 1, \lambda_1, \mu_1\rangle$. From (1.1), we have

(a) $\mu_1 = \theta > 1$.

Since $a(\mu_1) = F(\theta) = \theta'\theta'' = \theta^2 - 3 > b^2 - 3 \geq 1$, we obtain

(b) $a(\mu_1) > 1$.

By Proposition 2.1,(iii), we get

(c) $b(\mu_1) = Y_{\mu_1} = -\dfrac{1}{2}\theta < 0$.

From (1.1), we have

(d) $a(\mu_1) = \theta^2 - 3 > \dfrac{1}{2}\theta^2 = 2(b(\mu_1))^2$.

Since $|b(\mu_1)| > 1$, we have

(e) $(b(\mu_1))^2 > |b(\mu_1)|$.

From (b),(d),(e), we obtain

(f) $a(\mu_1) > \max(1, 2(b(\mu_1))^2, 2|b(\mu_1)|)$.

Since $\omega_1(\lambda_1, \mu_1) = \theta - b$, we have $0 < \omega_1 < 1$. By Proposition 2.1,(iii), we get $Y_{\lambda_1} = -\dfrac{1}{2}(\theta^2 + b\theta)$. Hence, $\omega_2(\lambda_1, \mu_1) = -Y_{\lambda_1} - \omega_1 Y_{\mu_1} = \dfrac{1}{2}(\theta^2 + b\theta) - (\theta - b)(-\dfrac{1}{2}\theta) = \theta^2$. From (1.2), we obtain $[\omega_2] = [\theta] = b^2 + 2$.

Since $\phi_1 = [\omega_2] + \lambda = b^2 + 2 + (-3 + b\theta + \theta^2) = b^2 - 1 + b\theta + \theta^2 > \theta = \mu_1$, we get

(g) $\phi_1 > \mu_1$.

By Proposition 2.1,(ii), we have $\phi_1 = b^2 - 1 + b\theta + \theta^2 = \dfrac{b + 2\theta}{-b + \theta}$. By Proposition 2.1,(i), we have

$$F(\phi_1) = (N_K \frac{b + 2\theta}{-b + \theta})/\frac{b + 2\theta}{-b + \theta} = \frac{N_K(b + 2\theta)}{N_K(-b + \theta)} \times \frac{-b + \theta}{b + 2\theta}$$
$$= \frac{b^3 - 3 \cdot b \cdot 2^2 + b^3 \cdot 2^3}{(-b)^3 - 3(-b) \cdot 1^2 + b^3 \cdot 1^3} \times \frac{-b + \theta}{b + 2\theta} = \frac{9b^3 - 12b}{3b} \times \frac{-b + \theta}{b + 2\theta}$$
$$= (3b^2 - 4) \times \frac{-b + \theta}{b + 2\theta} = (-b + \theta) \times \frac{3b^2 - 4}{b + 2\theta} < 1.$$ Hence,

(h) $F(\phi_1) < 1$.

Therefore, by Theorem 1.6A,(1),(i) and Remark 1.5, we obtain $\theta_g^{(1)} = \phi_1, \phi_3$ or ϕ_4, where $\theta_g^{(1)}$ is the minimal point adjacent to 1 of $\mathcal{R}_1$.

$\phi_4 = [\omega_2] - 1 + \lambda_1 = b^2 + 2 - 1 + (-3 + b\theta + \theta^2) = b^2 - 2 + b\theta + \theta^2 = \dfrac{2b + \theta}{-b + \theta}$.

$$F(\phi_4) = (N_K \frac{2b + \theta}{-b + \theta})/\frac{2b + \theta}{-b + \theta} = \frac{N_K(2b + \theta)}{N_K(-b + \theta)} \times \frac{-b + \theta}{2b + \theta}$$
$$= \frac{(2b)^3 - 3 \cdot 2b \cdot 1^2 + b^3 \cdot 1^3}{(-b)^3 - 3(-b) \cdot 1^2 + b^3 \cdot 1^3} \times \frac{-b + \theta}{2b + \theta} = \frac{9b^3 - 6b}{3b} \times \frac{-b + \theta}{2b + \theta}$$
$$= (3b^2 - 2) \times \frac{-b + \theta}{2b + \theta} = (-b + \theta) \times \frac{3b^2 - 2}{2b + \theta} < 1.$$

$\phi_3 = [\omega_2] + \lambda_1 - \mu_1 = b^2 + 2 + (-3 + b\theta + \theta^2) - \theta^2 = b^2 - 1 + b\theta.$

$F(\phi_3) = F(b^2 - 1 + b\theta) = \dfrac{N_K(b^2 - 1 + b\theta)}{b^2 - 1 + b\theta} = \dfrac{(b^2-1)^3 - 3\cdot(b^2-1)\cdot b^2 + b^3 \cdot b^3}{b^2 - 1 + b\theta}$

$= \dfrac{(b^2-1)^3 - 3\cdot(b^2-1)\cdot b^2 + b^3 \cdot b^3}{b^2 - 1 + b\theta} = \dfrac{2b^6 - 6b^4 + 6b^2 - 1}{b^2 - 1 + b\theta} = \dfrac{A}{B}.$

Since $A - B = 2b^4(b^2-3) + b(5b - \theta) > 0$, we have $A > B$. Hence, $F(\phi_3) > 1$.

Therefore, we have

(i) $\theta_g^{(1)} = \phi_4$. $N_{K/\mathbb{Q}}\theta_g^{(1)} = 3b^2 - 2 \neq 1$.

(V-2) Let $\mathcal{R}_2 := \dfrac{1}{\theta_g^{(1)}}\mathcal{R}_1 = \dfrac{1}{b^2 - 2 + b\theta + \theta^2}\langle 1, -3 + b\theta + \theta^2, \theta\rangle$

$= \dfrac{1}{b^2 - 2 + b\theta + \theta^2}\langle 1, b^2 - 2 + b\theta + \theta^2, \theta\rangle = \langle 1, \dfrac{1}{b^2 - 2 + b\theta + \theta^2}, \dfrac{\theta}{b^2 - 2 + b\theta + \theta^2}\rangle$

$= \langle 1, \dfrac{1}{3b^2-2}(-b^2 + 1 + 2b\theta - \theta^2), \dfrac{1}{3b^2-2}\{-b^3 - (b^2+2)\theta + 2b\theta^2\}\rangle$

$= \langle 1, \lambda_2, \nu_2\rangle = \langle 1, \lambda_2, \mu_2 = \nu_2 - (b-1)\lambda_2\rangle$

$= \langle 1, \dfrac{1}{3b^2-2}(-b^2 + 1 + 2b\theta - \theta^2), \dfrac{1}{3b^2-2}\{-b^2 - b + 1 - (3b^2 - 2b + 2)\theta + (3b-1)\theta^2\}\rangle.$

$0 < \lambda_2 = 1/\theta_g^{(1)} < 1$. From (1.1), (1.2),we get $\lambda_2 = \dfrac{1}{3b^2-2}A, A = -b^2 + 1 + 2b\theta - \theta^2$

$< -b^2 + 1 + 2b(b + \dfrac{1}{b}) - (b^2+2) = 1$. From this, we have $\lambda_2 = \dfrac{1}{3b^2-2}A < \dfrac{1}{3b^2-2} \leq \dfrac{1}{10}.$

From (1.1), (1.2),we have $B = -b^2 - b + 1 - (3b^2 - 2b + 2)\theta + (3b-1)\theta^2$

$> -b^2 - b + 1 - (3b^2 - 2b + 2)(b + \dfrac{1}{b^2}) + (3b-1)(b^2+2) = 5b - 6 + \dfrac{2}{b} - \dfrac{2}{b^2} > 0.$

From (1.1), (1.2),we obtain $B < -b^2 - b + 1 - (3b^2 - 2b + 2)(b + \dfrac{1}{b}) + (3b-1)(b^2+3) = 3b + 2 - \dfrac{2}{b}.$

Hence, $0 < \mu_2 = \dfrac{1}{3b^2-2} \times B < (3b + 2 - \dfrac{2}{b})/(3b^2-2) \leq 7/10 < 1$. Therefore, we see that

