Homeomorphisms and metrizations of function spaces with infinite-dimensional topology

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Chapter 1

Introduction and Main Results

1.1 An overview

Infinite-dimensional topology is an important area of topology. It relates to algebraic topology, functional analysis, descriptive set theory and other mathematical areas. It also provides a lot of techniques and methods for other areas of mathematics.

Almost function spaces and hyperspace are infinite-dimensional. To give the topological classifications of function spaces and hyperspaces is one of the most important works in infinite-dimensional topology. On the topological structure of hyperspaces, the most well-known result is perhaps the Curtis-Schori-West Hyperspace Theorem which states that the space $\mathrm{Cld}_V(X)$ of all non-empty closed sets in a space X with the Vietoris topology is homeomorphic to the Hilbert cube Q if and only if X is a non-degenerate connected and locally connected compact metrizable [13] (cf. [26, Theorem 8.4.5]). Since then, many researches have investigated this topic. For an infinite compact metrizable space X, the function space $\mathrm{C}_u(X, \mathbb{R})$ of all continuous maps from X to the real line $\mathbb{R}$ with the sup-norm is an infinite-dimensional separable Banach space. By the Anderson-Kadec Theorem [1, 19], $\mathrm{C}_u(X, \mathbb{R})$ is homeomorphic to the Hilbert space ℓ_2. For a countable Tychonoff s-pace X, the function space $\mathrm{C}_p(X, \mathbb{R})$ of all continuous maps from X to the real line $\mathbb{R}$ with the topology of pointwise convergence is a separable metrizable space. To give the topological classification of $\mathrm{C}_p(X, \mathbb{R})$ for all countable Tychonoff spaces

X is difficult, even impossible. However, in 1991, Dobrowolski, Marciszewski and Mogilski [14] proved that $\mathrm{C}_p(X, \mathbb{R})$ is homeomorphic to the space $\mathbf{c}_0$ of all sequences converging to 0 if X is a countable metrizable non-discrete space.

Using the hypo-graphs, topologies of hyperspaces may be induced to function spaces. For a space X and a poset L with a topology, every topology of hyperspace $\mathrm{Cld}(X \times L)$ may inherit to the set $\downarrow \mathrm{C}(X, L)$ of hypo-graphs of all continuous maps from X to L. In this thesis, we give some conditions for metrizable of the space $\downarrow \mathrm{C}(X, [0, 1])$ with the Fell topology. We also give conditions for $\downarrow \mathrm{C}(X, Y)$ with the Fell topology is homeomorphic to the space $\mathbf{c}_0$ in the case that Y is a dendrite.

The importance and usefulness of simplicial complexes lies in the fact the they can be used to approximate and explore topological spaces. A polyhedron is the underlying space of a simplicial complex, which has two typical topologies, the so-called weak (Whitehead) topology and the metric topology. The subdivision preserves the weak topology but does not the metric topology. On the other hand, the product preserves the metric topology but does not the weak topology. Like this, these topologies have good and bad points.

We first induce a new topology to a simplicial complex which is different from the two typical topologies. We discuss the comparison of topological property between the three types of topologies.

1.2 Concepts and symbols

In this section, we give some concepts and symbols we need. Those concepts are known and our symbols are the same as usual.

We use the following symbols:

- $\mathbb{R}$: the set of all real numbers with the usual topology and order;
- $\mathbf{I} = [0, 1]$;
- $\mathbb{Q}$: the set of rational numbers;
- ω: the set of all non-negative integers;

- $\mathbb{N} = \omega \setminus \{0\}$;
- $Q = [-1, 1]^{\mathbb{N}}$ with the product topology;
- $\ell_2 = \{(x_n) \in \mathbb{R}^{\mathbb{N}} |\ \sum_{n=1}^{\infty} x_n^2 < \infty\}$ with the norm topology;
- $\Sigma = \{(x_n) \in Q |\ \sup x_n < \infty\}$ with the subspace topology of Q;
- $\mathbf{c}_0 = \{(x_n) \in \Sigma |\ \lim x_n = 0\}$ with the subspace topology of Q;
- $Q_f = \{(x_n) \in \Sigma |\ x_n = 0 \text{ but finitely many } n\}$ with the subspace topology of Q.

All spaces are assumed to be Tychonoff topological spaces, but some function spaces are not. All maps are continuous, but functions are not necessarily continuous. For a space X, cl_X and int_X are the closure operator and the interior operator in X. We may omit the subscript if no confusion. For spaces X and Y, we define the following symbols:

- $\mathrm{Cld}(X)$: the family of all non-empty closed sets in X;
- $\mathrm{Cld}^*(X) = \mathrm{Cld}(X) \cup \{\emptyset\}$;
- $\mathrm{Comp}(X) = \{F \in \mathrm{Cld}(X) |\ F \text{ is compact}\}$;
- $\mathrm{Fin}(X) = \{F \in \mathrm{Cld}(X) |\ F \text{ is finite}\}$;
- $\mathrm{C}(X, Y)$: the set of all continuous maps from X to Y;
- $\mathrm{USC}(X, L)$: the set of all upper semi-continuous functions from X to L, where L is a subset of $\mathbb{R}$.
- $\mathrm{C}_u(X, Y)$: $\mathrm{C}(X, Y)$ with the sup-norm if Y is a normed space;
- $\mathrm{C}_p(X, Y)$: $\mathrm{C}(X, Y)$ with the topology of pointwise convergence.

For two subspaces A and B of spaces X and Y, respectively, the symbol $(X, A) \approx (Y, B)$ means that there exists a homeomorphism $h : X \to Y$ such that $h(A) = B$.

Of course, $X \approx Y$ means that X and Y are homeomorphic. Similarly, we may define $(X, A, S) \approx (Y, B, T)$.

For a subset U of X, define

$$U^- = \{A \in \mathrm{Cld}(X) |\ A \cap U \neq \emptyset\} \quad \text{and} \quad U^+ = \{A \in \mathrm{Cld}(X) |\ A \subset U\}.$$

The *Vietoris topology* on $\mathrm{Cld}(X)$ is the topology which is generated by

$$\{U^-, U^+ |\ U \text{ is open in } X\}.$$

Another topology on $\mathrm{Cld}(X)$ is the *Fell topology* which is generated by

$$\{U^-, (X \setminus K)^+ |\ U \text{ is open and } K \text{ is compact in } X\}.$$

Let $X = (X, d)$ be a metric space, $x \in X$ and $A, B \subset X$. We denote the *diameter* of A by

$$\mathrm{diam}\, A = \sup\{d(x, y) \mid x, y \in A\},$$

and the *distance* between A and B by

$$\mathrm{dist}(A, B) = \inf\{d(x, y) \mid x \in A, y \in B\}.$$

For each $\epsilon > 0$, let

- $B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$;
- $\overline{B}(x, \epsilon) = \{y \in X \mid d(x, y) \leq \epsilon\}$;
- $N(A, \epsilon) = \{x \in X \mid \mathrm{dist}(\{x\}, A) < \epsilon\}$.

The *Hausdorff metric* d_H is defined as follows:

$$d_H(A, B) = \inf\{r > 0 \mid A \subset N(B, r), B \subset N(A, r)\} \leq +\infty.$$

Using this "metric", we may define a topology on $\mathrm{Cld}(X)$ which is called the *Hausdorff topology*. If $X = (X, d)$ is a compact metric space, then $(\mathrm{Cld}(X), d_H)$ is a compact metric space.

The hyperspaces $\mathrm{Cld}(X)$ with the above three topologies are denoted by $\mathrm{Cld}_V(X)$, $\mathrm{Cld}_F(X)$ and $\mathrm{Cld}_H(X)$, respectively. Similarly, we may define $\mathrm{Comp}_V(X)$, $\mathrm{Fin}_F(X)$

and $\mathrm{Comp}_H(X)$ etc. It is well-known that $\mathrm{Cld}_V(X)$ (resp. $\mathrm{Cld}_F(X)$) is metrizable if and only if X is a compact (resp. locally compact and separable) metrizable space. Obviously, when X is compact, the Fell topology of $\mathrm{Cld}(X)$ is equal to the Vietoris topology. And when X is a compact metric space, the three topologies are the same. In this case, we may use $\mathrm{Cld}(X)$ to denote this space.

For every $f \in \mathrm{USC}(X, L)$, let

$$\downarrow f = \{(x, s) \in X \times L |\ s \leq f(x)\} \in \mathrm{Cld}(X \times L),$$

which is called the *hypo-graph* of f. For $A \subset \mathrm{USC}(X, L)$, let

$$\downarrow A = \{\downarrow f |\ f \in A\}.$$

By identifying each $f \in \mathrm{USC}(X, L)$ with $\downarrow f \in \mathrm{Cld}(X \times L)$, we can regard A as the subset $\downarrow A$. Let $\downarrow A_V$, $\downarrow A_F$ and $\downarrow A_H$ be the spaces with the topologies inherited from $\mathrm{Cld}_V(X \times L)$, $\mathrm{Cld}_F(X \times L)$ and $\mathrm{Cld}_H(X \times L)$, respectively.

A *continuum* is a compact connected metrizable space and a *Peano continuum* is a locally connected continuum. A metric space X is called an *absolute (neighborhood) retract for metrizable spaces*, briefly, *AR(ANR)*, if it is a (neighborhood) retract of every metric space Y concluding X as a closed space. A space X is called an *absolute (neighborhood) extensor for metrizable spaces,* briefly, *AE(ANE)*, if each map $f : A \to X$ from any closed set A in an arbitrary metrizable space Y extends over Y (some neighborhood of A in X). As we all know, a metrizable space is an ANE(AE) if and only if it is an ANR(AR). We call a metric space X a *uniform AR(ANR)* when the (neighborhood) retract is uniformly continuous at X.

A *dendrite* is a Peano continuum containing no simple closed curves, equivalently it is a 1-dimensional compact AR [10, Chapter V, Corollary 13.5]. An *end point* of a space has an arbitrarily small open neighborhood whose boundary is a singleton. Each pair of distinct points of a dendrite is connected by the unique arc [42, Chapter V, (1.2)]. So we denote the unique arc of two points x, y in a dendrite by $[x, y]$, where $[x, y]$ is the constant path if $x = y$.

For each function $f : X \to Y$ of X into a dendrite Y and $v \in Y$, we define the hypo-graph $\downarrow_v f$ of f with respect to v as follows:

$$\downarrow_v f = \bigcup_{x \in X} \{x\} \times [v, f(x)] \subset X \times Y.$$

When f is continuous, the hypo-graph $\downarrow_v f$ is a closed subset of the product space $X \times Y$. Hence we can regard

$$\downarrow_v \mathrm{C}_F(X,Y) = \{\downarrow_v f \mid f \in \mathrm{C}(X,Y)\}$$

as the subspace of the hyperspace $\mathrm{Cld}_F(X \times Y)$ with the Fell topology. Let $\overline{\downarrow_v \mathrm{C}_F(X,Y)}$ be the closure of $\downarrow_v \mathrm{C}_F(X,Y)$ in $\mathrm{Cld}_F(X \times Y)$. That is,

$$\overline{\downarrow_v \mathrm{C}_F(X,Y)} = \mathrm{cl}_{\mathrm{Cld}_F(X \times Y)} \downarrow_v \mathrm{C}_F(X,Y).$$

The vertices and polyhedron of a simplicial complex K are denoted by $K^{(0)}$ and $|K|$, respectively. Let $\mathbb{R}^{K^{(0)}}$ be the linear space of all real functions defined on $K^{(0)}$ with the operations defined coordinate-wise. For each $v \in K^{(0)}$, let $\mathbf{e}_v \in \mathbb{R}^{K^{(0)}}$ be the unit vector defined by $\mathbf{e}_v(v) = 1$ and $\mathbf{e}_v(u) = 0$ if $u \neq v$. By $\mathbb{R}^{K^{(0)}}_f$, we denote the linear subspace of $\mathbb{R}^{K^{(0)}}$ generated by $\mathbf{e}_v$, $v \in K^{(0)}$, i.e.,

$$\mathbb{R}^{K^{(0)}}_f = \left\{x \in \mathbb{R}^{K^{(0)}} \;\middle|\; x(v) = 0 \text{ except for finitely many } v \in K^{(0)}\right\}.$$

Identifying vertices $v \in K^{(0)}$ with $\mathbf{e}_v \in \mathbb{R}^{K^{(0)}}$, K can be realized in $\mathbb{R}^{K^{(0)}}_f$ and then $|K| \subset \mathbb{R}^{K^{(0)}}_f$.

The metric topology of $|K|$ is induced by the norm which is defined by

$$\|x\| = \sum_{v \in K^{(0)}} |x(v)|, \text{ for each } x \in \mathbb{R}^{K^{(0)}}_f.$$

Although the product topology on $\mathbb{R}^{K^{(0)}}_f$ is not generated by any norm, the topology of $|K|$ inherited from the product topology coincides with the metric topology of $|K|$.

The weak (*Whitehead*) *topology* of $|K|$ is no other than the relative (or subspace) topology of *the finite topology* of $\mathbb{R}^{K^{(0)}}_f$ which is the weak topology determined by the Euclidean topology on each finite-dimensional linear subspace (cf. Appendix One, A.4.2 and B.5 in [16]). However, $\mathbb{R}^{K^{(0)}}_f$ is not a topological linear space with respect to the finite topology. In fact, the addition is not continuous with respect to this topology [16, Appendix One, A.4.3]. In case K is locally finite, these topologies are identical.

On the other hand, $\mathbb{R}_f^{K^{(0)}}$ is a locally convex topological linear space with respect to *the box topology*, where the origin (the null element) $\mathbf{0} \in \mathbb{R}_f^{K^{(0)}}$ has the open neighborhood basis consisting of the following sets:

$$\mathbb{R}_f^{K^{(0)}} \cap \prod_{v \in K^{(0)}} (\varepsilon_v, -\varepsilon_v), \ \varepsilon_v > 0 \ (v \in K^{(0)}).$$

The box topology of $\mathbb{R}_f^{K^{(0)}}$ induces a new topology on $|K|$ which is also called *the box topology.*

By $|K|_{\mathrm{m}}$, $|K|_{\mathrm{w}}$ and $|K|_{\mathrm{b}}$ we denote the spaces $|K|$ with the metric topology, the weak topology and the box topology, respectively.

1.3 Literature survey

Vietoris in 1922 defined the Vietoris topology on the family of all non-empty closed sets [41]. Wojdysławski in [43] proved that for a compact space X, $\mathrm{Cld}_V(X)$ is an AR if and only if X is a Peano continuum. He conjectured that $\mathrm{Cld}_V(X) \approx Q$ if X is a non-degenerate Peano continuum. The Curtis-Schori-West Hyperspace Theorem gave an affirmative answer to this problem, see [13](cf. [26, Theorem 8.4.5]). For a non-compact space X, in [11], Curtis proved that $\mathrm{Comp}_V(X) \approx Q \setminus \{0\}$ if and only if X is a non-compact, locally compact, connected, locally connected metrizable space. In [12], Curtis and To Nhu proved that $\mathrm{Fin}_V(X) \approx Q_f$ if and only if X is a non-degenerate connected, locally path-connected metrizable space which is a countable union of finite-dimensional compact sets. Moreover, they also showed that if X is a non-degenerate strongly countable-dimensional[1] Peano continuum then

$$(\mathrm{Cld}_V(X), \mathrm{Fin}_V(X)) \approx (Q, Q_f).$$

[1]A space is called a *strongly countable-dimensional* if it is a countable union of finite-dimensional closed sets

The Fell topology was defined by many scholars with various names, for example, *H-topology* [17], *hit-or-miss topology* [6], *Choquet-Matheron topology* [38], since this topology was used in many areas of mathematics. In 2003, Sakai and Yang proved in [36] that $(\mathrm{Cld}^*_F(X), \mathrm{Comp}_F(X)) \approx (Q, \Sigma)$ if and only if X is a locally compact, locally connected, separable metrizable space with no compact components. They also proved in [36] that $(\mathrm{Cld}^*_F(X), \mathrm{Fin}_F(X)) \approx (Q, Q_f)$ if and only if X is a strongly countable-dimensional, locally compact, locally connected, separable metrizable s-pace with no compact components. Moreover, Banakh and Voytsitskyy in [5] proved that

$$(\mathrm{Cld}^*_F(X), \mathrm{Comp}_F(X), \mathrm{Fin}_F(X)) \approx (Q, \Sigma, Q_f)$$

if and only if X is a strongly countable-dimensional, locally compact, locally connected, separable metrizable space with no compact components. In [47] and [37], the authors also gave the topological structures of other pairs of hyperspaces with the Fell topology.

By the Anderson-Kadec Theorem [1] and [19], $\mathrm{C}_u(X, \mathbb{R})$ is homeomorphic to the Hilbert space ℓ_2 for an infinite compact metrizable space X. In 1991, Dobrowolski, Marciszewski and Mogilski [14] proved that $\mathrm{C}_p(X, \mathbb{R})$ is homeomorphic to the space $\mathbf{c}_0$ if X is a countable metrizable non-discrete space.

Using topologies on function spaces, we may define topologies on hyperspaces. For example, let a continuous function $f_A : X \to \mathbb{R}$ be defined by

$$X \ni x \longmapsto d(x, A)$$

for each $A \in \mathrm{Cld}(X)$. By this map, the set $\mathrm{Cld}(X)$ can be regarded as the subset of the set $\mathrm{C}(X, \mathbb{R})$. The *Attouch-Wets topology* of $\mathrm{Cld}(X)$ is inherited from $\mathrm{C}(X, \mathbb{R})$ with the topology of uniform convergence on bounded sets [2] and [3](cf. [35]). Conversely, as mentioned in §1.2, we may define the function spaces ${\downarrow}\mathrm{C}_V(X, L)$ and ${\downarrow}\mathrm{C}_F(X, L)$ using the Vietoris topology and the Fell topology on $\mathrm{Cld}(X \times L)$. Many scholars investigated properties on function spaces ${\downarrow}\mathrm{C}_V(X, L)$ and ${\downarrow}\mathrm{C}_F(X, L)$. For examples, see Beer and R. Lucchetti [8], Beer [7], Lin [22], McCoy and Ntantu [23]. In particular, McCoy and Ntantu in [23] gave a sufficient and necessary condition for the metrizable of the space ${\downarrow}\mathrm{C}_F(X, \mathbb{R})$.

In 2005, Yang [44] showed that $\downarrow\mathrm{USC}_V(X, L) \approx Q$ if X is an infinite locally connected compact metrizable space and L is a non-degenerate continuous lattice[2] with the connected Lawson topology[3]. Sakai and Uehara in [34] proved $\downarrow\mathrm{USC}_F(X, \mathbf{I}) \approx Q$ for an infinite locally compact separable metrizable space X. Yang, Zhou and Wu in [49, 46] proved that, for a compact metric space X,

$$(\downarrow\mathrm{USC}_V(X, \mathbf{I}), \downarrow\mathrm{C}_V(X, \mathbf{I})) \approx \begin{cases} (\mathbf{I}^{|X|}, \mathbf{I}^{|X|}) & \text{if } X \text{ is discrete;} \\ (Q, \mathbf{c}_0) & \begin{array}{l}\text{if the set of isolated}\\ \text{points is not dense;}\end{array} \\ (Q, \mathbf{c}_0 \cup (Q \setminus \Sigma)) & \text{otherwises.} \end{cases}$$

This work was generalized in [48, 45] to the following: For a locally compact separable metric space X,

$$(\downarrow\mathrm{USC}_F(X, \mathbf{I}), \downarrow\mathrm{C}_F(X, \mathbf{I})) \approx \begin{cases} (\mathbf{I}^{|X|}, \mathbf{I}^{|X|}) & \text{if } X \text{ is discrete;} \\ (Q, \mathbf{c}_0) & \begin{array}{l}\text{if the set of isolated}\\ \text{points is not dense;}\end{array} \\ (Q, \mathbf{c}_0 \cup (Q \setminus \Sigma)) & \text{otherwises.} \end{cases}$$

We will gave a generalized of above results in Chapter 3.

Infinite-dimensional topology give strong tools for proving results mentioned. Anderson in [1] defined a concept of capset and proved the Anderson Theorem which state that $\mathbb{R}^{\mathbb{N}} \approx \ell_2$. In 1980s, Torunczyk [39, 40] gave a characterization of Q-manifolds and a characterization of ℓ_2-manifolds. Based on those results, a lot of topological structures of function space and hyperspace could be given. For example, using Torunczyk's characterization of Q-manifolds, the authors in [12] gave a simple proof for the Curtis-Schori-West Hyperspace Theorem. We also use this theorem to prove my result in this paper. Since 1990s, as a generalized of capset, absorber of a class of spaces or pairs of spaces have defined. Using it, more topological structures of function spaces and hyperspaces have gotten. Absorber also is a main tool in

[2] A *continuous lattice* is a complete lattice satisfying the distributive law with respect to arbitrary infs and directed sups.

[3] The *Lawson topology* on a continuous lattice L is generated by the family of forms $U \setminus \uparrow \{x\}$, where U satisfies the condition (i) $U = \uparrow U$ and (ii) $L \setminus U$ is closed with directed sups, and $\uparrow A = \{x \in L \mid x \geq y \text{ for some } y \in A\}$ for $A \subset L$.

proving our main theorems. The above concepts will be defined in the following chapters.

For a long time, the field of simplicial complex investigation provides a lot of techniques and methods for other areas of mathematics. Topologists are glad to represent the topological spaces, using the simplicial complex. In 1937, Freudenthal [18] proved that every completely metrizable space can be represented as the inverse limit of locally finite-dimensional polyhedra with the metric toplogy. It follows from [31] that every paracompact space is homeomorphic to the inverse limit of an inverse system of polyhedra with the weak topology, but this does not imply that every metrizable space is homeomorphic to the inverse limit of an inverse sequence of polyhedra with the metric topology.

Now, we consider the following extension property:

(e_K) There exists $\alpha > 0$ such that for any locally finite countable simplicial complex K, each map $f : K^{(0)} \to Z$ extends to a map $\overline{f} : |K| \to Z$ such that $\operatorname{diam} \overline{f}(\sigma) \leq \alpha \operatorname{diam} f(\sigma^{(0)})$ for every $\sigma \in K$.

Using the property e_K of simplicial complex, a characterizations of uniform ANR was given by Michael [25] in 1979. The gap between the uniform ANR and homotopy dense was bridged in the paper Saki [32]. Using the connection between homotopy dense and simplicial complex, we prove that $\downarrow \mathrm{C}(X, Y)$ is homotopy dense in the closure of $\downarrow \mathrm{C}(X, Y)$ with Fell topology when X is a compact metric space and Y is a dendrite in Chapter 3.

The polyhedra with metric topology and weak topology is investigated by many topologists, for example the result of Freudenthal [18] and Morita [31]. A new book [33] written by Sakai gave a good survey. The box topology is important in study of topology of LF spaces (see [28, 29]). In the papers of Sakai and Mine [28, 29], it is proved that every open subset of LF-space with the box topology is homeomorphic to the product of $|K|$ and the LF-space for some locally finite-dimensional $|K|$ with the metric topology. But, no one ever induce the box topology to a simplicial complex before we did it. The polyhedra with the new topology will be discussed in Chapter 4.

1.4 The main theorems

In Chapter 2, we investigate conditions for metrizable of spaces $\downarrow\mathrm{C}_F(X, \mathbf{I})$. We prove

Main Theorem 1. *For a Tychonoff space X, the following conditions are equivalent:*

(a) $\downarrow\mathrm{C}_F(X, \mathbf{I})$ *is separable metrizable;*

(b) $\downarrow\mathrm{C}_F(X, \mathbf{I})$ *is metrizable.*

In case that X is first-countable, the above two conditions are equivalent to

(c) X is a locally compact and separable metrizable space.

We will give other results and examples. They show that some differences between spaces $\downarrow\mathrm{C}_F(X, \mathbf{I})$ and $\downarrow\mathrm{C}_V(X, \mathbf{I})$, see Theorems 2.1.2, 2.1.3, 2.1.4; and that there exist non-first-countable spaces X such that $\downarrow\mathrm{C}_F(X, \mathbf{I})$ are metrizable, see Theorem 2.1.1 and Corollary 2.1.1. All results in Chapter 2 are contained in my paper: Metrization of function spaces with the Fell topology [50].

In Chapter 3, we will give a topological structure of function space from a compact metric space to a dendrite. Our main theorem is

Main Theorem 2. *Let X be an infinite, locally connected, compact metrizable space without isolated points, Y a dendrite and $v \in Y$ an end point of Y. Then*

$$(\overline{\downarrow_v \mathrm{C}_F(X, Y)}, \downarrow_v \mathrm{C}_F(X, Y)) \approx (Q, \mathbf{c}_0).$$

To show Main Theorem 2, we need to use many techniques in infinite-dimensional topology and ANR theory. We also give an example to show that the space $\downarrow_v \mathrm{C}_F(X, Y)$ has a cluster point in $\mathrm{Cld}_F(X \times Y)$ which is not the hypo-graph of any map from X to Y. Our Chapter 3 comes from a joint paper "A function space from a compact metrizable space to a dendrite with the hypo-graph topology" [51] with Professors Sakai and Koshino.

