Infinite-dimensional manifolds and their pairs

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Introduction

Throughout this thesis, all spaces are Hausdorff and all maps are continuous, but functions are not necessarily continuous. We use often cardinals itself as a set. Given a space E, an *E-manifold* is a topological manifold modeled on E, that is, a paracompact space such that each point has an open neighborhood homeomorphic to an open subset of E, where E is called a model space. An E-manifold is an *infinite-dimensional manifold* if the model space E is infinite-dimensional. The Hilbert space of weight τ is denoted by $\ell_2(\tau)$, that is,

$$\ell_2(\tau) = \left\{ x = (x(\gamma))_{\gamma<\tau} \in \mathbb{R}^\tau \;\middle|\; \sum_{\gamma<\tau} x(\gamma)^2 < \infty \right\},$$

where τ is an infinite cardinal. We denote the Hilbert cube by $\mathbf{Q} = [-1,1]^{\mathbb{N}}$. They are the most typical model spaces of infinite-dimensional manifolds. The study of infinite-dimensional manifolds, which had risen in the late 1960s, reached the celebrated topological characterizations of $\ell_2(\tau)$-manifolds and $\mathbf{Q}$-manifolds by H. Toruńczyk [58, 59] in the early 1980s.

In this thesis, we study on characterizations of infinite-dimensional manifolds and their pairs modeled on Hilbert spaces, the Hilbert cube and the subspaces, and as applications, we detect infinite-dimensional manifolds among convex sets in topological linear spaces and function spaces.

In recent years, many researchers eagerly study infinite-dimensional manifolds modeled on incomplete metrizable spaces being universal for absolute Borel classes. The following concept plays a central role in topological characterizations of such infinite-dimensional manifolds. A space X is *strongly universal* for a class $\mathcal{C}$ if the following condition is satisfied:

(su) For each space $A \in \mathcal{C}$ and each closed subset B of A, every map $f : A \to X$, whose image $f(B)$ of B is a Z-set, is arbitrarily closely approximated by an embedding $g : A \to X$ such that $g(A)$ is a Z-set and the restriction $g|_B = f|_B$.

A closed subset A of a space X is said to be a *Z-set* (or a *strong Z-set*) in X if the identity map of X is arbitrarily closely approximated by a map $f : X \to X$ (the closure of) whose image misses A. Let $\ell_2^f(\tau)$ be the linear span of the canonical orthonormal basis of the Hilbert space $\ell_2(\tau)$, that is,

$$\ell_2^f(\tau) = \{x = (x(\gamma))_{\gamma<\tau} \in \ell_2(\tau) \mid x(\gamma) = 0 \text{ except for finitely many } \gamma < \tau\}.$$

In the case $\tau = \aleph_0$, the linear spaces $\ell_2(\aleph_0)$ and $\ell_2^f(\aleph_0)$ are simply denoted by ℓ_2 and ℓ_2^f, respectively. It is known that the spaces $\ell_2^f(\tau) \times \mathbf{Q}$ and $\ell_2^f(\tau)$ are strongly universal for the absolute F_σ class and its subclass, respectively. J. Mogilski [45] characterized ℓ_2^f-manifolds and $(\ell_2^f \times \mathbf{Q})$-manifolds. His result was extended to the non-separable case by K. Sakai and M. Yaguchi [52]. In Chapter 2, we shall improve their characterizations. It is difficult to adopt Sakai and Yaguchi's characterizations for detecting these manifolds because they use the strong universality for big and complicated classes in their characterizations. To give more useful characterizations, we shall introduce the *τ-discrete n-cells property*, that is defined as follows: For cardinals $\tau > 1$ and $n \leq \aleph_0$, a space X has the τ-discrete n-cells property if the following condition holds:

(dcp) Every map $f : \bigoplus_{\gamma<\tau} D_\gamma \to X$ of a discrete union of the n-cubes is arbitrarily closely approximated by a map $g : \bigoplus_{\gamma<\tau} D_\gamma \to X$ such that the family $\{g(D_\gamma) \mid \gamma < \tau\}$ is discrete in X.

Using this property, we can obtain a characterization of $\ell_2^f(\tau)$-manifolds as follows:

Theorem A (K. Koshino [37]). *For every infinite cardinal τ, a connected space X is an $\ell_2^f(\tau)$-manifold if and only if the following conditions hold:*

(1) *X is a strongly countable-dimensional, σ-locally compact ANR of weight τ;*

(2) *X has the τ-discrete n-cells property for every non-negative integer n;*

(3) *X is strongly universal for the class of finite-dimensional compact metrizable spaces;*

(4) *Every finite-dimensional compact subset of X is a strong Z-set in X.*

We say that a space is *strongly countable-dimensional* if it can be written as a countable union of finite-dimensional closed subsets, and a space is *σ-(locally)compact* if it can be written as a countable union of (locally) compact subsets. By the same argument, a characterization of $(\ell_2^f(\tau) \times \mathbf{Q})$-manifolds can be also obtained.

For spaces X and Y, writing (X, Y), we understand Y is a subspace of X. A pair (X, Y) of spaces is homeomorphic to (X', Y') if there exists a homeomorphism $f : X \to X'$ such that $f(Y) = Y'$. Considering how a subspace Y is embedded in a space X, we often investigate whether the pair (X, Y) is homeomorphic to a well-known pair of spaces. Given a pair (E, F), a pair (X, Y) of paracompact spaces is an (E, F)-*manifold pair* if each point of X has an open neighborhood U such that the pair $(U, U \cap Y)$ is homeomorphic to $(V, V \cap F)$ for some open subset V of E. R.D. Anderson [3] gave characterizations to the pairs (ℓ_2, ℓ_2^f) and $(\ell_2 \times \mathbf{Q}, \ell_2^f \times \mathbf{Q})$ by using the notions of f.d. cap sets and cap sets, respectively. These was generalized for (ℓ_2, ℓ_2^f)-manifold pairs and $(\ell_2 \times \mathbf{Q}, \ell_2^f \times \mathbf{Q})$-manifold pairs by T.A. Chapman in [17, 18]. J.E. West [61] characterized non-separable $(\ell_2(\tau), \ell_2^f(\tau))$-manifold pairs. Moreover, M. Bestvina and J. Mogilski [13] introduced the conception of absorbing sets in ℓ_2-manifolds and $\mathbf{Q}$-manifolds, which leads to the conception of absorbing pairs, see [5, 10]. Since these manifold pairs have certain topological uniqueness, the study of infinite-dimensional manifold pairs is a central role in infinite-dimensional topology. In Chapter 3, in order to use the later chapters, we modify West's characterization. In general, for pairs (X, Y) and (E, F), even if X is an E-manifold and Y is an F-manifold, the pair (X, Y) is not necessarily an (E, F)-manifold pair.

Probrem 1. Given a pair (X, Y) of an E-manifold and an F-manifold, when (X, Y) is an (E, F)-manifold pair?

Combining the modified West's characterization with the result in Chapter 2, we can establish the following theorem:

Theorem B (K. Koshino [37]). *Let τ be an infinite cardinal. A pair (X, Y) of spaces is an $(\ell_2(\tau), \ell_2^f(\tau))$-manifold pair if and only if X is an $\ell_2(\tau)$-manifold, Y is an $\ell_2^f(\tau)$-manifold and Y is homotopy dense in X.*

A subspace Y is *homotopy dense* in X if there exists a homotopy $h : X \times [0, 1] \to X$ such that $h(x, 0) = x$ and $h(x, t) \in Y$ for every $x \in X$ and $t \in (0, 1]$. We can also establish the similar characterization of $(\ell_2(\tau) \times \mathbf{Q}, \ell_2^f(\tau) \times \mathbf{Q})$-manifold pairs.

The theory of infinite-dimensional manifolds goes back to the topological classification of convex sets in linear spaces, that has been an important problem of infinite-dimensional topology. A *Fréchet space* is a locally convex completely metrizable linear space. The combined efforts of V. Klee [35], T. Dobrowolski [23], H. Toruńczyk [25, 26], T. Banakh and R. Cauty [9] gives the complete classification to closed convex sets in Fréchet spaces. D. Curtis, T. Dobrowolski and J. Mogilski [22] studied topological types of σ-compact convex sets in a topological linear space. The aim of Chapter 4 is to extend their result to the non-separable case. Using West's characterizations modified in Chapter 3, we will give sufficient and necessary conditions for a pair $(\operatorname{cl} C, C)$ of a σ-locally compact convex set and the closure in a Fréchet space to be homeomorphic to $(\ell_2(\tau), \ell_2^f(\tau))$ or $(\ell_2(\tau) \times \mathbf{Q}, \ell_2^f(\tau) \times \mathbf{Q})$ as follows:

Theorem C (I. Banakh, T. Banakh and K. Koshino [6, 38])**.** *Let C be a σ-locally compact convex set of weight $\tau > \aleph_0$ in a Fréchet space. Then the pair $(\operatorname{cl} C, C)$ is homeomorphic to $(\ell_2(\tau), \ell_2^f(\tau))$ if and only if C is strongly countable-dimensional, and $(\operatorname{cl} C, C)$ is homeomorphic to $(\ell_2(\tau) \times \mathbf{Q}, \ell_2^f(\tau) \times \mathbf{Q})$ if and only if C contains a topological copy of the Hilbert cube $\mathbf{Q}$.*

The study of topologies of function spaces plays an important role in functional analysis. Since function spaces are frequently infinite-dimensional, the theory of infinite-dimensional topology has made meaningful contributions to it. Chapters 5 and 6 are devoted to determining topological types of certain function spaces. For spaces X and Y, we denote by $\mathrm{C}(X, Y)$ the set of all maps from X to Y endowed with the compact-open topology. Let $\mathbf{s} = (-1, 1)^{\mathbb{N}}$ be the pseudo-interior of the Hilbert cube $\mathbf{Q}$. In the paper [36], it was shown that if X is an infinite, locally compact, locally connected, separable metrizable space, then the function space $\mathrm{C}(X, \mathbb{R})$ from X to the real line $\mathbb{R}$ has a natural compactification $\overline{\mathrm{C}(X, \mathbb{R})}$ such that the pair $(\overline{\mathrm{C}(X, \mathbb{R})}, \mathrm{C}(X, \mathbb{R}))$ is homeomorphic to $(\mathbf{Q}, \mathbf{s})$ (cf. the compact case was proved in [51]). In Chapter 5, we shall generalize this result by replacing $\mathbb{R}$ with a 1-dimensional locally compact AR as follows:

Theorem D (K. Koshino and K. Sakai [39])**.** *Let X be an infinite, locally compact, locally connected, separable metrizable space, and let Y be a 1-dimensional locally compact AR. Suppose that X is non-discrete or Y is non-compact. Then the function space $\mathrm{C}(X, Y)$ has a natural compactification $\overline{\mathrm{C}(X, Y)}$ such that the pair $(\overline{\mathrm{C}(X, Y)}, \mathrm{C}(X, Y))$ is homeomorphic to $(\mathbf{Q}, \mathbf{s})$.*

For a space X, let $\mathrm{Cld}_V(X)$ be the hyperspace of non-empty closed sets in X endowed with the Vietoris topology. A *dendrite* is a Peano continuum containing no simple closed curves. It is well known that any two distinct points of a dendrite is connected by the unique arc. Then we denote the unique arc between two points x and y in a dendrite by $[x, y]$, where it is the constant path if $x = y$. For each function $f : X \to Y$ into a dendrite Y and each point $v \in Y$, we can define the *hypo-graph* $\downarrow_v f$ of f with respect to v as follows:

$$\downarrow_v f = \bigcup_{x \in X} \{x\} \times [v, f(x)] \subset X \times Y.$$

When f is continuous, the hypo-graph $\downarrow_v f$ is a closed subset of the product space $X \times Y$. Hence we can regard

$$\downarrow_v \mathrm{C}(X, Y) = \{\downarrow_v f \mid f : X \to Y \text{ is continuous}\}$$

as the subspace of the hyperspace $\mathrm{Cld}_V(X \times Y)$. Let $\overline{\downarrow_v \mathrm{C}(X, Y)}$ be the closure of $\downarrow_v \mathrm{C}(X, Y)$ in $\mathrm{Cld}_V(X \times Y)$. In the case that $Y = [0, 1]$ and $v = 0$, Z. Yang and X. Zhou [63, 64] showed that for a compact metrizable space X whose set of isolated points is not dense, the pair $(\overline{\downarrow_0 \mathrm{C}(X, [0, 1])}, \downarrow_0 \mathrm{C}(X, [0, 1]))$ is homeomorphic to $(\mathbf{Q}, \mathbf{c}_0)$, where

$$\mathbf{c}_0 = \Big\{ x = (x(n))_{n \in \mathbb{N}} \in \mathbf{Q} \;\Big|\; \lim_{n \to \infty} x(n) = 0 \Big\}.$$

An *end point* of a space has an arbitrarily small open neighborhood whose boundary is a singleton. The aim of Chapter 6 is to generalize their result as follows:

Theorem E (K. Koshino, K. Sakai and H. Yang [40])**.** *Let X be an infinite, locally connected, compact metrizable space, Y a dendrite and $v \in Y$ an end point. Then the pair $(\overline{\downarrow_v \mathrm{C}(X, Y)}, \downarrow_v \mathrm{C}(X, Y))$ is homeomorphic to $(\mathbf{Q}, \mathbf{c}_0)$.*

Chapter 1

Preliminaries

In this chapter, we introduce some terminology and notation. We give several basic results on the ANR theory and the infinite-dimensional manifold theory for later use. In addition, we present some elementary information on hyperspaces and some properties of dendrites which are used in Chapters 5 and 6.

1.1 Terminology and notation

For the standard sets, we use the following notation:

- $\mathbb{N}$ is the set of positive integers;
- $\omega = \mathbb{N} \cup \{0\}$ is the set of non-negative integers;
- $\mathbb{R} = (-\infty, \infty)$ is the real line;
- $\mathbf{I} = [0, 1]$ is the closed unit interval.

We shall use the following symbols for subclasses of all metrizable spaces $\mathfrak{M}$:

- $\mathfrak{M}_0$ is the class of compact metrizable spaces;
- $\mathfrak{M}_0^f$ is the class of finite-dimensional compact metrizable spaces.