(a) $0 < \lambda_2 < 1/10,\ 0 < \mu_2 < 7/10.$

$b(\mu_2) = Y_{\mu_2} = \dfrac{1}{2(3b^2-2)}\{-2b^2 + 16b - 4 + (3b^2 - 2b + 2)\theta - (3b-1)\theta^2\}$

$= \dfrac{1}{2(3b^2-2)}C$. From (1.1), (1.2),we see that

(b) $C = -2b^2 + 16b - 4 + (3b^2 - 2b + 2)\theta - (3b-1)\theta^2$

$> -2b^2 + 16b - 4 + (3b^2 - 2b + 2)(b + \dfrac{1}{b^2}) - (3b-1)(b^2+3)$

$= -3b^2 + 9b + 2 - \dfrac{2}{b} + \dfrac{2}{b^2},$

(c) $C = -2b^2 + 16b - 4 + (3b^2 - 2b + 2)\theta - (3b-1)\theta^2$

$< -2b^2 + 16b - 4 + (3b^2 - 2b + 2)(b + \dfrac{1}{b}) - (3b-1)(b^2+2)$

$= -3b^2 + 16b - 4 + \dfrac{2}{b} < 0\ (b \geq 5).$

Hence, from (b),(c), we obtain $-3b^2 + 9b + 2 - \dfrac{2}{b} + \dfrac{2}{b^2} < C < -3b^2 + 16b - 4 + \dfrac{2}{b} < 0\ (b \geq 5).$

From this, we have $|-3b^2 + 9b + 2 - \dfrac{2}{b} + \dfrac{2}{b^2}| = 3b^2 - 9b - 2 + \dfrac{2}{b} - \dfrac{2}{b^2} < 3b^2 - 2.$

Hence, $|C| < 3b^2 - 2$. So we have $-1/2 < b(\mu_2) = Y_{\mu_2} = \dfrac{1}{2(3b^2-2)}C < 0$. Therefore, we see that

(d) $-1/2 < b(\mu_2) = Y_{\mu_2} < 0$.

For $b = 2, 3, 4$ we can easily check (d). By Proposition 2.1,(iv), we obtain
$\omega_1 = -\dfrac{2b+\theta}{-(3b^2-2b+2)-(3b-1)\theta} = \dfrac{2b+\theta}{3b^2-2b+2+(3b-1)\theta}$. It is easily seen that $0 < \omega_1 < 1$.
From $(1.1), (1.2)$,we have

(e) $\omega_1 = \dfrac{2b+\theta}{3b^2-2b+2+(3b-1)\theta} > \dfrac{2b+b+1/b^2}{3b^2-2b+2+(3b-1)(b+1/b)} = \dfrac{3b+1/b^2}{6b^2-3b+5-1/b}$
$> \dfrac{1}{2b}$.

$Y_{\lambda_2} = \dfrac{1}{2(3b^2-2)}(-2b^2-4-2b\theta+\theta^2)$. It is easily seen that $Y_{\lambda_2} < 0$. From $Y_{\lambda_2} < 0$ and $Y_{\mu_2} < 0$, we have $\omega_2 > 0$. By Proposition 2.1,(iv), we get
$\omega_2 = \dfrac{1}{3b^2-2}\{b^2+2+2b\theta+\dfrac{2b+\theta}{3b^2-2b+2+(3b-1)\theta}(b^2-8b+2-(3b^2-2b+2)\theta)\} = \dfrac{1}{3b^2-2}\times D$.
From $(1.1), (1.2)$ and (e), we obtain

$$
\begin{aligned}
D &= b^2+2+2b\theta+\omega_1(b^2-8b+2-(3b^2-2b+2)\theta) \\
&< b^2+2+2b(b+\frac{1}{b})+\omega_1(b^2-8b+2-(3b^2-2b+2)\theta) \\
&< b^2+2+2b(b+\frac{1}{b})+\frac{1}{2b}(b^2-8b+2-(3b^2-2b+2)\theta) \\
&< b^2+2+2b(b+\frac{1}{b})+\frac{1}{2b}(b^2-8b+2-(3b^2-2b+2)(b+\frac{1}{b^2})) \\
&= 3b^2+4+\frac{1}{2b}(-3b^3+3b^2-10b-1+2/b-2/b^2) \\
&= 3b^2+4-\frac{1}{2b}(3b^3-3b^2+10b+1-2/b+2/b^2) \\
&< 3b^2+4-\frac{1}{2b}(3b^3-3b^2+10b) = 3b^2+4-\frac{3}{2}b^2+\frac{3}{2}b-5 = \frac{3}{2}b^2+\frac{3}{2}b-1
\end{aligned}
$$

$< 3b^2-2$. Hence, $D < 3b^2-2$. From this, we get $\omega_2 = \dfrac{1}{3b^2-2}\times D < 1$. Therefore, we see that

(f) $[\omega_2] = 0$.

$$
\begin{aligned}
\phi_6 &= [\omega_2]+1+\lambda_2 = 1+\lambda_2 = \frac{1}{3b^2-2}(2b^2-1+2b\theta-\theta^2) \\
&= \frac{1}{3b^2-2}\frac{-3b^3+2b+(-6b^2+4)\theta}{-2b-\theta} = \frac{1}{3b^2-2}\frac{b(3b^2-2)+2(3b^2-2)\theta}{2b+\theta} = \frac{b+2\theta}{2b+\theta}.
\end{aligned}
$$

$$
\begin{aligned}
F(\phi_6) &= (N_K\frac{b+2\theta}{2b+\theta})/\frac{b+2\theta}{2b+\theta} = (N_K\frac{b+2\theta}{2b+\theta})\times\frac{2b+\theta}{b+2\theta} \\
&= \frac{9b^3-12b}{9b^3-6b}\times\frac{2b+\theta}{b+2\theta} = \frac{3b^2-4}{3b^2-2}\times\frac{2b+\theta}{b+2\theta} < \frac{3b^2-4}{3b^2-2}\times\frac{2b+b+1/b}{b+2(b+1/b^2)}
\end{aligned}
$$

$= \dfrac{3b^2-4}{3b^2-2}\times\dfrac{3b+1/b}{3b+2/b^2} < 1$. By Proposition 2.1,(iii), we get $X_{\mu_2} = \dfrac{3}{2(3b^2-2)}\{-2(3b-1) - (3b^2-2b+2)\theta+(3b-1)\theta^2\}$.

From $(1.1), (1.2)$, we see that

$$
-2(3b-1)-(3b^2-2b+2)\theta+(3b-1)\theta^2 > -2(3b-1)-(3b^2-2b+2)(b+\frac{1}{b})+(3b-1)(b^2+2)
$$

$= b^2 - 5b + 2 - \dfrac{2}{b} > 0$ $(b \geq 5)$. Hence, $X_{\mu_2} > 0$.
For $b = 2, 3, 4$ we can easily checked $X_{\mu_2} > 0$. We have
$X_{\lambda_2} = \dfrac{3}{2(3b^2-2)}\{2+2b\theta-\theta^2\} > 0,\ X_{\lambda_2} - X_{\mu_2} = \dfrac{3}{2(3b^2-2)}\{6b+(3b^2+2)\theta+3b\theta^2\} > 0$. Hence,
(g) $0 < X_{\mu_2} < X_{\lambda_2}$.
From (a) and (f), we get $\phi_2 = [\omega_2] + \lambda_2 + \mu_2 = \lambda_2 + \mu_2 < \dfrac{1}{10} + \dfrac{7}{10} < 1$. Therefore, by Theorem 1.5B,(3),(i), we have $\theta_g^{(2)} = \phi_6$. $N_{K/\mathbb{Q}}\theta_g^{(1)}\theta_g^{(2)} = (3b^2-2) \times \dfrac{3b^2-4}{3b^2-2} = 3b^2 - 4 \neq 1$.

Since the follwing procedures ((V-3) to (V-5)) are similar to (V-1),(V-2), we will present only these final results.