In Chapter 4, we will define a new topology– box topology—in a simplicial complex and give some results on it. It will be shown that if K is locally countable or $\dim K \leqslant 1$, then the box topology of $|K|$ coincides with the weak topology, i.e., $|K|_{\mathrm{b}} = |K|_{\mathrm{w}}$ as spaces (Theorem 4.2.1). In addition, it will be shown that even the barycentric subdivision does not preserve the box topology (Theorem 4.5.1) and that a simplicial map does not need to be continuous with respect to the box topology (Theorem 4.6.2). Our Chapter 4 comes from "The box topology of infinite simplicial complexes" [52] which is a joint paper with Professor Sakai.

Chapter 2

Metrization of Function Spaces with the Fell topology

2.1 The main results of Chapter 2

Yang and Zhou gave a characterization of metrizable of the space ${\downarrow}\mathrm{C}_V(X, \mathbf{I})$ [49].

Proposition 2.1.1. *For a Tychonoff space X, the following conditions are equivalent:*

(a) X is continuum;

(b) ${\downarrow}\mathrm{C}_V(X, \mathbf{I})$ is second-countable;

(c) ${\downarrow}\mathrm{C}_V(X, \mathbf{I})$ is metrizable.

We will give a characterization of metrizable of ${\downarrow}\mathrm{C}(X, \mathbf{I})$ in the Fell topology as follows.

Main Theorem 1. *For a Tychonoff space X, the following conditions are equivalent:*

(a) ${\downarrow}\mathrm{C}_F(X, \mathbf{I})$ is separable metrizable;

(b) ${\downarrow}\mathrm{C}_F(X, \mathbf{I})$ is metrizable.

In case X is first-countable, the above two conditions are equivalent to

(c) X is a locally compact and separable metrizable space.

The following theorem and corollary show that the first-countability of X is essential for the equivalence between (a) and (c) in Main Theorem 1. The following Theorem 2.1.2 tells us that, the non-compact case is very different from the compact case.

Theorem 2.1.1. *Let $\bigoplus_{s\in S} Y_s$ be the topological sum of Tychonoff spaces Y_s, $s \in S$, and $a_s \in Y_s$ a non-isolated point for every $s \in S$. And let Y be the quotient space of $\bigoplus_{s\in S} Y_s$ with the set $\{a_s |\ s \in S\}$ identified to a point. Then $\downarrow \mathrm{C}_F(Y, \mathbf{I})$ is homeomorphic to a subspace of the product space $\prod_{s\in S} \downarrow \mathrm{C}_F(Y_s, \mathbf{I})$.*

Applying this theorem, we show the following.

Corollary 2.1.1. *There exists a Tychonoff space X such that $\downarrow \mathrm{C}_F(X, \mathbf{I})$ is separable metrizable but X is not first-countable.*

Theorem 2.1.2. *There exists a countable Tychonoff space X such that $\downarrow \mathrm{C}_F(X, \mathbf{I})$ is Hausdorff and second-countable but not regular.*

In [6, 5.1.2 Proposition], it was proved that, for a Tychonoff space X, the following conditions are equivalent: (a) $\mathrm{Cld}_F(X)$ is Hausdorff, (b) $\mathrm{Cld}_F(X)$ is regular, (c) $\mathrm{Cld}_F(X)$ is Tychonoff, and (d) X is locally compact. Theorem 2.1.2 shows that we cannot replace $\mathrm{Cld}_F(X)$ by $\downarrow \mathrm{C}_F(X, \mathbf{I})$ in [6, 5.1.2 Proposition].

The following Theorem 2.1.3 states that, even for a compact space X, the regularity and the first-countability of $\downarrow \mathrm{C}_F(X, \mathbf{I})$ do not imply the metrizability of it.

Theorem 2.1.3. *There exists a compact space X such that $\downarrow \mathrm{C}_F(X, \mathbf{I})$ is Tychonoff, separable and first-countable but not metrizable.*

Finally, we will give a necessary condition for the metrizability of $\downarrow \mathrm{C}_F(X, \mathbf{I})$.

Theorem 2.1.4. *For a Tychonoff space X, if $\downarrow \mathrm{C}_F(X, \mathbf{I})$ is metrizable, then there exists a dense, locally compact, open and separable metrizable subspace of X. But the converse is not true.*

2.2 Preparatory results of metrization

For a closed set F in Y, let

$$F^* = (Y \setminus F)^+ = \{A \in \mathrm{Cld}(Y) |\ A \cap F = \emptyset\}.$$

By the definition, the topology of $\downarrow\mathrm{C}_F(X, \mathbf{I})$ is generated, as a base, by the following sets:

$$\bigcap_{i=1}^{n} G_i^- \cap K^* \cap \downarrow\mathrm{C}(X, \mathbf{I}),$$

where $G_1, G_2, \cdots, G_n$ are open sets in $X \times (0, 1]$ and K is a compact set in $X \times (0, 1]$. We use $\underline{s}$ to denote the constant function from X to $\mathbf{I}$ which maps all elements to $s \in \mathbf{I}$. $p : X \times \mathbf{I} \to X$ is the projection.

Specially,

$$\left\{\bigcap_{i=1}^{n} G_i^- \cap \downarrow\mathrm{C}_F(X, \mathbf{I}) |\ G_1, \cdots, G_n \text{ are nonempty open in } X \times (0, 1]\right\}$$

$$\text{and } \{K^* \cap \downarrow\mathrm{C}_F(X, \mathbf{I}) |\ K \text{ is compact in } X \times (0, 1]\}$$

are neighborhood bases at $\downarrow\underline{1}$ and $\downarrow\underline{0}$ in $\downarrow\mathrm{C}_F(X, \mathbf{I})$, respectively.

To prove theorems, we need some lemmas. At first, we show the following lemma.

Lemma 2.2.1. *For a space X, the following hold:*

(1) $\downarrow\mathrm{C}_F(X, \mathbf{I})$ is T_1;

(2) $\downarrow\mathrm{C}_F(X, \mathbf{I})$ is Hausdorff if and only if there exists a dense open subset U of X which is locally compact.

Proof. (1): Let $f \neq g \in \mathrm{C}(X, \mathbf{I})$. We may assume that $f(x_0) < g(x_0)$ for some $x_0 \in X$. Then x_0 has an open neighborhood W such that $f(x) < a < g(x)$ for every $x \in W$, where $a = \frac{f(x_0)+g(x_0)}{2}$. Thus $\downarrow f \in (\{x_0\} \times [a, 1])^* \not\ni \downarrow g$ and $\downarrow g \in (W \times (a, 1])^- \not\ni \downarrow f$.

(2): The "if"part: Take $f, g \in \mathrm{C}(X, \mathbf{I})$, $x_0 \in W$ and $a \in \mathbf{I}$ as the same as (1). Since f, g is continuous, we assume that $x_0 \in U$. Because U is locally compact, we have an open set V in X such that $x_0 \in V \subset \mathrm{cl}\, V \subset U \cap W$ and $\mathrm{cl}\, V$ is compact. Since

$f(x) < a < g(x)$ for $x \in \mathrm{cl}\, V$, $(\mathrm{cl}\, V \times [a,1])^* \cap \downarrow\mathrm{C}_F(X,\mathbf{I})$ and $(V \times (a,1])^- \cap \downarrow\mathrm{C}_F(X,\mathbf{I})$ are disjoint neighborhoods of $\downarrow f$ and $\downarrow g$, respectively.

The "only if" part: We define an open set

$$U = \bigcup \{\mathrm{int}\, K \mid K \text{ is compact in } X\} \subset X.$$

Then U is locally compact. We show that U is dense in X. Assume that U is not dense in X. Then there exists a nonempty open set V in X such that the interior of every compact subset of V is empty. Because X is Tychonoff, we can choose $f \in \mathrm{C}(X,\mathbf{I})$ such that $f(X \setminus V) \subset \{1\}$ and $f(x_0) = 0$ for some $x_0 \in V$. Since $\downarrow\mathrm{C}_F(X,\mathbf{I})$ is Hausdorff, there exist disjoint open sets $\mathcal{U}$ and $\mathcal{V}$ in $\downarrow\mathrm{C}_F(X,\mathbf{I})$ such that $\downarrow\underline{1} \in \mathcal{U}$ and $\downarrow f \in \mathcal{V}$. Then we can find nonempty open sets $G_1, G_2, \cdots, G_n, \cdots, G_m \subset X \times (0,1]$ and a compact set $K \subset X \times (0,1]$ such that

$$\downarrow\underline{1} \in G_1^- \cap G_2^- \cap \cdots \cap G_n^- \cap \downarrow\mathrm{C}_F(X,\mathbf{I}) \subset \mathcal{U} \quad \text{and}$$
$$\downarrow f \in G_{n+1}^- \cap \cdots \cap G_m^- \cap K^* \cap \downarrow\mathrm{C}_F(X,\mathbf{I}) \subset \mathcal{V}.$$

Since $f(X \setminus V) \subset \{1\}$, it follows that $p(K) \subset V$, which implies that $\mathrm{int}\, p(K) = \emptyset$. For every $i \le m$, $p(G_i) \setminus p(K) \neq \emptyset$ since $p(G_i)$ is a nonempty open set in X. Take $x_i \in p(G_i) \setminus p(K)$. Because X is Tychonoff, we have $g \in \mathrm{C}(X,\mathbf{I})$ satisfying

$$g(x_i) = 1 \ \text{ for } i \le m \text{ and } g(p(K)) = \{0\}.$$

Then $\downarrow g \in \mathcal{U} \cap \mathcal{V}$, which contradicts that $\mathcal{U} \cap \mathcal{V} = \emptyset$. □

Lemma 2.2.2. *If $\downarrow\mathrm{C}_F(X,\mathbf{I})$ is first-countable, then there exists compact sets $C_1 \subset C_2 \subset \cdots$ in X such that every compact set in X is contained in some C_n. In particular, $X = \bigcup_{n=1}^{\infty} C_n$.*

Proof. Because $\downarrow\mathrm{C}_F(X,\mathbf{I})$ is first-countable, we can find compact sets $K_1 \subset K_2 \subset \cdots$ in $X \times (0,1]$ such that $\{K_n^* \cap \downarrow\mathrm{C}_F(X,\mathbf{I}) \mid n = 1,2,\cdots\}$ is a neighborhood base of $\downarrow\underline{0}$ in $\downarrow\mathrm{C}_F(X,\mathbf{I})$. Then $C_n = p(K_n)$, $n = 1,2,\cdots$, are the desired compact sets in X. We verify that every compact set C in X is contained in some C_n. Otherwise, for every n, we can choose $x_n \in C \setminus C_n$ and define $f_n \in \mathrm{C}_F(X,\mathbf{I})$ such that $f_n(x_n) = 1$ and $f_n(C_n) = \{0\}$. Then $\downarrow f_n \in K_n^*$ for every n and hence $\downarrow f_n \to \downarrow\underline{0}$ in $\downarrow\mathrm{C}_F(X,\mathbf{I})$.

But every $\downarrow f_n$ is not contained in the neighborhood $(C \times \{1\})^*$ of $\downarrow \underline{0}$, which is a contradiction. □

Lemma 2.2.3. *If X and $\downarrow \mathrm{C}_F(X, \mathbf{I})$ are first-countable, then X is locally compact.*

Proof. Suppose there exists $x_0 \in X$, which has no compact neighborhood. Because X is first-countable, x_0 has a countable open neighborhood base $\{U_n | \ n = 1, 2, \cdots\}$, where $U_n \supset U_{n+1}$ for every n. Since $\downarrow \mathrm{C}_F(X, \mathbf{I})$ is also first-countable, we can find compact sets $K_1 \subset K_2 \subset \cdots$ in $X \times (0, 1]$ such that $\{K_n^* \cap \downarrow \mathrm{C}_F(X, \mathbf{I}) | \ n = 1, 2, \cdots\}$ is a neighborhood base at $\downarrow \underline{0}$ in $\downarrow \mathrm{C}_F(X, \mathbf{I})$. By the assumption, $p(K_n) \not\supset U_n$ for every $n = 1, 2, \cdots$, hence we can take $x_n \in U_n \setminus p(K_n)$. Then $x_n \to x_0$ in X. Since X is Tychonoff, we have $f_n \in \mathrm{C}(X, \mathbf{I})$ such that

$$f_n(x_n) = 1 \quad \text{and} \quad f_n(p(K_n) \cup (X \setminus U_n)) = \{0\}.$$

Then $\downarrow f_n \in K_n^*$ and hence $\downarrow f_n \to \downarrow \underline{0}$. On the contrary,

$$(\{x_n | \ n = 0, 1, 2, \cdots\} \times \{1\})^* \cap \downarrow \mathrm{C}_F(X, \mathbf{I})$$

is a neighborhood of $\downarrow \underline{0}$ in $\downarrow \mathrm{C}_F(X, \mathbf{I})$ which does not contain any $\downarrow f_n$. □

When X is locally compact and non-compact, let $\alpha X = X \cup \{\infty\}$ be the one-point compactification of X. Using Lemmas 2.2.2 and 2.2.3, we may prove the following

Proposition 2.2.1. *If X and $\downarrow \mathrm{C}_F(X, \mathbf{I})$ are first-countable, then*

(1) X is locally compact and αX is also first-countable;

(2) $\downarrow \mathrm{C}_F(\alpha X, \mathbf{I})$ is first-countable;

(3) $\downarrow \mathrm{C}_F(\alpha X, \mathbf{I})$ is second-countable if $\downarrow \mathrm{C}_F(X, \mathbf{I})$ is second-countable.

Proof. The assertion (1) directly follows from Lemmas 2.2.2 and 2.2.3. To show (2) and (3), we only consider the case that X is not compact. Let $\{U_n | \ n = 1, 2, \cdots\}$ be a countable open neighborhood base at ∞ in αX, and let $\phi : \mathrm{C}_F(\alpha X, \mathbf{I}) \to \mathrm{C}_F(X, \mathbf{I})$ be the restriction, that is,

$$\phi(f) = f|X \quad \text{for every } f \in \mathrm{C}_F(\alpha X, \mathbf{I}).$$

Then it is not hard to verify that $\downarrow\phi : \downarrow C_F(\alpha X, \mathbf{I}) \to \downarrow C_F(X, \mathbf{I})$ is a continuous injection. Unfortunately, it is not an embedding. However, the following $\mathcal{S}$ is a subbase of $\downarrow C_F(\alpha X, I)$:

$$\mathcal{S} = \{(\downarrow\phi)^{-1}(G) \mid G \in \mathcal{G}\} \cup \{(\mathrm{cl}_{\alpha X} U_n \times [r,1])^* \cap \downarrow C_F(\alpha X, \mathbf{I}) \mid r \in \mathbb{Q} \cap (0,1], n = 1, 2, \cdots\},$$

where $\mathcal{G}$ is an open base for $\downarrow C_F(X, \mathbf{I})$. Obviously, $\mathcal{S}$ is a subfamily of the topology of $\downarrow C_F(\alpha X, \mathbf{I})$. For every open set V in $\alpha X \times \mathbf{I}$, $V \cap (X \times \mathbf{I})$ is open in $X \times \mathbf{I}$ and

$$V^- \cap \downarrow C_F(\alpha X, \mathbf{I}) = (\downarrow\phi)^{-1}((V \cap (X \times \mathbf{I}))^- \cap \downarrow C_F(\alpha X, \mathbf{I})).$$

For every compact set K in $\alpha X \times (0,1]$, if $K \cap (\{\infty\} \times \mathbf{I}) = \emptyset$, then K is also compact in $X \times \mathbf{I}$ and

$$K^* \cap \downarrow C_F(\alpha X, \mathbf{I}) = (\downarrow\phi)^{-1}(K^* \cap \downarrow C_F(X, \mathbf{I})).$$

If $K \cap (\{\infty\} \times \mathbf{I}) \neq \emptyset$, then for every $\downarrow f \in K^* \cap \downarrow C_F(\alpha X, \mathbf{I})$, using the Wallace's Theorem, there exist n and a rational number $r \in (0,1]$ such that

$$(\mathrm{cl}_{\alpha X} U_n \times [r,1]) \cap \downarrow f = \emptyset \quad \text{and}$$
$$K \cap (\mathrm{cl}_{\alpha X} U_n \times \mathbf{I}) \subset \mathrm{cl}_{\alpha X} U_n \times [r,1].$$

Let

$$K_1 = (K \cap ((\alpha X \setminus U_n) \times \mathbf{I})) \cup (\mathrm{cl}_{\alpha X} U_n \times [r,1]).$$

Then K_1 is compact in $\alpha X \times (0,1]$, $K_1 \supset K$ and $K_1 \cap \downarrow f = \emptyset$. Thus, $\downarrow f \in K_1^* \subset K^*$. Note that

$$K_1^* \cap \downarrow C_F(\alpha X, \mathbf{I}) = (\downarrow\phi)^{-1}((K \cap ((\alpha X \setminus U_n) \times \mathbf{I}))^*) \cap (\mathrm{cl}(U_n) \times [r,1])^* \cap \downarrow C_F(\alpha X, \mathbf{I}),$$

that is, $K_1^* \cap \downarrow C_F(\alpha X, \mathbf{I})$ is an intersection of two elements of $\mathcal{S}$.

As a conclusion, $\mathcal{S}$ is a subbase for $\downarrow C_F(\alpha X, \mathbf{I})$. Therefore, $\downarrow C_F(\alpha X, \mathbf{I})$ is first-countable. Moreover, $\downarrow C_F(\alpha X, \mathbf{I})$ is second-countable if $\downarrow C_F(X, \mathbf{I})$ is second-countable. Hence (2) and (3) hold. □

Lemma 2.2.4. *We consider the following statements:*

(a) $\downarrow\mathrm{C}_F(X,\mathbf{I})$ *is first-countable.*

(b) $\downarrow\mathrm{C}_F(X,\mathbf{I})$ *has a countable neighborhood base at* $\downarrow\underline{1}$.

(c) There exists a countable family $\mathcal{U}$ *of nonempty open sets in* X *such that every nonempty open set in* X *includes an element of* $\mathcal{U}$*, that is,* $\mathcal{U}$ *is a countable* π*-base for* X.

(d) $\downarrow\mathrm{C}_F(X,\mathbf{I})$ *is separable.*

Then the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) *hold.*

Furthermore, when X *is compact, the implication* (c) $\Rightarrow$ (a) *holds and hence* (a), (b) *and* (c) *are equivalent.*

Proof. The implication (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (c): We may assume that

$$\{(G_1^n)^- \cap (G_2^n)^- \cap \cdots \cap (G_{k(n)}^n)^- \cap \downarrow\mathrm{C}_F(X,\mathbf{I}) |\ n = 1, 2, \cdots\}$$

is a countable neighborhood base at $\downarrow\underline{1}$ in $\downarrow\mathrm{C}_F(X,\mathbf{I})$. Let

$$\mathcal{U} = \{p(G_i^n) |\ i = 1, 2, \cdots, k(n), n = 1, 2, \cdots\}.$$

Then $\mathcal{U}$ is a countable family of nonempty open sets in X. We show that every nonempty open set U in X includes an element of $\mathcal{U}$. Take $f \in \mathrm{C}(X,\mathbf{I})$ such that $f(X \setminus U) \subset \{1\}$ and $f(x_0) = 0$ for some point $x_0 \in U$. Because $\downarrow\mathrm{C}_F(X,\mathbf{I})$ is T_1 by Lemma 2.2.1(1), $\downarrow f \notin \bigcap_{i=1}^{k(n)} (G_i^n)^-$ for some n, hence $\downarrow f \notin (G_i^n)^-$ for some $i \leq k(n)$. Then $\downarrow f \cap G_i^n = \emptyset$. Since $f(X \setminus U) \subset \{1\}$, we have $U \supset p(G_i^n)$, as required.

(c) $\Rightarrow$ (d): Let $\mathcal{U}$ be a countable π-base for X. For every $U \in \mathcal{U}$ and $r \in \mathbb{Q} \cap (0,1]$, we can take a continuous map $f_{(U,r)} : X \to [0,r]$ such that $f_{(U,r)}(X \setminus U) \subset \{0\}$ and $f_{(U,r)}(x) = r$ for some $x \in U$. Let

$$D = \{\max\{f_{(U,r)} | U \in \mathcal{F}, r \in F\} : \mathcal{F} \text{ and } F \text{ are finite subsets of } \mathcal{U} \text{ and } \mathbb{Q} \cap (0,1], \text{ resp.}\}.$$

Then $\downarrow D = \{\downarrow f \mid f \in D\}$ is a countable subset of $\downarrow C_F(X, \mathbf{I})$. It remains to verify that $\downarrow D$ is dense in $\downarrow C_F(X, \mathbf{I})$. Let $f \in C(X, \mathbf{I})$, K be compact in $X \times (0, 1]$ and G_i, $i \leq k$, open in $X \times (0, 1]$, such that

$$\downarrow f \in G_1^- \cap G_2^- \cap \cdots \cap G_k^- \cap K^* \cap \downarrow C_F(X, \mathbf{I}).$$

We have $x_1, \cdots, x_k \in$ X such that $\{x_i\} \times [0, f(x_i)] \cap G_i \neq \emptyset$ for each $i \leq k$. Because $\{x_i\} \times [0, f(x_i)] \cap K = \emptyset$, we have an open neighborhood W_i of x_i in X and $s_i < t_i$ such that $W_i \times (s_i, t_i) \subset G_i$ and $W_i \times [0, t_i] \cap K = \emptyset$. Thus, by (c), choose $r_i \in \mathbb{Q} \cap (s_i, t_i)$ and $U_i \in \mathcal{U}$ such that $U_i \subset W_i$. Then $\downarrow f_{(U_i, r_i)} \in G_i^- \cap K^*$ and hence

$$\downarrow \max\{f_{(U_i, r_i)} \mid i \leq k\} \in \downarrow D \cap G_1^- \cap G_2^- \cap \cdots \cap G_k^- \cap K^*.$$

Now, we show (c) $\Rightarrow$ (a) under the assumption that X is compact. Let $\mathcal{U}$ be a countable π-base of X. Then, $X \times \mathbf{I}$ has the following countable π-base:

$$\mathcal{G} = \{U \times (s, t) \mid U \in \mathcal{U}, s < t \in \mathbb{Q} \cap (0, 1)\}.$$

For every $f \in C(X, \mathbf{I})$ and $n = 1, 2, \cdots$, let

$$\mathcal{G}(f) = \{G \in \mathcal{G} \mid \downarrow f \in G^-\}, \quad K_n(f) = \{(x, t) \in X \times \mathbf{I} \mid t \geq f(x) + n^{-1}\}.$$

For every open set H in $X \times (0, 1]$ with $H^- \ni \downarrow f$, there exists $x_0 \in X$ such that $\{x_0\} \times [0, f(x_0)] \cap H \neq \emptyset$. Since $f(x_0) > 0$, we can find an open neighborhood V of x_0 in X and $s < t \in \mathbb{Q} \times (0, 1)$ such that $s < f(x_0)$, $V \times (s, t) \subset H$ and $s < f(x)$ for every $x \in V$. Since $\mathcal{U}$ is a π-base for X, V contains some $U \in \mathcal{U}$. Then we have $G = U \times (s, t) \in \mathcal{G}$ and $\downarrow f \in G^- \subset H^-$. Moreover, for every compact set K in $X \times \mathbf{I}$ with $K^* \ni \downarrow f$, by the compactness of X, there exists n such that $K_n(f) \supset K$ and hence $\downarrow f \in K_n(f)^* \subset K^*$. Therefore,

$$\{G_1^- \cap \cdots \cap G_k^- \cap K_n(f)^* \cap \downarrow C_F(X, \mathbf{I}) \mid G_i \in \mathcal{G}(f) \text{ for } i \leq k, k, n = 1, 2, \cdots\}$$

is a countable neighborhood base at $\downarrow f$ in $\downarrow C_F(X, \mathbf{I})$. $\square$

As a consequence of Lemma 4, we have the equivalence between (a) and (b) in Main Theorem 1, that is,

Proposition 2.2.2. *The space* $\downarrow\mathrm{C}_F(X,\mathbf{I})$ *is metrizable if and only if it is separable metrizable.*□

The following two propositions were proved in [48], [49], respectively.

Proposition 2.2.3. *If* V *is open in* X *such that* $\mathrm{cl}\,V$ *is compact, then the restriction* $\phi :\downarrow\mathrm{C}_F(X,\mathbf{I}) \to\downarrow\mathrm{C}_F(\mathrm{cl}\,V,\mathbf{I})$ *defined by* $\phi(\downarrow f) =\downarrow f|\,\mathrm{cl}\,V$ *is a continuous open surjection.*□

Proposition 2.2.4. *If* X *is compact and* $\downarrow\mathrm{C}_F(X,\mathbf{I}) =\downarrow\mathrm{C}_V(X,\mathbf{I})$ *is second-countable, then* X *is metrizable.*□

Metrizabliliy of $\mathrm{Cld}(X)$ with Fell topology was characterized by Flachsmeyer [6, Theorem 5.1.4]

Proposition 2.2.5. *Let* X *be a locally compact Hausdorff space. The following are equivalent:*

(a) X *is locally compact and second countable;*

(b) $\mathrm{Cld}_F(X)$ *is a Polish space;*

(c) $\mathrm{Cld}_F(X)$ *is metrizable;*

2.3 Proofs of main results in Chapter 2

In this section, we show our main results in Chapter 2.