Let X be a space, $x \in X$, $A, B \subset X$, and $\mathcal{A}, \mathcal{B}$ collections of subsets of X. The weight, the cardinality and the dimension of X are denoted by $\mathrm{w}(X)$, $\mathrm{card}(X)$ and $\dim(X)$, respectively. We denote the closure and the interior of A in X by $\mathrm{cl}_X A$ and $\mathrm{int}_X A$, respectively. By $\mathcal{A} \prec \mathcal{B}$ (or $\mathcal{A} \stackrel{\star}{\prec} \mathcal{B}$), it is meant that $\mathcal{A}$ is a refinement (or a star-refinement) of $\mathcal{B}$. The symbol id_X stands for the identity map of X. When $X = (X, d_X)$ is a metric space, we denote the diameter of A by $\mathrm{diam}_{d_X} A = \sup\{d_X(x, x') \mid x, x' \in A\}$, and the distance between A and B by $d_X(A, B) = \inf\{d_X(x, x') \mid x \in A, x' \in B\}$. For simplicity, we write $d_X(x, A) = d_X(\{x\}, A)$. For each $\epsilon > 0$, let $B_{d_X}(x, \epsilon) = \{x' \in X \mid d_X(x, x') < \epsilon\}$, $\overline{B_{d_X}}(x, \epsilon) = \{x' \in X \mid d_X(x, x') \leq \epsilon\}$ and $N_{d_X}(A, \epsilon) = \{x \in X \mid d_X(x, A) < \epsilon\}$. The mesh of $\mathcal{A}$ is denoted by $\mathrm{mesh}_{d_X} \mathcal{A} = \sup\{\mathrm{diam}_{d_X} A \mid A \in \mathcal{A}\}$. Let $f, g : X \to Y$ be maps. The restriction of f over A is denoted by $f|_A$. For an open cover $\mathcal{U}$ of Y, f is *$\mathcal{U}$-close* to g, which is denoted by $f \sim_{\mathcal{U}} g$, provided that for each $x \in X$, both $f(x)$ and $g(x)$ are contained in some member $U \in \mathcal{U}$. When $Y = (Y, d_Y)$ is a metric space, for each $\epsilon > 0$, it is said that f is *ϵ-close* to g if $d_Y(f(x), g(x)) < \epsilon$ for every $x \in X$. We write $f \simeq g$ if there is a homotopy $h : X \times \mathbf{I} \to Y$ linking f and g. A homotopy $h : X \times \mathbf{I} \to Y$ is called a $\mathcal{U}$-homotopy when $\{h(\{x\} \times \mathbf{I}) \mid x \in X\} \prec \mathcal{U}$, written as $f \simeq_{\mathcal{U}} g$. Then we say that f is $\mathcal{U}$-homotopic to g. Similarly, in the case that $Y = (Y, d_Y)$ is a metric space, we say that h is ϵ-homotopy and f is ϵ-homotopic to g, $\epsilon > 0$, if the diameter $\mathrm{diam}_{d_Y} h(\{x\} \times \mathbf{I}) < \epsilon$ for all $x \in X$. For each $t \in \mathbf{I}$, the map $h_t : X \to Y$ is defined by $h_t(x) = h(x, t)$ for all $x \in X$.

Let K be a simplicial complex and $\sigma, \sigma' \in K$ simplexes. For each $n \in \omega$, the n-skeleton of K is denoted by $K^{(n)}$. In particular, $K^{(0)}$ stands for the set of vertices. Similarly, the set of vertices of σ is denoted by $\sigma^{(0)}$. The symbol $\sigma' \preccurlyeq \sigma$ means that σ' is a face of σ. Let $\hat{\sigma}$ be the barycenter of σ. For a vertex $v \in K^{(0)}$, the star of v in K is denoted by $\mathrm{St}(v, K) = \{\sigma \in K \mid v \in \sigma\}$. We write $\operatorname{Sd} K$ as the barycentric subdivision of K. Note that $|K| = |\operatorname{Sd} K|$ as spaces. A simplicial complex K has two typical geometric realizations, the one of which is the polyhedron $|K|$ and the other is the metric polyhedron $|K|_m$. For an infinite cardinal τ, let

$$\ell_1(\tau) = \left\{ x = (x(\gamma))_{\gamma<\tau} \in \mathbb{R}^\tau \;\middle|\; \sum_{\gamma<\tau} |x(\gamma)| < \infty \right\},$$

which has the norm $\|\cdot\|_1$ defined by $\|x\|_1 = \sum_{\gamma<\tau} |x(\gamma)|$. For a simplicial complex K with $\operatorname{card}(K^{(0)}) \leq \tau$, the *metric polyhedron* $|K|_m$ of K is realized in $\ell_1(\tau)$ with the all vertices of K in one-to-one correspondence to the unit vectors of $\ell_1(\tau)$, where $|K|_m$ admits the metric induced by the norm $\|\cdot\|_1$. In general, $|K|$ and $|K|_m$ are not homeomorphic, but when K is locally finite, $|K| = |K|_m$ as spaces.

1.2 The ANR theory

A subset A of a space X is a *retract* of X if there exists a map $r : X \to A$ such that the restriction $r|_A = \mathrm{id}_A$, where r is called a retraction. Note that every retract is a closed subset. A closed subset A of X is a *neighborhood retract* of X, provided that it is a retract of some neighborhood of A in X. We say that a metrizable space X is an *absolute neighborhood retract*, briefly ANR, (or an *absolute retract*, briefly AR,) if X is a neighborhood retract (or a retract) of an arbitrary metrizable space that contains X as a closed subspace. A space Y is an *absolute neighborhood extensor*, briefly ANE, (or an *absolute extensor*, briefly AE,) if every map $f : A \to Y$ of a closed set A in a metrizable space X extends over some neighborhood of A in X (or over X). In this section, we list some results on the ANR theory, that will be often used in the rest of the thesis without mention. For more details, refer to [50, Chapter 6]. The following facts follow from the definitions immediately.

Fact 1 (cf. 6.2.10.(1), (2) and (3) of [50]). The following hold.

(1) A countable product of ARs is an AR and a finite product of ANRs is an ANR.

(2) A retract of an AR is an AR and a neighborhood retract of an ANR is an ANR.

(3) Every open subset of an ANR is also an ANR.

The following are basic properties of ANRs.

Proposition 1.2.1 (cf. 6.2.10.(4) of [50])**.** *Let X be a paracompact space. If each point of X has an ANR neighborhood, then X is an ANR.*

Proposition 1.2.2 (cf. Theorem 6.2.5 of [50])**.** *Let X be a metrizable space. Then X is an A(N)R if and only if X is an A(N)E.*

Proposition 1.2.3 (cf. Proposition 6.2.8 and Corollary 6.2.9 of [50])**.** . *Every ANR is locally contractible and every AR is contractible. A contractible ANR is an AR.*

The following extension theorem is very important among the ANR theory.

Theorem 1.2.4 (the Homotopy Extension Theorem [15] (cf. Theorem 6.4.1 of [50]))**.** *Let Y be an ANR, $\mathcal{U}$ an open cover of Y and $h : A \times \mathbf{I} \to Y$ be a $\mathcal{U}$-homotopy of a closed set A in a metrizable space X. If h_0 extends to a map $f : X \to Y$, then h extends to a $\mathcal{U}$-homotopy $\tilde{h} : X \times \mathbf{I} \to Y$ such that $\tilde{h}_0 = f$.*

The following proposition is very useful.

Proposition 1.2.5 (cf. Corollary 6.3.5 of [50])**.** *Let X be an ANR. For each open cover $\mathcal{U}$ of X, there is an open cover $\mathcal{V}$ of X such that $\mathcal{V} \prec \mathcal{U}$ and any two $\mathcal{V}$-close maps into X are $\mathcal{U}$-homotopic.*

Recall that a subspace Y is homotopy dense in X if there is a homotopy $h : X \times [0,1] \to X$ such that $h_0 = \mathrm{id}_X$ and $h(X \times (0,1]) \subset Y$. We have the following:

Proposition 1.2.6 (cf. Corollary 6.6.7 of [50])**.** *For each metrizable space X and each homotopy dense subset Y of X, X is an A(N)R if and only if Y is an A(N)R.*

The following lemma is very useful for detecting homotopy denseness of a dense set in a compact metric space, which is a generalization of Lemma 3 of [51] (cf. Corollary 4 of [49]) and will be used in Chapters 5 and 6.

Lemma 1.2.7. *Let $X = (X, d_X)$ be a compact metric space, and let Y be a dense subset of X which has the following property:*

(hd) There exists $\alpha > 0$ such that for any locally finite countable simplicial complex K, each map $f : K^{(0)} \to Y$ extends to a map $\tilde{f} : |K| \to Y$ such that

$$\mathrm{diam}_{d_X} \tilde{f}(\sigma) \leq \alpha \, \mathrm{diam}_{d_X} f(\sigma^{(0)}) \text{ for every } \sigma \in K.$$

Then Y is homotopy dense in X.

Proof. Since X is a compact metric space, we can find a finite open cover $\mathcal{U}_n$ of X for each $n \in \mathbb{N}$ so that $\mathrm{mesh}_{d_X} \mathcal{U}_n < (n+1)^{-1}$. Let $\mathcal{V}_1 = \{U \times (2^{-1}, 1] \mid U \in \mathcal{U}_1\}$ and $\mathcal{V}_n = \{U \times ((n+1)^{-1}, (n-1)^{-1}) \mid U \in \mathcal{U}_n\}$, $n \geq 2$. Note that $\mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ is a star-finite open cover of $X \times (0,1]$. Let K be the locally finite nerve of $\mathcal{V}$ and let K_n be the nerve of $\mathcal{V}_n \cup \mathcal{V}_{n+1}$ for each $n \in \mathbb{N}$, so each K_n is a finite subcomplex of $K = \bigcup_{n \in \mathbb{N}} K_n$.

Since Y is dense in X, we can choose $f(V) \in \mathrm{pr}_X(V) \cap Y$ for each $V \in \mathcal{V}$, where $\mathrm{pr}_X : X \times (0,1] \to X$ is the projection and $\mathcal{U}_n = \{\mathrm{pr}_X(W) \mid W \in \mathcal{V}_n\}$ for each $n \in \mathbb{N}$. Then $\mathrm{diam}_{d_X} f(\sigma^{(0)}) < 2(n+1)^{-1}$ for every $\sigma \in K_n$ and $n \in \mathbb{N}$. By (hd), we can obtain $\alpha > 0$ and a map $\tilde{f} : |K| \to Y$ so that $\mathrm{diam}_{d_X} \tilde{f}(\sigma) \leq \alpha \, \mathrm{diam}_{d_X} f(\sigma^{(0)})$ for each $\sigma \in K$. Taking a canonical map $\phi : X \times (0,1] \to |K|$ for K, we have the map $\tilde{f}\phi : X \times (0,1] \to Y$. It remains to show that $\tilde{f}\phi$ extends to the desired homotopy $h : X \times \mathbf{I} \to X$ by $h_0 = \mathrm{id}_X$. Fix any $(x,t) \in X \times (0,1]$. Then there exist $n \in \mathbb{N}$ and $V \in \mathcal{V}_n$ such that $(x,t) \in V$ and $(n+1)^{-1} < t \leq n^{-1}$. Since ϕ is a canonical map, we can choose $\sigma \in K_n$ so that $\phi(x,t) \in \sigma$ and $V \in \sigma^{(0)}$. Then $\tilde{f}(V) = f(V) \in \mathrm{pr}_X(V) \cap Y$ and $x \in \mathrm{pr}_X(V) \in \mathcal{U}_n$, hence

$$d_X(\tilde{f}(V), x) \leq \mathrm{diam}_{d_X} \mathrm{pr}_X(V) \leq \mathrm{mesh}_{d_X} \mathcal{U}_n < 1/(n+1).$$

It follows that

$$\begin{aligned} d_X(\tilde{f}\phi(x,t), x) &\leq d_X(\tilde{f}\phi(x,t), \tilde{f}(V)) + d_X(\tilde{f}(V), x) < \mathrm{diam}_{d_X} \tilde{f}(\sigma) + 1/(n+1) \\ &\leq \alpha \, \mathrm{diam}_{d_X} f(\sigma^{(0)}) + 1/(n+1) < (2\alpha + 1)/(n+1) < (2\alpha+1)t. \end{aligned}$$

Thus the proof is complete. □

1.3 The infinite-dimensional manifold theory

In this section, several results from the infinite-dimensional manifold theory will be presented. Recall that a closed subset A of a space X is a Z-set in X if for each open cover $\mathcal{U}$ of X, there is a map $f : X \to X$ such that f is $\mathcal{U}$-close to id_X and $f(X) \cap A = \emptyset$. This notion plays a central role in infinite-dimensional topology. A countable union of Z-sets (or a strong Z-set) is called a *Z_σ-set* (or a *strong Z_σ-set*). A *Z-embedding* is an embedding whose image is a Z-set in the range. A Z-set in an ANR is characterized as follows (cf. §2, 3 in [57]):

Proposition 1.3.1. *Let X be an ANR and A a closed subset of X. Then A is a Z-set if and only if $X \setminus A$ is homotopy dense in X.*

Recall that for cardinals $\tau > 1$ and $n \leq \aleph_0$, a space X has the τ-discrete n-cells property provided that the following condition is satisfied:

(dcp) Let $f : \bigoplus_{\gamma<\tau} D_\gamma \to X$ be a map of a discrete union of the n-cubes. For each open cover $\mathcal{U}$ of X, there exists a map $g : \bigoplus_{\gamma<\tau} D_\gamma \to X$ such that g is $\mathcal{U}$-close to f and $\{g(D_\gamma) \mid \gamma < \tau\}$ is discrete in X.

In particular, we say that X has the *disjoint cells property* if it has the 2-discrete n-cells property for all $n < \aleph_0$. As is easily observed, X has the disjoint cells property if and only if it has the 2-discrete $\aleph_0$-cells property, see [42, Lemma 7.3.1]. Using this notion, H. Toruńczyk [58] gave a characterization to $\mathbf{Q}$-manifolds as follows (cf. Theorem 7.8.3 and Corollary 7.8.4 of [42]):

Theorem 1.3.2. *A connected space is a $\mathbf{Q}$-manifold if and only if it is a locally compact ANR with the disjoint cells property.*

Corollary 1.3.3. *A space is homeomorphic to $\mathbf{Q}$ if and only if it is a compact AR with the disjoint cells property.*

H. Toruńczyk [59] also characterized $\ell_2(\tau)$-manifolds as follows:

Theorem 1.3.4. *For each infinite cardinal τ, a connected space X is an $\ell_2(\tau)$-manifold if and only if the following conditions are satisfied:*

(1) *X is a completely metrizable ANR of weight τ;*

(2) *X has the τ-discrete n-cells property for all $n < \aleph_0$;*

(3) *For each sequence $\{K_i\}_{i\in\mathbb{N}}$ of finite-dimensional simplicial complexes with $\operatorname{card}(K_i^{(0)}) \leq \tau$, each map $f : \bigoplus_{i\in\mathbb{N}} |K_i| \to X$ and each open cover $\mathcal{U}$ of X, there exists a map $g : \bigoplus_{i\in\mathbb{N}} |K_i| \to X$ such that g is $\mathcal{U}$-close to f and $\{g(|K_i|)\}_{i\in\mathbb{N}}$ is discrete in X.*

A space X has the *discrete approximation property* if the following condition is satisfied:

(dap) For each map $f : \bigoplus_{n\in\omega} \mathbf{I}^n \to X$ and each open cover $\mathcal{U}$ of X, there is a map $g : \bigoplus_{n\in\omega} \mathbf{I}^n \to X$ such that g is $\mathcal{U}$-close to f and $\{g(\mathbf{I}^n)\}_{n\in\omega}$ is discrete in X.

When $\tau = \aleph_0$, Theorem 1.3.4 can be restated as follows (Corollary 3.2 of [59]):

Theorem 1.3.5. *A connected space is an ℓ_2-manifold if and only if it is a separable completely metrizable ANR with the discrete approximation property.*

Concerning infinite products homeomorphic to Hilbert spaces, the following holds (cf. Theorem 5.1 of [59]):

Theorem 1.3.6. *Let $X = \prod_{i\in\mathbb{N}} X_i$ be a countable product of completely metrizable ARs. Suppose that $\mathrm{w}(X) = \tau = \sup_{j\geq i} \mathrm{w}(X_j)$ for every $i \in \mathbb{N}$. If infinitely many X_i's are non-compact, then X is homeomorphic to $\ell_2(\tau)$.*

As a corollary of the above, the pseudo-interior $\mathbf{s}$ is homeomorphic to the separable Hilbert space ℓ_2 (cf. [1]).