(V-3) $\mathcal{R}_3 = \dfrac{1}{\theta_g^{(2)}}\mathcal{R}_2 = \langle 1, \lambda_3, \mu_3\rangle$
$= \langle 1, \dfrac{1}{3b^2-4}\{-2b^3+b^2+4+(b^2+b-2)\theta+(b-2)\theta^2\}, \dfrac{1}{3b^2-4}(2b^2-8-b\theta+2\theta^2)\rangle$.
(a) $0 < \mu_3 < 1,\ 0 < b(\mu_3) = Y_{\mu_3} = \dfrac{1}{2(3b^2-4)}(4b^2-4+b\theta-2\theta^2) < 1/2$.
(b) $0 < \omega_1 = \dfrac{b^2+b-2-(b-2)\theta}{b+2\theta} < 1$.
(c) $[\omega_2] = b - 1$.
(d) $\phi_1 = [\omega_2] + \lambda_3 = \dfrac{(b-2)^2+2(b-1)\theta}{b^2+b-2-(b-2)\theta} > 1$.
(e) $F(\phi_1) = N_K(\dfrac{-(b-2)^2-2(b-1)\theta}{-b^2-b+2+(b-2)\theta}) \times \dfrac{b^2+b-2-(b-2)\theta}{(b-2)^2+2(b-1)\theta}$
$= \dfrac{3b^3-6b^2+6b-4}{3b^2-4} \times \dfrac{b^2+b-2-(b-2)\theta}{(b-2)^2+2(b-1)\theta} < 1$.
By Theorem 1.5A,(1),(ii), we obtain $\theta_g^{(3)} = \phi_1$ or ϕ_5.
(f) $\phi_5 = [\omega_2] - 1 + \lambda_3 - \mu_3 = \dfrac{-b^2+12b-10-(5b-9)\theta}{-b^2-2b+2+(b-4)\theta}$.
(g) $F(\phi_5) = N_K(\dfrac{-b^2+12b-10-(5b-9)\theta}{-b^2-2b+2+(b-4)\theta}) \times \dfrac{-b^2-2b+2+(b-4)\theta}{-b^2+12b-10-(5b-9)\theta}$
$= \dfrac{21b^3-66b^2-3b+65}{3b^2-4} \times \dfrac{b^2+2b-2-(b-4)\theta}{b^2-12b+10+(5b-9)\theta} > 1$ $(b \geq 3)$.
$\phi_5 = \dfrac{1}{4}(-4+3\theta-\theta^2) < 0$ $(b = 2)$.
Therefore, we see that $\theta_g^{(3)} = \phi_1$. $N_{K/\mathbb{Q}}\theta_g^{(1)}\theta_g^{(2)}\theta_g^{(3)} = (3b^2-4) \times \dfrac{3b^3-6b^2+6b-4}{3b^2-4} = 3b^3 - 6b^2 + 6b - 4 \neq 1$.

(V-4) $\mathcal{R}_4 = \dfrac{1}{\theta_g^{(3)}}\mathcal{R}_3 = \langle 1, \lambda_4, \mu_4\rangle = \langle 1, \dfrac{1}{3b^3-6b^2+6b-4}\{-b^3+b^2-2b+2-(b^2-4b+4)\theta + (2b-2)\theta^2\}, \dfrac{1}{3b^3-6b^2+6b-4}\{-b^3+3b^2-2b+2+(2b^2-3b+2)\theta-b\theta^2\}\rangle$.
(a) $\mu_4 < 0$.

(b) $-1/2 < Y_{\mu_4} = \dfrac{1}{2(3b^3-6b^2+6b-4)}\{-2b^3+6b^2-10b+4-(2b^2-3b+2)\theta+b\theta^2\} < 0.$

(c) $X_{\mu_4} = \dfrac{3}{2}\cdot\dfrac{1}{3b^3-6b^2+6b-4}\{2b+(2b^2-3b+2)\theta-b\theta^2\} > 0,$

$X_{\lambda_4} = \dfrac{3}{2}\cdot\dfrac{1}{3b^3-6b^2+6b-4}\{-4b+4-(b^2-4b+4)\theta+(2b-2)\theta^2\} > 0,$

$X_{\lambda_4}-X_{\mu_4} = \dfrac{3}{2}\cdot\dfrac{1}{3b^3-6b^2+6b-4}\{-6b+4+(-3b^2+7b-6)\theta+(3b-2)\theta^2\} > 0.$

(d) $0 < X_{\mu_4} < X_{\lambda_4}$.

(e) $0 < \omega_1 = \dfrac{b^2-4b+4+(2b-2)\theta}{2b^2-3b+2+b\theta} < 1.$

(f) $0 < \omega_2 < 1,\ [\omega_2] = 0.$

(g) $\phi_6 = 1+\lambda_4 = \dfrac{2b^2-3b+2+b\theta}{b^2-4b+4+(2b-2)\theta}.$

(h) $F(\phi_6) = N_K(\dfrac{2b^2-3b+2+b\theta}{b^2-4b+4+(2b-2)\theta}) \times \dfrac{b^2-4b+4+(2b-2)\theta}{2b^2-3b+2+b\theta}$

$= \dfrac{3b^3-6b^2+6b-2}{3b^3-6b^2+6b-4} \times \dfrac{b^2-4b+4+(2b-2)\theta}{2b^2-3b+2+b\theta} < 1.$

(i) $\phi_8 = [\omega_2]+1+\lambda_4+\mu_4 = \dfrac{-b-2\theta}{-b^2-b+2+(b-2)\theta}.$

(j) $F(\phi_8) = N_K(\dfrac{-b-2\theta}{-b^2-b+2+(b-2)\theta}) \times \dfrac{-b^2-b+2+(b-2)\theta}{-b-2\theta}$

$= \dfrac{3b}{3b^3-6b^2+6b-4} \times \dfrac{b^2+b-2-(b-2)\theta}{b+2\theta} < 1.$

Therefore, by Theorem 1.7B,(1), we obtain $\theta_g^{(4)} = \phi_8$. $N_K(\theta_g^{(1)}\theta_g^{(2)}\theta_g^{(3)}\theta_g^{(4)}) = (3b^3-6b^2+6b-4) \times \dfrac{3b}{3b^3-6b^2+6b-4} = 3b \neq 1.$

(V-5) $\mathcal{R}_5 = \dfrac{1}{\theta_g^{(4)}}\mathcal{R}_4 = \langle 1,\lambda_5,\mu_5\rangle = \langle 1, \dfrac{1}{3b}(-2b^2+3b-3+b\theta+\theta^2), \dfrac{1}{3b}(-b^2+3+2b\theta-\theta^2)\rangle.$

(a) $0 < \mu_5 < 1, \lambda_5 > 0, 0 < \omega_1 = \dfrac{-b+\theta}{2b+\theta} < 1.$

(b) $\mu_5 = \dfrac{-b^2+2+b\theta}{2b+\theta},\ a(\mu_5) = F(\mu_5) = N_K(\dfrac{-b^2+2+b\theta}{2b+\theta}) \times \dfrac{2b+\theta}{-b^2+2+b\theta}$

$= (3b^2-4) \times \dfrac{2b+\theta}{-b^2+2+b\theta} > 1.$

(c) $b(\mu_5) = Y_{\mu_5} = \dfrac{1}{2b} \times \dfrac{-b^3-(2b^2-1)\theta}{2b+\theta} < 0.$

(d) $a(\mu_5) > \max(1, 2(b(\mu_5))^2, 2|b(\mu_5)|).$

(e) $[\omega_2] = b-1.$

(f) $\phi_1 = [\omega_2]+\lambda_5 = b-1+\lambda_5 = \dfrac{1}{-b+\theta}.$

(g) $F(\phi_1) = N_K(\dfrac{1}{-b+\theta}) \times (-b+\theta) = \dfrac{1}{3b} \times (-b+\theta) < 1.$

Therefore, by Theorem 1.5A,(1),(i) and Remark 1.4, we have $\theta_g^{(5)} = \phi_1, \phi_3$ or ϕ_4.

(h) $\phi_3 = [\omega_2]+\lambda_5-\mu_5 = \dfrac{2b^2-2+b\theta}{b+2\theta}.$

(i) $F(\phi_3) = N_K(\dfrac{2b^2-2+b\theta}{b+2\theta}) \times \dfrac{b+2\theta}{2b^2-2+b\theta} = \dfrac{3b^4-6b^2+2}{3b} \times \dfrac{b+2\theta}{2b^2-2+b\theta} > 1.$

(j) $\phi_4 = [\omega_2] - 1 + \lambda_5 = \dfrac{b+1-\theta}{-b+\theta}$.