Proof of Main Theorem 1. The equivalence between (a) and (b) is Proposition 2.2.2. If X is first-countable, then X is locally compact by Proposition 2.2.1(1). Using Proposition 2.2.1(3), the condition (b) implies that $\downarrow\mathrm{C}(\alpha X)$ is second-countable. It follows from Lemma 2.2.4 that αX is metrizable. Hence the condition (c) holds. That is, the implication (b)⇒(c) holds under the assumption that X is first-countable. By the proposition 2.2.5, the condition (c) implies that $\mathrm{Cld}_F(X \times \mathbf{I})$ is metrizable, hence so is $\downarrow\mathrm{C}_F(X,\mathbf{I})$, i.e., (b) holds. Therefore, the implication (c)⇒(b) holds.□

Proof of Theorem 2.1.1. We may think that every Y_s is a subspace of Y. Define $\phi : \mathrm{C}(Y, \mathbf{I}) \to \prod_{s\in S} \mathrm{C}(Y_s, \mathbf{I})$ by

$$\phi(f) = (f|Y_s)_{s\in S} \quad \text{for each } f \in \mathrm{C}(Y, \mathbf{I}).$$

Evidently, ϕ is an injection and its image is

$$\phi(\mathrm{C}(Y, \mathbf{I})) = \left\{ g \in \prod_{s\in S} \mathrm{C}(Y_s, \mathbf{I})|\ g(s)(a_s) = g(s')(a_{s'}) \text{ for } s, s' \in S \right\}.$$

Now we show that $\downarrow\phi : \downarrow\mathrm{C}_F(Y, \mathbf{I}) \to \prod_{s\in S} \downarrow\mathrm{C}_F(Y_s, \mathbf{I})$ is an embedding. Let $p_s : \prod_{s\in S} \downarrow\mathrm{C}_F(Y_s, \mathbf{I}) \to \downarrow\mathrm{C}_F(Y_s, \mathbf{I})$ be the projection.

To show the continuity of $\downarrow\phi$, it is sufficient to verify that $p_s\circ \downarrow\phi$ is continuous for every $s \in S$. For every open set G in $Y_s \times (0, 1]$, $G \setminus (\{a_s\} \times \mathbf{I})$ is open in $Y \times (0, 1]$. Since a_s is a non-isolated point in Y_s,

$$(p_s\circ \downarrow\phi)^{-1}(G^- \cap \downarrow\mathrm{C}(Y_s, \mathbf{I})) = (G \setminus (\{a_s\} \times \mathbf{I}))^- \cap \downarrow\mathrm{C}_F(Y, \mathbf{I}).$$

For each compact set K in $Y_s \times (0, 1]$,

$$(p_s\circ \downarrow\phi)^{-1}(K^* \cap \downarrow\mathrm{C}(Y_s, \mathbf{I})) = K^* \cap \downarrow\mathrm{C}_F(Y, \mathbf{I}).$$

Hence, $p_s\circ \downarrow\phi : \downarrow\mathrm{C}_F(Y, \mathbf{I}) \to \downarrow\mathrm{C}_F(Y_s, \mathbf{I})$ is continuous for every $s \in S$.

Moreover, for every open set H in $Y \times (0, 1]$, if $\downarrow f \in H^- \downarrow\mathrm{C}_F(Y, \mathbf{I})$, then there exists $s \in S$ such that $\downarrow f|Y_s \in (H \cap (Y_s \times \mathbf{I}))^-$. Hence

$$\downarrow\phi(H^- \cap \downarrow\mathrm{C}_F(Y, \mathbf{I})) = \bigcup_{s\in S} \left((H \cap (Y_s \times \mathbf{I}))^- \times \prod_{t\in S\setminus\{s\}} \downarrow\mathrm{C}_F(Y_t, \mathbf{I}) \right) \cap \downarrow\phi(\downarrow(\mathrm{C}_F(Y, \mathbf{I}))).$$

It shows that $\downarrow\phi(H^- \cap \downarrow\mathrm{C}_F(Y, \mathbf{I}))$ is open in $\downarrow\phi(\downarrow(\mathrm{C}_F(Y, \mathbf{I})))$. For every compact set K in $Y \times (0, 1]$, there exists a finite subset S_0 of S such that $K \subset \bigcup_{s\in S_0} Y_s \times (0, 1]$. Then $K \cap Y_s \times (0, 1]$ is compact for every $s \in S_0$ and

$$\downarrow\phi(K^* \cap \downarrow\mathrm{C}_F(Y, \mathbf{I})) = \left(\prod_{s\in S_0} (K \cap Y_s \times (0, 1])^* \times \prod_{s\in S\setminus S_0} \downarrow\mathrm{C}_F(Y_s, \mathbf{I}) \right) \cap \downarrow\phi(\downarrow\mathrm{C}_F(Y, \mathbf{I})).$$

It follows that $\downarrow\phi(K^* \cap \downarrow\mathrm{C}_F(Y, \mathbf{I}))$ is open in $\downarrow\phi(\downarrow(\mathrm{C}_F(Y.\mathbf{I})))$. Since ϕ is one-to-one, we have that $\downarrow\phi$ maps every open set in $\downarrow\mathrm{C}_F(Y, \mathbf{I})$ to an open set in $\downarrow\phi(\downarrow(\mathrm{C}_F(Y, \mathbf{I})))$.

Therefore, $\downarrow\phi : \downarrow\mathrm{C}_F(Y, \mathbf{I}) \to \prod_{s\in S} \downarrow\mathrm{C}_F(Y_s, \mathbf{I})$ is an embedding. □

Remark 2.3.1. Even for a set S of two points, if a_s is an isolated point in Y_s for some s, the map $\downarrow\phi$ defined in the above proof must not be continuous. For example, let $Y_1 = \{1\} \times (\{0\} \cup [1,2]), Y_2 = \{2\} \times \mathbf{I}$ as subspaces of $\mathbb{R}^2$. If we think that $a_1 = (1,0), a_2 = (2,0)$, then $p_1\circ \downarrow\phi : \downarrow \mathrm{C}_F(Y,\mathbf{I}) \to \downarrow \mathrm{C}_F(Y_1,\mathbf{I})$ is not continuous. In fact, choose $f_n \in \mathrm{C}_F(Y,\mathbf{I})$ such that $f_n(2,0) = f_n(1,0) = 0$ and $f_n(x) = 1$ for every $x \in Y \setminus (\{2\} \times [0, n^{-1}])$. Then $\downarrow f_n \to \downarrow \underline{1}$ but $(p_1\circ \downarrow\phi)(\downarrow f_n) \not\to (p_1\circ \downarrow\phi)(\downarrow \underline{1})$.

Proof of Corollary 2.1.1. Let $\{Y_n : n = 1, 2, \cdots\}$ be a family of pair-disjoint locally compact separable metrizable spaces Y_n with a non-isolated point a_n. Then, by Theorems 2.1.1 and 2.1.2, the space Y defined in Theorem 2.1.2 is as required. □

Tychonoff Theorem [33, Theorem 2.1.1] was given in 1930.

Proposition 2.3.1. [Tychonoff] *The product of any collection of compact topological spaces is compact.*

We defined the evaluation map $e_X : X \to \mathbf{I}^{\mathrm{C}(\mathrm{X},\mathbf{I})}$ by $e_X(x) = (f(x))_{f\in \mathrm{C}(\mathrm{X},\mathbf{I})}$. From Tychonoff's Theorem, it follows that the product space $\mathbf{I}^{\mathrm{C}(\mathrm{X},\mathbf{I})}$ is compact. Then identifying X with $e_X(X)$, we define a compactification βX of X as follows:

$$\beta X = \mathrm{cl}_{\mathbf{I}^{\mathrm{C}(\mathrm{X},\mathbf{I})}}\, e_X(X),$$

which is called the *Stone-Čech compactification.*

The Stone-Čech compactification βX [33, Theorem 2.1.4] can be characterized as follows:

Proposition 2.3.2. [Stone-Čech] *Let X be a Tychonoff space. For any compactification γX of X, there exists the (unique) map $f : \beta X \to \gamma X$ such that $f|X = \mathrm{id}_X$. If a compactification $\beta' X$ has the same property as above, then there exists a homeomorphism $h : \beta X \to \beta' X$ such that $h|X = \mathrm{id}_X$.*

Using Stone-Čech compactification, we shall prove the Theorem 2.1.2.

Proof of Theorem 2.1.2. Let $\beta\omega$ be the Stone-Čech compactification of the discrete space ω of non-negative integers and $q \in \beta\omega \setminus \omega$. Then the subspace $X = \omega \cup \{q\}$ of $\beta\omega$ satisfies the conditions in Theorem 2.1.3. By Lemma 2.2.1(2), $\downarrow \mathrm{C}_F(X,\mathbf{I})$ is Hausdorff.

Before showing that $\downarrow C_F(X,\mathbf{I})$ is second-countable but not regular, we verify that every compact subset of X is finite. In fact, let C be an infinite compact subset of X. Then $q \in C$. Write $C = A \cup B \cup \{q\}$ such that A and B are disjoint infinite subsets of ω. Define a continuous map $f : \omega \to \{0,1\}$ as $f^{-1}(0) = A$. Then there exists a continuous extension $\overline{f} : X \to \{0,1\}$ since X is a subspace of $\beta\omega$. If $\overline{f}(q) = 0$, then B is closed in X and hence is compact. But it is impossible since B is infinite discrete. If $\overline{f}(q) = 1$, then A is closed in X and hence is compact. It is also impossible since A is also infinite discrete.

New, we define a product space $Y = \prod_{x\in X}\mathbf{I}_x$, where $\mathbf{I}_x$ is a copy of the unit interval $[0,1]$ with the usual topology for $x \in \omega$ and $\mathbf{I}_q$ is $[0,1]$ with the topology generated by $\{[0,r) : r \in [0,1]\cap\mathbb{Q}\} \cup \{[0,1]\}$. Then Y is second-countable. We may regard $\downarrow C(X,\mathbf{I}) \subset Y$ by identifying $\downarrow f$ with $(f(x))_{x\in X}$ for every $f \in C(X,\mathbf{I})$. To show that $\downarrow C_F(X,\mathbf{I})$ is second-countable, it suffices to verify that $\downarrow C_F(X,\mathbf{I})$ is the subspace of the space Y. It is easy to see that for each $x \in X$, the map $p_x :\downarrow C_F(Y,\mathbf{I}) \to \mathbf{I}_x$ defined by $p_x(\downarrow f) = f(x)$ is continuous. Hence the subspace topology is coarser than the Fell topology on $\downarrow C(X,\mathbf{I})$. Conversely, take a compact set $K \subset X\times(0,1]$ and $f \in C(X,\mathbf{I})$. Then $p(K)$ is compact in X . Then $p(K)$ is a finite set in X and $\downarrow f \cap K = \emptyset$ if and only if $f(x) < m(x) = \min\{s : (x,s)\in K\}$ for every $x \in p(X)$. Hence we can identify

$$K^*\cap \downarrow C(X,\mathbf{I}) = \left(\prod_{x\in p(K)}[0,m_x)\times\prod_{x\in X\setminus p(K)}\mathbf{I}_x\right)\cap \downarrow C(X,\mathbf{I})$$

is open in the subspace topology of Y. For every open set G in $X\times(0,1]$ and $f \in C(X,\mathbf{I})$, $\downarrow f\cap G \neq \emptyset$ if and only if $\downarrow f \cap G\setminus(\{q\}\times\mathbf{I}) \neq \emptyset$ if and only if $f(n) > s_n$ for some $n \in p(G)\cap\omega$, where $s_n = \inf\{s : (n,s)\in G\}$. Hence

$$G^{-}\cap \downarrow C(X,\mathbf{I}) = \left(\bigcup_{n\in p(G)\cap\omega} p_n^{-1}(s_n,1]\right)\cap \downarrow C(X,\mathbf{I}),$$

where $p_n : Y \to \mathbf{I}_n$ is the projection, is open in the subspace topology of Y. Therefore, $\downarrow C_F(X,\mathbf{I})$ is the subspace of Y.

To show that $\downarrow C_F(X,\mathbf{I})$ is not regular, we consider an open neighborhood $\mathcal{U} = (\{q\}\times[\frac{1}{2},1])^*\cap \downarrow C(X,\mathbf{I})$ of $\downarrow 0$. For every compact set K in $X\times(0,1]$, $p(K)$ is finite.

Define $f \in \mathrm{C}(X, \mathbf{I})$ such that $f^{-1}(0) = p(K) \cap \omega$ and $f^{-1}(1) = X \setminus (p(K) \cap \omega)$. Then $\downarrow f \in \mathrm{cl}_{\downarrow \mathrm{C}_F(X,\mathbf{I})}(K^* \cap \downarrow \mathrm{C}_F(X, \mathbf{I})) \setminus \mathcal{U}$. In fact, every neighborhood of $\downarrow f$ in $\downarrow \mathrm{C}_F(Y, \mathbf{I})$ contains the following neighborhood of $\downarrow f$:

$$\mathcal{G} = G_1^- \cap \cdots \cap G_k^- \cap G^- \cap C^* \cap \downarrow \mathrm{C}_F(X, \mathbf{I}),$$

where $G_i = \{n_i\} \times (s_i, t_i)$ for $1 \leq i \leq k$ and $G = (A \cup \{q\}) \times (s, t)$ are open and C is compact in $X \times (0, 1]$. Then A is an infinite subset of ω and hence we may choose $n_0 \in A \setminus p(K \cup C)$. Now, define $g \in \mathrm{C}(X, \mathbf{I})$ as

$$g(x) = \begin{cases} 0 & \text{if } x \in A \cup \{q\} \setminus \{n_i : 0 \leq i \leq k\}; \\ 1 & \text{if } x = n_0; \\ f(x) & \text{otherwise.} \end{cases}$$

Then it is easy to verify that $\downarrow g \in \mathcal{G} \cap K^*$. This shows that $\downarrow f \in \mathrm{cl}_{\downarrow \mathrm{C}_F(X,\mathbf{I})}(K^* \cap \downarrow \mathrm{C}_F(X, \mathbf{I}))$. Because $f(q) = 1$, we have $\downarrow f \notin \mathcal{U}$. Hence, $\mathrm{cl}_{\downarrow \mathrm{C}_F(X,\mathbf{I})}(K^* \cap \downarrow \mathrm{C}(X, \mathbf{I})) \not\subset \mathcal{U}$ for any compact K in $X \times (0, 1]$. Note that the family of all of such $K^* \cap \downarrow \mathrm{C}_F(X, \mathbf{I})$ is a neighborhood base at $\downarrow \underline{0}$ in $\downarrow \mathrm{C}_F(X, \mathbf{I})$. Therefore, $\downarrow \mathrm{C}_F(X, \mathbf{I})$ is not regular. □

We need an example in [20, Problem 5.M] to show our Theorem 2.1.3.

Example 2.3.1. We show there exist separable Hansdorff compact spaces with a countable π-base which are not metrizable. The first example is well-known $\beta\omega$. The second example is the Helly space H. Let

$$H = \{f \in \mathbf{I}^{\mathbf{I}} \mid f : \mathbf{I} \to \mathbf{I} \text{ is increasing functions}\}.$$

As a subspace of the product space $\mathbf{I}^{\mathbf{I}}$, H is as required.

If $f \in \mathbf{I}^{\mathbf{I}} \setminus H$, it is not nondecreasing. Thus there are pints $x, y \in \mathbf{I}$ such that $x < y$ but $f(x) > f(y)$. If $\epsilon < \frac{1}{2}[f(x) - f(y)]$, then $B(f(x), \epsilon) \times B(f(y), \epsilon) \times \prod_{x,y \notin \mathbf{I}} \mathbf{I}$ is a neighborhood of f. Because the neighborhood of f is disjoint from H, H is a closed subspaces of the compact set $\mathbf{I}^{\mathbf{I}}$, so H is a compact Hausdorff space.

Let

$$A_i = \{\frac{a}{2^i} | a \in [0, 2^i] \cap \mathbb{N}\}.$$

then $\{A_i\}_{i \in \mathbb{N}}$ is an increasing nested sequence of finite subsets of $\mathbf{I}$ and $\bigcup_{i \in \mathbb{N}} \{A_i\}$ is dense in $\mathbf{I}$.

Let

$$Y_i = \{f : \mathbf{I} \to \mathbf{I} \text{ is a map } | f(A_i) \subset \mathbb{Q} \cap \mathbf{I} \text{ and it is linear bewteen } A_i\}.$$

Each Y_i is contable, so is $\bigcup Y_i$. Furthermore, Y is dense in X, so X is separable.

For every $f \in H$, let D be the set of all discontinuous points of f. Choose a countable dense set A in $\mathbf{I} \setminus D$. Then $B = D \cup A$ is countable. For every finite subset F of B and $r \in \mathbb{Q}$, define

$$U(F, r) = \{g \in H \ \ | \ |g(x) - f(x)| < r \text{ for every } \ x \in F\}.$$

Then is not hard to verify that $\{U(F, r) \ |F \in \text{Fin}(B), r \in \mathbb{Q}\}$ is a countable neighborhood base at f. Hence, H is first countable.

Evidently, a first countable separable space has a countable π-base.

The collections of functions f_x defined by

$$f_x(t) = \begin{cases} 0 & \text{if } \ t < x; \\ \frac{1}{2} & \text{if } \ t = x; \\ 1 & \text{if } \ t > x; . \end{cases}$$

is an uncountable discrete subset of H. Thus, this subspace is not second countable, and so is H; furthermore, since H is separable, it is not metrizable.

Proof of Theorem 2.1.3. Choose a compact Hausdorff non-metrizable space X satisfying (c) in Lemma 2.2.2, for example, $\beta\omega$ or Helly space. Then, by Lemma 2.2.2, $\downarrow\mathrm{C}_F(X, \mathbf{I})$ is separable and first-countable. By [24](cf. [6, Proposition 5.1.2]), $\mathrm{Cld}_F(X \times \mathbf{I}) = \mathrm{Cld}_V(X \times \mathbf{I})$ is Tychonoff and hence so is $\downarrow\mathrm{C}_F(X, \mathbf{I})$. Since X is compact and non-metrizable, $\downarrow\mathrm{C}_F(X, \mathbf{I})$ is not second-countable because of Lemma 2.2.4. According to Proposition 2.2.2, if $\downarrow\mathrm{C}_F(X, \mathbf{I})$ is metrizable, then $\downarrow\mathrm{C}_F(X, \mathbf{I})$ is separable metrizable, hence second-countable. Therefore, $\downarrow\mathrm{C}_F(X, \mathbf{I})$ is not metrizable. $\square$

Proof of Theorem 2.1.4. Assume that $\downarrow\mathrm{C}_F(X, \mathbf{I})$ is metrizable, which means that $\downarrow\mathrm{C}_F(X, \mathbf{I})$ is separable metrizable by Lemma 2.2.2. Then $\downarrow\mathrm{C}_F(X, \mathbf{I})$ is second-countable. By Lemma 2.2.1(2), there exists a dense open set U in X such that U is locally compact. To complete the proof, it remains to verify that U is separable metrizable. By Lemma 2.2.2, there exists a countable family $\mathcal{C} = \{C_1, C_2, \cdots\}$ of compact sets in X such that every compact set in X is contained in some C_n. For each n, let $U_n = \operatorname{int}(U \cap C_n)$. Then, $\operatorname{cl} U_n$ is compact because $\operatorname{cl} U_n \subset C_n$. By Lemma 2.2.3, there exists a continuous open surjection from $\downarrow \mathrm{C}_F(X, \mathbf{I})$ onto $\downarrow \mathrm{C}_F(\operatorname{cl} U_n, \mathbf{I})$. Therefore, $\downarrow \mathrm{C}_F(\operatorname{cl} U_n, \mathbf{I})$ is second-countable, hence $\operatorname{cl} U_n$ is compact and metrizable by Lemma 2.2.4. Thus every U_n is also separable metrizable, hence it is second-countable. Moreover, for every $x \in U$, there exists an open set V such that $x \in V$, $\operatorname{cl} V$ is compact and $\operatorname{cl} V \subset U$. Hence there exists n such that $\operatorname{cl} V \subset C_n$. Then, $x \in V \subset \operatorname{int}(U \cap C_n) = U_n$. It follows that $U = \bigcup_{n=1}^{\infty} U_n$. Therefore, U is second-countable, hence it is separable metrizable.

As mentioned in proof of Theorem 2.1.3, $\beta\omega$ is a compact space and $\downarrow \mathrm{C}_F(\beta\omega, \mathbf{I})$ is not metrizable but ω is a dense, locally compact, open and separable metrizable subspace of $\beta\omega$. Namely, the converse is not true.□

Remark 2.3.2. The referee of my paper " Metrization of function spaces with the Fell topology" [51] pointed out that McCoy and Ntantu [23] obtained analogous results in 1992. For example, Theorem 4.12 in [23] is similar to our Main Theorem 1. Our Theorem 2.1.4 for $\downarrow \mathrm{C}_F(X, \mathbf{I})$ is true for $\uparrow \mathrm{C}_F(X, \mathbb{R})$ using Theorems 3.5, 3.7, 4.11 and Example 3.3 in [23] , where $\uparrow \mathrm{C}_F(X, \mathbb{R})$ is the subspace of $\mathrm{Cld}_F(X \times \mathbb{R})$ consisting of the epigraphs

$$\uparrow f = \{(x, s) \in X \times \mathbb{R} |\ f(x) \leq s\} \in \mathrm{Cld}(X \times \mathbb{R}),$$

of all $f \in \mathrm{C}(X, \mathbb{R})$. However our arguments are quite different from their arguments in [23] .

Chapter 3

Homeomorphisms of Function Spaces

3.1 Metric chosen on a dendrite

The following lemma can be easily proved.

Lemma 3.1.1. *Let A, A', B and B' be closed sets in a compact metric space $X = (X, d)$. Then*

$$d_H(A \cup B, A' \cup B') \leq \max\{d_H(A, A'), d_H(B, B')\}.$$

For a metric space $X = (X, d)$, the metric d is said to be *convex* if for each pair of points x and y in X, there exists a mid point $z \in X$ between x and y, i.e., $d(x, z) = d(y, z) = d(x, y)/2$. As is easily observed, when the metric d is convex and complete, there exists an arc from x to y isometric to the segment $[0, d(x, y)]$. It is well known that every Peano continuum admits a convex metric [9], [30], and hence every dendrite does so. The unique arcs in a dendrite have the following property with respect to the convex metric [21].

Lemma 3.1.2. *Let $Y = (Y, d)$ be a dendrite with a convex metric. Then there exists a map $\gamma : Y^2 \times \mathbf{I} \to Y$ such that for any distinct points $x, y \in Y$, the map $\gamma_{x,y} = \gamma(x, y, *) : \mathbf{I} \ni t \mapsto \gamma(x, y, t) \in Y$ is an arc from x to y and the following holds:*

(†) *For each* $x_i, y_i \in Y$, $i = 1, 2$, $d(\gamma_{x_1,y_1}(t), \gamma_{x_2,y_2}(t)) \leq \max\{d(x_1, x_2), d(y_1, y_2)\}$ *for all* $t \in \mathbf{I}$.

From now on, we consider any dendrite Y has an admissible convex metric d_Y and a distinguished end point $\mathbf{0} \in Y$. For simplicity, we write $\downarrow\mathrm{C}(X, Y) = \downarrow_{\mathbf{0}} \mathrm{C}_F(X, Y) = \downarrow_{\mathbf{0}} \mathrm{C}_V(X, Y)$, when X is compact space in this capter.

3.2 The closure of $\downarrow\mathrm{C}(X, Y)$ in $\mathrm{Cld}(X \times Y)$ is an AR

This section is devoted to proving the following theorem:

Theorem 3.2.1. *For every compact metrizable space X without isolated points and every dendrite Y, the space $\overline{\downarrow\mathrm{C}(X, Y)}$ is an AR.*

For each $A \in \mathrm{Cld}_F(X \times Y)$, we define a set-valued function $A : X \to \mathrm{Cld}^*_F(Y)$ as follows:

$$A(x) = \{y \in Y \mid (x, y) \in A\} \in \mathrm{Cld}^*_F(Y),$$

where $\mathrm{Cld}^*_F(Y) = \mathrm{Cld}_F(Y) \cup \{\emptyset\}$.

Remark 3.2.1. The space $\downarrow_v \mathrm{C}_F(X, Y)$ has a cluster point in $\mathrm{Cld}_F(X \times Y)$ which is not the hypo-graph of any map from X to Y. For example, let $X = \mathbf{I}$, $Y = \{0\} \times \mathbf{I} \cup [-1, 1] \times \{1\}$ a triod and $v = (0, 0) \in Y$. Define a closed set A in $X \times Y$ as follows:

$$A = \mathbf{I} \times \{0\} \times \mathbf{I} \cup \{0\} \times [-1, 1] \times \{1\} \cup \{(x, t\sin(\pi/x), 1) \mid x \in (0, 1], t \in \mathbf{I}\}.$$

For each $n \in \mathbb{N}$, let $f_n : X \to [-1, 1] \times \{1\} \subset Y$ be the map defined by

$$f_n(x) = \begin{cases} (\sin(\pi/x), 1) & \text{if } x \geq 1/2n, \\ (0, 1) & \text{if } x \leq 1/2n. \end{cases}$$

Then observe that

$$\downarrow_v f_n = \mathbf{I} \times \{0\} \times \mathbf{I} \cup \{(x, t\sin(\pi/x), 1) \mid x \in [1/2n, 1], t \in \mathbf{I}\}$$

and the sequence $(\downarrow_v f_n)_{n\in\mathbb{N}}$ converges to A in $\mathrm{Cld}_F(X \times \mathbf{I})$. However, the set A is not the hypo-graph of any map from X to Y.