1.4 Hyperspaces

Let $\mathrm{Cld}(X)$ be the set of all non-empty closed subsets of X and let $\mathrm{Cld}^*(X) = \mathrm{Cld}(X) \cup \{\emptyset\}$. For each subset Z of X, we write

$$Z^- = \{A \in \mathrm{Cld}^*(X) \mid A \cap Z \neq \emptyset\} \text{ and } Z^+ = \{A \in \mathrm{Cld}^*(X) \mid A \subset Z\}.$$

A hyperspace $\mathrm{Cld}^*(X)$ has the topology generated by families U^- and U^+, where U runs over the open sets in X. We call this topology the *Vietoris topology* and denote the hyperspace $\mathrm{Cld}^*(X)$ endowed with it and its subspace $\mathrm{Cld}(X)$ by $\mathrm{Cld}^*_V(X)$ and $\mathrm{Cld}_V(X)$, respectively. Note that the empty set $\emptyset$ is isolated in $\mathrm{Cld}^*_V(X)$. For a compact metric space $X = (X, d)$, the hyperspace $\mathrm{Cld}(X)$ admits the *Hausdorff metric* d_H defined as follows:

$$d_H(A, B) = \inf\{r > 0 \mid A \subset N_d(B, r), B \subset N_d(A, r)\} \text{ for each } A, B \in \mathrm{Cld}(X).$$

Then the Vietoris topology on $\mathrm{Cld}(X)$ coincides with the topology induced by d_H, refer to [50, Proposition 5.12.4].

1.5 Dendrites

Recall that a dendrite is a Peano continuum containing no simple closed curves, equivalently it is a 1-dimensional compact AR, see Corollary 13.5 in Chapter V of [16]. A *continuum* means a compact connected metrizable space, and a *Peano continuum* means a locally connected continuum. In this thesis, we shall use the following facts of dendrites.

Fact 2. Every dendrite D has the following properties.

(1) D is *uniquely arcwise connected*, that is, each pair of distinct points of D is connected by the unique arc [62, Chapter V, (1.2)].

(2) Every connected subset of D is arcwise connected [62, Chapter V, (1.3)].

For a metric space $X = (X, d_X)$, the metric d_X is *convex* if for each pair of points x and y, there exists a point $z \in X$ such that $d_X(x, z) = d_X(y, z) = d_X(x, y)/2$. As is easily observed, when the metric d_X is convex and complete, there exists an arc from x to y isometric to the segment $[0, d_X(x, y)]$.

Fact 3. Any Peano continuum admits a convex metric [14, 44]. Hence so any dendrite does.

Arcs in a dendrite have the following good property with respect to the convex metric.

Lemma 1.5.1. *Let $D = (D, d)$ be a dendrite with a convex metric. Then there exists a map $\gamma : D^2 \times \mathbf{I} \to D$ such that for any distinct points $x, y \in D$, the map $\gamma_{x,y} = \gamma(x, y, *) : \mathbf{I} \ni t \mapsto \gamma(x, y, t) \in D$ is an arc from x to y and the following holds:*

(†) *For each $x_i, y_i \in D$, $i = 1, 2$, $d(\gamma_{x_1,y_1}(t), \gamma_{x_2,y_2}(t)) \leq \max\{d(x_1, x_2), d(y_1, y_2)\}$ for all $t \in \mathbf{I}$.*

Proof. Since the metric d is convex, for each $x, y \in D$ there exists an isometric arc $\gamma'_{x,y} : [0, d(x, y)] \to D$ from x to y, which is uniquely determined due to Fact 2(1). We define a function $\gamma : D^2 \times \mathbf{I} \to D$ by

$$\gamma(x, y, t) = \gamma'_{x,y}(td(x, y)) \text{ for each } x, y \in D \text{ and } t \in \mathbf{I}.$$

Here, if $x \neq y$, then $\gamma_{x,y} : \mathbf{I} \to D$ is an arc from x to y, and if $x = y$, then $\gamma_{x,y}$ is the constant path. Note that

$$d(\gamma_{x,y}(t), \gamma_{x,y}(s)) = |t - s| d(x, y) \text{ for each } x, y \in D \text{ and } s, t \in \mathbf{I}.$$

Now, we will verify the condition (†), which implies the continuity of γ. Fix any $x_i, y_i \in D$, $i = 1, 2$, and denote the path γ_{x_i,y_i} from x_i to y_i by γ_i for the sake of convenience.

(Case I: $\gamma_1(\mathbf{I}) \cap \gamma_2(\mathbf{I}) = \emptyset$) We have the unique arc $\alpha : \mathbf{I} \to D$ linking the two paths such that $\alpha(\mathbf{I}) \cap \gamma_1(\mathbf{I}) = \{\alpha(0)\}$ and $\alpha(\mathbf{I}) \cap \gamma_2(\mathbf{I}) = \{\alpha(1)\}$. Then there uniquely exist points $s_i \in \mathbf{I}$, $i = 1, 2$, such that $\alpha(0) = \gamma_1(s_1)$ and $\alpha(1) = \gamma_2(s_2)$. We may assume that $s_1 \le s_2$ without loss of generality. When $t \le s_1$, both $\gamma_1(t)$ and $\gamma_2(t)$ are contained in the arc from x_1 to x_2, so we have $d(\gamma_1(t), \gamma_2(t)) \le d(x_1, x_2)$. When $t \ge s_2$, both $\gamma_1(t)$ and $\gamma_2(t)$ are contained in the arc from y_1 to y_2, hence $d(\gamma_1(t), \gamma_2(t)) \le d(y_1, y_2)$. When $s_1 \le t \le s_2$, since $\gamma_1(s_1)$ and $\gamma_2(s_2)$ sit on both of the arcs from x_1 to x_2 and from y_1 to y_2 in this order, we have

$$d(x_1, x_2) = d(x_1, \gamma_1(s_1)) + d(\gamma_1(s_1), \gamma_2(s_2)) + d(\gamma_2(s_2), x_2) \text{ and}$$
$$d(y_1, y_2) = d(y_1, \gamma_1(s_1)) + d(\gamma_1(s_1), \gamma_2(s_2)) + d(\gamma_2(s_2), y_2).$$

Then it follows that

$$\begin{aligned}
d(\gamma_1(t), \gamma_2(t)) &= d(\gamma_1(t), \gamma_1(s_1)) + d(\gamma_1(s_1), \gamma_2(s_2)) + d(\gamma_2(s_2), \gamma_2(t)) \\
&= (t - s_1)d(x_1, y_1) + d(\gamma_1(s_1), \gamma_2(s_2)) + (s_2 - t)d(x_2, y_2) \\
&\le (s_2 - s_1) \max_{i=1,2} d(x_i, y_i) + d(\gamma_1(s_1), \gamma_2(s_2)) \\
&\le \max\{(1 - s_1)d(x_1, y_1), s_2 d(x_2, y_2)\} + d(\gamma_1(s_1), \gamma_2(s_2)) \\
&= \max\{d(y_1, \gamma_1(s_1)), d(x_2, \gamma_2(s_2))\} + d(\gamma_1(s_1), \gamma_2(s_2)) \\
&\le \max\{d(y_1, y_2), d(x_1, x_2)\}.
\end{aligned}$$

(Case II: $\gamma_1(\mathbf{I}) \cap \gamma_2(\mathbf{I}) \ne \emptyset$) There exist $m_i \le n_i \in \mathbf{I}$, $i = 1, 2$, such that $\gamma_1(\mathbf{I}) \cap \gamma_2(\mathbf{I}) = \gamma_1([m_1, n_1]) = \gamma_2([m_2, n_2])$. Then we have two cases (i) $\gamma_1(m_1) = \gamma_2(m_2)$ and (ii) $\gamma_1(m_1) = \gamma_2(n_2)$. Remark that

$$(*) \quad (n_1 - m_1)d(x_1, y_1) = d(\gamma_1(m_1), \gamma_1(n_1)) = d(\gamma_2(m_2), \gamma_2(n_2)) = (n_2 - m_2)d(x_2, y_2).$$

(i) In the case that $\gamma_1(m_1) = \gamma_2(m_2)$, we have $\gamma_1(n_1) = \gamma_2(n_2)$. When $t \le \min\{m_1, m_2\}$, we get $d(\gamma_1(t), \gamma_2(t)) \le d(x_1, x_2)$ because the arc from x_1 to x_2 contains both $\gamma_1(t)$ and $\gamma_2(t)$. When $t \ge \max\{n_1, n_2\}$, the arc from y_1 to y_2 contains both $\gamma_1(t)$ and $\gamma_2(t)$, and hence $d(\gamma_1(t), \gamma_2(t)) \le d(y_1, y_2)$. When $\max\{m_1, m_2\} \le t \le \min\{n_1, n_2\}$, both of the points $\gamma_1(t)$ and $\gamma_2(t)$ are contained in the arc $\gamma_1([m_1, n_1]) = \gamma_2([m_2, n_2])$. By $(*)$, we have

$$\begin{aligned}
d(\gamma_1(t), \gamma_2(t)) &= |d(\gamma_1(t), \gamma_1(m_1)) - d(\gamma_2(t), \gamma_2(m_2))| \\
&= |(t - m_1)d(x_1, y_1) - (t - m_2)d(x_2, y_2)| \\
&= |m_2 d(x_2, y_2) - m_1 d(x_1, y_1) - t(d(x_2, y_2) - d(x_1, y_1))| \\
&= |m_2 d(x_2, y_2) - m_1 d(x_1, y_1) - t((1 - n_2 + n_2 - m_2 + m_2)d(x_2, y_2) \\
&\qquad\qquad - (1 - n_1 + n_1 - m_1 + m_1)d(x_1, y_1))| \\
&= |(1 - t)(m_2 d(x_2, y_2) - m_1 d(x_1, y_1)) + t((1 - n_1)d(x_1, y_1) - (1 - n_2)d(x_2, y_2))| \\
&\le (1 - t)|d(x_2, \gamma_2(m_2)) - d(x_1, \gamma_1(m_1))| + t|d(y_1, \gamma_1(n_1)) - d(y_2, \gamma_2(n_2))| \\
&\le (1 - t)d(x_1, x_2) + td(y_1, y_2) \le \max\{d(x_1, x_2), d(y_1, y_2)\}.
\end{aligned}$$

When $\min\{m_1, m_2\} \le t \le \max\{m_1, m_2\}$, let $m_i = \min\{m_1, m_2\}$ (so $m_{3-i} = \max\{m_1, m_2\}$). Then $\gamma_i(t)$ sits on the arc from $\gamma_i(m_i)$ to y_i and $\gamma_{3-i}(t)$ sits on the arc from x_{3-i} to $\gamma_{3-i}(m_{3-i})$. Hence

$$\begin{aligned}
d(\gamma_1(t), \gamma_2(t)) &= d(\gamma_i(t), \gamma_i(m_i)) + d(\gamma_{3-i}(m_{3-i}), \gamma_{3-i}(t)) \\
&= (t - m_i)d(x_i, y_i) + (m_{3-i} - t)d(x_{3-i}, y_{3-i}) \\
&= |(t - m_i)d(x_i, y_i) - (t - m_{3-i})d(x_{3-i}, y_{3-i})|.
\end{aligned}$$

By the same calculation as above, we get $d(\gamma_1(t), \gamma_2(t)) \le \max\{d(x_1, x_2), d(y_1, y_2)\}$. Similarly, when $\min\{n_1, n_2\} \le t \le \max\{n_1, n_2\}$, it follows that $d(\gamma_1(t), \gamma_2(t)) \le \max\{d(x_1, x_2), d(y_1, y_2)\}$.

(ii) In the case that $\gamma_1(m_1) = \gamma_2(n_2)$, we have $\gamma_1(n_1) = \gamma_2(m_2)$. When $t \le \min\{m_1, m_2\}$, we get $d(\gamma_1(t), \gamma_2(t)) \le d(x_1, x_2)$ since both $\gamma_1(t)$ and $\gamma_2(t)$ are contained in the arc from x_1 to x_2. When $t \ge \max\{m_1, m_2\}$, we have $d(\gamma_1(t), \gamma_2(t)) \le d(y_1, y_2)$ because both $\gamma_1(t)$ and $\gamma_2(t)$ are contained in the arc from y_1 to y_2. When $\min\{m_1, m_2\} \le t \le \max\{m_1, m_2\}$, let $m_i = \min\{m_1, m_2\}$. In the case $t \le n_i$, both $\gamma_1(t)$ and $\gamma_2(t)$ are contained in the arc from x_1 to x_2, hence $d(\gamma_1(t), \gamma_2(t)) \le d(x_1, x_2)$. In the case $t \ge n_i$, the point $\gamma_i(t)$ is on the arc from $\gamma_i(n_i)$ to y_i and the point $\gamma_{3-i}(t)$ is on the arc from x_{3-i} to $\gamma_{3-i}(m_{3-i})$. It follows that

$$\begin{aligned} d(\gamma_1(t), \gamma_2(t)) &= d(\gamma_i(t), \gamma_i(n_i)) + d(\gamma_{3-i}(m_{3-i}), \gamma_{3-i}(t)) \\ &= (t - n_i) d(x_i, y_i) + (m_{3-i} - t) d(x_{3-i}, y_{3-i}) \\ &\le (t - m_i) d(x_i, y_i) - (t - m_{3-i}) d(x_{3-i}, y_{3-i}). \end{aligned}$$

As is observed in (i), we have $d(\gamma_1(t), \gamma_2(t)) \le \max\{d(x_1, x_2), d(y_1, y_2)\}$. Thus the proof is complete. $\square$

Chapter 2

Characterizing infinite-dimensional manifolds modeled on sigma-locally compact metrizable spaces

In this chapter, we shall give characterizations to $\ell_2^f(\tau)$-manifolds and $(\ell_2^f(\tau) \times \mathbf{Q})$-manifolds for each infinite cardinal τ. Throughout the chapter, all spaces are assumed to be paracompact. In 1984, J. Mogilski [45] characterized ℓ_2^f-manifolds as follows:

Theorem 2.0.1. *A connected space X is an ℓ_2^f-manifold if and only if the following conditions are satisfied:*

(1) *X is an ANR and a countable union of finite-dimensional compact metrizable spaces;*

(2) *X is strongly universal for the class of finite-dimensional compact metrizable spaces;*

(3) *Every finite-dimensional compact subset of X is a strong Z-set in X.*

Recall that a space X is strongly universal for a class $\mathcal{C}$ when the following condition is satisfied:

(su) For each space $A \in \mathcal{C}$, each closed subset B of A, each map $f : A \to X$ such that the restriction $f|_B$ is a Z-embedding, and each open cover $\mathcal{U}$ of X, there exists a Z-embedding $g : A \to X$ such that $g \sim_{\mathcal{U}} f$ and $g|_B = f|_B$.

By removing "finite-dimensionality" from the above conditions in Theorem 2.0.1, a characterization of $(\ell_2^f \times \mathbf{Q})$-manifolds can be obtained, see [45]. In 2003, Theorem 2.0.1 was generalized to the non-separable case by K. Sakai and M. Yaguchi [52].

Theorem 2.0.2. *Let τ be an infinite cardinal. A connected space X is an $\ell_2^f(\tau)$-manifold if and only if the following conditions hold:*

(1) *X is an ANR of weight τ and a strongly countable-dimensional, σ-locally compact, strong Z_σ-space;*

(2) *X is strongly universal for the class of strongly countable-dimensional, locally compact metrizable spaces of weight $\leq \tau$.*

Similar to the characterizations of J. Mogilski, removing "strongly countable-dimensionality" from the above allows us to characterize $(\ell_2^f(\tau) \times \mathbf{Q})$-manifolds, see [52].