(k) $F(\phi_4) = N_K(\dfrac{b+1-\theta}{-b+\theta}) \times \dfrac{-b+\theta}{b+1-\theta} = \dfrac{3b^2-2}{3b} \times \dfrac{-b+\theta}{b+1-\theta} > 1$.

Therefore, we have $\theta_g^{(5)} = \phi_1$. $N_K(\theta_g^{(1)}\theta_g^{(2)}\theta_g^{(3)}\theta_g^{(4)}\theta_g^{(5)}) = 3b \times \dfrac{1}{3b} = 1$.

$\epsilon = \theta_g^{(1)}\theta_g^{(2)}\theta_g^{(3)}\theta_g^{(4)}\theta_g^{(5)}$

$= \dfrac{2b+\theta}{-b+\theta} \times \dfrac{b+2\theta}{2b+\theta} \times \dfrac{(b-2)^2+2(b-1)\theta}{b^2+b-2-(b-2)\theta} \times \dfrac{b+2\theta}{b^2+b-2-(b-2)\theta} \times \dfrac{1}{-b+\theta}$

$= b^4 - b^2 + 1 + (b^3+b)\theta + b^2\theta^2 = \dfrac{1}{b^2+1-b\theta} = \dfrac{1}{1-b(\theta-b)}$.

Therefore, we obtained the following Theorem:

Theorem 2.1. Let θ be the real root of the polynomial $f(X)$, $K = \mathbb{Q}(\theta)$, and $\mathcal{O} = \mathbb{Z}[\theta]$. Then

$$\epsilon = b^4 - b^2 + 1 + (b^3+b)\theta + b^2\theta^2 = \frac{1}{1-b(\theta-b)}(>1)$$

is the fundamental unit of $\mathcal{O}$ and Voronoi-algorithm expansion period length is $\ell = 5$.

2.2. Integral bases

In this section we refer to Voronoi's Theorem and Llorente and Nart [27](cf. [13]) to find integral bases. For our convenience we quote a part of Voronoi's Theorem which is well known as Theorem 2.2.

Theorem 2.2(cf. Section 17 in [9])**.** If δ is a primitive integer in a cubic field satisfying the equation $F(\delta) = \delta^3 - q\delta - n = 0$, and if there is no integer τ whose square divides q and whose cube divides n, then an integral basis of the field $\mathbb{Q}(\delta)$can be found as follows:
If the congruences $3 - q \equiv 0 (\mathrm{mod}\ 9), n + q - 1 \equiv 0 (\mathrm{mod}\ 27)$, $n - q + 1 \equiv 0 (\mathrm{mod}\ 27)$ are not satisfied and if the integer a is the greatest square factor of the discriminant $D_\delta (= D_F)$ of δ

for which the congruences $\begin{cases} F'(X) \equiv 0 (\mathrm{mod}\ a) \\ F(X) \equiv 0 (\mathrm{mod}\ a^2) \end{cases}$ have a solution t,

then $\{1, \delta, \dfrac{t^2 - q + t\delta + \delta^2}{a}\}$ is an integral basis of $\mathbb{Q}(\delta)$ and D_δ / a^2 is the discriminant of $\mathbb{Q}(\delta)$.

Theorem 2.3. Let $b(\neq 0) \in \mathbb{Z}$ and $f(\theta) = \theta^3 - 3\theta - b^3 = 0$. Let $K = \mathbb{Q}(\theta)$ and D_K be the discriminant of K.
Let $b^3 - 2 = 2^e \cdot 3^{g_1} \cdot k_1^2 \ell_1, b^3 + 2 = 2^e \cdot 3^{g_2} \cdot k_2^2 \ell_2$, where ℓ_1, ℓ_2 are squarefree, $GCD(k_1\ell_1, k_2\ell_2) = GCD(k_1\ell_1 k_2\ell_2, 2 \cdot 3) = 1$ and $e, g_1, g_2 = 0$ or 1.
(i) If $b \equiv \pm 1 (\text{mod } 3)$, then $\{1, \theta, \dfrac{t^2 - 3 + t\theta + \theta^2}{k_1 k_2}\}$is an integral basis of K,

where t is a solution of the following congruences $\begin{cases} X \equiv 1 (\text{mod } k_2) \\ X \equiv -1 (\text{mod } k_1) \end{cases}$.

(ii) If $b \equiv 0 (\text{mod } 3)$, then $\{1, \theta, \dfrac{t^2 - 3 + t\theta + \theta^2}{3k_1 k_2}\}$is an integral basis of K,

where t is a solution of the following congruences $\begin{cases} X \equiv 1 (\text{mod } k_2) \\ X \equiv -1 (\text{mod } k_1) \\ X \equiv 0 (\text{mod } 3) \end{cases}$.

Proof. At first, we note that $GCD(b^3 - 2, b^3 + 2) = 1$ or 2. Next, $e = 1$ if and only if b is even. If b is even, then $D_\theta / 2^2 \equiv 3 (\text{mod } 4)$.
Therefore, by Theorem 2(or 1) in [27], if e = 1, then $2^2 | D_K$.
According to Theorem 2.2, we must find the greatest square factor a of $3^g k_1^2 k_2^2 (g = 3$ or $4)$ such that the congruences

$$\begin{cases} f'(X) = 3(X-1)(X+1) \equiv 0 (\text{mod } a) \\ f(X) = X^3 - 3X - b^3 \equiv 0 (\text{mod } a^2) \end{cases} \text{ have a solution } t.$$

(i) The case $b \equiv \pm 1 (\text{mod } 3)$:
By Remark 2.2 and Theorem 2 in [27] we have $GCD(3, a) = 1$.
Let t be a solution of

the following congruences $\begin{cases} X \equiv 1 (\text{mod } k_2) \\ X \equiv -1 (\text{mod } k_1) \end{cases}$.

Then it is easily seen that the integer t satisfies the following congruences

$$\begin{cases} f'(X) = 3(X-1)(X+1) \equiv 0 (\text{mod } k_1 k_2) \\ f(X) = X^3 - 3X - b^3 \equiv 0 (\text{mod } k_1^2 k_2^2) \end{cases}. \text{ Therefore, we have that } a = k_1 k_2.$$

(ii) The case $b \equiv 0 (\text{mod } 3)$:

From Theorem 2 in [27] we have that $3||D_K$.

Let t be a solution of the following congruences

$\begin{cases} X \equiv 1 (\text{mod } k_2) \\ X \equiv -1 (\text{mod } k_1) \\ X \equiv 0 (\text{mod } 3) \end{cases}$. Then it is easily seen that the integer t satisfies

the following congruences

$\begin{cases} f'(X) = 3(X-1)(X+1) \equiv 0 (\text{mod } 3k_1k_2) \\ f(X) = X^3 - 3X - b^3 \equiv 0 (\text{mod } 3^2k_1^2k_2^2) \end{cases}$. Therefore, we have that $a = 3k_1k_2$. $\square$

2.3. Fundamental units

Lee and Spearman [25] proved the following Lemma 2.1. Here, we shall give another proof.

Lemma 2.1([25,Theorem 1.1])**.** The integer solutions (A, B, b) of the following diophantine system are $(0, -3, \pm 1), (-1, -1, 0), (3, 3, 0)$ and $(8, 17, \pm 3)$:

$$\begin{cases} A^2 - 2B = 3(b^2 + 1) & (1.1) \\ B^2 - 2A = 3(b^4 + b^2 + 1). & (1.2) \end{cases}$$

Proof. Without loss of generality, we may suppose $b \geq 0$.

Since $b^2 + 1 \equiv \pm 1 \bmod 3$, from (1.1) we have $B \neq 0$.

From $(1.1), (1.2)$, we have

$$B^2 - 2(2A^2 - 3)B + A^4 - 3A^2 + 6A + 9 = 0. \tag{1.3}$$

If $b = 0$, then from $(1.1), (1.2)$ we have only the following integer solusions :

$$(A, B, b) = (-1, -1, 0), (3, 3, 0).$$

If $A = -1, 0$ or 2, then from $(1.3), (1.1), (1.2)$ we have only the following integer solusions :

$$(A, B, b) = (0, -3, \pm 1), (-1, -1, 0).$$

Hence, we shall suppose $A \neq -1, 0, 2$ and $b \neq 0$.