Tieze Extension Theorem [33, Theorem 2.2.2] is well known.

Proposition 3.2.1. [Tieze Extension Theorem] *Let A be a closed set in a normal space X. Then, every map $f : A \to \mathbf{I}$ extends over X.*

As a corollary, we have Urysohn's Lemma [33, Corollary 2.2.3]:

Proposition 3.2.2. [Urysohn map] *For each disjoint pair of closed sets A and B in normal space X, there exists a map $f : X \to I$ such that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$.*

Using the Urysohn map, we can prove the following lemma.

Lemma 3.2.1. *Let $X = (X, d_X)$ be a compact metric space without isolated points and $Y = (Y, d_Y)$ a dendrite. Then*

$$\overline{\downarrow\mathrm{C}(X,Y)} = \{A \in \mathrm{Cld}_F(X\times Y) \mid A(x) \neq \emptyset \textit{ for all } x \in X \textit{ and } y \in A(x) \Rightarrow [\mathbf{0}, y] \subset A(x)\}.$$

Proof. For convenience sake, let F be the set of the right side of the above equality. Then observe that $\downarrow\mathrm{C}(X,Y) \subset F$.

First, we prove that F is closed in $\mathrm{Cld}_F(X\times Y)$. Let A be the limit of a sequence $(A_n)_{n\in\mathbb{N}}$ in F. We shall show that $A(x) \neq \emptyset$ for every $x \in X$. For $n \in \mathbb{N}$, we can take $y_n \in A_n(x) \neq \emptyset$. Because of the compactness of Y, we can assume that $(y_n)_{n\in\mathbb{N}}$ converges to some $y \in Y$. Since $\rho_H(A_n, A) \to 0$ as $n \to \infty$ and

$$\mathrm{dist}(\{(x,y)\}, A_n) \leq \rho((x,y),(x,y_n)) = d_Y(y,y_n) \to 0 \text{ as } n \to \infty,$$

it follows that $(x,y) \in A$. Hence $A(x) \neq \emptyset$. To show that $[\mathbf{0}, y] \subset A(x)$ for each $y \in A(x)$, take any $z \in [\mathbf{0}, y]$. Since $(x,y) \in A$, we can choose $(x_n, y_n) \in A_n$, $n \in \mathbb{N}$, so that $(x_n, y_n) \to (x,y)$ as $n \to \infty$. Accoding to Lemma 3.1.2, we can find $z_n \in [\mathbf{0}, y_n]$, $n \in \mathbb{N}$, such that $d_Y(z, z_n) \leq d_Y(y, y_n)$. Since $y_n \to y$ as $n \to \infty$, we have $z_n \to z$ as $n \to \infty$. Then $z_n \in [\mathbf{0}, y_n] \subset A_n(x_n)$, so $(x_n, z_n) \in A_n$ for every $n \in \mathbb{N}$. Because $(x_n, z_n) \to (x,z)$ as $n \to \infty$, it follows that $(x,z) \in A$, so $z \in A(x)$. Thus we have $[\mathbf{0}, y] \subset A(x)$. Consequently, $A \in F$, so F is closed in $\mathrm{Cld}_F(X \times Y)$.

Next, we will show that $\downarrow\mathrm{C}(X,Y)$ is dense in F. For each $\epsilon > 0$ and $A \in F$, because of the compactness of A, A has finite points (x_i, y_i), $i = 1, \cdots, n$, such that

$A \subset \bigcup_{i=1}^n B((x_i, y_i), \epsilon/2)$, where we can take $x_i \neq x_j$ if $i \neq j$ because X has no isolated points. Let $A_0 = \bigcup_{i=1}^n \{x_i\} \times [\mathbf{0}, y_i] \subset A$. Then $A \subset N(A_0, \epsilon/2)$, which implies that $\rho_H(A_0, A) < \epsilon/2$. Let $\delta = \min\{\epsilon, d_X(x_i, x_j) \mid i \neq j\}/3 > 0$. Note that $\overline{B}(x_i, \delta) \cap \overline{B}(x_j, \delta) = \emptyset$ for every $i \neq j$. X is compact metric space. Hence using Urysohn maps, we can construct a map $f : X \to Y$ such that $f\big(X \setminus \bigcup_{i=1}^n B(x_i, \delta)\big) = \{0\}$, $f(B(x_i, \delta)) \subset [\mathbf{0}, y_i]$ and $f(x_i) = y_i$ for each $i = 1, \cdots, n$. Then $\rho_H(\downarrow f, A_0) < \delta \leq \epsilon/3$. It follows that

$$\rho_H(\downarrow f, A) \leq \rho_H(\downarrow f, A_0) + \rho_H(A_0, A) \leq \epsilon/3 + \epsilon/2 < \epsilon.$$

Therefore $\downarrow\mathrm{C}(X \times Y)$ is dense in F. $\square$

Let $X = (X, d_X)$ be a compact metric space $Y = (Y, d_Y)$ a dendrite. We define $r : Y \times \mathbf{I} \to Y$ by $r(y, t) = \gamma(\mathbf{0}, y, t)$ for each $y \in Y$ and $t \in \mathbf{I}$, where γ is the map as in Lemma 3.1.2. Note that $r_0(Y) = \{\mathbf{0}\}$ and $r_1 = \mathrm{id}_Y$. Using this map r, we can define the homotopy $\overline{r} : \overline{\downarrow\mathrm{C}(X,Y)} \times \mathbf{I} \to \overline{\downarrow\mathrm{C}(X,Y)}$ as follows:

$$\overline{r}(A, t) = (\mathrm{id}_X \times r_t)(A) = \{(x, r_t(y)) \mid (x, y) \in A\}.$$

Then $\overline{r}_0(\overline{\downarrow\mathrm{C}(X,Y)}) = X \times \{\mathbf{0}\}$ and $\overline{r}_1 = \mathrm{id}_{\overline{\downarrow\mathrm{C}(X,Y)}}$. We shall verify the uniform continuity of $\overline{r}$. Take any $\epsilon > 0$. According to Lemma 3.1.2, the map r is uniform continuous. Hence we can choose $\epsilon > \delta > 0$ so that for each $y, y' \in Y$ and $t, t' \in \mathbf{I}$, if $d_Y(y, y') < \delta$ and $|t - t'| < \delta$, then $d_Y(r(y, t), r(y', t')) < \epsilon$. Now, let $A, A' \in \overline{\downarrow\mathrm{C}(X,Y)}$ and $t, t' \in \mathbf{I}$ such that $\rho_H(A, A') < \delta$ and $|t - t'| < \delta$. For each $(x, z) \in \overline{r}_t(A)$, there is a point $y \in A(x)$ such that $z = r_t(y)$. Since $\mathrm{dist}(\{(x, y)\}, A') < \delta$, we can find $(x', y') \in A'$ such that $\rho((x, y), (x', y')) < \delta$, which means that $d_X(x, x') < \delta$ and $d_Y(y, y') < \delta$. Let $z' = r_{t'}(y') \subset A'(x')$. Then $(x', z') \in \overline{r}_{t'}(A')$ and $d_Y(z, z') = d_Y(r_t(y), r_{t'}(y')) < \epsilon$, and hence $\rho((x, z), (x', z')) = \max\{d_X(x, x'), d_Y(z, z')\} < \epsilon$. Thus we have $\mathrm{dist}(\{(x, z)\}, \overline{r}_{t'}(A')) < \epsilon$. By the same argument, we can show that $\mathrm{dist}(\{(x', z')\}, \overline{r}_t(A)) < \epsilon$ for each $(x', z') \in r_{t'}(A')$. Therefore $\rho_H(\overline{r}_t(A), \overline{r}_{t'}(A)) < \epsilon$. Consequently, the map $\overline{r}$ is uniformly continuous. Then $\overline{r}$ is a contraction of $\overline{\downarrow\mathrm{C}(X,Y)}$.

We show the uniformly local path-connectedness of $\overline{\downarrow\mathrm{C}(X,Y)}$ as follows:

Lemma 3.2.2. *For each compact metric space* $X = (X, d_X)$ *and each dendrite* $Y = (Y, d_Y)$*, the space* $\overline{\downarrow \mathrm{C}(X,Y)}$ *is uniformly locally path-connected with respect to* ρ_H*.*

Proof. Let $\epsilon > 0$ and $A, A' \in \overline{\downarrow \mathrm{C}(X,Y)}$ such that $\rho_H(A, A') < \epsilon/2$. We define a path $h : \mathbf{I} \to \overline{\downarrow \mathrm{C}(X,Y)}$ from A to $A \cup A'$ by $h(t) = A \cup \overline{r}_t(A')$. The continuity of h follows from the one of $\overline{r}$ and Lemma 3.1.1. In fact,

$$\rho_H(h(t), h(t')) = \rho_H(A \cup \overline{r}_t(A'), A \cup \overline{r}_{t'}(A')) \leq \rho_H(\overline{r}_t(A'), \overline{r}_{t'}(A')).$$

Moreover, $A \subset h(t), h(t') \subset A \cup A'$, and hence

$$\rho_H(h(t), h(t')) \leq \rho_H(A, A \cup A') = \rho_H(A, A') < \epsilon/2.$$

It follows that $\operatorname{diam} h(\mathbf{I}) \leq \rho_H(A, A') < \epsilon/2$. Consequently, A is connected with $A \cup A'$ by an $\epsilon/2$-path. Similarly, A' is connected with $A \cup A'$ by an $\epsilon/2$-path. Therefore A and A' are connected by an ϵ-path. Thus the proof is complete. □

Based on the following Proposition 3.2.3 [26, Theorem 5.3.14] and Proposition 3.2.4 [26, Proposition 5.3.6], we shall prove Theorem 3.2.1.

Proposition 3.2.3. [Wojdysławski] *Let* X *be a compact space. The following statements are equivalent:*

(a) X *is a Peano continuum;*

(b) $\mathrm{Cld}_F(X)$ *is a Peano continuum;*

(c) $\mathrm{Cld}_F(X)$ *is an AR;*

Proposition 3.2.4. *Let* X *be a compact space. Then the union-operator*

$$\bigcup : \mathrm{Cld}_F(\mathrm{Cld}_F(X)) \to \mathrm{Cld}_F(X)$$

is continuous.

Proof of Theorem 3.2.1. By Lemma 3.2.2, $\overline{\downarrow \mathrm{C}(X,Y)}$ is a Peano continuum. Then, according to the Proposition 3.2.3, we have $\mathrm{Cld}_F(\overline{\downarrow \mathrm{C}(X,Y)})$ is an AR. Identifying

$A \in \mathrm{Cld}_F(X \times Y)$ with $\{A\} \in \mathrm{Cld}_F(\mathrm{Cld}_F(X \times Y))$, we can regard $\mathrm{Cld}_F(X \times Y) \subset \mathrm{Cld}_F(\mathrm{Cld}_F(X \times Y))$. Then the union operator

$$\bigcup : \mathrm{Cld}_F(\mathrm{Cld}_F(X \times Y)) \ni \mathcal{A} \mapsto \bigcup \mathcal{A} \in \mathrm{Cld}_F(X, Y)$$

is a retraction, see Proposition 3.2.4. As is easily observed due to Lemma 3.2.1, the image $\bigcup(\mathrm{Cld}_F(\overline{{\downarrow}\mathrm{C}(X,Y)})) = \overline{{\downarrow}\mathrm{C}(X,Y)}$. It follows that $\overline{{\downarrow}\mathrm{C}(X,Y)}$ is a retract of the AR $\mathrm{Cld}_F(\overline{{\downarrow}\mathrm{C}(X,Y)})$. Therefore $\overline{{\downarrow}\mathrm{C}(X,Y)}$ is an AR. □

3.3 The homotopy denseness of ${\downarrow}\mathrm{C}(X,Y)$ in $\overline{{\downarrow}\mathrm{C}(X,Y)}$

A subset A of a space X is said to be *homotopy dense* in X if there exists a homotopy $h : X \times \mathbf{I} \to X$ such that $h_0 = \mathrm{id}_X$ and $h_t(X) \subset A$ for every $t > 0$. In this section, we will prove the following theorem:

Theorem 3.3.1. *Let $X = (X, d_X)$ be a compact metric space without isolated points and $Y = (Y, d_Y)$ a dendrite. Then ${\downarrow}\mathrm{C}(X,Y)$ is homotopy dense in $\overline{{\downarrow}\mathrm{C}(X,Y)}$.*

In general setting, we can restate Lemma 3 of [34], refer to Corollary 4 of [32] and Lemma 4.2 of [21], as follows:

Lemma 3.3.1. *Let $X = (X, d)$ be a compact metric space and Z be a dense subset of X that has the following property:*

(hd) *There exists $\alpha > 0$ such that for any locally finite countable simplicial complex K, each map $f : K^{(0)} \to Z$ extends to a map $\overline{f} : |K| \to Z$ such that $\mathrm{diam}\,\overline{f}(\sigma) \leq \alpha\,\mathrm{diam}\,f(\sigma^{(0)})$ for every $\sigma \in K$.*

Then Z is homotopy dense in X.

Proof of Theorem 3.3.1. We only need to verify condition (hd) with respect to $\alpha = 10$ in Lemma 3.3.1. Let K be a locally finite countable simplicial complex and $f : K^{(0)} \to {\downarrow}\mathrm{C}(X,Y)$. We shall construct a map $\overline{f} : |K| \to {\downarrow}\mathrm{C}(X,Y)$ such that the restriction $\overline{f}|_{K^{(0)}} = f$ and $\mathrm{diam}\,\overline{f}(\sigma) \leq 10\,\mathrm{diam}\,f(\sigma^{(0)})$ for every $\sigma \in K$. For simplicity, let $\epsilon_\sigma = \operatorname{diam} f(\sigma^{(0)}) \geq 0$ for each $\sigma \in K \setminus K^{(0)}$. Let K_0 be the full subcomplex of K such that

$$K_0^{(0)} = \{v \in K^{(0)} \mid f(\mathrm{St}(v,K)^{(0)}) \text{ is a singleton}\},$$

where $\mathrm{St}(v,K)$ is the star at v in K. Note that $f(\sigma^{(0)})$ is a singleton if $\sigma \in K$ and $\sigma \cap |K_0| \neq \emptyset$. We define $K_1 = \{\sigma \in K \mid \sigma \cap |K_0| = \emptyset\}$. For every $v \in K_1^{(0)}$, since $\operatorname{diam} f(\mathrm{St}(v,K)^{(0)}) > 0$, we can define

$$\epsilon_v = \min\{\epsilon_\sigma \mid \sigma \in \mathrm{St}(v,K), \epsilon_\sigma > 0\} > 0.$$

Let $f_0 : |K_0| \to \downarrow\mathrm{C}(X,Y)$ be the map such that $f_0(\sigma) = f(\sigma^{(0)})$ for each $\sigma \in K_0$.

Since K is locally finite and X has no isolated points, we can choose a finite sets $A_v \subset X$ and $\delta_v > 0$, $v \in K_1^{(0)}$, so that

(1) $\rho_H(f(v)|_{A_v}, f(v)) < \epsilon_\sigma$,

(2) $B(a,\delta_v) \cap B(a',\delta_{v'}) = \emptyset$ if $v \neq v' \in K_1^{(0)}$, v and v' are contained in some $\sigma \in K$, $a \in A_v$, and $a' \in A_{v'}$,

(3) $B(a,\delta_v) \cap B(a',\delta_v) = \emptyset$ if $a \neq a' \in A_v$ and $v \in K_1^{(0)}$,

where $f(v)|_{A_v} = \bigcup_{a \in A_v} \{a\} \times [0, f(v)(a)]$. First, we will construct a map $f_1 : |K_1| \to \downarrow\mathrm{C}(X,Y)$ such that $\rho_H(f_1(v), f(v)) < \epsilon_v$ for each $v \in K_1^{(0)}$ and $\operatorname{diam} f_1(\sigma) < 7\epsilon_\sigma$ for each $\sigma \in K_1$. For every $v \in K_1^{(0)}$, we define $f_1(v) \in \downarrow\mathrm{C}(X,Y)$ as follows:

$$f_1(v)(x) = \begin{cases} r(f(v)(x) \times \{(\delta_v - \operatorname{dist}(\{x\}, A_v))/\delta_v\}) & \text{if } \operatorname{dist}(\{x\}, A_v) \leq \delta_v, \\ \{\mathbf{0}\} & \text{if } \operatorname{dist}(\{x\}, A_v) \geq \delta_v. \end{cases}$$

Since $f(v)|_{A_v} \subset f_1(v) \subset f(v)$, it follows that $\rho_H(f(v), f_1(v)) \leq \rho_H(f(v)|_{A_v}, f(v)) < \varepsilon_v$. Denote the barycenter of $\sigma \in K_1$ by $\hat{\sigma}$. For $\sigma \in K_1$, let

$$f_1(\hat{\sigma}) = \bigcup_{v \in \sigma^{(0)}} f_1(v) \in \downarrow\mathrm{C}(X,Y).$$

For each $z \in \sigma$, there exist faces $\sigma_0 \prec \sigma_1 \prec \cdots \prec \sigma_n \prec \sigma$ of σ such that $z = \sum_{i=0}^n t_i \hat{\sigma}_i$, where $\sum_{i=0}^n t_i = 1$ and $t_i > 0$. Then we can define

$$f_1(z) = \bigcup_{i=0}^n \overline{r}\left(f_1(\hat{\sigma}_i), \sum_{i=j}^n t_j \right) \in \downarrow\mathrm{C}(X,Y).$$

For each $\sigma \in K_1$ and $v \in \sigma^{(0)}$, the continuity of $f_1|_{\mathrm{St}(v,\mathrm{Sd}\,K)\cap\sigma}$ follows from the ones of both the map $\overline{r}$ and the union operator on $\mathrm{Cld}_F(X \times Y)$, where $\mathrm{Sd}\,K$ is the barycentric subdivision of K. Since K_1 is locally finite, it follows that f_1 is continuous. Thus we have a map $f_1 : |K_1| \to \downarrow\mathrm{C}(X,Y)$. For each $\sigma \in K_1$, let $v \in \sigma^{(0)}$ and $z \in |\,\mathrm{St}(v, \mathrm{Sd}\,K)| \cap \sigma$. By the definition of f_1, we have

$$f_1(v) \subset f_1(z) \subset f_1(\hat{\sigma}) = \bigcup_{v' \in \sigma^{(0)}} f(v').$$

Then it follows that

$$\begin{aligned}\rho_H(f_1(z), f_1(v)) &\leq \rho_H\Bigl(f_1(v), \bigcup_{v'\in\sigma^{(0)}} f(v')\Bigr) \leq \rho_H(f_1(v), f(v)) + \rho_H\Bigl(f(v), \bigcup_{v'\in\sigma^{(0)}} f(v')\Bigr)\\ &\leq \rho_H(f_1(v), f(v)) + \max\{\rho_H(f(v), f(v')) \mid v' \in \sigma^{(0)}\}\\ &\leq \rho_H(f_1(v), f(v)) + \mathrm{diam}\, f(\sigma^{(0)}) \leq \epsilon_v + \epsilon_\sigma \leq 2\epsilon_\sigma.\end{aligned}$$

For each $z, z' \in \sigma \in K_1$, we can choose vertices $v, v' \in \sigma^{(0)}$ such that $z \in |\,\mathrm{St}(v, \mathrm{Sd}\,K)|$ and $z' \in |\,\mathrm{St}(v'), \mathrm{Sd}\,K|$. Then we have

$$\begin{aligned}\rho_H(f_1(z), f_1(z')) &\leq \rho_H(f_1(z), f_1(v)) + \rho_H(f_1(v), f(v)) + \rho_H(f(v), f(v'))\\ &\qquad + \rho_H(f(v'), f_1(v')) + \rho_H(f_1(v'), f_1(z'))\\ &< 2\epsilon_\sigma + \epsilon_v + \epsilon_\sigma + \epsilon_{v'} + 2\epsilon_\sigma \leq 7\epsilon_\sigma.\end{aligned}$$

Consequently, $\mathrm{diam}\, f_1(\sigma) < 7\epsilon_\sigma$ for each $\sigma \in K_1$.

Next, we construct a map $f_* : |K| \cup K^{(0)} \times \mathbf{I} \to \downarrow\mathrm{C}(X,Y)$, where $|K|$ is identified with $|K| \times \{0\} \subset |K| \times \mathbf{I}$. Let $f_*|_{|K_0|} = f_0$ and $f_*|_{|K_1|} = f_1$. For each $z \in |K| \setminus |K_0 \cup K_1|$, there exits $\sigma_0 \in K_0$ and $\sigma_1 \in K_1$ such that z is contained in the join of σ_0 and σ_1, and hence z can be uniquely written as follows: $z = tz_0 + (1-t)z_1$ for some $z_0 \in \sigma_0$, $z_1 \in \sigma_1$ and $t \in \mathbf{I}$. Then we can define

$$f_*(z) = \overline{r}(f_0(z_0), t) \cup f_1(z_1) \in \downarrow\mathrm{C}(X,Y).$$

Observe that $f_*(z_0) = f_0(z_0)$ and $f_*(z_1) = f_1(z_1)$. For each $(v,t) \in K^{(0)} \times \mathbf{I}$, we define

$$f_*(v,t) = \overline{r}(f(v), t) \cup f_1(v),$$

where $f_*(v,0)=f_1(v)$ and $f_*(v,1)=f(v)$.

Thirdly, we can obtain a map $g:|K|\to|K|\cup K^{(0)}\times\mathbf{I}$ so that $g(v)=(v,1)$ for each $v\in K^{(0)}$ and $g(\sigma)=\sigma\cup\sigma^{(0)}\times\mathbf{I}$ for each $\sigma\in K\setminus K^{(0)}$. In fact, let $v\in K^{(0)}$ and $z=\sum_{i=0}^n t_i\hat{\sigma_i}\in|\operatorname{St}(v,\operatorname{Sd}K)|$, where $\sigma_0\prec\sigma_1\prec\cdots\prec\sigma_n\in K$, $\sum_{i=0}^n t_i=1$ and $t_i\geq 0$. We define

$$g(z)=\begin{cases}(1-2t_0)z+2t_0v & \text{if } t_0\leq 1/2,\\ (v,2t_0-1) & \text{if } t_0\geq 1/2.\end{cases}$$

Now, the desired map $\overline{f}:|K|\to\downarrow\mathrm{C}(X,Y)$ can be defined by $\overline{f}=f_*g$. As is easily observed, $\overline{f}|_{K^{(0)}}=f$. We will show that $\operatorname{diam}\overline{f}(\sigma)\leq 10\epsilon_\sigma$ for every $\sigma\in K$. When $\sigma\in K_0$, we have $\operatorname{diam}\overline{f}(\sigma)=\operatorname{diam}f(\sigma^{(0)})=0$. For each $\sigma\in K_1$, since $\overline{f}(\sigma)=f_1(\sigma)\cup f_*(\sigma^{(0)}\times\mathbf{I})$, it follows that

$$\begin{aligned}\operatorname{diam}\overline{f}(\sigma)&\leq\operatorname{diam}f_1(\sigma)+\operatorname{diam}f_*(\sigma^{(0)}\times\mathbf{I})\\&\leq\operatorname{diam}f_1(\sigma)+\operatorname{diam}f(\sigma^{(0)})+2\max\{\rho_H(f_1(v),f(v))\mid v\in\sigma^{(0)}\}\\&<7\epsilon_\sigma+\epsilon_\sigma+2\epsilon_\sigma=10\epsilon_\sigma.\end{aligned}$$

When $\sigma\in K\setminus(K_0\cup K_1)$, we can take $\sigma_0\in K_0$ and $\sigma_1\in K_1$ so that σ is the join of σ_0 and σ_1. Since $\sigma\in\operatorname{St}(v_0,K)$ for any $v_0\in\sigma_0^{(0)}\subset K_0^{(0)}$, $f(\sigma^{(0)})$ is a singleton. For each $z=tz_0+(1-t)z_1\in\sigma$, where $z_0\in\sigma_0$, $z_1\in\sigma_1$ and $0\leq t\leq 1$, choose $v\in\sigma_1^{(0)}$ such that $z_1\in|\operatorname{St}(v,\operatorname{Sd}K)|$. Then $f(\sigma^{(0)})=\{f(v)\}$, $f_1(v)\subset f_1(z_1)\subset f(v)$ and $f_*(z)=\overline{r}(f_0(z_0),t)\cup f_1(z_1)\subset f(v)$. Hence we get

$$\operatorname{dist}(f_*(z),f(\sigma^{(0)}))=\rho_H(f_*(z),f(v))\leq\rho_H(f_1(v),f(v))<\epsilon_v\leq\epsilon_\sigma.$$

Therefore for each $z,z'\in\sigma$,

$$\rho_H(f_*(z),f_*(z'))\leq\operatorname{dist}(f_*(z),f(\sigma^{(0)}))+\operatorname{dist}(f(\sigma^{(0)}),f_*(z'))+\operatorname{diam}f(\sigma^{(0)})<\epsilon_\sigma+\epsilon_\sigma=2\epsilon_\sigma.$$

Consequently, $\operatorname{diam}f_*(\sigma)\leq 2\epsilon_\sigma$. Since

$$\operatorname{diam}f_*(\sigma^{(0)}\times\mathbf{I})\leq\operatorname{diam}f(\sigma^{(0)})+\max\{\rho_H(f(v),f_1(v))\mid v\in\sigma_1^{(0)}\}\leq\epsilon_{\sigma_1}\leq\epsilon_\sigma,$$

it follows that

$$\operatorname{diam}\overline{f}(\sigma)\leq\operatorname{diam}f_*(\sigma)+\operatorname{diam}f_*(\sigma^{(0)}\times\mathbf{I})\leq 2\epsilon_\sigma+\epsilon_\sigma=3\epsilon_\sigma.$$

Thus the proof is complete. □

3.4 The disjoint cells property of $\overline{\downarrow \mathrm{C}(X,Y)}$

Using the following Proposition [33, Theorem 2.7.6], we prove the Lemma 3.4.1.