Clearly, the strong universality for the class of strongly countable-dimensional, locally compact metrizable spaces (the condition (2) of Theorem 2.0.2) is more difficult to verify than the one for the class of finite-dimensional compact metrizable spaces (the condition (2) of Theorem 2.0.1). The aim of this chapter is to improve Theorem 2.0.2 as follows:

Main Theorem. *For every infinite cardinal τ, a connected space X is an $\ell_2^f(\tau)$-manifold if and only if the following conditions hold:*

(1) *X is a strongly countable-dimensional, σ-locally compact ANR of weight τ;*

(2) *X has the τ-discrete n-cells property for every non-negative integer n;*

(3) *X is strongly universal for the class of finite-dimensional compact metrizable spaces;*

(4) *Every finite-dimensional compact subset of X is a strong Z-set in X.*

A characterization of $(\ell_2^f(\tau) \times \mathbf{Q})$-manifolds can be also obtained by the same argument as the above, see Theorem 2.4.3.

2.1 Preliminaries

In this section, we shall present some notation and results which are used later. Let X be a space. The symbol $\operatorname{cov}(X)$ means the collection of all open covers of X. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of X. We define $\operatorname{st}(\mathcal{A},\mathcal{B}) = \{A \cup \bigcup\{B \in \mathcal{B} \mid A \cap B \neq \emptyset\} \mid A \in \mathcal{A}\}$ and write $\operatorname{st}\mathcal{A} = \operatorname{st}(\mathcal{A},\mathcal{A})$. Inductively, we define $\operatorname{st}^n \mathcal{A} = \operatorname{st}(\operatorname{st}^{n-1}\mathcal{A},\mathcal{A})$ for each $n \geq 2$. Let $\mathcal{A} \wedge \mathcal{B} = \{A \cap B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. For a subset $C \subset X$, the collection $\mathcal{A} \wedge \{C\}$ is denoted by $\mathcal{A}|_C$. The following proposition can be proved by the same way as Corollary 1.8 of [13], which is useful to us for detecting Z-sets in ANRs.

Proposition 2.1.1. *Let X be an ANR. If X has the $\aleph_0$-discrete n-cells property for every $n \in \omega$, then every compact subset of X is a Z-set.*

The following properties of (strong) Z-sets in ANRs are well-known.

Proposition 2.1.2. *Let X be an ANR.*

(1) *For every (strong) Z-set A in X and every open subset U of X, $A \cap U$ is a (strong) Z-set in U.*

(2) *A locally finite union of (strong) Z-sets in X is a (strong) Z-set.*

We shall use the following lemma to construct a homeomorphism which approximates a map in the next section. Refer to (D) of §2 in [45].

Lemma 2.1.3. *Let X and $Y = (Y,d)$ be metric spaces and $\{Y_n\}_{n\in\mathbb{N}}$ be a closed cover of Y such that $Y_1 \subset Y_2 \subset \cdots$. Suppose that $\{g_n : X \to Y\}_{n\in\mathbb{N}}$ is a sequence of surjective maps satisfying the following conditions:*

(I) *$g_n|_{g_n^{-1}(Y_n)} : g_n^{-1}(Y_n) \to Y_n$ is bijective and for every point $y \in Y_n$ and every neighborhood V of $g_n^{-1}(y)$ in X, there exists an open neighborhood U of y in Y such that $g_n^{-1}(U) \subset V$;*

(II) *$g_{n+1}|_{g_n^{-1}(Y_n)} = g_n|_{g_n^{-1}(Y_n)}$;*

(III) *$d(g_{n+1}(x), g_n(x)) < \alpha_n(g_n(x))$ for all $x \in X \setminus g_n^{-1}(Y_n)$, where $\alpha_n(y) = 2^{-n}\min\{1, d(y,Y_n)\}$, $n \in \mathbb{N}$, and $\alpha_0(y) = 1$.*

Then, a homeomorphism $g : \bigcup_{n\in\mathbb{N}} g_n^{-1}(Y_n) \to Y$ can be defined as follows:

$$g(x) = \lim_{n\to\infty} g_n(x) \text{ for all } x \in \bigcup_{n\in\mathbb{N}} g_n^{-1}(Y_n),$$

where $d(g(x), g_1(x)) < 1$ for each $x \in \bigcup_{n\in\mathbb{N}} g_n^{-1}(Y_n)$.

Let X and Y be spaces and A be a closed subset of X. The product of X and Y *reduced over* A, which is denoted by $(X \times Y)_A$, is the space $((X \setminus A) \times Y) \cup A$ endowed with the topology generated by open subsets of the product space $(X \setminus A) \times Y$ and sets $((U \setminus A) \times Y) \cup (U \cap A)$, where U is an open subset of X. Then, the product space $(X \setminus A) \times Y$ is an open subspace in $(X \times Y)_A$. Moreover, the projection $\mathrm{pr}_X : X \times Y \to X$ is factored into the two natural maps $q : X \times Y \to (X \times Y)_A$ and $p : (X \times Y)_A \to X$ defined as follows:

$$\begin{cases} q(x,y) &= (x,y) \quad \text{if } (x,y) \in (X \setminus A) \times Y, \\ q(x,y) &= x \qquad\quad \text{if } (x,y) \in A \times Y, \end{cases}$$

$$\begin{cases} p(x,y) &= x \quad \text{if } (x,y) \in (X \setminus A) \times Y, \\ p(x) &= x \quad \text{if } x \in A. \end{cases}$$

Note that if both X and Y are metrizable spaces, then $(X \times Y)_A$ is also a metrizable space by the Bing Metrization Theorem (Theorem 4.4.8 of [30]). We shall prove the following lemma used the next section.

Lemma 2.1.4. *Let X and Y be metrizable spaces and let $A_1 \subset A_2$ be closed subsets in X. Then, there exists $\mathcal{U} \in \mathrm{cov}(X \setminus A_1)$ with the following property:*

(∗) *For a subspace B of $(X \setminus A_1) \times Y$ and an embedding $g : B \to (X \times Y)_{A_2} \setminus A_1$, if $g \sim_{p^{-1}(\mathcal{U})} q|_B$, then g extends to the embedding $\tilde{g} : B \cup A_1 \to (X \times Y)_{A_2}$ by $\tilde{g}|_{A_1} = \mathrm{id}_{A_1}$,*

where p, q are the natural maps, that is,

$$p : (X \times Y)_{A_2} \setminus A_1 = ((X \setminus A_1) \times Y)_{A_2 \setminus A_1} \to X \setminus A_1,$$
$$q : (X \setminus A_1) \times Y \to ((X \setminus A_1) \times Y)_{A_2 \setminus A_1} = (X \times Y)_{A_2} \setminus A_1.$$

Moreover, if g is a closed embedding, that is, $g(B)$ is closed in $(X \times Y)_{A_2} \setminus A_1$, then $\tilde{g}$ is also a closed embedding.

Proof. Taking an admissible metric d for X, we can define the desired open cover $\mathcal{U}$ as follows:

$$\mathcal{U} = \{B_d(x, d(x, A_1)/2) \mid x \in X \setminus A_1\} \in \mathrm{cov}(X \setminus A_1).$$

To show that $\mathcal{U}$ has the property (∗), let $g : B \to (X \times Y)_{A_2} \setminus A_1$ be an embedding of $B \subset (X \setminus A_1) \times Y$, which is $p^{-1}(\mathcal{U})$-close to $q|_B$. We extend g to $\tilde{g}$ by $\tilde{g}|_{A_1} = \mathrm{id}_{A_1}$. Then, it is enough to show the continuity of both $\tilde{g}$ and $\tilde{g}^{-1} : g(B) \cup A_1 \to B \cup A_1$. Since $(X \setminus A_1) \times Y$ and $(X \times Y)_{A_2} \setminus A_1$ are respectively open subspaces of $(X \times Y)_{A_1}$ and $(X \times Y)_{A_2}$, we need to check that both $\tilde{g}$ and $\tilde{g}^{-1}$ are continuous at each $a \in A_1$.

First, to verify that $\tilde{g}$ is continuous at $a \in A_1$, let $\epsilon > 0$. Fix a point $x \in B_d(a, \epsilon/3) \subset X$. In case $x \in A_1$, we have

$$\tilde{g}(x) = x \in B_d(a, \epsilon/3) \cap A_1 \subset B_d(a, \epsilon) \cap A_2.$$

In case $x \notin A_1$, we have $\tilde{g}(x,y) = g(x,y)$ for all $y \in Y$ with $(x,y) \in B$. Since $g \sim_{p^{-1}(\mathcal{U})} q|_B$, there exists a point $x_0 \in X \setminus A_1$ such that both $p\tilde{g}(x,y) = pg(x,y)$ and $pq(x,y) = x$ are contained in $B_d(x_0, d(x_0, A_1)/2)$. Then, we get

$$d(x_0, A_1) \le d(x_0, a) \le d(x_0, x) + d(x, a) < \frac{1}{2} d(x_0, A_1) + \frac{\epsilon}{3},$$

hence $d(x_0, A_1) < 2\epsilon/3$. It follows that

$$d(p\tilde{g}(x,y), a) \le d(pg(x,y), x) + d(x, a) \le d(x_0, A_1) + \frac{\epsilon}{3} \le \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

so $\tilde{g}(x,y) \in (B_d(a, \epsilon) \setminus A_2) \times Y \cup (B_d(a, \epsilon) \cap A_2)$. Therefore

$$\tilde{g}((((B_d(a, \epsilon/3) \setminus A_1) \times Y) \cap B) \cup (B_d(a, \epsilon/3) \cap A_1)) \subset (B_d(a, \epsilon) \setminus A_2) \times Y \cup (B_d(a, \epsilon) \cap A_2),$$

which implies that $\tilde{g}$ is continuous at a.

Next, we show that $\tilde{g}^{-1}$ is continuous at $a \in A_1$. Given $\epsilon > 0$, take any point

$$x \in ((B_d(a, \epsilon/3) \setminus A_2) \times Y \cup (B_d(a, \epsilon/3) \cap A_2)) \cap (g(B) \cup A_1).$$

When $x \in A_1$, we get

$$\tilde{g}^{-1}(x) = x \in B_d(a, \epsilon/3) \cap A_1 \subset B_d(a, \epsilon) \cap A_1.$$

When $x \in g(B) \subset (X \times Y)_{A_2} \setminus A_1$, we have $\tilde{g}(x', y') = g(x', y') = x$ for the unique point $(x', y') \in B$. We can choose a point $x_0 \in X \setminus A_1$ so that both of the points $p(x) = p\tilde{g}(x', y') = pg(x', y')$ and $pq(x', y') = x'$ are contained in $B_d(x_0, d(x_0, A_1)/2)$ because $g \sim_{p^{-1}(\mathcal{U})} q|_B$. It follows that

$$d(x_0, A_1) \leq d(x_0, a) \leq d(x_0, p(x)) + d(p(x), a) < \frac{1}{2} d(x_0, A_1) + \frac{\epsilon}{3},$$

so $d(x_0, A_1) \leq 2\epsilon/3$. Therefore, we have

$$d(x', a) \leq d(x', p(x)) + d(p(x), a) < d(x_0, A_1) + \frac{\epsilon}{3} \leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

that is, $\tilde{g}^{-1}(x) = (x', y') \in (B_d(a, \epsilon) \setminus A_1) \times Y$. Hence

$$\tilde{g}^{-1}(((B_d(a, \epsilon/3) \setminus A_2) \times Y \cup (B_d(a, \epsilon/3) \cap A_2)) \cap (g(B) \cup A_1)) \subset (B_d(a, \epsilon) \setminus A_1) \times Y \cup (B_d(a, \epsilon) \cap A_1),$$

so $\tilde{g}^{-1}$ is continuous at a.

To prove the additional assertion, assume that $g(B)$ is closed in $(X \times Y)_{A_2} \setminus A_1$. Then we have $\mathrm{cl}_{(X \times Y)_{A_2}} g(B) \cap ((X \times Y)_{A_2} \setminus A_1) = g(B)$. Therefore

$$\tilde{g}(B \cup A_1) = g(B) \cup A_1 = (\mathrm{cl}_{(X \times Y)_{A_2}} g(B) \cap ((X \times Y)_{A_2} \setminus A_1)) \cup A_1 = \mathrm{cl}_{(X \times Y)_{A_2}} g(B) \cup A_1,$$

that is, $\tilde{g}(B \cup A_1)$ is closed in $(X \times Y)_{A_2}$. Hence $\tilde{g}$ is a closed embedding. $\square$

Remark 1. In the above lemma, if g is a continuous map, then so the extension $\tilde{g}$ is. When $B = (X \setminus A_1) \times Y$ and $g : (X \setminus A_1) \times Y \to (X \times Y)_{A_2} \setminus A_1$ is a homeomorphism, $\tilde{g} : (X \times Y)_{A_1} \to (X \times Y)_{A_2}$ is a homeomorphism.

2.2 E-manifold factors being E-manifolds

Throughout the section, let $\mathfrak{C}$ be a class of spaces which has the following properties:

- $(*)$ $\mathfrak{C}$ is *topological*, that is, every space homeomorphic to some member of $\mathfrak{C}$ is also a member of $\mathfrak{C}$;
- $(**)$ $\mathfrak{C}$ is *closed hereditary*, that is, every closed subspace of a member of $\mathfrak{C}$ is also a member of $\mathfrak{C}$.

Moreover, let E be a locally convex topological linear metric space such that E is homeomorphic to the countable product $E^{\mathbb{N}}$ or

$$E^{\mathbb{N}}_f = \{x = (x(n))_{n \in \mathbb{N}} \in E^{\mathbb{N}} \mid x(n) = \mathbf{0} \text{ except for finitely many } n \in \mathbb{N}\},$$

and E satisfies the following conditions:

- $(\star)$ E is a countable union of closed subspaces which belong to $\mathfrak{C}$;
- $(\star\star)$ For any closed subset C of E, if $C \in \mathfrak{C}$, then C is a strong Z-set.

Recall that $\mathfrak{M}_0$ means the class of compact metrizable spaces, and $\mathfrak{M}_0^f$ means the class of finite-dimensional compact metrizable spaces. In addition, we use the symbol $\mathfrak{M}_0(n)$ as the class of compact metrizable spaces of dimension $\leq n$. For a cardinal τ and a class $\mathcal{C}$, we denote by $\bigoplus_\tau \mathcal{C}$, the class of spaces $X = \bigoplus_{\gamma<\tau} X_\gamma$ which are discrete unions of spaces $X_\gamma \in \mathcal{C}$. Note that the classes $\bigoplus_\tau \mathfrak{M}_0$, $\bigoplus_\tau \mathfrak{M}_0^f$ and $\bigoplus_\tau \mathfrak{M}_0(n)$ are topological and closed hereditary. It is known that the locally convex topological linear metric space $\ell_2^f(\tau)$ is homeomorphic to $(\ell_2^f(\tau))_f^{\mathbb{N}}$. Let ℓ_2^Q be the linear subspace in ℓ_2 spanned by $\prod_{n\in\mathbb{N}}[-2^{-n}, 2^{-n}]$. Then, it is also known that $\ell_2^f(\tau)\times\mathbf{Q}$ is homeomorphic to the locally convex topological linear metric space $\ell_2^f(\tau)\times\ell_2^Q$, which is homeomorphic to $(\ell_2^f(\tau)\times\ell_2^Q)_f^{\mathbb{N}}$. Furthermore, $\ell_2^f(\tau)$ (respectively, $\ell_2^f(\tau)\times\mathbf{Q}$) satisfies the conditions $(\star)$ and $(\star\star)$ with respect to $\bigoplus_\tau \mathfrak{M}_0^f$ (respectively, $\bigoplus_\tau \mathfrak{M}_0$), which will be seen in the proof of Theorem 2.4.2 (cf. Remark 4).