The discriminant D_B of the quadratic equation (1.3) is

$$D_B = 3A(A+1)^2(A-2). \tag{1.4}$$

Hence, we have

$$D_B > 0 \quad \Leftrightarrow \quad A < 0 \text{ or } 2 < A. \tag{1.5}$$

Under the condition (1.5), we have

$$\begin{aligned} B \in Z \quad &\Leftrightarrow \quad \sqrt{D_B} = |A+1|\sqrt{3A(A-2)} \ \in \mathbb{Z} \\ &\Leftrightarrow \quad A(A-2) = 3C_1^2 \text{ for some } C_1(>0) \ \in \mathbb{Z} \\ &\Leftrightarrow \quad A^2 - 2A - 3C_1^2 = 0 \text{ for some } C_1(>0) \ \in \mathbb{Z}. \end{aligned}$$

From this and (1.1) we have $B = 2A^2 - 3 - 3C_1|A+1|$.

Next, we consider the quadratic equation

$$A^2 - 2A - 3C_1^2 = 0. \tag{1.6}$$

Since the discriminant D_A of (1.6) is $D_A = 1 + 3C_1^2$, we have

$$\begin{aligned} A \in Z \quad &\Leftrightarrow \quad 1 + 3C_1^2 = C_2^2 \text{ for some } C_2(>0) \ \in \mathbb{Z} \\ &\Leftrightarrow \quad C_2^2 - 3C_1^2 = 1 \text{ for some } C_2(>0) \ \in \mathbb{Z}. \end{aligned}$$

From this we have $A = 1 \pm C_2$. Note that the equation $C_2^2 - 3C_1^2 = 1$ has infinitely many integer solutions. Therefore, as a necessary condition, the integer solusion (A, B) of (1.3) is

$$\text{(I)} \quad \begin{cases} A = 1 + C_2(C_2 > 0) \\ B = 2A^2 - 3C_1A - 3C_1 - 3(C_1 > 0) \\ C_2^2 - 3C_1^2 = 1 \end{cases}$$

or

$$\text{(II)} \quad \begin{cases} A = 1 - C_2(C_2 > 0) \\ B = 2A^2 + 3C_1A + 3C_1 - 3(C_1 > 0) \\ C_2^2 - 3C_1^2 = 1 \end{cases} .$$

Now we shall consider the equation (1.1).

The case (I): (1.1) become

$$b^2 + (C_2 - C_1 + 1)^2 = (C_1 + 1)^2. \tag{1.7}$$

We may consider positive integer solutions of (1.7).

Hence, we can put

(Ia) $b = (u^2 - v^2)t, C_2 - C_1 + 1 = 2uvt, C_1 + 1 = (u^2 + v^2)t$,

or

(Ib) $b = 2uvt, C_2 - C_1 + 1 = (u^2 - v^2)t, C_1 + 1 = (u^2 + v^2)t$,

where u, v and t are positive integers such that $u > v, GCD(u, v) = 1, u \not\equiv v (\text{mod } 2)$.

The case (Ia): From $C_1 = (u^2+v^2)t-1, C_2 = t(u+v)^2-2$ and $C_2^2 - 3C_1^2 = 1$,
we have

$$t(u+v)^4 - (u+v)^2 - 6tuv(u+v)^2 + 6tu^2v^2 + 6uv = 0. \tag{1.8}$$

We put $u+v = X, uv = Y$, then (1.8) become

$$(X^2-6Y)(tX^2-1) = -6tY^2. \tag{1.9}$$

Since $GCD(X,Y) = 1$, we have $GCD(X^2-6Y, Y^2) = GCD(tX^2-1, t) = 1$.
From this and (1.9) we have

$$\begin{cases} X^2 - 6Y = -pt \\ tX^2 - 1 = qY^2 \end{cases} \tag{1.10}$$

where p and q are positive integers such that $pq = 6$.
From (1.10) we have

$$X^4 - 6X^2Y + 6Y^2 = -p. \tag{1.11}$$

From (1.11) we have

$$u^4 + v^4 - 2uv(u^2+v^2) = -p. \tag{1.12}$$

It is well known that the diophantine equation (1.12) has only finite solutions.
The case (Ib): From $C_1 = (u^2+v^2)t-1, C_2 = 2u^2t-2$ and $C_2^2 - 3C_1^2 = 1$,
we have

$$(u^2-3v^2)\{(u^2-3v^2)t-2\} = 12v^4t. \tag{1.13}$$

Since $GCD(u^2-3v^2, v) = 1, GCD((u^2-3v^2)t-2, t) = 1$ or 2, we have

(i) t:even$(t = 2t')\begin{cases} u^2 - 3v^2 = p't' \\ (u^2-3v^2)t - 2 = q'v^4 \end{cases}$,

(ii) t:odd$\begin{cases} u^2 - 3v^2 = pt \\ (u^2-3v^2)t - 2 = qv^4 \end{cases}$,

where p, q, p' and $q'$are positive integers such that $pq = 12, p'q' = 24$.
From (i),(ii) we have

$$u^4 - 6u^2v^2 - 3v^4 = p'(t{:}even),\ u^4 - 6u^2v^2 - 3v^4 = 2p\ (t{:}odd). \tag{1.14}$$

These diophantine equations have only finite solutions.
The case (II): As the process is almost the same as in the case (I),
we only mention the corresponding equations.

$$b^2 + (C_2 - C_1 - 1)^2 = (C_1 - 1)^2. \tag{1.7$'$}$$

(IIa) $b = (u^2 - v^2)t, C_2 - C_1 - 1 = 2uvt, C_1 - 1 = (u^2 + v^2)t,$

$$u^4 + v^4 - 2uv(u^2 + v^2) = p. \tag{1.12$'$}$$

(IIb) $b = 2uvt, C_2 - C_1 - 1 = (u^2 - v^2)t, C_1 - 1 = (u^2 + v^2)t,$

$$u^4 - 6u^2v^2 - 3v^4 = -p'(t{:}\,even),\ u^4 - 6u^2v^2 - 3v^4 = -2p(t{:}\,odd) \tag{1.14$'$}$$

At this stage, all we have to do is to solve the following diophantine equations:

$$u^4 + v^4 - 2uv(u^2 + v^2) = -p, \tag{1.12}$$

$$u^4 - 6u^2v^2 - 3v^4 = q, \tag{1.14}$$

$$u^4 + v^4 - 2uv(u^2 + v^2) = p, \tag{1.12$'$}$$

$$u^4 - 6u^2v^2 - 3v^4 = -q, \tag{1.14$'$}$$

where $p \in \{1, 2, 3, 6\}, q \in \{1, 2, 3, 4, 6, 8, 12, 24\}, u > v > 0, GCD(u, v) = 1, u \not\equiv v (\text{mod } 2)$.
From the condition $u \not\equiv v (\text{mod } 2)$, we obtain that $p, q \in \{1, 3\}$.
Using the KASH 2.5 command *ThueSolve*,
(1.12) has the solution $(u, v) = (2, 1)$ for $p = 3$,
$(1.14), (1.12)'$ and $(1.14)'$ all have no solution.
For (1.12), we shall find (A, B, b) from the following relation:
(1) $b = (u^2 - v^2)t,\ C_2 - C_1 + 1 = 2uvt,\ C_1 + 1 = (u^2 + v^2)t,$
(2) $A = 1 + C_2 (C_2 > 0),\ B = 2A^2 - 3C_1A - 3C_1 - 3 (C_1 > 0),\ C_2^2 - 3C_1^2 = 1.$
From (1) we have $b = 3t,\ C_2 = 9t - 2,\ C_1 = 5t - 1.$
From this and $C_2^2 - 3C_1^2 = 1$, we have $t(t - 1) = 0$. Since t is a positive integer, we have $t = 1$. Hence, from $b = 3t$ and (2), we have $(A, B, b) = (8, 17, 3)$. Therefore, the integer solusions (A, B, b) such that $b \neq 0, A \neq -1, 0, 2$ are $(A, B, b) = (8, 17, \pm 3)$. □

Lemma 2.2. The integer solutions (A, B, b) of the following diophantine system are $(0, 0, 0), (3, 3, 0)$ and $(-3, 6, \pm 3)$:

$$\begin{cases} A^3 - 3AB + 3 = 3(b^2 + 1) \\ B^3 - 3AB + 3 = 3(b^4 + b^2 + 1) \end{cases}.$$

Proof. We have

$$A^3 - 3AB = 3b^2, \tag{2.1}$$

$$B^3 - 3AB = 3(b^4 + b^2). \tag{2.2}$$

(i) The case $b = 0$: If $A = 0$, then we have $B = 0$. If $A \neq 0$, then we have $B \neq 0$. And easily we have $A = B = 3$. Therefore, in this case, we have $(A, B, b) = (0, 0, 0), (3, 3, 0)$.