Proposition 3.4.1. *Let $g, h : X \to \mathbb{R}$ be real-valued functions on a paracompact space X such that h is lower semi-continuous, g is upper semi-continuous and $g(x) < h(x)$ for each $x \in X$. Then, there exists a map $f : X \to \mathbb{R}$ such that $g(x) < f(x) < h(x)$ for each $x \in X$. Moreover, given a map $f_0 : A \to \mathbb{R}$ of a closed set A in X such that $g(x) < f_0(x) < h(x)$ for each $x \in A$, the map f can be an extension of f_0.*

The following lemma will play the important role for the rest of this paper.

Lemma 3.4.1. *Let $X = (X, d_X)$ be a compact metric space and $Y = (Y, d_Y)$ a dendrite. Suppose that $Z = (Z, d_Z)$ is a metric space, $\phi : Z \to \downarrow \mathrm{C}(X,Y)$ is a map, and $a \in X$ is a non-isolated point. Then for each map $\epsilon : Z \to (0,1)$, there exist maps $\psi : Z \to \downarrow \mathrm{C}(X,Y)$ and $\delta : Z \to (0,1)$ such that for each $z \in Z$,*

(a) $\rho_H(\phi(z), \psi(z)) < \epsilon(z)$,

(b) $\psi(z)(B(a, \delta(z))) = \{\mathbf{0}\}$.

Proof. For each $z \in Z$, let $\xi(z) = \sup\{\eta > 0 \mid \rho_H(\phi(z), \phi(z)|_{X \setminus B(a,\eta)}) < \epsilon(z)\}$. Since a is not isolated and $\phi(z) \in \downarrow \mathrm{C}(X,Y)$, we have $\xi(z) > 0$. We shall prove $\xi : Z \to (0, \infty)$ is a lower semi-continuous function. Fix any $z \in Z$ and $\eta \in (0, \xi(z))$. From the definition of $\xi(z)$,

($\star$) $\rho_H(\phi(z), \phi(z)|_{X \setminus B(a, \xi(z) - \eta/2)}) < (n-1)\epsilon(z)/n$ for some $n \in \mathbb{N}$.

Let $t = \min\{\eta/2, \epsilon(z)/3n\}$. Since ϕ and ϵ are continuous, there exists $s > 0$ such that if $d_Z(z, z') < s$, then $\rho_H(\phi(z), \phi(z')) < t$ and $|\epsilon(z) - \epsilon(z')| < \epsilon(z)/3n$. We shall show that for every $z' \in Z$ with $d_Z(z, z') < s$, $\xi(z') \geq \xi(z) - \eta$. Take any $(x, y) \in \phi(z')|_{B(a, \xi(z) - \eta)}$. Since $\rho_H(\phi(z), \phi(z')) < t$, we can choose $(x', y') \in \phi(z)$ so that $\rho((x, y), (x', y')) < t \leq \eta/2$. Then $d_X(x, x') < \eta/2$, that is, $(x', y') \in \phi(z)|_{B(a, \xi(z) - \eta/2)}$. Due to ($\star$), there exists $(x'', y'') \in \phi(z)|_{X \setminus B(a, \xi(z) - \eta/2)}$ such that $\rho((x', y'), (x'', y'')) < (n-1)\epsilon(z)/n$. Since $\rho_H(\phi(z), \phi(z')) < t$, we can find a point $(x''', y''') \in \phi(z')$ such that $\rho((x'',y''),(x''',y''')) < t \leq \eta/2$, which implies that $x''' \in X \setminus B(a, \xi(z)-\eta)$. Then it follows that

$$\begin{aligned}\rho((x,y),(x''',y''')) &\leq \rho((x,y),(x',y')) + \rho((x',y'),(x'',y'')) + \rho((x'',y''),(x''',y''')) \\ &< t + (n-1)\epsilon(z)/n + t \leq (2/3n + (n-1)/n)\epsilon(z) \\ &= \epsilon(z) - \epsilon(z)/3n < \epsilon(z').\end{aligned}$$

Thus ξ is lower semi-continuous.

By Proposition 3.4.1, we can obtain a map $\delta : Z \to (0,1)$ so that $\delta(z) < \xi(z)/2$ for each $z \in Z$. Now, we can define the desired map $\psi : Z \to \downarrow\mathrm{C}(X,Y)$ as follows:

$$\begin{aligned}\psi(z) = \phi(z)|_{X\setminus B(a,2\delta(z))} &\cup B(a,\delta(z)) \times \{\mathbf{0}\} \\ &\cup \{(x,y) \in X \times Y \mid \delta(z) \leq d_X(x,a) \leq 2\delta(z), \\ &\qquad y \in [\mathbf{0}, r(\max\phi(z)(x), d_X(x,a)/\delta(z) - 1)]\}.\end{aligned}$$

Remark that $\phi(z) \in\downarrow\mathrm{C}(X,Y)$ is the hypo-graph of the map $X \ni x \mapsto \max\phi(z)(x) \in Y$. By the definition of ψ, it is easy to show that ψ satisfies conditions (a) and (b).

Claim. *The function ψ is continuous.*

For every $z \in Z$ and $\epsilon > 0$, by Lemma 3.1.2, there exists $\delta_1 > 0$ such that $\delta_1 < 1/2$ and

$$d_Y(y,y_1) < \delta_1 \text{ and } |t - t_1| < \delta_1 \Rightarrow d_Y(r(y,t), r(y_1,t_1)) < \epsilon.$$

Take $\delta_2 > 0$ such that $\delta_2 \leq \delta_1/2$ and $\delta_2 \operatorname{diam} Y < \epsilon$. We can choose $\delta_3 > 0$ so that $\delta_3 < \delta(z)$ and

$$a, b \in [\delta(z)/2, 5\delta(z)/2] \text{ and } |a-b| < \delta_3 \Rightarrow |b/a - 1| < \delta_2.$$

Since ϕ and δ are continuous, there exists a neighborhood U of z such that for each $z' \in U$, $\rho_H(\phi(z),\phi(z')) < \min\{\epsilon, \delta(z)\delta_1/2, \delta_3/4\}$, $|1/\delta(z) - 1/\delta(z')| < 2\delta_1/9\delta(z)$ and $|\delta(z) - \delta(z')| < \delta_3/8$. We shall verify that $\rho_H(\psi(z),\psi(z')) < \epsilon$ for each $z' \in U$. Take any $(x,y) \in \psi(z)$. It is sufficient to show that $(x,y) \in N(\psi(z'),\epsilon)$.

Case I. $d_X(x,a) \leq \delta(z)$

Then we have $y = \mathbf{0}$. So $(x,y) = (x,\mathbf{0}) \in \psi(z')$.

Case II. $\delta(z) < d_X(x,a) < \delta(z) + \delta_3$
Then $|d_X(x,a)/\delta(z) - 1| < \delta_2$, so

$$\begin{aligned} d_Y(\mathbf{0}, y) &\leq d_Y(\mathbf{0}, r(\max \phi(z)(x), d_X(x,a)/\delta(z) - 1)) \\ &= (d_X(x,a)/\delta(z) - 1) d_Y(\mathbf{0}, \max \phi(z)(x)) \\ &< \delta_2 \operatorname{diam} Y < \epsilon. \end{aligned}$$

Therefore $\rho((x,y),(x,\mathbf{0})) = d_Y(\mathbf{0}, y) < \epsilon$.

Case III. $d_X(x,a) \geq \delta(z) + \delta_3$
Since $\rho_H(\phi(z), \phi(z')) < \min\{\epsilon, \delta(z)\delta_1/2, \delta_3/4\}$, there exists a point $(x_1, y_1) \in \phi(z')$ such that

$$\rho((x, \max \phi(z)(x)), (x_1, y_1)) < \min\{\epsilon, \delta(z)\delta_1/2, \delta_3/4\}.$$

Then we have

$$d_X(x, x_1) \leq \rho((x, \max \phi(z)(x)), (x_1, y_1)) < \min\{\epsilon, \delta(z)\delta_1/2, \delta_3/4\}.$$

Moreover, $|\delta(z) - \delta(z')| < \delta_3/8$, and hence

$$\begin{aligned} &d_X(x_1, a) \\ \geq &d_X(x,a) - d_X(x, x_1) \\ > &\delta(z) + \delta_3 - \delta_3/4 \\ > &\delta(z') - \delta_3/8 + \delta_3 - \delta_3/4 \\ > &\delta(z'). \end{aligned}$$

If $d_X(x_1, a) \geq 2\delta(z')$, we get $(x_1, y_1) \in \psi(z')$. Since $y \in [\mathbf{0}, \max \phi(z)(x)]$, by Lemma 3.1.2, we can find $y_2 \in [\mathbf{0}, y_1]$ such that $d_Y(y, y_2) \leq d_Y(\max \phi(z)(x), y_1) < \epsilon$. It follows that $(x_1, y_2) \in \psi(z')$ and

$$\rho((x,y),(x_1,y_2)) = \max\{d_X(x,x_1), d_Y(y,y_2)\} < \epsilon.$$

Now, we need only to consider the case that $\delta(z') < d_X(x_1, a) < 2\delta(z')$. Let $y_3 = r(y_1, d_X(x_1,a)/\delta(z') - 1)$. Then $y_3 \in [\mathbf{0}, r(\max \phi(z')(x_1), d_X(x_1,a)/\delta(z') - 1)]$, so $(x_1, y_3) \in \psi(z')$.

Case III-i. $\delta(z)+\delta_3 \le d_X(x,a) < 2\delta(z)$.
Then we have

$$\begin{aligned}
&|d_X(x,a)/\delta(z) - 1 - (d_X(x_1,a)/\delta(z') - 1)| \\
&\quad \le |1/\delta(z) - 1/\delta(z')| d_X(x_1,a) + |d_X(x,a) - d_X(x_1,a)|/\delta(z) \\
&\quad \le |1/\delta(z) - 1/\delta(z')|(d_X(x,x_1) + d_X(x,a)) + d_X(x,x_1)/\delta(z) \\
&\quad < 2\delta_1(\delta(z)/4 + 2\delta(z))/9\delta(z) + \delta(z)\delta_1/2\delta(z) \\
&\quad = \delta_1/2 + \delta_1/2 = \delta_1.
\end{aligned}$$

On the other hand, we get

$$d_Y(\max\phi(z)(x), y_1) \le \rho((x, \max\phi(z)(x)), (x_1,y_1)) < \delta(z)\delta_1/2 < \delta_1.$$

It follows that

$$\begin{aligned}
&d_Y(r(\max\phi(z)(x), d_X(x,a)/\delta(z) - 1), y_3) \\
&= d_Y(r(\max\phi(z)(x), d_X(x,a)/\delta(z) - 1), r(y_1, d_X(x_1,a)/\delta(z') - 1)) \\
&< \epsilon.
\end{aligned}$$

Using Lemma 3.1.2, we can choose $y_4 \in [\mathbf{0}, y_3]$ so that

$$d_Y(y, y_4) \le d_Y(r(\max\phi(z)(x), d_X(x,a)/\delta(z) - 1), y_3) < \epsilon.$$

Then $(x_1,y_4) \in \psi(z')$ and $\rho((x,y),(x_1,y_4)) = \max\{d_X(x,x_1), d_Y(y,y_4)\} < \epsilon$.

Case III-ii. $2\delta(z) \le d_X(x,a) < 2\delta(z) + \delta_3/2$.
It follows that

$$\begin{aligned}
&|2\delta(z') - d_X(x_1,a)| \\
\le &|2\delta(z') - 2\delta(z)| + |2\delta(z) - d_X(x,a)| + |d_X(x,a) - d_X(x_1,a)| \\
< &\delta_3/4 + \delta_3/2 + \delta_3/4 \\
= &\delta_3.
\end{aligned}$$

Therefore we have

$$|1 - (d_X(x_1,a)/\delta(z') - 1)| = |2 - d_X(x_1,a)/\delta(z')| < 2\delta_2 < \delta_1.$$

Observe that

$$
\begin{aligned}
&d_Y(\max\psi(z)(x), y_3)\\
=&d_Y(\max\phi(z)(x), y_3)\\
=&d_Y(r(\max\phi(z)(x), 1), r(y_1, d_X(x_1, a)/\delta(z') - 1))\\
<&\epsilon.
\end{aligned}
$$

Due to Lemma 3.1.2, there exists $y_5 \in [\mathbf{0}, y_3]$ such that $d_Y(y, y_5) \leq d_Y(\max\psi(z)(x), y_3)$ $< \epsilon$. Then $(x_1, y_5) \in \psi(z')$ and $\rho((x, y), (x_1, y_5)) = \max\{d_X(x, x_1), d_Y(y, y_5)\} < \epsilon$.

Case III-iii. $d_X(x, a) \geq 2\delta(z) + \delta_3/2$.

Note that

$$
\begin{aligned}
&d_X(x_1, a)\\
\geq&d_X(x, a) - d_X(x, x_1)\\
\geq&2\delta(z) + \delta_3/2 - \delta_3/4\\
>&2\delta(z') - \delta_3/4 + \delta_3/2 - \delta_3/4\\
=&2\delta(z'),
\end{aligned}
$$

which is a contradiction.

Consequently, $(x, y) \in N(\psi(z'), \epsilon)$. Similarly, $\psi(z') \subset N(\psi(z), \epsilon)$. Thus $\rho_H(\psi(z),$ $\psi(z')) < \epsilon$, and hence ψ is continuous. □

A space X has the *disjoint cells property* provided that for any maps $f, g : Q \to X$ of the Hilbert cube and open cover $\mathcal{U}$ of X, there exist maps $f', g' : Q \to X$ such that f' and g' are $\mathcal{U}$-close to f and g, respectively, and $f'(Q) \cap g'(Q) = \emptyset$.

Proposition 3.4.2. *Let $X = (X, d_X)$ be a compact metric space without isolated points and $Y = (Y, d_Y)$ a dendrite. Then $\overline{\downarrow\mathrm{C}(X, Y)}$ has the disjoint cells property.*

Proof. Let $f, g : Q \to \overline{\downarrow\mathrm{C}(X, Y)}$ be maps and $0 < \epsilon < \operatorname{diam} Y$. Since $\downarrow\mathrm{C}(X, Y)$ is homotopy dense in $\overline{\downarrow\mathrm{C}(X, Y)}$ by Theorem 3.3.1, we can obtain maps $f' : Q \to \downarrow\mathrm{C}(X, Y)$ that is ϵ-close to f, and $g' : Q \to \downarrow\mathrm{C}(X, Y)$ that is $\epsilon/3$-close to g. Take a non-isolated point $x_0 \in X$. Using Lemma 3.4.1, we can find a map $g'' : Q \to \downarrow\mathrm{C}(X, Y)$ such that g''

is $\epsilon/3$-close to g' and $g''(z)(x_0) = \{\mathbf{0}\}$ for all $z \in Q$. Define a map $g''' : Q \to \overline{\downarrow\mathrm{C}(X,Y)}$ as follows:

$$g'''(z) = \overline{r}(g''(z), 1 - \epsilon/(3 \operatorname{diam} Y)) \cup \{x_0\} \times \overline{B}(0, \epsilon/3).$$

Then $\rho_H(g(z), g'''(z)) < \epsilon$ and $g'''(z) \notin \downarrow\mathrm{C}(X,Y)$ for each $z \in Q$. Indeed, since

$$\overline{r}(g''(z), 1 - \epsilon/(3 \operatorname{diam} Y)) \subset g'''(z) \subset g''(z) \cup \{x_0\} \times \overline{B}(0, \epsilon/3),$$

it follows that

$$\begin{aligned}
&\rho_H(g''(z), g'''(z))\\
\leq&\rho_H(g''(z) \cup \{x_0\} \times \overline{B}(0, \epsilon/3), \overline{r}(g''(z), 1 - \epsilon/(3 \operatorname{diam} Y)))\\
\leq&\sup\{d_Y(y, r(y, 1 - \epsilon/(3 \operatorname{diam} Y))), \epsilon/3 \mid y \in Y\} \leq \epsilon/3.
\end{aligned}$$

Therefore g''' is $\epsilon/3$-close to g'', so it is ϵ-close to g. Moreover, we have

$$\begin{aligned}
&\overline{r}(g''(z), 1 - \epsilon/(3 \operatorname{diam} Y))(x_0)\\
=&r(g''(z)(x_0) \times \{1 - \epsilon/(3 \operatorname{diam} Y)\})\\
\subsetneq&g''(z)(x_0) \cup \overline{B}(0, \epsilon/3)\\
=&g'''(z)(x_0).
\end{aligned}$$

Take $y \in g'''(z)(x_0) \setminus \overline{r}(g''(z), 1 - \epsilon/(3 \operatorname{diam} Y))(x_0)$, so we can choose $\delta > 0$ so that

$$B((x_0, y), \delta) \cap \overline{r}(g''(z), 1 - \epsilon/(3 \operatorname{diam} Y)) = \emptyset,$$

which implies that $g'''(z)$ is not the hypo-graph of any map because x_0 is a non-isolated point. Hence $g'''(z) \notin \downarrow\mathrm{C}(X,Y)$. Consequently, $f'(Q) \cap g'''(Q) = \emptyset$. Thus $\overline{\downarrow\mathrm{C}(X,Y)}$ has the disjoint cells property. □

From [39], the following Proposition shows a characterization of Hilbert cube.

Proposition 3.4.3. *[Toruńczyk's characterization] A space X is homeomorphic to Hilbert cube Q if and only if it is a compact AR with disjoint-cells property.*

Combining Theorem 3.2.1, Proposition 3.4.2 and Proposition 3.4.3 [39], we can immediately obtain the following:

Corollary 3.4.1. *Let X be a compact metrizable space without isolated points and Y a dendrite. Then $\overline{\downarrow\mathrm{C}(X,Y)}$ is homeomorphic to the Hilbert cube Q.*

3.5 The space $\downarrow\mathrm{C}(X,Y)$ is an $F_{\sigma\delta}$-set in $\overline{\downarrow\mathrm{C}(X,Y)}$

A dendrite Y has an order $\leq$ defined as follows: $x \leq y$ if $x \in [\mathbf{0}, y]$. For each $\delta, \epsilon > 0$, let $\mathcal{A}(\delta,\epsilon)$ be the set which consists of $A \in \overline{\downarrow\mathrm{C}(X,Y)}$ such that the following condition is satisfied:

> For all $x, x' \in X$, if $d_X(x,x') < \delta$ and $y, y' \in Y$ are maximal points of $A(x), A(x')$, respectively, then $d_Y(y,y') \leq \epsilon$.

To prove that $\downarrow\mathrm{C}(X,Y)$ is an $F_{\sigma\delta}$-set in $\overline{\downarrow\mathrm{C}(X,Y)}$, we need the following lemma.

Lemma 3.5.1. *Let $X = (X, d_X)$ be a compact metric space and $Y = (Y, d_Y)$ a dendrite. For each $\delta, \epsilon > 0$, the set $\mathcal{A}(\delta,\epsilon)$ is closed in $\overline{\downarrow\mathrm{C}(X,Y)}$.*

Proof. Take any sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}(\delta,\epsilon)$ that converges to A in $\overline{\downarrow\mathrm{C}(X,Y)}$. To show that $A \in \mathcal{A}(\delta,\epsilon)$, let $(x,y),(x',y') \in A$ such that $d_X(x,x') < \delta$ and y, y' are maximal in $A(x), A(x')$, respectively. Since $A_n \to A$, there exist $(x_n,y_n),(x'_n,y'_n) \in A_n$ such that $(x_n,y_n) \to (x,y)$ and $(x'_n,y'_n) \to (x',y')$, see [26, Lemma 5.3.1]. Without loss of generality, we may assume that $d_X(x_n,x'_n) < \delta$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, there exist maximal points $z_n \in A_n(x_n)$ and $z'_n \in A_n(x'_n)$ such that $z_n \geq y_n$ and $z'_n \geq y'_n$. Because Y is compact, replacing $(z_n)_{n\in\mathbb{N}}$ and $(z'_n)_{n\in\mathbb{N}}$ with subsequences, we can assume that $z_n \to z \in Y$ and $z'_n \to z' \in Y$. Using Lemma 5.3.1 of [26] again, we have $z \in A(x)$ and $z' \in A(x')$. Then y is contained in the arc $[\mathbf{0}, z]$ from $\mathbf{0}$ to z. Indeed, if not, we have $\mathrm{dist}(\{y\},[\mathbf{0},z]) > 0$. Since $y_n \to y$ and $z_n \to z$, we can choose $m \in \mathbb{N}$ so that $d_Y(y,y_m), d_Y(z,z_m) < \mathrm{dist}(\{y\},[\mathbf{0},z])/2$. Note that $y_m \in [\mathbf{0}, z_m]$. Then there exists a point $p \in [\mathbf{0}, z]$ such that $d_Y(y_m,p) \leq d_Y(z,z_m) < \mathrm{dist}(\{y\},[\mathbf{0},z])/2$ by Lemma 3.1.2. It follows that

$$
\begin{aligned}
&d_Y(y,p)\\
\leq &d_Y(y,y_m) + d_Y(y_m,p)\\
< &\mathrm{dist}(\{y\},[\mathbf{0},z])/2 + \mathrm{dist}(\{y\},[\mathbf{0},z])/2\\
= &\mathrm{dist}(\{y\},[\mathbf{0},z]),
\end{aligned}
$$

which is a contradiction. Hence $y \in [\mathbf{0}, z]$. By the maximality of y in $A(x)$, we have $y = z$. Similarly, $y' = z'$.

Since each $A_n \in \mathcal{A}(\delta,\epsilon)$, $d_X(x_n,x'_n) < \delta$ and z_n, z'_n are maximal in $A(x_n), A(x'_n)$, respectively, it follows that $d_Y(z_n,z'_n) \leq \epsilon$. Recall that $z_n \to z = y$ and $z'_n \to z' = y'$, so $d_Y(y,y') \leq \epsilon$. Consequently, we have $A \in \mathcal{A}(\delta,\epsilon)$. Thus the proof is complete. □

Now, we show the following:

Proposition 3.5.1. *For each compact metric space $X = (X, d_X)$ and each dendrite $Y = (Y, d_Y)$, the space $\downarrow\mathrm{C}(X,Y)$ is an $F_{\sigma\delta}$-set in $\overline{\downarrow\mathrm{C}(X,Y)}$.*

Proof. By virtue of Lemma 3.5.1, it suffices to show that

$$\downarrow\mathrm{C}(X,Y) = \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \mathcal{A}(1/m, 1/n).$$

From the definition, we need only to prove that $A(x)$ has the unique maximal point in Y for every $A \in \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \mathcal{A}(1/m,1/n)$ and $x \in X$. Let $y, y' \in Y$ be maximal points in $A(x)$. For each $n \in \mathbb{N}$, we can choose $m \in \mathbb{N}$ such that $A \in \mathcal{A}(1/m,1/n)$, which implies that $d_Y(y,y') < 1/n$. It follows that $d_Y(y,y') = 0$, that is, $y = y'$. Therefore the maximal point of $A(x)$ is unique. This completes the proof. □

Remark 3.5.1. Combining Corollary 3.4.1 and Proposition 3.5.1, we have that for each compact metrizable space X with no isolated points and each dendrite Y, $\downarrow\mathrm{C}(X,Y)$ is an $F_{\sigma\delta}$-set in $\overline{\downarrow\mathrm{C}(X,Y)}$, which is homeomorphic to Q. As is easily observed, the space $\downarrow\mathrm{C}(X,Y)$ is an absolute $F_{\sigma\delta}$-set.

3.6 Detecting a Z_σ-set in $\overline{\downarrow\mathrm{C}(X,Y)}$ containing $\downarrow\mathrm{C}(X,Y)$

A closed subset A of X is said to be a *Z-set* in X if for each open cover $\mathcal{U}$ of X, there exists a map $f : X \to X$ such that f is $\mathcal{U}$-close to the identity id_X and $f(X) \cap A = \emptyset$. A countable union of Z-sets in X is called a *Z_σ-set*. In addition, a *Z-embedding* is an embedding whose image is a Z-set in the range. We can easily prove the following:

Lemma 3.6.1. *Let Z be a Z-set in M that is homotopy dense in N. Then the closure $\overline{Z}$ of Z in N is a Z-set in N.*

The next lemma is very useful for detecting Z-sets in $\overline{\downarrow\mathrm{C}(X,Y)}$.