Remark 2. Let M be a connected E-manifold. Then M is a countable union of strong Z-sets which belong to the class $\mathfrak{C}$. Indeed, Theorem 4 of [32] allows us to regard an E-manifold M as an open subspace in E, that is, an F_σ set, so we have $M = \bigcup_{m\in\mathbb{N}} D_m$, where each D_m is regarded as a closed subspace in E. On the other hand, by the conditions $(\star)$ and $(\star\star)$ of E, we can write $E = \bigcup_{n\in\mathbb{N}} E_n$ such that every E_n is a strong Z-set belonging to $\mathfrak{C}$. Since $\mathfrak{C}$ is closed hereditary, $D_m \cap E_n \in \mathfrak{C}$ for all $m, n \in \mathbb{N}$. Furthermore, $D_m \cap E_n$ is a strong Z-set in M due to $(\star\star)$ and Proposition 2.1.2(1). Therefore $M = \bigcup_{m,n\in\mathbb{N}} D_m \cap E_n$ is a countable union of strong Z-sets which are members of $\mathfrak{C}$.

The following proposition, which was proved by H. Toruńczyk in Theorem B1 of [60] (cf. Proposition 5.1 of [57]), shall play an important role in the proof of Theorem 2.2.3.

Proposition 2.2.1. *Suppose that A is a strong Z-set in a space X. If $X \times E$ is an E-manifold, then for each open cover $\mathcal{U} \in \operatorname{cov}((X \times E)_A)$, there exists a homeomorphism $h : X \times E \to (X \times E)_A$ such that $h \sim_{\mathcal{U}} q$ and $h(x, 0) = x$ for all $x \in A$, where $q : X \times E \to (X \times E)_A$ is the natural map.*

Lemma 2.2.2. *Let X be a strongly universal ANR for a class $\mathfrak{C}$. Suppose that $f : A \to X$ is a map from a space $A \in \mathfrak{C}$ to X and U is an open subset of X. Given any open cover $\mathcal{U}$ of U, there exists a Z-embedding $g : f^{-1}(U) \to U$ such that $g \sim_{\mathcal{U}} f|_{f^{-1}(U)}$.*

Proof. We write $U = \bigcup_{n\in\omega} C_n$, where C_n is a closed subset of X and

$$\emptyset = C_0 \subset \operatorname{int}_X C_1 \subset C_1 \subset \operatorname{int}_X C_2 \subset C_2 \subset \cdots .$$

Let $A_n = f^{-1}(C_n)$ and $B_n = f^{-1}(X \setminus \operatorname{int}_X C_{n+1})$ for each $n \in \mathbb{N}$. Then $A_1 \subset A_2 \subset \cdots$ and $B_1 \supset B_2 \supset \cdots$ are closed in A, $A_n \cap B_n = \emptyset$ for each $n \in \mathbb{N}$, $f^{-1}(U) = \bigcup_{n\in\mathbb{N}} A_n$ and $A \setminus f^{-1}(U) = \bigcap_{n\in\mathbb{N}} B_n$.

Let $\mathcal{V} \in \operatorname{cov}(U)$ be a star-refinement of $\mathcal{U}$. Give an admissible metric for X and take a sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of open covers of X so that $\operatorname{mesh}\mathcal{U}_n \leq 2^{-n}$ and

$$\mathcal{U}_n \prec (\mathcal{V} \wedge \{\operatorname{int}_X C_{i+1} \setminus C_{i-1} \mid i \in \mathbb{N}\}) \bigcup \{X \setminus C_{n+2}\}.$$

By induction, we shall construct a sequence $\{f_n : A \to X\}_{n\in\mathbb{N}}$ so as to satisfy the following conditions:

$(1)_n$ $f_n|_{B_n} = f|_{B_n}$;

$(2)_n$ $f_n|_{A_n} : A_n \to U$ is a Z-embedding;

$(3)_n$ $f_n|_{A_{n-1}\cup B_n} = f_{n-1}|_{A_{n-1}\cup B_n}$;

$(4)_n$ $f_n \sim_{\mathcal{U}_n} f_{n-1}$;

$(5)_n$ $f_n(A_n \setminus \operatorname{int}_A A_{n-1}) \subset \operatorname{int}_X C_{n+2} \setminus C_{n-3}$, where $A_0 = C_{-1} = C_{-2} = \emptyset$, $B_0 = A$ and $f_0 = f$. Assume that f_m has been constructed for all $m \leq n-1$. Since X is an ANR and X is strongly universal for $\mathfrak{C}$, we can obtain a $\mathcal{U}_n$-homotopy $h : A \times \mathbf{I} \to X$ such that $h_0 = f_{n-1}$, h_1 is a Z-embedding and $h_1|_{A_{n-1}} = f_{n-1}|_{A_{n-1}}$. Taking an Urysohn map $k : A \to \mathbf{I}$ so that $k(B_n) = 0$ and $k(A_n) = 1$, we define the map $f_n : A \to X$ by $f_n(x) = h(x, k(x))$. Immediately, the conditions $(1)_n$, $(3)_n$ and $(4)_n$ hold from the definition. Observe that

$$A_n \setminus \operatorname{int}_A A_{n-1} = A_n \setminus \operatorname{int}_A f^{-1}(C_{n-1}) \subset A_n \setminus f^{-1}(\operatorname{int}_X C_{n-1}) \subset A_n \cap B_{n-2}.$$

By the inductive assumption $(1)_{n-2}$,

$$f_{n-2}(A_n \cap B_{n-2}) = f(A_n \cap B_{n-2}) \subset f(A_n) \cap f(B_{n-2}) \subset C_n \setminus \operatorname{int}_X C_{n-1},$$

where $B_{-1} = A$ and $f_{-1} = f$. Furthermore, $f_n(A_n \cap B_{n-2}) \subset \operatorname{int}_X C_{n+2} \setminus C_{n-3}$ due to the conditions $(4)_{n-1}$ and $(4)_n$. It follows that

$$f_n(A_n \setminus \operatorname{int}_A A_{n-1}) \subset f_n(A_n \cap B_{n-2}) \subset \operatorname{int}_X C_{n+2} \setminus C_{n-3},$$

hence $(5)_n$ holds. Since $f_n|_{A_n} = h_1|_{A_n}$ is a Z-embedding into X and $f_n(A_n) \subset \operatorname{int}_X C_{n+2} \subset U$, it follows from Proposition 2.1.2(1) that $f_n(A_n)$ is a Z-set in U, that is, $(2)_n$ also holds.

Now, we can define the desired map $g : f^{-1}(U) \to U$ by $g|_{A_n} = f_n|_{A_n}$ because of $(3)_n$, where the continuity of g is guaranteed by $(4)_n$ and the condition $\operatorname{mesh} \mathcal{U}_n < 2^{-n}$ for all $n \in \mathbb{N}$. To verify that $g \sim_{\mathcal{U}} f|_{f^{-1}(U)}$, let $x \in f^{-1}(U)$. Then, we have $x \in A_n \setminus \operatorname{int}_A A_{n-1} \subset A_n \cap B_{n-2}$ for some $n \in \mathbb{N}$, so $f_{n-2}(x) = f(x) \in C_n$ and $g(x) = f_n(x) \in \operatorname{int}_X C_{n+2}$. Since $f_{n-1} \sim_{\mathcal{U}_{n-1}} f_{n-2}$ and $f_n \sim_{\mathcal{U}_n} f_{n-1}$ by $(4)_{n-1}$ and $(4)_n$, respectively, we can choose $V, V' \in \mathcal{V}$ so that $f_{n-2}(x), f_{n-1}(x) \in V$ and $f_{n-1}(x), f_n(x) \in V'$. Therefore,

$$f(x), g(x) \in V \cup V' \subset W \in \mathcal{U} \text{ for some } W \in \mathcal{U}$$

because $\mathcal{V}$ is a star-refinement of $\mathcal{U}$, and hence $g \sim_{\mathcal{U}} f|_{f^{-1}(U)}$. It remains to show that g is a Z-embedding into U. It is clear that g is injective because $f^{-1}(U) = \bigcup_{n \in \mathbb{N}} A_n$ and $g|_{A_n} = f_n|_{A_n}$ is injective. For any closed subset $D \subset f^{-1}(U)$ and $n \in \mathbb{N}$, due to $(5)_n$,

$$g(D \cap A_n \setminus \operatorname{int}_A A_{n-1}) = f_n(D \cap A_n \setminus \operatorname{int}_A A_{n-1}) \subset \operatorname{int}_X C_n + 2 \setminus C_{n-3}.$$

It follows from $(2)_n$ that

$$g(D) = \bigcup_{n \in \mathbb{N}} g(D \cap A_n \setminus \operatorname{int}_A A_{n-1}) = \bigcup_{n \in \mathbb{N}} f_n(D \cap A_n \setminus \operatorname{int}_A A_{n-1})$$

is a locally finite union of closed sets in U, that is, a closed subset of $g(f^{-1}(U))$. Thus, the map $g : f^{-1}(U) \to g(f^{-1}(U))$ is a closed map. Moreover,

$$g(f^{-1}(U)) = \bigcup_{n \in \mathbb{N}} g(A_n \setminus \operatorname{int}_X A_{n-1}) = \bigcup_{n \in \mathbb{N}} f_n(A_n \setminus \operatorname{int}_X A_{n-1})$$

is a locally finite union of Z-sets in U, that is, a Z-set by Proposition 2.1.2(2). As a result, g is a Z-embedding. □

A map $f : X \to Y$ is a *near-homeomorphism* provided that for each open cover $\mathcal{U} \in \operatorname{cov}(Y)$, there exists a homeomorphism $h : X \to Y$ with $h \sim_{\mathcal{U}} f$. The following theorem is proved by analogy with Theorem 4 of [45].

Theorem 2.2.3. *Suppose that X is a connected ANR satisfying the following conditions:*

(i) X is a countable union of closed subspaces which belong to $\mathfrak{C}$;

(ii) X is strongly universal for $\mathfrak{C}$;

(iii) For every closed subset $C \subset X$, if $C \in \mathfrak{C}$, then C is a strong Z-set in X.

If $X \times E$ is an E-manifold, then the projection $\mathrm{pr}_X : X \times E \to X$ is a near-homeomorphism, and hence X is an E-manifold.

Proof. According to Remark 2 and the conditions (i) and (iii), we can write $X \times E = \bigcup_{n\in\mathbb{N}} A_n$ and $X = \bigcup_{n\in\mathbb{N}} B_n$, where A_n and B_n are strong Z-sets which belong to $\mathfrak{C}$. For any open cover $\mathcal{U} \in \mathrm{cov}(X)$, X admits a metric d such that $\{B_d(x,1) \mid x \in X\} \prec \mathcal{U}$ due to Theorem 4.1 in Chapter II of [12]. Then, it is sufficient to construct a homeomorphism $k : X \times E \to X$ which is 1-close to the projection pr_X.

To begin with, we shall inductively construct a sequence of strong Z-sets $C_1 \subset C_2 \subset \cdots \subset X$ with $X = \bigcup_{n\in\omega} C_n$ and homeomorphisms $h_n : X \times E \to (X \times E)_{C_n}$, $n \in \mathbb{N}$, such that

$(1)_n$ $B_n \cup C_{n-1} \subset C_n$,

$(2)_n$ $h_n(A_n) \subset C_n$,

$(3)_n$ $h_n|_{h_{n-1}^{-1}(C_{n-1})} = h_{n-1}|_{h_{n-1}^{-1}(C_{n-1})}$ and

$(4)_n$ $d(p_n h_n(x), p_{n-1} h_{n-1}(x)) < \alpha_{n-1}(p_{n-1} h_{n-1}(x))$
for all $x \in (X \times E) \setminus h_{n-1}^{-1}(C_{n-1})$,

where $C_0 = \emptyset$, $h_0 : X \times E \to X \times E$ is the identity map, $p_0 : X \times E \to X$ is the projection onto X, $p_n : (X \times E)_{C_n} \to X$ is the natural map, and $\alpha_n : X \setminus C_n \to (0,1)$ is the map defined by $\alpha_n(y) = 2^{-n}\min\{1, d(y, C_n)\}$, $n \in \mathbb{N}$, and $\alpha_0(y) = 1$.

Suppose that C_i and h_i satisfying $(1)_i$, $(2)_i$, $(3)_i$ and $(4)_i$ have been obtained for all $i \le n$. We define the map $\alpha_n : X \setminus C_n \to (0,1)$ by $\alpha_n(y) = 2^{-n}\min\{1, d(y, C_n)\}$. Due to Lemma 2.1.4, we can choose $\mathcal{U}_n \in \mathrm{cov}(X \setminus C_n)$ so that the following conditions are satisfied:

(a) For a map $f : (X \times E) \setminus h_n^{-1}(C_n) \to X$, if $f \sim_{\mathrm{st}^2 \mathcal{U}_n} p_n h_n|_{(X\times E)\setminus h_n^{-1}(C_n)}$, then $d(f(x), p_n h_n(x)) < \alpha_n(p_n h_n(x))$ for all $x \in (X \times E) \setminus h_n^{-1}(C_n)$;

(b) For a homeomorphism $f' : (X \setminus C_n) \times E \to (X \setminus C_n) \times E$, if $f' \sim_{p_n^{-1}(\mathrm{st}\,\mathcal{U}_n)} \mathrm{id}_{(X\setminus C_n)\times E}$, then f' extends to the homeomorphism $f : (X \times E)_{C_n} \to (X \times E)_{C_n}$ by $f|_{C_n} = \mathrm{id}_{C_n}$;

(c) For a closed embedding $v : h_n(A_{n+1}) \setminus C_n \to X \setminus C_n$, if $v \sim_{\mathrm{st}\,\mathcal{U}_n} p_n|_{h_n(A_{n+1})\setminus C_n}$, then v extends to the closed embedding $\tilde{v} : h_n(A_{n+1}) \cup C_n \to X$ by $v|_{C_n} = \mathrm{id}_{C_n}$.