(ii) The case $b \neq 0$: We obviously see $A \neq 0, B \neq 0$ and $3|A, B, b$. We put $A = 3A_0, B = 3B_0, b = 3b_0$. From $(2.1), (2.2)$ we have

$$A_0^3 - A_0B_0 = b_0^2, \tag{2.3}$$

$$B_0^3 - A_0B_0 = 9b_0^4 + b_0^2. \tag{2.4}$$

From $(2.3), (2.4)$ we have

$$B_0^3 - A_0^3 = 9b_0^4. \tag{2.5}$$

From $(2.3), (2.5)$ we have $B_0^3 - A_0^3 = 9(A_0^3 - A_0B_0)^2$. From this we have

$$B_0^3 = A_0^2(9(A_0^2 - B_0)^2 + A_0). \tag{2.6}$$

We put $A_0 = A_1m, B_0 = B_1m$, where $m = GCD(A_0, B_0)(\geq 1), GCD(A_1, B_1) = 1$. Hence, from (2.6) we have $B_1^3m^3 = A_1^2m^2(9(A_1^2m^2 - B_1m)^2 + A_1m)$. From this we have

$$B_1^3 = A_1^2(9m(A_1^2m - B_1)^2 + A_1). \tag{2.7}$$

Since $GCD(A_1, B_1) = 1$, we have $A_1 = \pm 1$. Hence, from (2.7) we have

$$B_1^3 = 9m(m - B_1)^2 \pm 1. \tag{2.8}$$

From (2.8) we have

$$B_1^3 - 9B_1^2m + 18B_1m^2 - 9m^3 = \pm 1. \tag{2.9}$$

Using the KASH 2.5 command *ThueSolve*, the solutions of (2.9) are

$$(B_1, m) = (\mp 2, \mp 1), (\pm 1, 0), (\pm 1, \pm 1). \tag{2.10}$$

Since $m \geq 1$, we have $(B_1, m) = (2, 1), (1, 1)$.

Hence, we have $(A_1, B_1, m) = (-1, 2, 1), (1, 1, 1)$.

Since $A_0 = A_1m, B_0 = B_1m$, we have $(A_0, B_0) = (-1, 2), (1, 1)$.

By (2.3), $b_0^2 = A_0^3 - A_0B_0 = 1$ or 0.

Since $b_0 \neq 0$, we have $(A_0, B_0, b_0) = (-1, 2, \pm 1)$.

Hence, we have $(A, B, b) = (3A_0, 3B_0, 3b_0) = (-3, 6, \pm 3)$. $\square$

Theorem 2.4. Let $b(\neq 0, \pm 1, \pm 3) \in \mathbb{Z}$ and let $\theta^3 - 3\theta - b^3 = 0$.
Then, if $4(4b^4)^{\frac{3}{5}} + 24 < |D_K|$, $\epsilon = \dfrac{1}{1 - b(\theta - b)}(> 1)$ is the fundamental unit of $\mathbb{Q}(\theta)$.

Proof. First, we note that

$$F(\epsilon) = \epsilon^3 - 3(b^4 + b^2 + 1)\epsilon^2 + 3(b^2 + 1)\epsilon - 1 = 0.$$

If ϵ is not a fundamental unit of $\mathbb{Q}(\theta)$, there exists a unit $\epsilon_0(> 1)$ of $\mathbb{Q}(\theta)$ such that $\epsilon = \epsilon_0^n$, with some $n \in \mathbb{Z}, n > 1$.

The case $n = 2$(i.e. $\epsilon = \epsilon_0^2$): Let ϵ_0 be a root of the equation

$$\epsilon_0^3 - B\epsilon_0^2 + A\epsilon_0 - 1 = 0(A, B \in \mathbb{Z}) .$$

Then we have the relation

$$\begin{cases} A^2 - 2B = 3(b^2 + 1) \\ B^2 - 2A = 3(b^4 + b^2 + 1). \end{cases} \tag{2.11}$$

By Lemma 2.1, the diophantine system (2.11) has the integer solutions $(A, B, b) = (0, -3, \pm 1)$, $(-1, -1, 0), (3, 3, 0)$ and $(8, 17, \pm 3)$.

The case $n = 3$(i.e. $\epsilon = \epsilon_0^3$): Let ϵ_0 be a root of the equation

$$\epsilon_0^3 - B\epsilon_0^2 + A\epsilon_0 - 1 = 0(A, B \in \mathbb{Z}) .$$

Then we have the relation

$$\begin{cases} A^3 - 3AB + 3 = 3(b^2 + 1) \\ B^3 - 3AB + 3 = 3(b^4 + b^2 + 1). \end{cases} \tag{2.12}$$

By Lemma 2.2, the diophantine system (2.12) has the integer solutions $(A, B, b) = (0, 0, 0), (3, 3, 0)$ and $(-3, 6, \pm 3)$. Therefore, we obtained the fact that there exist no units $\epsilon_0(> 1)$ such that $\epsilon = \epsilon_0^2, \epsilon_0^3$ or ϵ_0^4. Next we shall show that, for any unit $\epsilon_0(> 1)$, if $4(4b^4)^{\frac{3}{5}} + 24 < |D_K|$, then $\epsilon < \epsilon_0^5$. Since $F(4b^4) > 0$, we have $\epsilon < 4b^4$. From Artin's Lemma ([15], Lemma 2), if $4(4b^4)^{\frac{3}{5}} + 24 < |D_K|$, then we have $(4b^4)^{\frac{1}{5}} < \epsilon_0$, where $\epsilon_0(> 1)$ is any unit of $\mathbb{Q}(\theta)$. Hence, we have that for any unit $\epsilon_0(> 1)$ if $4(4b^4)^{\frac{3}{5}} + 24 < |D_K|$, then $\epsilon < \epsilon_0^5$. Therefore, if $4(4b^4)^{\frac{3}{5}} + 24 < |D_K|$, then $\epsilon(> 1)$ is the fundamental unit of $\mathbb{Q}(\theta)$. $\square$

Remark 2.3. Lee and Spearman [25] point out that ϵ is the sixth power of the fundamental unit of $\mathbb{Q}(\theta)$ for the case $b = \pm 3$.

Corollary 2.1. Let $b(\neq 0, \pm 1, \pm 3) \in \mathbb{Z}$ and let $\theta^3 - 3\theta - b^3 = 0$.
Then, if $b^3 - 2$ or $b^3 + 2$ is squarefree,
$\epsilon = \dfrac{1}{1 - b(\theta - b)}(> 1)$ is the fundamental unit of $\mathbb{Q}(\theta)$.

Proof. Suppose $b^3 - 2$ is squarefree. Then by Theorem 2.3 we have
$|D_K| = 27(b^3 - 2) \times 2^e \cdot 3^{g_2} \cdot \ell_2 > 27(b^3 - 2)$. It is easilly seen that $4(4b^4)^{\frac{3}{5}} + 24 < 27(b^3 - 2)$.
Therefore, from Theorem 2.4 ϵ is the fundamental unit of $\mathbb{Q}(\theta)$.
The case that $b^3 + 2$ is squarefree is similar to the case that $b^3 - 2$ is squarefree. $\square$

Corollary 2.2. Let $b(\neq 0, \pm 1, \pm 3) \in \mathbb{Z}$ and let $\theta^3 - 3\theta - b^3 = 0$.
Then, there exist infinitely many cubic fields $\mathbb{Q}(\theta)$ such that
$\epsilon = \dfrac{1}{1 - b(\theta - b)}(> 1)$ is the fundamental unit of $\mathbb{Q}(\theta)$.