Lemma 3.6.2. *Let $X = (X, d_X)$ be a compact metric space and $Y = (Y, d_Y)$ a dendrite. Suppose that $F = E \cup Z$ is a closed set in $\overline{\downarrow\mathrm{C}(X,Y)}$ such that Z is a Z-set in $\overline{\downarrow\mathrm{C}(X,Y)}$, and for each $A \in E$, there exists a point $a \in X$ with $A(a) = \{\mathbf{0}\}$. Then F is a Z-set in $\overline{\downarrow\mathrm{C}(X,Y)}$.*

Proof. Let $\epsilon : \overline{\downarrow\mathrm{C}(X,Y)} \to (0,1)$. It suffices to construct a map $\phi : \overline{\downarrow\mathrm{C}(X,Y)} \to \overline{\downarrow\mathrm{C}(X,Y)}$ such that $\phi(\overline{\downarrow\mathrm{C}(X,Y)}) \cap F = \emptyset$ and $\rho_H(\phi(A), A) < \epsilon(A)$ for each $A \in \overline{\downarrow\mathrm{C}(X,Y)}$. Since Z is a Z-set, there exists a map $\psi : \overline{\downarrow\mathrm{C}(X,Y)} \to \overline{\downarrow\mathrm{C}(X,Y)} \setminus Z$ such that $\rho_H(\psi(A), A) < \epsilon(A)/2$ for each $A \in \overline{\downarrow\mathrm{C}(X,Y)}$. Fix a point $y_0 \in Y \setminus \{\mathbf{0}\}$. We define a map $\phi : \overline{\downarrow\mathrm{C}(X,Y)} \to \overline{\downarrow\mathrm{C}(X,Y)}$ by

$$\phi(A) = \psi(A) \cup \overline{r}([\mathbf{0}, y_0], t(A)),$$

where $t(A) = \min\{\epsilon(A), \rho_H(\psi(A), Z)\}/(2 \operatorname{diam} Y) > 0$. Obviously, $\phi(A)(x) \neq \mathbf{0}$ for each $x \in X$, that is, $\phi(A) \notin E$. Observe that

$$\rho_H(\phi(A), \psi(A)) \leq t(A) d_Y(\mathbf{0}, y_0) \leq t(A) \operatorname{diam} Y \leq \min\{\epsilon(A), \rho_H(\psi(A), Z)\}/2.$$

Hence $\phi(A) \notin Z$ and

$$\rho_H(\phi(A), A) \leq \rho_H(\phi(A), \psi(A)) + \rho_H(\psi(A), A) < \epsilon(A)/2 + \epsilon(A)/2 = \epsilon(A).$$

The continuity of ϕ follows from the ones of $\overline{r}$, ψ and t, and Lemma 3.1.1. This completes the proof. □

Proposition 3.6.1. *Let $X = (X, d_X)$ be a compact metric space with no isolated points and $Y = (Y, d_Y)$ be a dendrite. Then $\downarrow\mathrm{C}(X,Y)$ is contained in some Z_σ-set in $\overline{\downarrow\mathrm{C}(X,Y)}$.*

Proof. Take a countable dense set $D = \{d_n \mid n \in \mathbb{N}\}$ in X. For each $n, m \in \mathbb{N}$, let

$$F_{n,m} = \{\downarrow f \in \downarrow\mathrm{C}(X,Y) \mid d_Y(f(d_n), \mathbf{0}) \geq 1/m\}.$$

As is easily observed, $F_{n,m}$ is closed in $\downarrow\mathrm{C}(X,Y)$. For each map $\epsilon : \downarrow\mathrm{C}(X,Y) \to (0,1)$, by Lemma 3.4.1, we have $\phi : \downarrow\mathrm{C}(X,Y) \to \downarrow\mathrm{C}(X,Y)$ such that $\rho_H(\downarrow f, \phi(\downarrow f)) < \epsilon(\downarrow f)$

and $\phi(\downarrow f)(d_n) = \mathbf{0}$ for $\downarrow f \in \downarrow\mathrm{C}(X,Y)$. Obviously, $\phi(\downarrow\mathrm{C}(X,Y)) \cap F_{n,m} = \emptyset$. Thus each $F_{n,m}$ is a Z-set in $\downarrow\mathrm{C}(X,Y)$. It follows from Theorem 3.3.1 and Lemma 3.6.1 that the closure $\overline{F_{n,m}}$ is a Z-set in $\overline{\downarrow\mathrm{C}(X,Y)}$.

Let $F = \bigcap_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} (\downarrow\mathrm{C}(X,Y) \setminus F_{n,m})$. It remains to prove that the closure $\overline{F}$ of F in $\overline{\downarrow\mathrm{C}(X,Y)}$ is a Z-set. Observe that

$$F = \{\downarrow f \in \downarrow\mathrm{C}(X,Y) \mid f(d_n) = \mathbf{0} \text{ for each } n \in \mathbb{N}\} = \{\downarrow\mathbf{0}\},$$

where $\mathbf{0} : X \to \{\mathbf{0}\} \subset Y$ is the constant map. Hence $\overline{F} = \{\downarrow\mathbf{0}\} = \{X \times \{\mathbf{0}\}\}$. According to Lemma 3.6.2, $\overline{F}$ is a Z-set in $\overline{\downarrow\mathrm{C}(X,Y)}$. Consequently, $\downarrow\mathrm{C}(X,Y)$ is contained in the Z_σ-set $\overline{F} \cup \bigcup_{m,n\in\mathbb{N}} \overline{F_{n,m}}$. □

3.7 The strong $(\mathfrak{M}_0, \mathcal{F}_{\sigma\delta})$-universality of $(\overline{\downarrow\mathrm{C}(X,Y)}, \downarrow\mathrm{C}(X,Y))$

In this section, we shall prove the main theorem. Let (X_1, X_2) be a pair of spaces with $X_2 \subset X_1$ and $(\mathcal{C}_1, \mathcal{C}_2)$ be a pair of classes. We say that (X_1, X_2) is *strongly $(\mathcal{C}_1, \mathcal{C}_2)$-universal* if the following condition holds:

(su) Let $(Z_1, Z_2) \in (\mathcal{C}_1, \mathcal{C}_2)$ with $Z_2 \subset Z_1$, K a closed subset of Z_1, and $f : Z_1 \to X_1$ a map such that $f|_K$ is a Z-embedding. Then for every open cover $\mathcal{U}$ of X_1, there exists a Z-embedding $g : Z_1 \to X_1$ such that g is $\mathcal{U}$-close to f, $g|_K = f|_K$ and $g^{-1}(X_2) \setminus K = Z_2 \setminus K$.

A pair (X_1, X_2) with $X_2 \subset X_1$ is $(\mathcal{C}_1, \mathcal{C}_2)$-*absorbing*[1] provided that the following conditions are satisfied:

(i) $(X_1, X_2) \in (\mathcal{C}_1, \mathcal{C}_2)$,

(ii) X_2 is contained in a Z_σ-set in X_1,

(iii) (X_1, X_2) is strongly $(\mathcal{C}_1, \mathcal{C}_2)$-universal.

[1]We modify the definition of [4] for this paper.

Denote the class of compact metrizable spaces by $\mathfrak{M}_0$, and the one of separable metrizable absolute $F_{\sigma\delta}$-spaces by $\mathcal{F}_{\sigma\delta}$. According to Theorem 1.7.6 of [4], the following can be established.

Theorem 3.7.1. *Let X_1 and Z_1 be topological copies of the Hilbert cube Q. If pairs (X_1, X_2) and (Z_1, Z_2) are $(\mathfrak{M}_0, \mathcal{F}_{\sigma\delta})$-absorbing, then there exists a homeomorphism $f : X_1 \to Z_1$ such that $f(X_2) = Z_2$.*

Let $\mathbf{c}_1 = \{(x_i)_{i\in\mathbb{N}} \in Q \mid \lim_{i\to\infty} x_i = 1\}$. The following fact is well known.

Fact 3.7.1. *The pairs $(Q, \mathbf{c}_0)$ and $(Q, \mathbf{c}_1)$ are $(\mathfrak{M}_0, \mathcal{F}_{\sigma\delta})$-absorbing. In particular, $(Q, \mathbf{c}_0)$ is homeomorphic to $(Q, \mathbf{c}_1)$.*

We needs the following lemma to verify the strong $(\mathfrak{M}_0, \mathcal{F}_{\sigma\delta})$-universality of $(\overline{{\downarrow}\mathrm{C}(X, Y)}, {\downarrow}\mathrm{C}(X, Y))$.

Lemma 3.7.1. *Let $X = (X, d_X)$ be a compact metric space with $x_m, x_\infty \in X$, $m \in \mathbb{N}$, such that $\{r_m = d_X(x_m, x_\infty)\}_{m\in\mathbb{N}}$ is a strictly decreasing sequence conversing to 0, and let $Y = (Y, d_Y)$ be a dendrite with a distinguished point $y_0 \in Y \setminus \{\mathbf{0}\}$ such that $d_Y(\mathbf{0}, y_0) \leq 1$. Suppose that $g : Z \to Q$ is an injection from a space Z to the Hilbert cube Q and $\delta : Z \to (0, 1)$ is a map. Then there exists a map $\Phi : Z \to \overline{{\downarrow}\mathrm{C}(X, [\mathbf{0}, y_0])}$ satisfying the following conditions:*

(1) *Φ is injective,*

(2) *$\rho_H(\Phi(z), X \times \{\mathbf{0}\}) \leq \delta(z)$ for all $z \in Z$,*

(3) *$\Phi(z)(X \setminus B(x_\infty, r_{2k})) = \{\mathbf{0}\}$ for all $z \in Z$ with $2^{-k} \leq \delta(z) \leq 2^{-k+1}$, $k \in \mathbb{N}$,*

(4) *$z \in g^{-1}(\mathbf{c}_1)$ if and only if $\Phi(z) \in {\downarrow}\mathrm{C}(X, [\mathbf{0}, y_0])$,*

(5) *$\Phi(z)(x_\infty) = [\mathbf{0}, r(y_0, \delta(z))]$ for all $z \in Z$.*

Proof. For each $k, m \in \mathbb{N}$, let $Z_k = \{z \in Z \mid 2^{-k} \leq \delta(z) \leq 2^{-k+1}\}$ and $S_m = \{x \in X \mid r_m \leq d_X(x, x_\infty) \leq r_{m-1}\}$. Note that $Z = \bigcup_{k\in\mathbb{N}} Z_k$, $x_{m-1}, x_m \in S_m$, $\bigcup_{m\in\mathbb{N}} S_m = X \setminus \{x_\infty\}$, and $S_m \cap S_{m'} \neq \emptyset$ if and only if $|m - m'| \leq 1$. We define maps $\phi_k : Z_k \to \mathbf{I}$ and $\psi_m : S_m \to \mathbf{I}$ for each $k, m \in \mathbb{N}$ by $\phi_k(z) = 2 - 2^k\delta(z)$

and $\psi_m(x) = (d_X(x, x_\infty) - r_m)/(r_{m-1} - r_m)$, respectively. Then $\psi_m(x_{m-1}) = 1$ and $\psi_m(x_m) = 0$. For each $i, k \in \mathbb{N}$, let $f_i^k : Z_k \to \mathbf{I}$ be a map defined by

$$f_i^k(z) = \begin{cases} 0 & \text{if } i = 1, \\ (1 - \phi_k(z))\delta(z) & \text{if } i = 2, \\ (1 - \phi_k(z))\delta(z)g(z)(1) & \text{if } i = 3, \\ \delta(z) & \text{if } i = 2j, j \geq 2, \\ \delta(z)((1 - \phi_k(z))g(z)((i-1)/2) + \phi_k(z)g(z)((i-3)/2)) & \text{if } i = 2j+1, j \geq 2. \end{cases}$$

Remark that $f_i^k(z) \leq \delta(z)$ for every $z \in Z$. We define a map $\Phi_k : Z_k \to \overline{{\downarrow}\mathrm{C}(X, [\mathbf{0}, y_0])}$, $k \in \mathbb{N}$, as follows:

$$\begin{aligned} \Phi_k(z) = \{x \in X \mid d_X(x, x_\infty) \geq r_{2k}\} \times \{\mathbf{0}\} \cup \{x_\infty\} \times [\mathbf{0}, r(y_0, \delta(z))] \\ \cup \bigcup_{i \in \mathbb{N}} \{(x, y) \in X \times Y \mid x \in S_{2k+i}, y \in [\mathbf{0}, r(y_0, \alpha_i^k(x, z))]\}, \end{aligned}$$

where $\alpha_i^k(x, z) = \psi_{2k+i}(x) f_i^k(z) + (1 - \psi_{2k+i}(x)) f_{i+1}^k(z)$. Then $\Phi_k(z) = \Phi_{k+1}(z)$ for every $z \in Z_k \cap Z_{k+1}$. Indeed, take any $z \in Z_k \cap Z_{k+1}$. Since $\delta(z) = 2^{-k}$, we have $\phi_k(z) = 1$ and $\phi_{k+1}(z) = 0$. Observe that $f_1^k(z) = f_2^k(z) = f_3^k(z) = 0$. Hence for each $x \in X$,

$$\begin{aligned} \alpha_1^k(x, z) = \psi_{2k+1}(x) f_1^k(z) + (1 - \psi_{2k+1}(x)) f_2^k(z) = 0 \quad \text{and} \\ \alpha_2^k(x, z) = \psi_{2k+2}(x) f_2^k(z) + (1 - \psi_{2k+2}(x)) f_3^k(z) = 0. \end{aligned}$$

It follows that

$$\begin{aligned} &\Phi_k(z)(\{x \in X \mid d_X(x, x_\infty) \geq r_{2k+2}\}) \\ =&\{\mathbf{0}\} \\ =&\Phi_{k+1}(z)(\{x \in X \mid d_X(x, x_\infty) \geq r_{2k+2}\}). \end{aligned}$$

We see $f_3^k(z) = 0 = f_1^{k+1}(z)$, $f_{2j+3}^k(z) = \delta(z)g(z)(j) = f_{2j+1}^{k+1}(z)$ and $f_{2j+2}^k(z) = \delta(z) = f_{2j}^{k+1}(z)$ for all $j \geq 1$, that is, $f_{i+2}^k(z) = f_i^{k+1}(z)$ for all $i \geq 1$. Therefore for each $x \in S_{2k+i+2}$, $i \geq 1$,

$$\Phi_k(z)(x) = [\mathbf{0}, r(y_0, \alpha_{i+2}^k(x, z))] = [\mathbf{0}, r(y_0, \alpha_i^{k+1}(x, z))] = \Phi_{k+1}(z)(x).$$

Moreover, $\Phi_k(z)(x_\infty) = [\mathbf{0}, r(y_0, \delta(z))] = \Phi_{k+1}(z)(x_\infty)$. Thus $\Phi_k(z) = \Phi_{k+1}(z)$.

Now, we can obtain the desired map $\Phi : Z \to \overline{\downarrow \mathrm{C}(X, [\mathbf{0}, y_0])}$ defined by $\Phi(z) = \Phi_k(z)$ if $z \in Z_k$. It follows from the definition that Φ satisfies conditions (2), (3) and (5). So it remains to verify that conditions (1) and (4) hold.

Condition (1) Φ is injective.
Let $z_1, z_2 \in Z$ such that $\Phi(z_1) = \Phi(z_2)$. Then

$$[\mathbf{0}, r(y_0, \delta(z_1))] = \Phi(z_1)(x_\infty) = \Phi(z_2)(x_\infty) = [\mathbf{0}, r(y_0, \delta(z_2))],$$

which implies that $\delta(z_1) = \delta(z_2)$. Hence both of z_1 and z_2 are contained in Z_k for some $k \in \mathbb{N}$ and

$$\phi_k(z_1) = 2 - 2^k\delta(z_1) = 2 - 2^k\delta(z_1) = \phi_k(z_2).$$

Since $\psi_{2k+i}(x_{2k+i}) = 0$ for all $i \in \mathbb{N}$, we have

$$[\mathbf{0}, r(y_0, f^k_{i+1}(z_1))] = \Phi_k(z_1)(x_{2k+i}) = \Phi_k(z_2)(x_{2k+i}) = [\mathbf{0}, r(y_0, f^k_{i+1}(z_2))],$$

which implies that $f^k_j(z_1) = f^k_j(z_2)$ for every $j \geq 2$. In the case $\phi_k(z_1) = 1$, for each $j \in \mathbb{N}$, we have

$$g(z_1)(j) = f^k_{2j+3}(z_1) = f^k_{2j+3}(z_2) = g(z_2)(j),$$

In the case $\phi_k(z_1) \neq 1$, we have

$$(1 - \phi_k(z_1))\delta(z_1)g(z_1)(1) = f^k_3(z_1) = f^k_3(z_2) = (1 - \phi_k(z_2))\delta(z_2)g(z_2)(1),$$

which implies that $g(z_1)(1) = g(z_2)(1)$. Assume that $g(z_1)(i) = g(z_2)(i)$ for $i \in \mathbb{N}$. Then

$$\begin{aligned}
&\delta(z_1)((1 - \phi_k(z_1))g(z_1)(i+1) + \phi_k(z_1)g(z_1)(i)) \\
=&f^k_{2i+3}(z_1) = f^k_{2i+3}(z_2) \\
=&\delta(z_2)((1 - \phi_k(z_2))g(z_2)(i+1) + \phi_k(z_2)g(z_2)(i)),
\end{aligned}$$

so $g(z_1)(i+1) = g(z_2)(i+1)$. By induction, for all $j \in \mathbb{N}$, we get $g(z_1)(j) = g(z_2)(j)$. It follows that $g(z_1) = g(z_2)$. Since g is injective, $z_1 = z_2$. Therefore Φ is injective.

Condition (4) $z \in g^{-1}(\mathbf{c}_1)$ if and only if $\Phi(z) \in \downarrow \mathrm{C}(X, [\mathbf{0}, y_0])$.

We define a function $h(z) : X \to [\mathbf{0}, y_0] \subset Y$ for each $z \in Z_k$ and $k \in \mathbb{N}$ as follows:

$$h(z)(x) = \begin{cases} 0 & \text{if } d_X(x, x_\infty) \geq r_{2k}, \\ r(y_0, \alpha_i^k(x,z)) & \text{if } x \in S_{2k+i}, i \in \mathbb{N}, \\ r(y_0, \delta(z)) & \text{if } x = x_\infty. \end{cases}$$

Observe that $\downarrow h(z) = \Phi(z)$ and $h(z)$ is continuous on $X \setminus \{x_\infty\}$. When $h(z)$ is continuous at the point x_∞, $\Phi(z) = \downarrow h(z) \in \downarrow \mathrm{C}(X, [\mathbf{0}, y_0])$. So we need only to show that $z \in g^{-1}(\mathbf{c}_1)$ if and only if $h(z)$ is continuous at x_∞.

First, we shall prove the only if part. Take any $\epsilon > 0$. We may assume that $\epsilon < \delta(z)$. Since $g(z) \in \mathbf{c}_1$, there exists $i_0 \in \mathbb{N}$ such that for every $i \geq i_0$, $g(z)(i) > 1 - \epsilon/\delta(z)$. Fix any point $x \neq x_\infty$ in the neighborhood $\{x_\infty\} \cup \bigcup_{i \geq 2i_0+3} S_{2k+i}$ of x_∞ in X, where $z \in Z_k$. Then $x \in S_{2k+i}$ for some $i \geq 2i_0 + 3$. When i is even, $f_i^k(z) = \delta(z)$. When i is odd,

$$\begin{aligned} & f_i^k(z) \\ =& \delta(z)((1 - \phi_k(z))g(z)((i-1)/2) + \phi_k(z)g(z)((i-3)/2)) \\ >& \delta(z)((1 - \phi_k(z))(1 - \epsilon/\delta(z)) + \phi_k(z)(1 - \epsilon/\delta(z))) \\ >& \delta(z) - \epsilon. \end{aligned}$$

Hence we have

$$\begin{aligned} & \alpha_i^k(x,z) \\ =& \psi_{2k+i}(x) f_i^k(z) + (1 - \psi_{2k+i}(x)) f_{i+1}^k(z) \\ >& \psi_{2k+i}(x)(\delta(z) - \epsilon) + (1 - \psi_{2k+i}(x))(\delta(z) - \epsilon) \\ =& \delta(z) - \epsilon. \end{aligned}$$

It follows that

$$\begin{aligned} & d_Y(h(z)(x_\infty), h(z)(x)) \\ =& d_Y(r(y_0, \delta(z)), r(y_0, \alpha_i^k(z))) \\ =& (\delta(z) - \alpha_i^k(z)) d_Y(\mathbf{0}, y_0) \\ <& \delta(z) - (\delta(z) - \epsilon) \\ =& \epsilon. \end{aligned}$$

Consequently, $h(z)$ is continuous.

Next, we shall show the if part. Let $\epsilon \in (0,1)$ and $\epsilon' = \epsilon\phi_k(z)\delta(z)$, where $z \in Z_k$ with $\phi_k(z) > 0$. Since $h(z)$ is continuous at x_∞, we can choose $i_0 \geq 5$ so that for any $x \in X$,

$$d_X(x, x_\infty) \leq r_{2k+i_0} \Rightarrow d_Y(h(z)(x), h(z)(x_\infty)) < \epsilon' d_Y(\mathbf{0}, y_0).$$

Recall that $\psi_m(x_m) = 0$ for all $m \in \mathbb{N}$. Therefore for every $i \geq i_0$,

$$\begin{aligned}
&d_Y(r(y_0, f_{i+1}^k(z)), r(y_0, \delta(z))) \\
=&d_Y(r(y_0, \psi_{2k+i}(x_{2k+i})f_i^k(z) + (1 - \psi_{2k+i}(x_{2k+i}))f_{i+1}^k(z)), r(y_0, \delta(z))) \\
=&d_Y(r(y_0, \alpha_i^k(x_{2k+i}, z)), r(y_0, \delta(z))) \\
=&d_Y(h(z)(x_{2k+i}), h(z)(x_\infty)) \\
<&\epsilon' d_Y(\mathbf{0}, y_0).
\end{aligned}$$

Note that for all $i \geq i_0 + 1$,

$$\delta(z) - f_i^k(z) = d_Y(r(y_0, f_i^k(z)), r(y_0, \delta(z)))/d_Y(\mathbf{0}, y_0) < \epsilon'.$$

It follows that for any $j \geq (i_o - 2)/2$,

$$\begin{aligned}
&g(z)(j) \\
=&(f_{2j+3}^k(z)/\delta(z) - (1 - \phi_k(z))g(z)(j+1))/\phi_k(z) \\
\geq&(f_{2j+3}^k(z)/\delta(z) - (1 - \phi_k(z)))/\phi_k(z) \\
>&((\delta(z) - \epsilon')/\delta(z) - (1 - \phi_k(z)))/\phi_k(z) \\
=&((\delta(z) - \epsilon\phi_k(z)\delta(z))/\delta(z) - (1 - \phi_k(z)))/\phi_k(z) \\
=&1 - \epsilon.
\end{aligned}$$

Hence $g(z) \in \mathbf{c}_1$. Thus the proof is complete. □

Proposition 3.7.1. *Let $X = (X, d_X)$ be a compact metric space with no isolated points and $Y = (Y, d_Y)$ a dendrite. Then the pair $(\overline{\downarrow\mathrm{C}(X,Y)}, \downarrow\mathrm{C}(X,Y))$ is strongly $(\mathfrak{M}_0, \mathcal{F}_{\sigma\delta})$-universal.*

Proof. Let $(Z, C) \in (\mathfrak{M}_0, \mathcal{F}_{\sigma\delta})$, K a closed subset of Z, $\epsilon > 0$ and $\Phi : Z \to \overline{\downarrow\mathrm{C}(X,Y)}$ a map such that the restriction $\Phi|_K$ is a Z-embedding. We shall construct a Z-embedding $\Psi : Z \to \overline{\downarrow\mathrm{C}(X,Y)}$ so that Ψ is ϵ-close to Φ, $\Psi|_K = \Phi|_K$ and $\Psi^{-1}(\downarrow\mathrm{C}(X,Y)) \setminus K = C \setminus K$. Since $\Phi(K)$ is a Z-set in $\overline{\downarrow\mathrm{C}(X,Y)}$, we may assume that $\Phi(K) \cap \Phi(Z \setminus K) = \emptyset$. Define a map $\delta : Z \to [0,1)$ by $\delta(z) = \min\{\epsilon, \rho_H(\Phi(z), \Phi(K))\}/4$. Observe that $\delta(z) = 0$ if and only if $z \in K$. Since $\downarrow\mathrm{C}(X,Y)$ is homotopy dense in $\overline{\downarrow\mathrm{C}(X,Y)}$ by Theorem 3.3.1, there exists a homotopy $H : \overline{\downarrow\mathrm{C}(X,Y)} \times \mathbf{I} \to \overline{\downarrow\mathrm{C}(X,Y)}$ such that $H_0 = \mathrm{id}_{\overline{\downarrow\mathrm{C}(X,Y)}}$, $H_t(\overline{\downarrow\mathrm{C}(X,Y)}) \subset \downarrow\mathrm{C}(X,Y)$ for all $t \in (0,1]$ and $\rho_H(H_t(\downarrow A), \downarrow A) \leq t$ for each $\downarrow A \in \overline{\downarrow\mathrm{C}(X,Y)}$ and $t \in \mathbf{I}$. Let $h : Z \to \overline{\downarrow\mathrm{C}(X,Y)}$ be a map defined by $h(z) = H(\Phi(z), \delta(z))$. Remark that $\rho_H(h(z), \Phi(z)) = \rho_H(H(\Phi(z), \delta(z)), \Phi(z)) \leq \delta(z)$ for every $z \in Z$, in particular, $h(z) = \Phi(z)$ for all $z \in K$, and $h(Z \setminus K) \subset \downarrow\mathrm{C}(X,Y)$. Take a non-isolated point $x_\infty \in X$. According to Lemma 3.4.1, we can obtain maps $\psi : Z \setminus K \to \downarrow\mathrm{C}(X,Y)$ and $r : Z \setminus K \to (0,1)$ so that for each $z \in Z \setminus K$,

(a) $\rho_H(h(z), \psi(z)) \leq \delta(z)$,

(b) $\psi(z)(B(x_\infty, r(z))) = \{\mathbf{0}\}$.