Since h_n is a homeomorphism and $\mathfrak{C}$ is topological, $h_n(A_{n+1}) \in \mathfrak{C}$ is a strong Z-set in $(X \times E)_{C_n}$. Applying Lemma 2.2.2 to the map $p_n|_{h_n(A_{n+1})} : h_n(A_{n+1}) \to X$ and the open subset $X \setminus C_n \subset X$, we can find a Z-embedding $v : h_n(A_{n+1}) \setminus C_n \to X \setminus C_n$ such that $v \simeq_{\mathcal{U}_n} p_n|_{h_n(A_{n+1})\setminus C_n}$. Let $i : X \setminus C_n \to (X \setminus C_n) \times \{0\} \subset (X \setminus C_n) \times E$ be the natural inclusion. Then $iv(h_n(A_{n+1}) \setminus C_n)$ is a Z-set in $(X \setminus C_n) \times E$. Hence $iv : h_n(A_{n+1}) \setminus C_n \to (X \setminus C_n) \times E$ is a Z-embedding such that $iv \simeq_{p_n^{-1}(\mathcal{U}_n)} \mathrm{id}_{h_n(A_{n+1})\setminus C_n}$ in $(X \setminus C_n) \times E$ because $v \simeq_{\mathcal{U}_n} p_n|_{h_n(A_{n+1})}$ and E is contractible. On the other hand, $(X \setminus C_n) \times E$ is an E-manifold as an open subspace of the E-manifold $X \times E$. By Proposition 2.1.2(1), $h_n(A_{n+1}) \setminus C_n = h_n(A_{n+1}) \cap (X \setminus C_n) \times E$ is a strong Z-set in $(X \setminus C_n) \times E$. Applying the Z-set Unknotting Theorem (cf. Theorem 2 of [19] [1]) to the E-manifold $(X \setminus C_n) \times E$ and using the condition (b), we can obtain a homeomorphism $f : (X \times E)_{C_n} \to (X \times E)_{C_n}$ so that

$$f|_{h_n(A_{n+1})\setminus C_n} = iv,\ f|_{(X\setminus C_n)\times E} \simeq_{p_n^{-1}(\mathrm{st}\,\mathcal{U}_n)} \mathrm{id}_{(X\setminus C_n)\times E}$$

[1]Theorem 2 of [19] holds for a locally convex topological linear metric space E not only such that E is homeomorphic to $E^{\mathbb{N}}$ but also such that E is homeomorphic to $E_f^{\mathbb{N}}$.

and $f|_{C_n} = \mathrm{id}_{C_n}$. Then $f \sim_{p_n^{-1}(\mathrm{st}\,\mathcal{U}_n)} \mathrm{id}_{(X\times E)_{C_n}}$.

By the way, due to (c), the Z-embedding v extends to a closed embedding $\tilde{v} : h_n(A_{n+1}) \cup C_n \to X$ by $v|_{C_n} = \mathrm{id}_{C_n}$, so $\tilde{v}(h_n(A_{n+1})) \in \mathfrak{C}$ is a closed subspace in X, which implies that $\tilde{v}(h_n(A_{n+1}))$ is a strong Z-set in X by (iii). Since C_n and B_{n+1} are strong Z-sets, it follows from Proposition 2.1.2 that $C_{n+1} = \tilde{v}(h_n(A_{n+1}) \cup C_n \cup B_{n+1}$ is a strong Z-set in X, so $C_{n+1} \setminus C_n$ is a strong Z-set in $X \setminus C_n$. Let $q : (X \times E)_{C_n} \to (X \times E)_{C_{n+1}}$ be the natural map defined by $p_n = p_{n+1}q$. Lemma 2.1.4 allows us to choose $\mathcal{V}_n \in \mathrm{cov}(X \setminus C_n)$ so that

(d) $\mathcal{V}_n \prec \mathcal{U}_n$ and

(e) for a homeomorphism $g' : (X \setminus C_n) \times E \to (X \times E)_{C_{n+1}} \setminus C_n$, if $g' \sim_{p_{n+1}^{-1}(\mathcal{V}_n)} q|_{(X\setminus C_n)\times E}$, then g' extends to the homeomorphism $g : (X \times E)_{C_n} \to (X \times E)_{C_{n+1}}$ by $g|_{C_n} = \mathrm{id}_{C_n}$.

Then, applying Proposition 2.2.1 and (e), we can find a homeomorphism $g : (X \times E)_{C_n} \to (X \times E)_{C_{n+1}}$ such that

$$g|_{(X\setminus C_n)\times E} \sim_{p_{n+1}^{-1}(\mathcal{V}_n)} q|_{(X\setminus C_n)\times E},\ g(x, 0) = x \text{ for all } x \in C_{n+1} \setminus C_n$$

and $g|_{C_n} = \mathrm{id}_{C_n}$. Then $g \sim_{p_{n+1}^{-1}(\mathcal{U}_n)} q$ by (d).

Now, we have the homeomorphism $h_{n+1} = gfh_n : X \times E \to (X \times E)_{C_{n+1}}$. By the definition of C_{n+1}, we have $(1)_{n+1}$. It follows that

$$\begin{aligned} h_{n+1}(A_{n+1}) = gfh_n(A_{n+1}) &= g(v(h_n(A_{n+1}) \setminus C_n) \times \{0\}) \cup (h_n(A_{n+1}) \cap C_n) \\ &\subset g((C_{n+1} \setminus C_n) \times \{0\}) \cup C_n = (C_{n+1} \setminus C_n) \cup C_n = C_{n+1}, \end{aligned}$$

that is, $(2)_{n+1}$ holds. Moreover, we get

$$h_{n+1}(x) = gfh_n(x) = h_n(x) \text{ for every } x \in h_n^{-1}(C_n),$$

which means $(3)_{n+1}$. Observe that

$$\begin{aligned} p_{n+1}h_{n+1}|_{(X\times E)\setminus h_n^{-1}(C_n)} &= p_{n+1}gfh_n|_{(X\times E)\setminus h_n^{-1}(C_n)} \\ &\sim_{\mathcal{U}_n} p_{n+1}qfh_n|_{(X\times E)\setminus h_n^{-1}(C_n)} \\ &= p_nfh_n|_{(X\times E)\setminus h_n^{-1}(C_n)} \\ &\sim_{\mathrm{st}\,\mathcal{U}_n} p_nh_n|_{(X\times E)\setminus h_n^{-1}(C_n)}, \end{aligned}$$

and hence $p_{n+1}h_{n+1}|_{(X\times E)\setminus h_n^{-1}(C_n)} \sim_{\mathrm{st}^2\,\mathcal{U}_n} p_nh_n|_{(X\times E)\setminus h_n^{-1}(C_n)}$. By (a), we have

$$d(p_{n+1}h_{n+1}(x), p_nh_n(x)) < \alpha_n(p_nh_n(x)) \text{ for every } x \in (X \times E) \setminus h_n^{-1}(C_n),$$

so $(4)_{n+1}$ holds. Thus, we complete the inductive step.

Finally, we shall construct the desired homeomorphism $k : X \times E \to X$ by using Lemma 2.1.3. Define the surjective maps $k_n = p_nh_n : X \times E \to X$, $n \in \omega$. Since $B_n \subset C_n$ by $(1)_n$ for all $n \in \mathbb{N}$, the increasing sequence $\{C_n\}_{n\in\omega}$ is a closed cover of X. It follows from $(2)_n$ that

$$A_n \subset h_n^{-1}(C_n) = h_n^{-1}p_n^{-1}(C_n) = k_n^{-1}(C_n),$$

which means that $X \times E = \bigcup_{n\in\omega} k_n^{-1}(C_n)$. It remains to show that the sequence $\{k_n\}_{n\in\omega}$ satisfies the conditions (I), (II) and (III) of Lemma 2.1.3.

(I): Note that $k_n|_{k_n^{-1}(C_n)} = p_nh_n|_{k_n^{-1}(C_n)} = h_n|_{k_n^{-1}(C_n)}$, so $k_n|_{k_n^{-1}(C_n)}$ is bijective. Given a point $x \in C_n$

and a neighborhood V of $k_n^{-1}(x)$ in $X \times E$, $h_n(V)$ is a neighborhood of $h_n(k_n^{-1}(x)) = p_n^{-1}(x) = x$ in $(X \times E)_{C_n}$. Then, there exists an open neighborhood U of x in X such that

$$p_n^{-1}(U) = (U \cap C_n) \cup (U \setminus C_n) \times E \subset h_n(V),$$

hence it follows that $k_n^{-1}(U) = h_n^{-1}p_n^{-1}(U) \subset V$.
(II): By $(3)_n$, we have

$$k_{n+1}|_{k_n^{-1}(C_n)} = p_{n+1}h_{n+1}|_{h_n^{-1}p_n^{-1}(C_n)} = p_{n+1}h_{n+1}|_{h_n^{-1}(C_n)} = p_n h_n|_{h_n^{-1}(C_n)} = k_n|_{k_n^{-1}(C_n)}.$$

(III): It follows from $(4)_{n+1}$ that for all $x \in (X \times E) \setminus k_n^{-1}(C_n)$,

$$d(k_{n+1}(x), k_n(x)) = d(p_{n+1}h_{n+1}(x), p_n h_n(x)) < \alpha_n(p_n h_n(x)) = \alpha_n(k_n(x)).$$

In conclusion, we can obtain the desired homeomorphism $k : X \times E \to X$ as follows:

$$k(x) = \lim_{n\to\infty} k_n(x) \text{ for every } x \in X \times E,$$

where k is 1-close to $k_0 = p_0 h_0 = \mathrm{pr}_X$. The proof is complete. □

2.3 The discrete approximation property for a class of spaces

For a cardinal $\tau > 1$, a space X has the *τ-discrete approximation property* (or the *τ-locally finite approximation property*) for a class $\mathcal{C}$ if the following condition is satisfied:

- Let $A = \bigoplus_{\gamma<\tau} A_\gamma$ be a discrete union of a collection $\{A_\gamma \in \mathcal{C} \mid \gamma < \tau\}$ and $f : A \to X$ be a map. Then, for each open cover $\mathcal{U}$ of X, there exists a map $g : A \to X$ such that $g \sim_{\mathcal{U}} f$ and $\{g(A_\gamma) \mid \gamma < \tau\}$ is discrete (or locally finite) in X.

For the sake of convenience, we abbreviate the τ-discrete approximation property for $\mathcal{C}$ and the τ-locally finite approximation property for $\mathcal{C}$ to τ-DAP($\mathcal{C}$) and τ-LFAP($\mathcal{C}$), respectively. When $\mathcal{C} = \{C\}$, we simply write τ-DAP(C) and τ-LFAP(C). The τ-discrete n-cells property is no other than τ-DAP($\mathbf{I}^n$). Moreover, τ-DAP($\{\mathbf{I}^n \mid n \in \omega\}$) is called the *$\tau$-discrete cells property*. The τ-discrete cells property is stronger than the τ-discrete n-cells property for all $n \in \omega$, but the same as τ-DAP($\mathbf{Q}$), namely, the τ-discrete $\aleph_0$-cells property.

Lemma 2.3.1. *For a cardinal $\tau > 1$, a space X has the τ-discrete cells property if and only if X has τ-DAP($\mathbf{Q}$).*

Proof. Let $\mathbf{Q}_\gamma$ be a copy of $\mathbf{I}^{\mathbb{N}}$ for all $\gamma < \tau$ and $\mathcal{U} \in \mathrm{cov}(X)$, where each $\mathbf{Q}_\gamma$ admits the following metric d defined by

$$d(x, y) = \sup_{i\in\mathbb{N}} i^{-1}|x(i) - y(i)| \text{ for } x = (x(i))_{i\in\mathbb{N}}, y = (y(i))_{i\in\mathbb{N}} \in \mathbf{Q}_\gamma.$$

For each $n \in \mathbb{N}$, the inclusion $i_n : \mathbf{I}^n \to \mathbf{I}^{\mathbb{N}}$ and the projection $p_n : \mathbf{I}^{\mathbb{N}} \to \mathbf{I}^n$ are respectively defined as follows:

$$i_n(x) = (x(1), \cdots, x(n), 0, 0, \cdots) \text{ for } x = (x(i))_{1\le i\le n} \in \mathbf{I}^n \text{ and}$$
$$p_n(x) = (x(1), \cdots, x(n)) \text{ for } x = (x(i))_{i\in\mathbb{N}} \in \mathbf{I}^{\mathbb{N}}.$$

Moreover, let $i_0 : \mathbf{I}^0 = \{0\} \ni 0 \mapsto (0, 0, \cdots) \in \mathbf{I}^{\mathbb{N}}$ and $p_0 : \mathbf{I}^{\mathbb{N}} \ni x \mapsto 0 \in \mathbf{I}^0 = \{0\}$.

First, to show the "if" part, take any map $f : D = \bigoplus_{\gamma<\tau} \mathbf{I}^{n(\gamma)} \to X$, where $n(\gamma) \in \omega$ for all $\gamma < \tau$. Define a map $g : \bigoplus_{\gamma<\tau} \mathbf{Q}_\gamma \to X$ by $g|_{\mathbf{Q}_\gamma} = f|_{\mathbf{I}^{n(\gamma)}} p_{n(\gamma)}$ for each $\gamma < \tau$. Since X has τ-DAP($\mathbf{Q}$), there is a map $g' : \bigoplus_{\gamma<\tau} \mathbf{Q}_\gamma \to X$ such that $g' \sim_{\mathcal{U}} g$ and $\{g'(\mathbf{Q}_\gamma) \mid \gamma < \tau\}$ is discrete in X. Then, we define a map $f' : D \to X$ by $f'|_{\mathbf{I}^{n(\gamma)}} = g'|_{\mathbf{Q}_\gamma} i_{n(\gamma)}$ for each $\gamma < \tau$. It follows that

$$f'|_{\mathbf{I}^{n(\gamma)}} = g'|_{\mathbf{Q}_\gamma} i_{n(\gamma)} \sim_{\mathcal{U}} g|_{\mathbf{Q}_\gamma} i_{n(\gamma)} = f|_{\mathbf{I}^{n(\gamma)}} p_{n(\gamma)} i_{n(\gamma)} = f|_{\mathbf{I}^{n(\gamma)}} \text{ for every } \gamma < \tau,$$

and hence $f' \sim_{\mathcal{U}} f$. Moreover, $f'(\mathbf{I}^{n(\gamma)}) = g'|_{\mathbf{Q}_\gamma} i_{n(\gamma)}(\mathbf{I}^{n(\gamma)}) \subset g'(\mathbf{Q}_\gamma)$ for each $\gamma < \tau$, so the collection $\{f'(\mathbf{I}^{n(\gamma)}) \mid \gamma < \tau\}$ is discrete in X. As a result, X has the τ-discrete cells property.