Proof. By Erdös [10], there are infinitely many natural numbers b for which $b^3 - 2$ is squarefree. The Corollary 2.2 is obtained from this and Corollary 2.1. □

2.4. A family of biquadratic fields

We need two lemmas. As for class field tower, refer to Yoshida [39].
Let K be a non-Galois cubic extension of $\mathbb{Q}$; let L be the normal closure of K; and let k be the quadratic field containd in L. Note that no primes are totally ramified in the cubic field $K \Leftrightarrow L/k$ is an unramified extension.
Assume that $3|D_k(D_k$ is the discriminant of $k)$ and that L/k is an unramified extension. By [13,§1,(1)] (or [27,Theorem 3]), $D_K = D_k\mathfrak{f}^2(\exists \mathfrak{f} \in \mathbb{Z})$. From this and $3|D_k$, the decomposition of 3 at K is $3 = \mathfrak{p}_1\mathfrak{p}_2^2$, where $\mathfrak{p}_1, \mathfrak{p}_2$ are distinct prime ideals lying above 3.

Lemma 2.3([13,Lemma 8])**.** Let K, k be as above. If there exists a unit ϵ in K such that

1. ϵ is not a cube of any unit of K and
2. $\epsilon^2 \equiv 1 \pmod{\mathfrak{p}_1^2\mathfrak{p}_2^3}$,

then the length of the 3-class field tower of $k(\sqrt{-3})$ is greater than 1.

Lemma 2.4([40,p134])**.** Let K, k be as Lemma 3.1. Let $X^3 + AX^2 + BX - 1$ be the minimal polynomial of a unit η in K. Then
$\eta \equiv 1 \pmod{\mathfrak{p}_1^2\mathfrak{p}_2^3} \Leftrightarrow 27|A + 3, 3^5|A + B$.

Let $b(\neq 0, \pm 3) \in \mathbb{Z},\ 3|b$ and let θ be the real root of the irreducible cubic polynomial

$$f(X) = X^3 - 3X - b^3 \in \mathbb{Z}[X].$$

The discriminant of $f(X)$ is $D_f = -3^3(b^6 - 4) = -3^3(b^3 - 2)(b^3 + 2)$ and $D_f < 0$.
Let $K := \mathbb{Q}(\theta), k := \mathbb{Q}(\sqrt{D_f}) = \mathbb{Q}(\sqrt{-3(b^6 - 4)})$. We shall consider a family of biquadratic fields $F_b := \mathbb{Q}(\sqrt{-3(b^6 - 4)}, \sqrt{-3}) = \mathbb{Q}(\sqrt{b^6 - 4}, \sqrt{-3})$. We can show that $\#\{F_b;\ b(\neq 0, \pm 3) \in \mathbb{Z},\ 3|b\} = \infty$. In fact, let S be a finite set of primes. By Dirichlet's theorem on arithmetical progressions, we can find a prime p such that $p(\neq 2) \notin S$ and $p \equiv 2 \pmod 3$. For such p, we can find $c \in \mathbb{Z}$ such that $p||c^3 - 2$. Then, for $b \in \mathbb{Z}$ with $b \equiv 0 \pmod 3$ and $b \equiv c \pmod{p^2}$, we

have $p||b^3-2$ and $3|b$. Since $GCD(b^3-2, b^3+2) = 1$ or 2, we have $p||D_f$. Hence, we have $p|D_k$. Therefore, p is ramified in F_b. (cf. [32,Hilfssatz 1]).

Theorem 2.5. Assume that $b(\neq 0, \pm 3) \in \mathbb{Z},\ 3|b$. Then the length of the 3-class field tower of $F_b = \mathbb{Q}(\sqrt{b^6-4}, \sqrt{-3})$ is greater than 1.

Proof. (a) Since $3 \nmid b^6-4$, we have $3|D_k$.

(b) We shall consider the minimal splitting field Kk of $f(X)$.

By [27,Theorem 1] no primes are totally ramified in the cubic field K.

From this, Kk/k is an unramified cyclic cubic extension.

(c) From (a),(b), the decomposition of 3 at K is $3 = \mathfrak{p}_1\mathfrak{p}_2^2$, where $\mathfrak{p}_1, \mathfrak{p}_2$ are distinct prime ideals lying above 3.

(d) By the proof of Theorem 2.4, $\epsilon = \dfrac{1}{1-b(\theta-b)}$ is not a cube of any unit of K.

(e) Let $F(X) = X^3+AX^2+BX-1$ be the minimal polynomial of ϵ. Then $A = -3(b^4+b^2+1), B = 3(b^2+1)$. Hence, we have $27|-3(b^4+b^2) = A+3,\ 3^5|-3b^4 = A+B$. By Lemma 2.4, we have $\epsilon \equiv 1 \pmod{\mathfrak{p}_1^2\mathfrak{p}_2^3}$.

Therefore, from (d),(e) and Lemma 2.3, the length of the 3-class field tower of $k(\sqrt{-3}) = F_b$ is greater than 1. □

Remark 2.4. By the same reason as [40,p.334,example], the 3-rank of the ideal class group of F_b is greater than 1.

3 A two-parameter family of cubic fields

Levesque and Rhin [26] introduced two families of complex cubic fields $\mathbb{Q}(\alpha)$, each of which depends on two parameters. Adam [1] obtained the Voronoi-algorithm expansions of the order $\mathbb{Z}[\alpha]$ for these two families, for one of which Kühner [23] also found the Voronoi-algorithm expansions.

In this chapter we shall consider a new family of complex cubic fields, similar but different from those families above i.e. $\mathbb{Q}(\alpha)$, where α is the real root of the irreducible cubic polynomial $f(X)$ in Proposition 3.1.

Using the similar method in chapter 1, we obtain the following results :

the Voronoi-algorithm expansions of the order $\mathbb{Z}[\alpha]$,

the period length of these expansions goes to infinity,

the fundamental units of the order $\mathbb{Z}[\alpha]$.

The precise proof of the Theorem 3.1 is given in [19].

Proposition 3.1. Let $f(X) = X^3 - c^m X^2 + (c+1)X - c^m$, where m, c are intergers such that $m \geq 1$ and $c \geq 2$. Then $f(X)$ has only one real root α and $f(X)$ is irreducible except the case $m = 1, c = 2$.

Moreover, if $m \geq 2$, then α satisfies

$$c^m - \frac{1}{c^{m-1}} - \frac{1}{c^{m+2}} < \alpha < c^m - \frac{1}{c^{m-1}}.$$

Proof. Since the discriminant of $f(X)$ is

$$D_f = -\{4c^{4m} - (c^2 + 20c - 8)c^{2m} + 4(c+1)^3\} < 0,$$

$f(X)$ has only one real root α.

Since

$$f(c^m - \frac{2}{c^{m-1}}) = -c^{m+1} + \frac{6}{c^{m-2}} - \frac{2}{c^{m-1}} - \frac{8}{c^{3m-3}} < 0 \text{ and}$$
$$f(c^m - \frac{1}{c^{m-1}}) = \frac{c-1}{c^{m-1}} - \frac{1}{c^{3m-3}} > 0 \quad ((m,c) \neq (1,2)),$$
$$c^m - \frac{2}{c^{m-1}} < \alpha < c^m - \frac{1}{c^{m-1}} \quad ((m,c) \neq (1,2)).$$

Therefore, if $(m, c) \neq (1, 2)$, then $f(X)$ is irreducible.

Furthermore, we have

$$f(c^m - \frac{1}{c^{m-1}} - \frac{1}{c^{m+2}})$$
$$= -c^{m-2} + \frac{1}{c^{m-2}} - \frac{1}{c^{m-1}} + \frac{3}{c^{m+1}} - \frac{1}{c^{m+2}} + \frac{2}{c^{m+4}} - \frac{1}{c^{3m-3}} - \frac{3}{c^{3m}} - \frac{3}{c^{3m+3}} - \frac{1}{c^{3m+6}} < 0 \ \ (m \geq 2).$$

Hence, if $m \geq 2$, then
$$c^m - \frac{1}{c^{m-1}} - \frac{1}{c^{m+2}} < \alpha < c^m - \frac{1}{c^{m-1}}. \ \square$$

Let $f(X) = X^3 - c^m X^2 + (c+1)X - c^m$, where m, c are intergers such that $m \geq 2$ and $c \geq 2$. By Proposition 3.1 $f(X)$ is irreducible and has only one real root.