Let $Z_k = \{z \in Z \mid 2^{-k} \leq \delta(z) \leq 2^{-k+1}\} \subset Z \setminus K$ for each $k \in \mathbb{N}$. Then each Z_k is compact and $Z \setminus K = \bigcup_{k \in \mathbb{N}} Z_k$. Since x_∞ is a non-isolated point, there exists a point $x_1 \in X \setminus \{x_\infty\}$ such that $d_X(x_1, x_\infty) < \min\{1, r(z) \mid z \in Z_1\}$. By induction, we can choose $x_m \in X \setminus \{x_\infty\}$ for each $m \geq 2$ so that $d_X(x_m, x_\infty) < \min\{1/m, d_X(x_{m-1}, x_\infty), r(z) \mid z \in Z_m\}$. Let $r_m = d_X(x_m, x_\infty)$ for each $m \in \mathbb{N}$, so r_m converges to 0 as m intends to ∞. Note that for every $z \in Z_k$ and $k \in \mathbb{N}$, $\psi(z)(B(x_\infty, r_k)) = \{\mathbf{0}\}$. Since the pair $(Q, \mathbf{c}_1)$ is strongly $(\mathfrak{M}_0, \mathcal{F}_{\sigma\delta})$-universal due to Fact 3.7.1, we can take am embedding $g : Z \to Q$ so that $g^{-1}(\mathbf{c}_1) = C$. Choose $y_0 \in Y \setminus \{\mathbf{0}\}$ with $d_Y(\mathbf{0}, y_0) \leq 1$.

Using Lemma 3.7.1, we can obtain a map $\psi' : Z \setminus K \to \overline{\downarrow\mathrm{C}(X, [\mathbf{0}, y_0])}$ satisfying the following conditions:

(1) ψ' is injective,

(2) $\rho_H(\psi'(z), X \times \{\mathbf{0}\}) \leq \delta(z)$ for all $z \in Z \setminus K$,

(3) $\psi'(z)(X \setminus B(x_\infty, r_{2k})) = \{\mathbf{0}\}$ for all $z \in Z_k$, $k \in \mathbb{N}$,

(4) $z \in C \setminus K$ if and only if $\psi'(z) \in \downarrow\mathrm{C}(X, [\mathbf{0}, y_0])$,

(5) $\psi'(z)(x_\infty) = [\mathbf{0}, r(y_0, \delta(z))]$ for all $z \in Z \setminus K$.

Define $\psi'' : Z \setminus K \to \overline{\downarrow\mathrm{C}(X,Y)}$ by $\psi''(z) = \psi(z) \cup \psi'(z)$. The continuity of ψ'' follows from the ones of ψ and ψ', and Lemma 3.1.1. By conditions (a) and (2), and Lemma 3.1.1, for each $z \in Z \setminus K$,

$$\begin{aligned}&\rho_H(h(z), \psi''(z))\\ =&\rho_H(h(z) \cup X \times \{\mathbf{0}\}, \psi(z) \cup \psi'(z))\\ \leq&\max\{\rho_H(h(z), \psi(z)), \rho_H(X \times \{\mathbf{0}\}, \psi'(z))\}\\ \leq&\delta(z).\end{aligned}$$

According to conditions (b), (3) and (4), we have $z \in C \setminus K$ if and only if $\psi''(z) \in \downarrow\mathrm{C}(X, Y)$. Moreover, ψ'' is injective. Indeed, take any $z_1, z_2 \in Z \setminus K$ with $\psi''(z_1) = \psi''(z_2)$. Then there exist $k_1, k_2 \in \mathbb{N}$ such that $z_1 \in Z_{k_1}$ and $z_2 \in Z_{k_2}$, respectively. It follows from (b) and (5) that

$$\begin{aligned}&[\mathbf{0}, r(y_0, \delta(z_1))] = \psi'(z_1)(x_\infty) = \psi''(z_1)(x_\infty)\\ =& \psi''(z_2)(x_\infty) = \psi'(z_2)(x_\infty) = [\mathbf{0}, r(y_0, \delta(z_2))],\end{aligned}$$

which implies that $\delta(z_1) = \delta(z_2)$. Hence $z_1, z_2 \in Z_k$, where $k = k_1 = k_2$. Since $\psi(z_1)(B(x_\infty, r_k)) = \{\mathbf{0}\} = \psi(z_2)(B(x_\infty, r_k))$ by (b), we have

$$\psi'(z_1)(x) = \psi''(z_1)(x) = \psi''(z_2)(x) = \psi'(z_2)(x)$$

for every $x \in B(x_\infty, r_{2k})$. On the other hand, by (3), $\psi'(z_1)(X \setminus B(x_\infty, r_{2k})) = \{\mathbf{0}\} = \psi'(z_2)(X \setminus B(x_\infty, r_{2k}))$. Therefore $\psi'(z_1) = \psi'(z_2)$. Due to (1), we get $z_1 = z_2$, so ψ'' is injective.

We can extend ψ'' to the desired map $\Psi : Z \to \overline{\downarrow\mathrm{C}(X,Y)}$ by $\Psi|_K = \Phi|_K$. Then for each $z \in Z$,

$$\begin{aligned}&\rho_H(\Phi(z), \Psi(z))\\ \leq&\rho_H(\Phi(z), h(z)) + \rho_H(h(z), \Psi(z)) \leq 2\delta(z)\\ \leq&\min\{\epsilon, \rho_H(\Phi(z), \Phi(K))\}/2,\end{aligned}$$

which means that Ψ is continuous. Moreover, it follows that $\rho_H(\Phi(z), \Psi(z)) \leq \epsilon$ for all $z \in Z$, and $\Psi(z) \in \overline{\downarrow\mathrm{C}(X,Y)} \setminus \Phi(K)$ for all $z \in Z \setminus K$. Since $z \in C \setminus K$ if and only if $\psi''(z) \in \downarrow\mathrm{C}(X,Y)$, we have $\Psi^{-1}(\downarrow\mathrm{C}(X,Y)) \setminus K = C \setminus K$. It remains to show that Ψ is a Z-embedding. It is easy to see that Ψ is an embedding. Recall that $\Psi(K) = \Phi(K)$ is a Z-set in $\overline{\downarrow\mathrm{C}(X,Y)}$. Since $x_{2k} \in B(x_\infty, r_k) \setminus B(x_\infty, r_{2k})$ for every $k \in \mathbb{N}$, it follows from (b) and (3) that

$$\Psi(z)(x_{2k}) = \psi''(z)(x_{2k}) = \psi(z)(x_{2k}) \cup \psi'(z)(x_{2k}) = \{\mathbf{0}\}$$

for each $z \in Z_k$. Applying Lemma 3.6.2, $\Psi(Z) = \Psi(Z \setminus K) \cup \Psi(K)$ is a Z-set in $\overline{\downarrow\mathrm{C}(X,Y)}$. Consequently, Ψ is a Z-embedding. □

3.8 Proof of Main Theorem 2

Finally, we prove the main theorem in Chapter 3.

Main Theorem 2. *Let X be an infinite, locally connected, compact metrizable space without isolated points, Y a dendrite and $v \in Y$ an end point of Y. Then*

$$(\overline{\downarrow_v \mathrm{C}_F(X,Y)}, \downarrow_v \mathrm{C}_F(X,Y)) \approx (Q, \mathbf{c}_0).$$

Proof. We can write $X = \bigoplus_{i=1}^n X_i$, where each X_i is a component of X. Note that the pair $(\overline{\downarrow_v \mathrm{C}(X,Y)}, \downarrow_v \mathrm{C}(X,Y))$ is homeomorphic to $\left(\prod_{i=1}^n \overline{\downarrow_v \mathrm{C}(X_i,Y)}, \prod_{i=1}^n \downarrow_v \mathrm{C}(X_i,Y)\right)$, refer to Lemma 6.8 of [21]. Since X is infinite, there exists at least one component that is non-degenerate. When X_i is a singleton, $(\overline{\downarrow_v \mathrm{C}(X_i,Y)}, \downarrow_v \mathrm{C}(X_i,Y))$ is homeomorphic to (Y,Y). When X_i is non-degenerate, it is compact and has no isolated points. Combining Corollary 3.4.1, Proposition 3.5.1, Proposition 3.6.1 and Proposition 3.7.1, we can obtain that $(\overline{\downarrow_v \mathrm{C}(X_i,Y)}, \downarrow_v \mathrm{C}(X_i,Y))$ is $(\mathfrak{M}_0, \mathcal{F}_{\sigma\delta})$-absorbing. It follows from Theorem 3.7.1 and Fact 3.7.1 that $(\overline{\downarrow_v \mathrm{C}(X_i,Y)}, \downarrow_v \mathrm{C}(X_i,Y))$ is homeomorphic to $(Q, \mathbf{c}_0)$. On the other hand, using Theorem 3.7.1, we can easily show that the pairs $(Q \times Q, \mathbf{c}_0 \times \mathbf{c}_0)$ and $(Q \times Y, \mathbf{c}_0 \times Y)$ are homeomorphic to $(Q, \mathbf{c}_0)$. This means that $\left(\prod_{i=1}^n \overline{\downarrow_v \mathrm{C}(X_i,Y)}, \prod_{i=1}^n \downarrow_v \mathrm{C}(X_i,Y)\right)$ is homeomorphic to $(Q, \mathbf{c}_0)$. Thus the proof is complete. □

Chapter 4

A New Topology of a Simplicial Complex

4.1 The infinite power of real lines

For each non-empty set Γ, let $\mathbb{R}^\Gamma$ be the set of all functions from Γ to $\mathbb{R}$. In other word, $\mathbb{R}^\Gamma$ is the product space $\prod_{\gamma\in\Gamma}\mathbb{R}_\gamma$ of $\mathbb{R}_\gamma$, $\gamma \in \Gamma$, which are copies of the real line $\mathbb{R}$. Then, $\mathbb{R}^\Gamma$ is a locally convex topological linear space with respect to the product topology. When $\Gamma \leq \aleph_0$, $\mathbb{R}^\Gamma$ is metrizable. On the other hand, $\mathbb{R}^\Gamma$ is not normable whenever Γ is infinite. In fact, the following is well-known:

Proposition 4.1.1. *Let Γ be an infinite set. Any linear subspace of $\mathbb{R}^\Gamma$ is not normable if it contains the subspace*

$$\mathbb{R}^\Gamma_f = \{x \in \mathbb{R}^\Gamma \mid x(\gamma) = 0 \text{ except for finitely many } \gamma \in \Gamma\}.$$

The Banach spaces $\ell_1(\Gamma)$, $\ell_2(\Gamma)$ and $\ell_\infty(\Gamma)$ are the following spaces:

$$\ell_1(\Gamma) = \left\{ x \in \mathbb{R}^\Gamma \,\middle|\, \sum_{\gamma\in\Gamma} |x(\gamma)| < \infty \right\},\ \|x\|_1 = \sum_{\gamma\in\Gamma} |x(\gamma)|;$$

$$\ell_2(\Gamma = \left\{ x \in \mathbb{R}^\Gamma \,\middle|\, \sum_{\gamma\in\Gamma} x(\gamma)^2 < \infty \right\},\ \|x\|_2 = \sqrt{\sum_{\gamma\in\Gamma} x(\gamma)^2} \text{ and}$$

$$\ell_\infty(\Gamma) = \left\{ x \in \mathbb{R}^\Gamma \,\middle|\, \sup_{\gamma\in\Gamma} |x(\gamma)| < \infty \right\},\ \|x\|_\infty = \sup_{\gamma\in\Gamma} |x(\gamma)|.$$

Note that $\ell_1(\Gamma) \subset \ell_2(\Gamma) \subset \ell_\infty(\Gamma)$ as sets, and $\mathbb{R}^\Gamma_f$ is contained in these spaces. Regarding $\mathbb{R}^\Gamma_f$ as their normed linear subspaces, we write $\ell^f_1(\Gamma)$, $\ell^f_2(\Gamma)$ and $\ell^f_\infty(\Gamma)$, respectively, i.e., $\ell^f_1(\Gamma) = (\mathbb{R}^\Gamma_f, \|\cdot\|_1)$, $\ell^f_2(\Gamma) = (\mathbb{R}^\Gamma_f, \|\cdot\|_2)$ and $\ell^f_\infty(\Gamma) = (\mathbb{R}^\Gamma_f, \|\cdot\|_\infty)$. In addition, $\|x\|_\infty \le \|x\|_2 \le \|x\|_1$ for every $x \in \mathbb{R}^\Gamma_f$ and these norms induce different topologies on $\mathbb{R}^\Gamma_f$. In fact, the unit sphere $\mathbf{S}_{\ell_1(\Gamma)}$ of $\ell_1(\Gamma)$ is closed in $\ell_1(\Gamma)$ but not in $\ell_2(\Gamma)$ nor in $\ell_\infty(\Gamma)$. The unit sphere $\mathbf{S}_{\ell_2(\Gamma)}$ of $\ell_2(\Gamma)$ is not closed in $\ell_\infty(\Gamma)$. Nevertheless , as is well known, $\ell_1(\Gamma)$ is homeomorphic to $\ell_2(\Gamma)$. By the same homeomorphism, $\ell^f_1(\Gamma)$ is also homeomorphic to $\ell^f_2(\Gamma)$.

The following proposition is also well known:

Proposition 4.1.2. *For $p = 1, 2,$ and $x \in \ell_p(\Gamma)$, a sequence $(x_n)_{n\in\mathbb{N}}$ in $\ell_p(\Gamma)$ converges to x in $\ell_p(\Gamma)$ if and only if*

$$\|x\|_p = \lim_{n\to\infty} \|x_n\| \ \textit{ and } \ x(\gamma) = \lim_{n\to\infty} x_n(\gamma) \ \textit{ for each } \gamma \in \Gamma.$$

As a corollary, we have

Corollary 4.1.1. *On the unit sphere $\mathbf{S}_{\ell_1(\Gamma)}$ of $\ell_1(\Gamma)$, the norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ induce the same topology.*

Proposition 4.1.2 is valid even for a net. Then, the following proposition holds:

Proposition 4.1.3. *The topologies on the unit spheres $\mathbf{S}_{\ell_p(\Gamma)}$, $p = 1, 2$, coincide with the relative topology of the product topology. In other words, $\mathbf{S}_{\ell_2(\Gamma)}$, $p = 1, 2$, are subspaces of the product space $\mathbb{R}^\Gamma$.*

The *finite topology* of $\mathbb{R}^\Gamma_f$ is the weak topology determined by the Euclidean topology on each finite-dimensional linear subspace. To be accurate, $U \subset \mathbb{R}^\Gamma_f$ is open with respect to the finite topology provided that for every finite-dimensional linear subspace F of $\mathbb{R}^\Gamma_f$, $U \cap F$ is open in F with the Euclidean topology. According to Appendix One A.4.3 in the book "Topology" by Dugundji[16], $\mathbb{R}^\Gamma_f$ with finite topology is not topological linear space. In fact, the addition is not continuous if the cardinality of Γ is not less than $2^\aleph_0$. We use the idea of its proof.

The box topology of $\mathbb{R}^\Gamma$ is defined as follows: a set U is open in $\mathbb{R}^\Gamma$ with the box topology if and only if for every $x \in U$ there exist $a_\gamma < b_\gamma \in \mathbb{R}$, $\gamma \in \Gamma$, such that $x \in \prod_{\gamma\in\Gamma}(a_\gamma, b_\gamma) \subset U$. The space $\mathbb{R}^\Gamma$ with the box topology is a locally convex topological linear space. The box topology of $\mathbb{R}^\Gamma_f$ is the relative topology inherited from the box topology of $\mathbb{R}^\Gamma$.

According to the Dugundji Extension Theorem [15], every locally convex topological linear space is an absolute extensor for metric spaces. Hence, $\mathbb{R}^\Gamma$ and $\mathbb{R}^\Gamma_f$ with the box topology are absolute extensors for metric spaces.

4.2 Comparison between topologies

Let K be a simplicial complex. The vertices of a simplex $\sigma \in K$ is denoted by $\sigma^{(0)}$. The boundary of σ is denoted by $\partial\sigma$ and $\operatorname{rint}\sigma = \sigma \setminus \partial\sigma$ is the radial interior of σ. For each $x \in |K|$, let $c_K(x)$ be the smallest simplex of K containing x, which is called the *carrier* of x in K. Let $c_K(x)^{(0)} = \{v_1, \dots, v_n\}$. Then, we can find $t_1, \dots, t_n > 0$ such that $\sum_{i=1}^n t_i = 1$ and $x = \sum_{i=1}^n t_i v_i$. We define

$$\beta_v^K(x) = \begin{cases} t_i & \text{if } v = v_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have $\beta^K : |K| \to \mathbb{R}_f^{K^{(0)}}$ defined by $\beta^K(x) = (\beta_v^K(x))_{v\in K^{(0)}}$. Note that

$$\mathbb{R}_f^{K^{(0)}} \subset \ell_1(K^{(0)}) \subset \ell_2(K^{(0)}) \subset \ell_\infty(K^{(0)}) \subset \mathbb{R}^{K^{(0)}}.$$

The weak topology of $|K|$ is induced by the relative (or subspace) topology of the finite topology of $\mathbb{R}_f^{K^{(0)}}$ which is the weak topology determined by the Euclidean topology on each finite-dimensional linear subspace (cf. Appendix One, A.4.2 and B.5 in [16]). However, $\mathbb{R}_f^{K^{(0)}}$ is not a topological linear space with respect to the finite topology. In fact, the addition is not continuous with respect to this topology [16, Appendix One, A.4.3].

The metric topology is induced by the metric ρ_K defined by the norm $\|\cdot\|_1$ of $\ell_1(K^{(0)})$, that is,

$$\rho_K(x, y) = \left\|\beta^K(x) - \beta^K(y)\right\|_1 = \sum_{v\in K^{(0)}} \left|\beta_v^K(x) - \beta_v^K(y)\right|.$$

The space $|K|$ with this topology is denoted by $|K|_m$. Since $\sum_{v\in K^{(0)}} \beta^K_v(x) = 1$, the image $\beta^K(|K|)$ is contained in the unit sphere $\mathbf{S}_{\ell_1(K^{(0)})}$ of $\ell_1(K^{(0)})$. Then, it follows from Proposition 4.1.3 that $|K|_m$ coincides with the relative topology inherited from the product topology of $\mathbb{R}^{K^{(0)}}$. According to Corollary 4.1.1, the following metrics are admissible:

$$\rho^K_2(x,y) = \left\|\beta^K(x) - \beta^K(y)\right\|_2 = \sqrt{\sum_{v\in K^{(0)}} \left|\beta^K_v(x) - \beta^K_v(y)\right|};$$

$$\rho^K_\infty(x,y) = \left\|\beta^K(x) - \beta^K(y)\right\|_\infty = \sup_{v\in K^{(0)}} \left|\beta^K_v(x) - \beta^K_v(y)\right|.$$

Regarding $\beta^K(|K|)$ as a subspace of the space $\mathbb{R}^{K^{(0)}}$ with the box topology, we introduce a new topology on $|K|$ and the space $|K|$ with this topology is denoted by $|K|_\mathrm{b}$. We call this topology the *box topology on* $|K|$. For each $x \in |K|$, $U \subset |K|$ is a neighborhood of x in $|K|_\mathrm{b}$ if and only if there are $\varepsilon_v > 0$, $v \in K^{(0)}$, such that

$$\left\{y \in |K| \;\middle|\; \forall v \in K^{(0)},\ \left|\beta^K_v(x) - \beta^K_v(y)\right| < \varepsilon_v\right\} \subset U.$$

As is easily observed, $\mathrm{id} : |K|_\mathrm{w} \to |K|_\mathrm{b}$ and $\mathrm{id} : |K|_\mathrm{b} \to |K|_\mathrm{m}$ are continuous. In other words,

$$\text{the metric topology} \subset \text{the box topology} \subset \text{the weak topology}.$$

In case K is locally finite, these topologies are equal because the metric topology coincides with the weak topology.

Proposition 4.2.1. *Let K be a simplicial complex and L a subcomplex of K. Then, $|L|_\mathrm{b}$ is a closed subspace of $|K|_\mathrm{b}$.*

Proof. For each $x \in |K| \setminus |L|$, $c_K(x) \notin L$, which means that $\operatorname{rint} c_K(x) \cap |L| = \emptyset$. Then,

$$U = \left\{y \in |K| \;\middle|\; \beta^K_v(y) > \beta^K_v(y)/2 \ \text{ for every } v \in c_K(x)^{(0)}\right\}.$$

is a neighborhood of x in $|K|_\mathrm{b}$. For every $y \in U$, since $c_K(x) \leqslant c_K(y)$, it follows that $c_K(y) \notin L$, hence $\operatorname{rint} c_K(y) \cap |L| = \emptyset$. Consequently, $U \cap |L| = \emptyset$, i.e., $U \subset |K| \setminus |L|$. Therefore, $|K| \setminus |L|$ is open in $|K|_\mathrm{b}$. □

The star $\mathrm{St}(\sigma, K)$ of $\sigma \in K$ is the subcomplex of K consisting of all faces of simplexes of K containing σ as a face. For simplicity, we write $\mathrm{St}(x, K) = \mathrm{St}(c_K(x), K)$ for a point $x \in |K|$. For each vertex $v \in K^{(0)}$, $(\beta_v^K)^{-1}((0,1])$ is denoted by $O(v, K)$ and called the *open star* at v in K. For every $x \in |K|$, $O(x, K) = \bigcap_{v \in c_K(x)^{(0)}} O(v, K)$ is called the *open star* at x in K, which is open in $|K|_{\mathrm{m}}$, and hence in both $|K|_{\mathrm{b}}$ and $|K|_{\mathrm{w}}$.

Lemma 4.2.1. *For each $x \in |K|$, $U \subset |K|$ is a neighborhood of x in $|K|_{\mathrm{b}}$ if and only if there are $\varepsilon_v > 0$, $v \in \mathrm{St}(x, K)^{(0)}$, such that*

$$\left\{y \in |K| \;\middle|\; \forall v \in \mathrm{St}(x, K)^{(0)},\ \left|\beta_v^K(x) - \beta_v^K(y)\right| < \varepsilon_v\right\} \subset U.$$

In particular, the open star $O(x, K)$ is open in $|K|_{\mathrm{b}}$, and hence $|\,\mathrm{St}(x, K)|$ is a neighborhood of x in $|K|_{\mathrm{b}}$.

Proof. The "if" part is trivial. To prove the "only if" part, let $U \subset |K|$ be a neighborhood of x in $|K|_{\mathrm{b}}$. By the definition of the b-topology, there are $\varepsilon_v > 0$, $v \in K^{(0)}$, such that

$$\left\{y \in |K| \;\middle|\; \forall v \in K^{(0)},\ \left|\beta_v^K(x) - \beta_v^K(y)\right| < \varepsilon_v\right\} \subset U.$$

For each $v \in c_K(x)^{(0)}$, replacing $\varepsilon_v > 0$, we may assume that $\varepsilon_v < \beta_v^K(x)$. Let

$$V = \left\{y \in |K| \;\middle|\; \forall v \in \mathrm{St}(x, K)^{(0)},\ \left|\beta_v^K(x) - \beta_v^K(y)\right| < \varepsilon_v\right\}.$$

For each $y \in V$, $\beta_v^K(y) > 0$ for every $v \in c_K(x)^{(0)}$. This means that $c_K(x) \leqslant c_K(x)$, hence $c_K(x) \in \mathrm{St}(x, K)$. Therefore, $\beta_v^K(y) = 0 < \varepsilon_v$ for every $v \in K^{(0)} \setminus \mathrm{St}(x, K)^{(0)}$. Then, it follows that

$$V = \left\{y \in |K| \;\middle|\; \forall v \in K^{(0)},\ \left|\beta_v^K(x) - \beta_v^K(y)\right| < \varepsilon_v\right\} \subset U.$$

This completes the proof. □

Theorem 4.2.1. *Let K be a simplicial complex. If K is* (1) *locally countable or* (2) $\dim K \leqslant 1$*, then $|K|_{\mathrm{b}} = |K|_{\mathrm{w}}$ as spaces.*

Proof. (1): Due to Lemma 4.2.1, it suffices to show the case K is countable. In this case, $|K|_{\mathrm{b}}$ can be regarded as a subspace of the space $\mathbb{R}^{\mathbb{N}}$ with the box topology, where $|K| \subset \mathbb{R}^{\mathbb{N}}_f$. As is well know, the subspace $\mathbb{R}^{\mathbb{N}}_f$ of the space $\mathbb{R}^{\mathbb{N}}$ with the box topology is the direct limit of the tower $\mathbb{R}^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \cdots$ of Euclidean spaces, that is, its topology is the weak topology with respect to this tower. Then, $|K|_{\mathrm{b}}$ has the weak topology with respect to $|K| \cap \mathbb{R}^n$, $n \in \mathbb{N}$. Since each simplex of K is contained in some $|K| \cap \mathbb{R}^n$, it follows that $|K|_{\mathrm{b}}$ has the weak topology with respect to K, that is $|K|_{\mathrm{b}} = |K|_{\mathrm{w}}$.