Next, to prove the "only if" part, take any map $f : \bigoplus_{\gamma<\tau} \mathbf{Q}_\gamma \to X$. Let $\mathcal{V} \in \operatorname{cov}(X)$ be a star-refinement of $\mathcal{U}$ and ϵ_γ be a Lebesgue number for $(f|_{\mathbf{Q}_\gamma})^{-1}(\mathcal{V}) \in \operatorname{cov}(\mathbf{Q}_\gamma)$. Then, we can choose $n(\gamma) \in \mathbb{N}$ so that $n(\gamma)^{-1} < \epsilon_\gamma$. It is easy to see that $\mathrm{id}_{\mathbf{Q}_\gamma}$ is $n(\gamma)^{-1}$-close to $i_{n(\gamma)} p_{n(\gamma)}$, hence $f|_{\mathbf{Q}_\gamma} \sim_{\mathcal{V}} f|_{\mathbf{Q}_\gamma} i_{n(\gamma)} p_{n(\gamma)}$. Define a map $g : D = \bigoplus_{\gamma<\tau} \mathbf{I}^{n(\gamma)} \to X$ by $g|_{\mathbf{I}^{n(\gamma)}} = f|_{\mathbf{Q}_\gamma} i_{n(\gamma)}$ for each $\gamma < \tau$. Due to the τ-discrete cells property of X, we can find a map $g' : D \to X$ such that $g' \sim_{\mathcal{V}} g$ and $\{g'(\mathbf{I}^{n(\gamma)}) \mid \gamma < \tau\}$ is discrete in X. Then, we define a map $f' : \bigoplus_{\gamma<\tau} \mathbf{Q}_\gamma \to X$ by $f'|_{\mathbf{Q}_\gamma} = g'|_{\mathbf{I}^{n(\gamma)}} p_{n(\gamma)}$ for each $\gamma < \tau$. Observe that for every $\gamma < \tau$,

$$f'|_{\mathbf{Q}_\gamma} = g'|_{\mathbf{I}^{n(\gamma)}} p_{n(\gamma)} \sim_{\mathcal{V}} g|_{\mathbf{I}^{n(\gamma)}} p_{n(\gamma)} = f|_{\mathbf{Q}_\gamma} i_{n(\gamma)} p_{n(\gamma)} \sim_{\mathcal{V}} f|_{\mathbf{Q}_\gamma},$$

which means that $f' \sim_{\mathcal{U}} f$. Furthermore, $f'(\mathbf{Q}_\gamma) = g'|_{\mathbf{I}^{n(\gamma)}} p_{n(\gamma)}(\mathbf{Q}_\gamma) = g'(\mathbf{I}^{n(\gamma)})$ for all $\gamma < \tau$, so the collection $\{f'(\mathbf{Q}_\gamma) \mid \gamma < \tau\}$ is discrete in X. Consequently, X has τ-DAP($\mathbf{Q}$). □

For a topological subclass $\mathcal{C} \subset \mathfrak{M}_0$, by the same argument as Lemma 4 of [7] (cf. [21]) we can show that τ-LFAP($\mathcal{C}$) coincides with τ-DAP($\mathcal{C}$), that is:

Lemma 2.3.2. *Let τ be an infinite cardinal and let $\mathcal{C}$ be a topological subclass of $\mathfrak{M}_0$. A space X has τ-LFAP($\mathcal{C}$) if and only if X has τ-DAP($\mathcal{C}$).*

Proof. The "if" part is clear. So we shall show "the only if" part. Let $f : A = \bigoplus_{\gamma<\tau} A_\gamma \to X$ be a map, where $A_\gamma \in \mathcal{C}$. As τ is infinite, $\operatorname{card}(\tau \times \tau) \le \tau$. For each $(\gamma, \gamma') \in \tau \times \tau$, we define

$$A_{(\gamma,\gamma')} = A_\gamma \times \{\gamma'\} \subset A \times \tau,$$

where τ is considered as a discrete space. Then, $A \times \tau$ is a discrete union of $\{A_{(\gamma,\gamma')} \mid (\gamma, \gamma') \in \tau \times \tau\}$. Take any open cover $\mathcal{U} \in \operatorname{cov}(X)$. Applying τ-LFAP($\mathcal{C}$) of X to the map $\tilde{f} = f \operatorname{pr}_A : A \times \tau \to X$, where $\operatorname{pr}_A : A \times \tau \to A$ is the projection onto A, we can obtain a map $\tilde{g} : A \times \tau \to X$ such that $\tilde{g} \sim_{\mathcal{U}} \tilde{f}$ and $\{\tilde{g}(A_{(\gamma,\gamma')}) \mid (\gamma, \gamma') \in \tau \times \tau\}$ is locally finite in X. Then, each $\tilde{g}(A_{(\gamma,\gamma')})$ meets only finitely many $\tilde{g}(A_{(\delta,\delta')})$'s because $\tilde{g}(A_{(\gamma,\gamma')})$ is compact.

By transfinite induction, we can choose $\delta(\gamma) < \tau$ for each $\gamma < \tau$ so as to satisfy the following:

(∗) $\tilde{g}(A_{(\gamma,\delta(\gamma))}) \cap \tilde{g}(A_{(\gamma',\delta(\gamma'))}) = \emptyset$ for all $\gamma' < \gamma$.

Indeed, suppose that $\delta(\gamma') < \tau$ has been chosen for each $\gamma' < \gamma$. Then, as observed in the above,

$$\operatorname{card}(\{\delta < \tau \mid \tilde{g}(A_{(\gamma,\delta)}) \cap \tilde{g}(A_{(\gamma',\delta(\gamma'))}) \neq \emptyset\}) < \infty \text{ for all } \gamma' < \gamma.$$

So we have

$$\operatorname{card}\left(\left\{\delta < \tau \;\middle|\; \tilde{g}(A_{(\gamma,\delta)}) \cap \left(\bigcup_{\gamma'<\gamma} \tilde{g}(A_{(\gamma',\delta(\gamma'))})\right) \neq \emptyset\right\}\right) \le \aleph_0 \gamma < \tau,$$

which allows us find $\delta(\gamma) < \tau$ satisfying (∗). It follows from the local finiteness of $\{\tilde{g}(A_{(\gamma,\gamma')}) \mid (\gamma, \gamma') \in \tau \times \tau\}$ and (∗) that $\{\tilde{g}(A_{(\gamma,\delta(\gamma))}) \mid \gamma < \tau\}$ is discrete in X. Then, we define the map $g : A \to X$ by $g(x) = \tilde{g}(x, \delta(\gamma))$ for each $x \in A_\gamma$ and $\gamma \in \tau$. It is easy to see that $g \sim_{\mathcal{U}} f$ and $\{g(A_\gamma) \mid \gamma < \tau\}$ is discrete. As a result, X has τ-DAP($\mathcal{C}$). □

Proposition 2.3.3. *Let τ be a cardinal > 1 and $n \in \omega$. Suppose that W is an open set in an ANR X which is contractible in X. If X has the τ-discrete cells property (respectively, the τ-discrete $(2n+1)$-cells property), then W has τ-DAP($\mathfrak{M}_0$) (respectively, τ-DAP($\mathfrak{M}_0(n)$)).*

Proof. We may only prove the case when X has the τ-discrete $(2n+1)$-cells property because the other case is similarly proved by virtue of Lemma 2.3.1. Suppose that $f : A = \bigoplus_{\gamma<\tau} A_\gamma \to W$ is a map, where $A_\gamma \in \mathfrak{M}_0(n)$ for all $\gamma < \tau$, and $\mathcal{U} \in \operatorname{cov}(X)$. Due to Lemma 2.3.2, we may construct a map $h : A \to W$ such that $h \sim_{\mathcal{U}} f$ and $\{h(A_\gamma) \mid \gamma < \tau\}$ is locally finite in W. Denote $D = \bigoplus_{\gamma<\tau} D_\gamma$, where $D_\gamma = \mathbf{I}^{2n+1}$ for each $\gamma < \tau$. We may assume that $A_\gamma \subset D_\gamma$ for all $\gamma < \tau$.

Since W is an ANR, f extends to a map $\tilde{f} : V \to W$ from an open neighborhood V of A in D to W. Take an open neighborhood V' of A in D so that $\operatorname{cl} V' \subset V$ and let $k : D \to \mathbf{I}$ be an Urysohn map such that $k^{-1}(0) = A$ and $k^{-1}(1) = D \setminus V'$. By the hypothesis, we have a contraction $\phi : W \times \mathbf{I} \to X$ so that $\phi_0 = \mathrm{id}_W$ and $\phi_1(W) = \{x_0\}$ for some $x_0 \in X$. Then, we can define the map $\bar{f} : D \to X$ as follows:

$$\bar{f}(x) = \phi(\tilde{f}(x), k(x)) \text{ for each } x \in V \text{ and } \bar{f}(D \setminus V) = \{x_0\}.$$

Now, we can write $W = \bigcup_{i\in\mathbb{N}} W_i$, where W_i is an open set in X and $\operatorname{cl} W_i \subset W_{i+1}$ for every $i \in \mathbb{N}$. Let $\mathcal{U}_0 \in \operatorname{cov}(X)$ such that $\mathcal{U}_0 \prec^\star \mathcal{U}$. We define closed subsets $R_i \subset A$, $i \in \mathbb{N}$, an open cover $\mathcal{U}' \in \operatorname{cov}(W)$ and open covers $\mathcal{U}_i \in \operatorname{cov}(X)$, $i \in \mathbb{N}$, as follows:

$$R_i = f^{-1}(\operatorname{cl} W_i \setminus W_{i-1}),\ \mathcal{U}' = \bigcup_{i\in\mathbb{N}} \mathcal{U}_0|_{W_i \setminus \operatorname{cl} W_{i-2}} \text{ and } \mathcal{U}_i = \mathcal{U}'|_{W_{2i}} \cup \{X \setminus \operatorname{cl} W_{2i-1}\},$$

where $W_{-1} = W_0 = \emptyset$. Using the τ-discrete $(2n+1)$-cells property of X, we can obtain a map $g_i : D \to X$ such that $g_i \simeq_{\mathcal{U}_i} \bar{f}$ and $\{g_i(D_\gamma) \mid \gamma < \tau\}$ is discrete in X. Then $g_i|_{R_{2i-1}} \simeq_{\mathcal{U}'} f|_{R_{2i-1}}$ for all $i \in \mathbb{N}$. By the Homotopy Extension Theorem 1.2.4, we can take a map $g : A \to W$ such that $g \simeq_{\mathcal{U}'} f$ and $g|_{R_{2i-1}} = g_i|_{R_{2i-1}}$ for each $i \in \mathbb{N}$. It is easy to see that $\{g(A_\gamma \cap R_{2i-1}) \mid \gamma < \tau\}$ is discrete in $W_{2i} \setminus \operatorname{cl} W_{2i-3}$. Therefore $\{g(A_\gamma \cap R_{2i-1}) \mid \gamma < \tau, i \in \mathbb{N}\}$ is locally finite in W.

Next, we can find an open refinement $\mathcal{V} \in \operatorname{cov}(W)$ of $\mathcal{U}_0$ so as to satisfy the following:

- For every map $h : A \to W$, $h \sim_{\mathcal{V}} g$ implies that $\{h(A_\gamma \cap R_{2i-1}) \mid \gamma < \tau, i \in \mathbb{N}\}$ is locally finite in W.

By the same construction as g, we can obtain a map $h : A \to W$ so that $h \simeq_{\mathcal{V}} g$ and $\{h(A_\gamma \cap R_{2i}) \mid \gamma < \tau, i \in \mathbb{N}\}$ is locally finite in W. It is follows from the definition of $\mathcal{V}$ that $\{h(A_\gamma \cap R_{2i-1}) \mid \gamma < \tau, i \in \mathbb{N}\}$ is locally finite in W. Therefore $\{h(A_\gamma \cap R_i) \mid \gamma < \tau, i \in \mathbb{N}\}$ is locally finite in W, which means that $\{h(A_\gamma) \mid \gamma < \tau\}$ is locally finite in W. Moreover, $h \sim_{\mathcal{V}} g \sim_{\mathcal{U}'} f$, and hence $h \sim_{\mathcal{U}} f$. Thus, the proof is complete. □

A little stronger condition than τ-DAP will be introduced in the following proposition.

Proposition 2.3.4. *Let τ be a cardinal > 1 and $\mathcal{C}$ be a topological and closed hereditary subclass of $\mathfrak{M}_0$. Suppose that X is an ANR with τ-DAP($\mathcal{C}$) and that any closed set $C \in \mathcal{C}$ in X is a strong Z-set. Then, for every map $f : A = \bigoplus_{\gamma<\tau} A_\gamma \to X$ from a discrete union of A_γ's to X, where $A_\gamma \in \mathcal{C}$, for every closed subset $B \subset A$ such that the restriction $f|_B$ is a closed embedding, and for every $\mathcal{U} \in \operatorname{cov}(X)$, there exists a map $g : A \to X$ such that $g \sim_{\mathcal{U}} f$, $g|_B = f|_B$ and the collection $\{g(A_\gamma) \mid \gamma < \tau\}$ is discrete in X.*

Proof. We take $\mathcal{U}_1, \mathcal{U}_2 \in \operatorname{cov}(X)$ so that $\mathcal{U} \succ^\star \mathcal{U}_1 \succ^\star \mathcal{U}_2$. Let $B_\gamma = A_\gamma \cap B$ for each $\gamma < \tau$. Since $f|_B$ is a closed embedding, $\{f(B_\gamma) \mid \gamma < \tau\}$ is a discrete collection in X. Then, we can find a pairwise disjoint collection $\{U_\gamma \mid \gamma < \tau\}$ of open subsets of X so that $f(B_\gamma) \subset U_\gamma$ for each $\gamma < \tau$.

Take $\mathcal{U}_2' \in \operatorname{cov}(X)$ such that $\mathcal{U}_2' \prec \mathcal{U}_2 \wedge \{U_\gamma, X \setminus f(B) \mid \gamma < \tau\}$. Since $f(B_\gamma) \in \mathcal{C}$ for every $\gamma < \tau$, it follows from Proposition 2.1.2(2) that $f(B) = \bigcup_{\gamma<\tau} f(B_\gamma)$ is a strong Z-set in X. Then, we can obtain a $\mathcal{U}_2'$-homotopy $h' : X \times \mathbf{I} \to X$ and an open neighborhood W of $f(B)$ in X such that $h_0' = f$ and

$h'_1(X) \subset X \setminus W$. We write $W_\gamma = W \cap U_\gamma$ for each $\gamma < \tau$. Let $h = h'(f \times \mathrm{id}_{\mathbf{I}}) : A \times \mathbf{I} \to X$, so h is a $\mathcal{U}'_2$-homotopy and $h_0 = h'_0 f = f$. Observe that $h(B_\gamma \times \mathbf{I}) \subset U_\gamma$ for each $\gamma < \tau$. Since each B_γ is compact, we can find an open neighborhood V_γ of B_γ in A_γ so that $h(V_\gamma \times \mathbf{I}) \subset U_\gamma$. Take an Urysohn map $k : A \to \mathbf{I}$ such that $k^{-1}(0) = B$ and $k^{-1}(1) = A \setminus \bigcup_{\gamma<\tau} V_\gamma$ and define the map $f' : A \to X$ by $f'(x) = h(x, k(x))$ for $x \in A$. It is easy to see that $f' \sim_{\mathcal{U}'_2} f$ and $f'|_B = h_0|_B = f|_B$. Moreover, f' satisfies the following condition:

(1) $f'(A \setminus V_\gamma) \cap W_\gamma = \emptyset$ for any $\gamma < \tau$.

Indeed, take any point $x \in A \setminus V_\gamma$. When $x \in A \setminus \bigcup_{\gamma<\tau} V_\gamma$,

$$f'(x) = h_1(x) = h'_1 f(x) \in X \setminus W \subset X \setminus W_\gamma.$$

When $x \in V_{\gamma'}$ for some $\gamma' \neq \gamma$, we have

$$f'(x) = h_{k(x)}(x) \in U_{\gamma'} \subset X \setminus U_\gamma \subset X \setminus W_\gamma.$$

We take an open neighborhood W'_γ of $f(B_\gamma)$ for each $\gamma < \tau$ so that $\operatorname{cl} W'_\gamma \subset W_\gamma$. Let $\mathcal{U}'_1 \in \operatorname{cov}(X)$ such that

$$\mathcal{U}'_1 \prec \mathcal{U}_1 \wedge \left\{ W'_\gamma, W_\gamma \setminus f(B_\gamma), X \setminus \bigcup_{\gamma' \in \tau} \operatorname{cl} W'_{\gamma'} \;\middle|\; \gamma < \tau \right\}.$$

Applying τ-DAP($\mathcal{C}$) of X to f', we can obtain a $\mathcal{U}'_1$-homotopy $h'' : A \times \mathbf{I} \to X$ so that $h''_0 = f'$ and

(2) $\{h''_1(A_\gamma) \mid \gamma < \tau\}$ is discrete in X.