Theorem 3.1. Let α be the real root of the polynomial $f(X),\ K = \mathbb{Q}(\alpha)$ and $\mathcal{O} = \mathbb{Z}[\alpha]$. Then

(i) The chain of the minimal points of $\mathcal{O}$ is : for $1 \leq s \leq m-1$

$\theta_0 = 1,\ \theta_{3s-2} = (c^s + \alpha - c^m)\left(\frac{\alpha}{c^m - \alpha}\right)^s,\ \theta_{3s-1} = \left(\frac{c\alpha}{c^m-\alpha}\right)^s,\ \theta_{3s} = \alpha\left(\frac{\alpha}{c^m-\alpha}\right)^s,$

$\theta_{3m-2} = \alpha(1 + \alpha - c^m)\left(\frac{\alpha}{c^m-\alpha}\right)^m$ and $\theta_{3m-1} = \alpha\left(\frac{\alpha}{c^m-\alpha}\right)^m.$

(ii) $\varepsilon = \alpha\left(\frac{\alpha}{c^m-\alpha}\right)^m$ is the fundamental unit of $\mathcal{O}$ and Voronoi-algorithm expansion period length is $\ell = 3m-1$.

Remark 3.1. The following relation holds among the minimal points of $\mathcal{O}$:

$\theta_2 = \alpha\theta_0 + \theta_1,\ \theta_{3s-1} = \theta_{3s-3} + \theta_{3s-2}$ for $2 \leq s \leq m-1,\ \theta_{3m-1} = \alpha\theta_{3m-3} + \theta_{3m-2}.$

Remark 3.2. In fact, (ii) in Theorem 3.1 is valid for $m = 1$ provided $c \geq 4$.

References

[1] B. Adam, *Voronoi-algorithm expansion of two families with period length going to infinity*, Math.Comp. **64**(1995),1687-1704.

[2] B. Adam, *Generalisation d'une familie de Shanks*, Acta Arith. **84**,No.1(1998),43-58.

[3] E. Benjamin, F. Lemmermeyer and C. Snyder, *Real quadratic fields with abelian 2-class field tower*, J. Number Theory **73**(1998),182-194.

[4] L. Bernstein, *The Jacobi-Perron Algorithm, Its Theory and Application*, Lecture Notes in Math.**207**, Springer(1971).

[5] W.E.H. Berwick, *The classification of ideal numbers that depend on a cubic irrationality*, Proc. London Math.Soc.(2)**12**(1913),393-429.

[6] T. Callahan, *The 3-class groups of non-Galois cubic fields*, I, Mathematica **21**(1974),72-89.

[7] J.W.S. Cassels and A. Fröhlich, *Algebraic Number Theory*, Academic Press, London, 1967.

[8] I.V.L. Clarkson, J.E. Perkins and I.M.Y. Mareels, *An algorithm for best approximation of a line by lattice points in three dimensions*, In CANT '95 Conference Abstracts and Other Information(1995),77-91.

[9] B.N. Delone and D.K. Faddeev, *The theory of irrationalities of the third degree*, Transl.Math. Monographs, vol.10,Amer.Math. Soc.,Providence, RI,1964.

[10] P. Erdös, *Arithmetical properties of polynomials*, Jour. London Math.Soc.**28**(1953),416-425.

[11] Ph. Furtwängler, *Über die simultane Approximztion von Irrationalzahlen*, Math.Ann.,**99** (1927),71-83.

[12] F. Gerth III, *Rank of 3-class group of non-Galois cubic fields*, Acta Arith. **30**(1976),308-322.

[13] H. Hasse, *Arithmetische Theorie der kubischen Zahlkörper auf klassenkörpertheoretischer Grundlage*, Math.Z.**31**(1930),565-582.

[14] H. Hasse, *Vorlesungen über Klassenkörpertheorie*, Physica-Verlag, Würzburg, 1967.

[15] M. Ishida, *Fundamental Units of Certain Algebraic Number Field*, Abh.Math.Sem.Univ. Hamburg, Bd.**39**(1973),245-250.

[16] M. Ishida, *The genus fields of algebraic number fields*, Lecture Notes in Math.**555**, Springer, (1976).

[17] M. Ishida, *Existence of an unramified cyclic extension and congruence conditions*, Acta Arith.**51**(1988),75-84.

[18] K. Kaneko, *On the cubic fields* $\mathbb{Q}(\theta)$ *defined by* $\theta^3-3\theta+b^3=0$, Sut J.Math.vol.**32**,No.2(1996),

141-147.

[19] K. Kaneko, *Voronoi-algorithm expansion of a family with period length going to infinity*, Sut J.Math.vol.**34**,No.1(1998),49-62.

[20] K. Kaneko, *Integral bases and fundamental units of certain cubic number fields*, Sut J.Math.vol.**39**,No.2(2003),117-124.

[21] K. Kaneko, *A method for finding a minimal point of the lattice in cubic number fields*, to appear Tsukuba J.Math.

[22] K. Kaneko, *On units of a family of cubic number fields*, submitted to Sut J.Math.

[23] J. Kühner, *On a family of generalized continued fraction expansions with period length going to infinity*, J.Number Theory **53**(1995),1-12.

[24] O. Lahlou and A. Farhane, *Sur les points extrémaux dans un ordre cubique*, Bull.Belg.Math.Soc. **12**(2005),449-459.

[25] Paul D. Lee and Blair K. Spearman, *A Diophantine System and a Problem on Cubic Fields*, International Mathematical Forum, Vol.6,(2011),141-146.

[26] C. Levesque and G. Rhin, *Two families of periodic Jacobi algorithms with period lengths going to infnity*, J. Number Theory **37**(1991),173-180.

[27] P. Llorente and E. Nart, *Effective determination of the decomposition of the rational primes in a cubic field*, Proc.Amer.Math.Soc.**87**(1983),579-585.

[28] D.J. Madden and W.Y. Vélez, *A note on the normality of unramified, abelian extensions of quadratic extensions*, Manuscripta Math. **30**(1980),343-349.

[29] D.C. Mayer, *Multiplicities of diheadral discriminants*, Math.Comp. **58**(1992),831-847.

[30] R. Morikawa, *On Units of certain Cubic Number Fields*, Abh.Math.Sem.Univ.Hamburg, Bd. **42**(1974),72-77.

[31] N. Nakagoshi, *On the class number relations of abelian extensions whose Galois groups are of type* (p,p), Math. Rep. Toyama Univ. **4**(1981),91-106.

[32] H. Reichart, *Arithmetische Theorie der kubischen Körper als Radikalkörper*, Monatshefte Math.Phys., **40**(1933),323-350.

[33] A. Scholz, *Über die Beziehung der Klassenzahlen quadratischer Körper zueinander*, J. Reine Angew. Math. **166**(1932),201-203.

[34] H.C. Williams, G. Cormack and E. Seah, *Calculation of the regulator of a pure cubic field*, Math.Comp. **34**(1980),567-611.

[35] H.C. Williams, G.W. Dueck and B.K. Schmid, *A rapid method of evaluating the regulator*

and class number of a pure cubic field, Math.Comp. **41**(1983),235-286.

[36] H.C. Williams and G.W. Dueck, *An analogue of the nearest integer continued fraction for certain cubic irrationalities*, Math.Comp. **42**(1984),683-705.

[37] H.C. Williams, *Continued fractions and number-theoretic computations*, Rocky Mountain J.Math. **15**(1985),621-655.

[38] H.C. Williams, *The period length of Voronoi's algorithm for certain cubic orders*, Publ,Math.Debrecen **37**(1990),245-265.

[39] E. Yoshida, *On the 3-class tower of some biquadratic fields*, Acta Arith.,**107**,No.4(2003), 327-336.

[40] E. Yoshida, *On the unit groups and the ideal class groups of certain cubic number fields*, Sut J.Math.vol.**39**,No.2(2003),125-136.