(2): Since the case $\dim K = 0$ is trivial, we prove the case $\dim K = 1$. Let $x \in |K|$. In case $x = v_0 \in K^{(0)}$, each $y \in |\operatorname{St}(v_0, K)|$ is contained in some 1-simplex $\langle v_0, v \rangle \in K$, where note that

$$0 \leqslant \beta^K_{v_0}(v_0) - \beta^K_{v_0}(y) = 1 - \beta^K_{v_0}(y) = \beta^K_v(y).$$

It follows from Lemma 4.2.1 that $U \subset |K|$ is a neighborhood of x in $|K|_{\mathrm{b}}$ if and only if there are $\delta_v > 0$, $v \in \operatorname{Lk}(v_0, K)^{(0)}$ such that

$$\left\{ y \in |\operatorname{St}(v_0, K)| \;\middle|\; \beta^K_v(y) < \delta_v \right\} \subset U.$$

When $c_K(x)$ is a 1-simplex $\langle v_0, v_1 \rangle \in K$, we can write $x = (1-t)v_0 + tv_1$ for some $0 < t < 1$. Since $|\operatorname{St}(x, K)| = c_K(x)$, it follows from Lemma 4.2.1 that $U \subset |K|$ is a neighborhood of x in $|K|_{\mathrm{b}}$ if and only if there is some $0 < \delta < \min\{t,\ 1-t\}$ such that

$$\left\{ (1-s)v_0 + sv_1 \;\middle|\; 0 < t - \delta < s < t + \delta \right\} \subset U.$$

Therefore, $U \subset |K|$ is a neighborhood of x in $|K|_{\mathrm{b}}$ if and only if U is neighborhood of x in $|K|_{\mathrm{w}}$. Thus, we have the result. □

The following 1-dimensional countable simplicial complex J is a well known example of a simplicial complex such that $|J|_{\mathrm{b}} \neq |J|_{\mathrm{m}}$:

$$J = \left\{ v_0,\ v_n,\ \langle v_0, v_n \rangle \;\middle|\; n \in \mathbb{N} \right\},$$

where $v_n \neq v_m$ if $n \neq m \in \omega$. Note that if a simplicial complex K is not locally finite, then K contains a subcomplex which is simplicially isomorphic to J. Then, we have the following:

Corollary 4.2.1. *Let K be a simplicial complex such that K is locally countable or 1-dimensional. If K is not locally finite, then $|K|_{\mathrm{b}} \neq |K|_{\mathrm{m}}$ as spaces.* □

4.3 Absolute neighborhood extensors for metrizable spaces

As mentioned in §4.1, for a full simplicial complex K, $|K|_{\mathrm{b}}$ is an absolute extensor for metrizable spaces. This fact extends as follows:

Theorem 4.3.1. *For every simplicial complex K, $|K|_{\mathrm{b}}$ is an absolute neighborhood extensor for metrizable spaces.*

To prove this theorem, we need the following proposition:

Proposition 4.3.1. *Let K be a simplicial complex and X a metrizable space. Then, $f : X \to |K|_{\mathrm{b}}$ is continuous if and only if $f : X \to |K|_{\mathrm{w}}$ is continuous.*

Proof. The "if" part is obvious. To show the "only if" part, assume that $f : X \to |K|_{\mathrm{w}}$ is not continuous at $x_0 \in X$, that is, $f(x_0)$ has a neighborhood V in $|K|_{\mathrm{w}}$ such that $f(U) \not\subset V$ for any neighborhood U of x_0 in X. Then, we can find a sequence $x_1, x_2, \cdots \in X$ such that $x_0 = \lim_{n\to\infty} x_n$ and $f(x_n) \notin V$ for every $n \in \mathbb{N}$. Let

$$L = \left\{\sigma \in K \;\middle|\; \exists n \in \mathbb{N} \text{ such that } \sigma \leqslant c_K(f(x_n))^{(0)}\right\}.$$

Then, L is a countable subcomplex of K and $\{f(x_n) \mid n \in \omega\} \subset |L|$. Hence, $f(x_0) \neq \lim_{n\to\infty} f(x_n)$ in $|L|_{\mathrm{w}}$. On the other hand, the countablity of L implies $|L|_{\mathrm{w}} = |L|_{\mathrm{b}}$ by Theorem 4.2.1(1). Consequently, $f : X \to |K|_{\mathrm{w}}$ is not continuous. □

Proof of Theorem 4.3.1. Let X be a metrizable space and $f : A \to |K|_{\mathrm{b}}$ a map from a closed set A in X. By Proposition 4.3.1, $f : A \to |K|_{\mathrm{w}}$ is also continuous. Since $|K|_{\mathrm{w}}$ is an absolute neighborhood extensor for metrizable spaces, there exist a neighborhood U of A in X with a continuous extension $\tilde{f} : U \to |K|_{\mathrm{w}}$ of f. Then, $\tilde{f} : U \to |K|_{\mathrm{b}}$ is also continuous. Thus, $f : A \to |K|_{\mathrm{b}}$ extends over a neighborhood U of A in X. □

4.4 The simplicial complex B

We define a simplicial complex B as follows:

$$B^{(0)} = \{v_n \mid n \in \omega\} \cup \{v_\lambda \mid \lambda \in \mathbb{N}^{\mathbb{N}}\}$$
$$B = \{\sigma \mid \exists n \in \mathbb{N},\ \exists \lambda \in \mathbb{N}^{\mathbb{N}} \text{ such that } \sigma \leqslant \langle v_0, v_n, v_\lambda\rangle\}$$

Then, $\operatorname{card} B^{(0)} = 2^{\aleph_0}$ and $\dim B = 2$. For each $n \in \mathbb{N}$ and $\lambda \in \mathbb{N}^{\mathbb{N}}$, let

$$a_{n,\lambda} = \left(1 - \frac{2}{\lambda(n)}\right)v_0 + \frac{1}{\lambda(n)}v_n + \frac{1}{\lambda(n)}v_\lambda \in \langle v_0, v_n, v_\lambda\rangle.$$

Theorem 4.4.1. *The box topology on $|B|$ is different from the weak topology, that is, $|B|_{\mathrm{b}} \neq |B|_{\mathrm{w}}$ as spaces.*

Proof. Let $A = \{a_{n,\lambda} \mid n \in \mathbb{N}, \lambda \in \mathbb{N}^{\mathbb{N}}\} \subset |B|$. For each $n \in \mathbb{N}$ and $\lambda \in \mathbb{N}^{\mathbb{N}}$, A meets the simplex $\langle v_0, v_n, v_\lambda\rangle$ at the point $a_{n,\lambda}$. Hence, A is closed in $|B|_{\mathrm{w}}$. On the other hand, A is not closed in $|B|_{\mathrm{b}}$. On the contrary, assume that A is closed in $|B|_{\mathrm{b}}$, that is, $|B| \setminus A$ is open in $|B|_{\mathrm{b}}$. Since $v_0 \in |B| \setminus A$, we can find $\varepsilon_n > 0$, $n \in \omega$ and $\varepsilon_\lambda > 0$, $\lambda \in \mathbb{N}^{\mathbb{N}}$, such that

$$(\beta^B_{v_0})^{-1}((1-\varepsilon_0, 1]) \cap \bigcap_{n\in\mathbb{N}} (\beta^B_{v_n})^{-1}([0,\varepsilon_n)) \cap \bigcap_{\lambda\in\mathbb{N}^{\mathbb{N}}} (\beta^B_{v_\lambda})^{-1}([0,\varepsilon_\lambda)) \subset |B| \setminus A.$$

Let $\lambda_0 \in \mathbb{N}^{\mathbb{N}}$ be the map defined by

$$\lambda_0(n) = \max\left\{n,\ \left[\frac{2}{\varepsilon_0}\right] + 1,\ \left[\frac{1}{\varepsilon_n}\right] + 1\right\} \text{ for each } n \in \mathbb{N},$$

where $[t] \in \mathbb{Z}$ is the largest integer such that $[t] \leqslant t$, hence $[t]+1$ is smallest integer such that $t < [t]+1$. Choose $n_0 \in \mathbb{N}$ so that $1/n_0 < \varepsilon_{\lambda_0}$. Then, it follows that

$$\beta^B_{v_{\lambda_0}}(a_{n_0,\lambda_0}) = \frac{1}{\lambda_0(n_0)} \leqslant \frac{1}{n_0} < \varepsilon_{\lambda_0},$$
$$\beta^B_{v_0}(a_{n_0,\lambda_0}) = 1 - \frac{2}{\lambda_0(n_0)} > 1 - \varepsilon_0 \text{ and}$$
$$\beta^B_{v_{n_0}}(a_{n_0,\lambda_0}) = \frac{1}{\lambda_0(n_0)} < \varepsilon_{n_0}.$$

Moreover, $\beta^B_{v_n}(a_{n_0,\lambda_0}) = 0 < \varepsilon_n$ for each $n \in \mathbb{N} \setminus \{n_0\}$ and $\beta^B_{v_\lambda}(a_{n_0,\lambda_0}) = 0 < \varepsilon_\lambda$ for each $\lambda \in \mathbb{N}^{\mathbb{N}} \setminus \{\lambda_0\}$. Therefore, $a_{n_0,\lambda_0} \in |B| \setminus A$, which is a contradiction. □

Due to Proposition 4.2.1, if L is a subcomplex of K then $|L|_{\mathrm{b}}$ is a subspace of $|K|_{\mathrm{b}}$. Then, we have the following corollary:

Corollary 4.4.1. *If a simplicial complex K contains a subcomplex which is simplicially isomorphic to B defined as above, then $|K|_{\mathrm{b}} \neq |K|_{\mathrm{w}}$ as spaces.* $\square$

4.5 The box topology of subdivisions.

Let K' be a simplicial subdivision of a simplicial complex K. For each $x \in |K|$, $c_K(v') \subset c_K(x)$ for each $v' \in c_{K'}(x)^{(0)}$ and

$$x = \sum_{v' \in c_{K'}(x)^{(0)}} \beta^{K'}_{v'}(x) v' = \sum_{v \in c_K(x)^{(0)}} \left(\sum_{v' \in c_{K'}(x)^{(0)}} \beta^{K'}_{v'}(x) \beta^K_v(v') \right) v,$$

which implies that

$$(*) \qquad \sum_{v' \in c_{K'}(x)^{(0)}} \beta^{K'}_{v'}(x) \beta^K_v(v') = \beta^K_v(x) \quad \text{for each } v \in c_K(x)^{(0)}.$$

Proposition 4.5.1. *For every simplicial subdivision K' of a simplicial complex K, $\mathrm{id} : |K'|_{\mathrm{b}} \to |K|_{\mathrm{b}}$ is continuous.*

Proof. To prove the continuity of $\mathrm{id} : |K'|_{\mathrm{b}} \to |K|_{\mathrm{b}}$ at a point $x \in |K'|$, given $\varepsilon_v > 0$, $v \in K^{(0)}$, we define $\delta_{v'} > 0$, $v' \in K'^{(0)}$, as follows:

$$\delta_{v'} = \begin{cases} \dfrac{\min\{\varepsilon_v \mid v \in c_K(v')^{(0)}\}}{2(\dim c_{K'}(x) + 1)} & \text{if } v' \in c_{K'}(x)^{(0)}, \\ \dfrac{1}{2} \min\{\beta^{K'}_{v'}(x) \mid v' \in c_{K'}(x)^{(0)}\} & \text{otherwise.} \end{cases}$$

Assume that $x' \in |K'|$ and $|\beta^{K'}_{v'}(x) - \beta^{K'}_{v'}(x')| < \delta_{v'}$ for every $v' \in K'^{(0)}$. Then, $\beta^{K'}_{v'}(x) > 0$ implies $\beta^{K'}_{v'}(x') > 0$, which means that $c_{K'}(x) \leqslant c_{K'}(x')$. Moreover, it follows that

$$\begin{aligned}\sum_{v' \in c_{K'}(x')^{(0)} \setminus c_{K'}(x)^{(0)}} \beta^{K'}_{v'}(x') &= 1 - \sum_{v' \in c_{K'}(x)^{(0)}} \beta^{K'}_{v'}(x') \\ &= \sum_{v' \in c_{K'}(x)^{(0)}} \beta^{K'}_{v'}(x) - \sum_{v' \in c_{K'}(x)^{(0)}} \beta^{K'}_{v'}(x') \\ &\leqslant \sum_{v' \in c_{K'}(x)^{(0)}} \left|\beta^{K'}_{v'}(x) - \beta^{K'}_{v'}(x')\right| \\ &< \sum_{v' \in c_{K'}(x)^{(0)}} \delta_{v'}.\end{aligned}$$

By (*), for each $v \in K^{(0)}$,

$$\begin{aligned}\left|\beta^{K}_{v}(x) - \beta^{K}_{v}(x')\right| &\leqslant \sum_{v' \in c_{K'}(x')^{(0)}} \left|\beta^{K'}_{v'}(x) - \beta^{K'}_{v'}(x')\right| \beta^{K}_{v}(v') \\ &\leqslant \sum_{v' \in c_{K'}(x)^{(0)}} \left|\beta^{K'}_{v'}(x) - \beta^{K'}_{v'}(x')\right| \\ &\qquad + \sum_{v' \in c_{K'}(x')^{(0)} \setminus c_{K'}(x)^{(0)}} \beta^{K'}_{v'}(x') \\ &< 2 \sum_{v' \in c_{K'}(x)^{(0)}} \delta_{v'} \\ &\leqslant \min\{\varepsilon_u \mid u \in c_K(v')^{(0)}\} \leqslant \varepsilon_v.\end{aligned}$$

This completes the proof. □

Any simplicial subdivision of a simplicial complex K preserves the weak topology and the barycentric subdivision of K preserves the metric topology.

Theorem 4.5.1. *The barycentric subdivision of the simplicial complex B does not preserve the* b*-topology, that is,* $|\operatorname{Sd} B|_{\mathrm{b}} \neq |B|_{\mathrm{b}}$ *as spaces.*

Proof. We prove that $\mathrm{id} : |B|_{\mathrm{m}} \to |\operatorname{Sd} B|_{\mathrm{m}}$ is not continuous at v_0. For each $n \in \mathbb{N}$ and $\lambda \in \mathbb{N}^{\mathbb{N}}$, let $\sigma_{n,\lambda} = \langle v_0, v_n, v_\lambda \rangle$ and $\varepsilon_{\hat{\sigma}_{n,\lambda}} = 2/\lambda(n) > 0$. Then, for any $\delta_v > 0$, $v \in B^{(0)}$, we can find $n_0 \in \mathbb{N}$ and $\lambda_0 \in \mathbb{N}^{\mathbb{N}}$ with $x \in \sigma_{n_0,\lambda_0}$ such that

$$\left|\beta^{B}_{v}(v_0) - \beta^{B}_{v}(x)\right| < \delta_v \text{ for every } v \in B^{(0)} \text{ but}$$
$$\left|\beta^{\operatorname{Sd} B}_{\hat{\sigma}_{n_0,\lambda_0}}(v_0) - \beta^{\operatorname{Sd} B}_{\hat{\sigma}_{n_0,\lambda_0}}(x)\right| \geqslant \varepsilon_{\hat{\sigma}_{n_0,\lambda_0}}.$$

In fact, we define $\lambda_0 \in \mathbb{N}^{\mathbb{N}}$ by

$$\lambda_0(n) = \max\left\{n,\ \left[\frac{1}{\delta_{v_n}}\right]+1,\ \left[\frac{2}{\delta_{v_0}}\right]+1\right\},$$

and choose $n_0 \in \mathbb{N}$ so that $n_0^{-1} < \delta_{\lambda_0}$. Let

$$\begin{aligned} x &= \left(1 - \frac{2}{\lambda_0(n_0)}\right)v_0 + \frac{1}{\lambda_0(n_0)}v_{n_0} + \frac{1}{\lambda_0(n_0)}v_{\lambda_0} \\ &= \left(1 - \frac{3}{\lambda_0(n_0)}\right)v_0 + \frac{3}{\lambda_0(n_0)}\left(\frac{1}{3}v_0 + \frac{1}{3}v_{n_0} + \frac{1}{3}v_{\lambda_0}\right) \\ &= \left(1 - \frac{3}{\lambda_0(n_0)}\right)v_0 + \frac{3}{\lambda_0(n_0)}\hat{\sigma}_0 \end{aligned}$$

Then, $|\beta^B_v(v_0) - \beta^B_v(x)| < \delta_v$ for every $v \in B^{(0)}$ because

$$\begin{gathered} 1 - \beta^B_{v_0}(x) = \frac{2}{\lambda_0(n_0)} < \delta_{v_0},\ \beta^B_{v_{n_0}}(x) = \frac{1}{\lambda_0(n_0)} < \delta_{v_{n_0}}, \\ \beta^B_{v_{\lambda_0}}(x) = \frac{1}{\lambda_0(n_0)} \leqslant \frac{1}{n_0} < \delta_{v_{\lambda_0}} \text{ and} \\ \beta^B_v(x) = 0 < \delta_v \text{ for } v \neq v_0, v_{n_0}, v_{\lambda_0}. \end{gathered}$$

On the other hand,

$$\left|\beta^{\operatorname{Sd} B}_{\hat{\sigma}_{n_0,\lambda_0}}(v_0) - \beta^{\operatorname{Sd} B}_{\hat{\sigma}_{n_0,\lambda_0}}(x)\right| = \beta^{\operatorname{Sd} B}_{\hat{\sigma}_0}(x) = \frac{3}{\lambda_0(n_0)} > \varepsilon_{\hat{\sigma}_{n_0,\lambda_0}}.$$

This completes the proof. □

4.6 The continuity of simplicial maps

Every simplicial map is continuous with respect to both the metic topology and the weak topology. Concerning the box topology, we have the following result:

Theorem 4.6.1. *Let K and L be simplicial complexes and $f : K \to L$ a simplicial map. In the following cases, $f : |K|_{\mathrm{b}} \to |L|_{\mathrm{b}}$ is continuous.*

(1) *K is locally countable,*

(2) *K is locally finite-dimensional,*

(3) *f is proper.*

In the above (3), a simplicial map $f : K \to L$ is *proper* if $f^{-1}(v) \cap K^{(0)}$ ($=(f||K^{(0)}|)^{-1}(v)$) is finite for each $v \in L^{(0)}$.

Proof. (1): Note that $f|\sigma$ is continuous for every $\sigma \in K$ because both simplexes σ and $f(\sigma)$ have the Euclidean topology. Because of the local countability of K, $|K|_{\mathrm{b}} = |K|_{\mathrm{w}}$ as spaces. Then, it follows that f is continuous.

(2): We show the continuity of f at $x = \sum_{v \in K^{(0)}} \beta_v^K(x)v \in |K|$. Let $k = \dim \mathrm{St}(x, K) < \infty$ (the local finite-dimensionality of K). Given $\varepsilon_u > 0$, $u \in \mathrm{St}(f(x), L)^{(0)}$, we define

$$\delta_v = \frac{\varepsilon_{f(v)}}{k+1} > 0, \ v \in \mathrm{St}(x, K)^{(0)}.$$

Take $x' \in |\mathrm{St}(x, K)|$ such that $|\beta_v^K(x) - \beta_v^K(x')| < \delta_v$ for each $v \in \mathrm{St}(x, K)^{(0)}$. Then, $\operatorname{card} c_K(x')^{(0)} \leqslant k+1$. For each $u \in \mathrm{St}(f(x), L)^{(0)}$,

$$\begin{aligned}
\left|\beta_u^L(f(x)) - \beta_u^L(f(x')\right| &= \left| \sum_{v \in c_K(x')^{(0)} \cap f^{-1}(u)} \beta_v^K(x) - \sum_{v \in c_K(x')^{(0)} \cap f^{-1}(u)} \beta_v^K(x') \right| \\
&\leqslant \sum_{v \in c_K(x')^{(0)} \cap f^{-1}(u)} \left|\beta_v^K(x) - \beta_v^K(x')\right| \\
&< \sum_{v \in c_K(x')^{(0)} \cap f^{-1}(u)} \delta_v \leqslant \sum_{v \in c_K(x')^{(0)}} \frac{\varepsilon_{f(v)}}{k+1} \leqslant \varepsilon_u.
\end{aligned}$$

This shows that $f : |K|_{\mathrm{b}} \to |L|_{\mathrm{b}}$ is continuous.

(3): As (2), we show the continuity of f at $x = \sum_{v \in K^{(0)}} \beta_v^K(x)v \in |K|$. For each $u \in L^{(0)}$, let $k_u = \operatorname{card} f^{-1}(v) \cap K^{(0)} \in \omega$ (the properness of f). Given $\varepsilon_u > 0$, $u \in \mathrm{St}(f(x), L)^{(0)}$, define

$$\delta_v = \frac{\varepsilon_{f(v)}}{k_{f(v)}} > 0, \ v \in \mathrm{St}(x, K)^{(0)}.$$

Take $x' \in |\mathrm{St}(x, K)|$ such that $|\beta_v^K(x) - \beta_v^K(x')| < \delta_v$ for each $v \in \mathrm{St}(x, K)^{(0)}$. As the case (2), for each $u \in \mathrm{St}(f(x), L)^{(0)}$,

$$\begin{aligned}
\left|\beta_u^L(f(x)) - \beta_u^L(f(x')\right| &\leqslant \sum_{v \in c_K(x')^{(0)} \cap f^{-1}(u)} \left|\beta_v^K(x) - \beta_v^K(x')\right| \\
&< \sum_{v \in c_K(x')^{(0)} \cap f^{-1}(u)} \delta_v \leqslant \sum_{v \in K^{(0)} \cap f^{-1}(u)} \frac{\varepsilon_{f(v)}}{k_{f(v)}} = \varepsilon_u.
\end{aligned}$$

This shows that $f : |K|_{\mathrm{b}} \to |L|_{\mathrm{b}}$ is continuous. □

Finally, we show the following:

Theorem 4.6.2. *Let K and L be simplicial complexes such that K contains an uncountable full simplicial complex and L contains an infinite full simplicial complex. Then, there exists a simplicial map $f : K \to L$ which is not continuous.*

Proof. Without loss of generality, we may assume that L itself is a full simplicial complex with $L^{(0)} = \{u_n \mid n \in \omega\}$, where $u_n \neq u_{n'}$ if $n \neq n'$. Let F be an uncountable full simplicial complex contained in K. Take a vertex $v_0 \in F^{(0)}$ and let

$$F^{(0)} \setminus \{v_0\} = \{v_{n,\lambda} \mid (n, \lambda) \in \mathbb{N} \times \Lambda\},$$

where Λ is uncountable and $v_{n,\lambda} \neq v_{n',\lambda'}$ if $(n, \lambda) \neq (n', \lambda')$. We define a simplicial complex $f : K \to L$ by

$$f(v) = \begin{cases} u_n & \text{if } v = v_{n,\lambda},\ (n, \lambda) \in \mathbb{N} \times \Lambda, \\ u_0 & \text{if } v = v_0 \text{ or } v \in K^{(0)} \setminus F^{(0)}. \end{cases}$$

Then, $f : |K|_{\mathrm{b}} \to |L|_{\mathrm{b}}$ is not continuous at v_0. Indeed, $V_0 = \bigcap_{n\in\mathbb{N}} (\beta^L_{u_n})^{-1}([0, 2^{-n}))$ is an open neighborhood of $u_0 = f(v_0)$ in $|L|_{\mathrm{b}}$. For each neighborhood of U in $|K|_{\mathrm{b}}$, we have $\delta_v > 0$, $v \in K^{(0)}$, such that

$$(\beta^K_{v_0})^{-1}((1 - \delta_{v_0}, 1]) \cap \bigcap_{v\in K^{(0)}\setminus\{v_0\}} (\beta^K_v)^{-1}([0, \delta_v)) \subset U.$$

Choose an $n_0 \in \mathbb{N}$ so that $2^{-n_0} < \delta_{v_0}$. Since Λ is uncountable, we have $n_1 > n_0$ such that $\delta_{v_{n_0,\lambda_i}} > 2^{-n_1}$ for infinitely many distinct $\lambda_i \in \Lambda$, $i \in \mathbb{N}$. Let

$$p = \bigl(1 - 2^{-n_0}\bigr)v_0 + \sum_{i=1}^{2^{n_1-n_0}} 2^{-n_1} v_{n_0,\lambda_i} \in |F| \subset |K|.$$

Then, it follows that

$$\beta^K_{v_0}(p) = 1 - 2^{-n_0} > 1 - \delta_{v_0},$$
$$\beta^K_{v_{n_0,\lambda_i}}(p) = 2^{-n_1} < \delta_{n_0,\lambda_i} \text{ for each } i = 1, 2, \ldots, 2^{n_1-n_0} \text{ and}$$
$$\beta^K_v(p) = 0 < \delta_v \text{ if } v \neq v_0, v_{n_0,\lambda_1}, v_{n_0,\lambda_2}, \ldots, v_{n_0,\lambda_{2^{n_1-n_0}}},$$

which means that $p \in U$. On the other hand,

$$f(p) = \big(1 - 2^{-n_0}\big)u_0 + 2^{n_1 - n_0}2^{-n_1}u_{n_0} = \big(1 - 2^{-n_0}\big)u_0 + 2^{-n_0}u_{n_0}.$$

Then, $\beta^L_{u_{n_0}}(f(p)) = 2^{-n_0}$, which implies that $f(p) \notin V_0$. Hence, f is not continuous at v_0. $\square$

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