Since h'' is a $\mathcal{U}'_1$-homotopy and $h''_0|_B = f'|_B = f|_B$, it follows that $h''(B_\gamma \times \mathbf{I}) \subset W'_\gamma$ for each $\gamma < \tau$. Because of the compactness, each B_γ has an open neighborhood G_γ in A_γ such that $h''(G_\gamma \times \mathbf{I}) \subset W'_\gamma$. Let $k' : A \to \mathbf{I}$ be an Urysohn map such that $(k')^{-1}(0) = B$ and $(k')^{-1}(1) = A \setminus \bigcup_{\gamma<\tau} G_\gamma$. Now, we can define the desired map $g : A \to X$ by $g(x) = h''(x, k'(x))$ for all $x \in A$. Observe that $g \sim_{\mathcal{U}'_1} f'$ and the restriction $g|_B = h''_0|_B = f'|_B$, and hence $g \sim_{\mathcal{U}} f$ and $g|_B = f|_B$. Thus, it remains to show that $\{g(A_\gamma) \mid \gamma < \tau\}$ is discrete in X.

Fix a point $x \in X$. Due to (2), the collection $\{g(A_\gamma \setminus G_\gamma) \mid \gamma < \tau\}$ is discrete in X, and hence there exists an open neighborhood U_x of x in X such that $\operatorname{card}(\{\gamma < \tau \mid g(A_\gamma \setminus G_\gamma) \cap U_x \neq \emptyset\}) \leq 1$.

(Case 1) $\operatorname{card}(\{\gamma < \tau \mid g(A_\gamma \setminus G_\gamma) \cap U_x \neq \emptyset\}) = 0$.

When $x \in X \setminus \bigcup_{\gamma<\tau} \operatorname{cl} W'_\gamma$, the subset $U'_x = U_x \setminus \bigcup_{\gamma<\tau} \operatorname{cl} W'_\gamma$ is an open neighborhood of x in X. Since $g(G_\gamma) \subset W'_\gamma$, we have $U'_x \cap g(G_\gamma) = \emptyset$, so $U'_x \cap g(A_\gamma) = \emptyset$ for any $\gamma < \tau$. When $x \in \bigcup_{\gamma<\tau} \operatorname{cl} W'_\gamma$, $x \in \operatorname{cl} W'_{\gamma_0}$ for the unique $\gamma_0 \in \tau$. Then $U'_x = U_x \setminus \bigcup_{\gamma \neq \gamma_0} \operatorname{cl} W'_\gamma$ is an open neighborhood of x in X such that $U'_x \cap g(A_\gamma) = \emptyset$ for all $\gamma \neq \gamma_0$.

(Case 2) $\operatorname{card}(\{\gamma < \tau \mid g(A_\gamma \setminus G_\gamma) \cap U_x \neq \emptyset\}) = 1$.

We may assume that $g(A_{\gamma_0} \setminus G_{\gamma_0}) \cap U_x \neq \emptyset$ for the unique $\gamma_0 \in \tau$. Note that $g(A_{\gamma_0} \setminus G_{\gamma_0})$ is a closed set in X because of the compactness of A_{γ_0}, so we can turn the case when $x \notin g(A_{\gamma_0} \setminus G_{\gamma_0})$ into Case 1 by replacing U_x by $U_x \setminus g(A_{\gamma_0} \setminus G_{\gamma_0})$. When $x \in g(A_{\gamma_0} \setminus G_{\gamma_0})$, we have $x \in X \setminus \bigcup_{\gamma \neq \gamma_0} \operatorname{cl} W'_\gamma$. Otherwise $x \in \operatorname{cl} W'_{\gamma_1}$ for some $\gamma_1 \neq \gamma_0$. As $x \in g(A_{\gamma_0} \setminus G_{\gamma_0})$, the point $x = g(a)$ for a point $a \in A_{\gamma_0} \setminus G_{\gamma_0}$. Then $f'(a) \in W_{\gamma_1}$ because $g \sim_{\mathcal{U}'_1} f'$. On the other hand, since $A_{\gamma_0} \subset A \setminus V_{\gamma_1}$, it follows from (1) that $f'(A_{\gamma_0}) \cap W_{\gamma_1} = \emptyset$, which is a contradiction. Now x has the open neighborhood $U'_x = U_x \setminus \bigcup_{\gamma \neq \gamma_0} \operatorname{cl} W'_\gamma$ in X such that $U'_x \cap g(A_\gamma) = \emptyset$ for every $\gamma \neq \gamma_0$. $\square$

2.4 Proof of Main Theorem

This section is devoted to proving Main Theorem. The following proposition follows from Stone's Theorem (Theorem 4.4.1 of [30]).

Proposition 2.4.1. *Let X be a metrizable space. Then the following conditions are equivalent:*

(1) *X is strongly countable-dimensional and σ-locally compact;*

(2) *X is strongly countable-dimensional and a countable union of closed locally compact subsets;*

(3) *X is a countable union of locally compact locally finite-dimensional closed subsets;*

(4) *X is a countable union of closed subsets which are discrete unions of finite-dimensional compact metrizable spaces.*

Proof. The implication (2) $\Rightarrow$ (3) is obvious. First, we prove the implication (1) $\Rightarrow$ (2). It is sufficient to show that any σ-locally compact metrizable space X can be written as a countable union of closed locally compact subsets. We can write $X = \bigcup_{n\in\omega} X_n$, where each X_n is locally compact. According to [54, Theorem 2], each X_n is an absolute F_σ set. Hence we have $X_n = \bigcup_{m\in\omega} A^n_m$, where A^n_m is closed in X for all $m, n \in \omega$. Since X_n is locally compact, so A^n_m is. Therefore $X = \bigcup_{m,n\in\omega} A^n_m$ is a countable union of closed locally compact subsets.

To prove the implication (3) $\Rightarrow$ (4), we assume that $X = \bigcup_{n\in\omega} X_n$, where X_n is a locally compact locally finite-dimensional closed subsets for all $n \in \omega$. By the local compactness and the local finite-dimensionality, each X_n has an open cover $\mathcal{U}_n$ such that for every $U \in \mathcal{U}_n$, the closure of U is compact and finite-dimensional. Due to Stone's Theorem, each $\mathcal{U}_n$ has a σ-discrete open refinement $\mathcal{V}_n = \bigcup_{m\in\omega} \mathcal{V}^m_n \in \operatorname{cov}(X_n)$, where $\mathcal{V}^m_n$ is discrete in X_n. Then, $A^m_n = \bigcup_{V\in\mathcal{V}^m_n} \operatorname{cl} V$ is a closed subset of X_n which is a discrete union of finite-dimensional compact metrizable spaces. Evidently $X = \bigcup_{n,m\in\omega} A^m_n$, which implies that X satisfies the condition (4).

Finally, we show the implication (4) $\Rightarrow$ (1). As is easily observed, we can write $X = \bigcup_{n\in\omega} X_n$, where each X_n is a closed subspace which is discrete unions of compact metrizable spaces of dimension $\leq n$. Hence X is a countable union of finite-dimensional locally compact closed subsets, which means that it is strongly countable-dimensional and σ-locally compact. The proof is complete. □

Remark 3. As is seen in the above proof, when a metrizable space X satisfies the above conditions, we can write $X = \bigcup_{n\in\omega} X_n$, where each X_n is a closed subspace which is discrete unions of compact metrizable spaces of dimension $\leq n$.

Now, we shall show the following characterization.

Theorem 2.4.2. *Let τ be an infinite cardinal. For a connected space X, the following conditions* (1), (2) *and* (3) *are equivalent:*

(1) *X is an $\ell^f_2(\tau)$-manifold;*

(2) (a) *X is an ANR of weight τ and a countable union of closed sets which are discrete unions of finite-dimensional compact metrizable spaces;*

  (b) *X is strongly universal for $\bigoplus_\tau \mathfrak{M}_0(n)$ for all $n \in \omega$;*

  (c) *For every subset $C \subset X$, if $C \in \mathfrak{M}^f_0$, then C is a strong Z-set in X;*

(3) (a) *X is an ANR of weight τ and a countable union of closed sets which are discrete unions of finite-dimensional compact metrizable spaces;*

  (b) (i) *X has τ-DAP($\mathfrak{M}_0(n)$) for all $n \in \omega$;*

(ii) *X is strongly universal for $\mathfrak{M}_0^f$;*

(c) *For every subset $C \subset X$, if $C \in \mathfrak{M}_0^f$, then C is a strong Z-set in X.*

Proof. The implication (2) $\Rightarrow$ (3) is clear. According to Proposition 2.3.4, the condition (b) of (3) implies the condition (b) of (2), so the implication (3) $\Rightarrow$ (2) also holds. Now, we shall show the equivalence (1) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (2): Due to Proposition 4.5 of [56], X is an ANR which is a countable union of locally compact locally finite-dimensional closed subsets. By Proposition 2.4.1, X is a countable union of closed subsets which are discrete unions of finite-dimensional compact metrizable spaces. Moreover, since X is connected, we have $\mathrm{w}(X) = \mathrm{w}(\ell_2^f(\tau)) = \tau$. Therefore X satisfies the condition (a).

By 1.1 of [56], every space in $\bigoplus_\tau \mathfrak{M}_0(n)$, $n \in \omega$, can be embedded into $\ell_2^f(\tau)$ as a closed subspace. Hence, the condition (b) follows from the Strong Universality Theorem (cf. Lemma 5.1 of [19] [2]). Furthermore, since the condition (b) implies that X has the τ-discrete n-cells property for all $n \in \omega$, any finite-dimensional compact subset $C \subset X$ is a Z-set in X by Proposition 2.1.1. Then C is a strong Z-set in X due to A1 of [60], which means that the condition (c) holds.

(2) $\Rightarrow$ (1): Obviously, the class $\mathfrak{C} = \bigcup_{n\in\omega} \bigoplus_\tau \mathfrak{M}_0(n)$ is topological and closed hereditary. As is seen in the proof of (1) $\Rightarrow$ (2), the model space $\ell_2^f(\tau)$ satisfies the condition (2). Due to the condition (a) and Remark 3, with respect to $\mathfrak{C}$ the locally convex topological linear metric space $\ell_2^f(\tau)$ and the connected ANR X satisfy ($\star$) in Section 2.2 and (i) in Theorem 2.2.3, respectively. Combining the condition (c) with Proposition 2.1.2(2) implies that $\ell_2^f(\tau)$ and X satisfy ($\star\star$) in Section 2.2 and (iii) in Theorem 2.2.3 with respect to $\mathfrak{C}$, respectively. The condition (b) is no other than the condition (ii) in Theorem 2.2.3. On the other hand, since X is an ANR of weight τ and a countable union of locally compact locally finite-dimensional closed subsets, applying Theorem 4.3 of [56] to $X \times \ell_2^f(\tau)$, we have $X \times \ell_2^f(\tau)$ is an $\ell_2^f(\tau)$-manifold. According to Theorem 2.2.3, X is homeomorphic to $X \times \ell_2^f(\tau)$, that is, it is an $\ell_2^f(\tau)$-manifold. □

Remark 4. As is seen in the above, the space $\ell_2^f(\tau)$ has the properties ($\star$) and ($\star\star$) in Section 2.2 with respect to the class $\mathfrak{C} = \bigcup_{n\in\omega} \bigoplus_\tau \mathfrak{M}_0(n)$. Then, it follows from $\mathfrak{C} \subset \bigoplus_\tau \mathfrak{M}_0^f$ that $\ell_2^f(\tau)$ satisfies ($\star$) with respect to $\bigoplus_\tau \mathfrak{M}_0^f$, immediately. Moreover, combining (c) of Theorem 2.4.2 with Proposition 2.1.2(2) implies the stronger assertion that $\ell_2^f(\tau)$ satisfies ($\star\star$) with respect to $\bigoplus_\tau \mathfrak{M}_0^f$, actually. In addition, removing "finite-dimensionality", we have $\ell_2^f(\tau) \times \mathbf{Q}$ satisfies ($\star$) and ($\star\star$) with respect to the class $\bigoplus_\tau \mathfrak{M}_0$.

Using the above characterization, we shall prove Main Theorem.

Proof of Main Theorem. Using the condition (3) of Theorem 2.4.2 and Proposition 2.4.1, we can obtain the "only if" part immediately. Now, we shall prove the "if" part. Since X is locally contractible, each point $x \in X$ has an open neighborhood W which is contractible in X. It is enough to show that W is an $\ell_2^f(\tau)$-manifold, that is, W satisfies (3) of Theorem 2.4.2.

It follows from Proposition 2.1.2(1) that W satisfies the condition (c). To verify the condition (b-ii), suppose that $f : A \to W$ is a map from $A \in \mathfrak{M}_0^f$ such that the restriction $f|_B$ on a closed subset B of A is a Z-embedding. For each open cover $\mathcal{W} \in \mathrm{cov}(W)$, the collection $\mathcal{U} = \mathcal{W} \cup \{X \setminus f(A)\} \in \mathrm{cov}(X)$ because A is compact. Then, applying the strong universality of X to f allows us to find a Z-embedding $g : A \to X$ such that $g \sim_\mathcal{U} f$ and $g|_B = f|_B$. Due to the definition of $\mathcal{U}$, we have $g(A) \subset W$ and $g \sim_\mathcal{W} f$. Thus, W satisfies (b-ii). The contractibility of W in X and the τ-discrete n-cells property of X, $n \in \omega$, imply that W has τ-DAP($\mathfrak{M}_0(n)$) for all $n \in \omega$ by Proposition 2.3.3, namely, the condition (b-i) is satisfied. It

[2]Lemma 5.1 of [19] holds for a locally convex topological linear metric space E not only such that E is homeomorphic to $E^\mathbb{N}$ but also such that E is homeomorphic to $E_f^\mathbb{N}$.

remains to check the condition (a). It follows from τ-DAP($\mathfrak{M}_0(n)$) of W that $\tau \le \mathrm{w}(W) \le \mathrm{w}(X) = \tau$, hence $\mathrm{w}(W) = \tau$. Since W is an open subset in X, it is an ANR and an F_σ set in X. Then, because X is a countable union of closed subsets which are discrete unions of finite-dimensional compact metrizable space by Proposition 2.4.1, so an F_σ set W is. Therefore, the condition (a) holds. □

By removing "finite-dimensionality" from the characterization of $\ell_2^f(\tau)$-manifolds, we can similarly prove the following characterization of $(\ell_2^f(\tau) \times \mathbf{Q})$-manifolds.

Theorem 2.4.3. *Let τ be an infinite cardinal. A connected space X is an $(\ell_2^f(\tau) \times \mathbf{Q})$-manifold if and only if the following conditions are satisfied:*

(1) *X is a σ-locally compact ANR of weight τ;*

(2) *X has the τ-discrete cells property;*

(3) *X is strongly universal for $\mathfrak{M}_0$;*

(4) *For every subset $C \subset X$, if $C \in \mathfrak{M}_0$, then C is a strong Z-set in X.*

Chapter 3

Characterizations of infinite-dimensional manifold pairs

In this chapter, we assume that spaces are paracompact. Combining West's characterization [61] with the main theorem in Chapter 2, we shall prove the following:

Main Theorem. *A pair (X, Y) of spaces is an $(\ell_2(\tau), \ell_2^f(\tau))$-manifold pair if and only if X is an $\ell_2(\tau)$-manifold, Y is an $\ell_2^f(\tau)$-manifold and Y is homotopy dense in X.*

For an infinite cardinal τ, the *hedgehog* $J(\tau)$ is the closed subspace in $\ell_1(\tau)$ defined as follows:

$$J(\tau) = \{x = (x(\gamma))_{\gamma<\tau} \in \ell_1(\tau) \cap \mathbf{I}^\tau \mid x(\gamma) \neq 0 \text{ at most one } \gamma < \tau\}.$$

It is well known that the countable product $J(\tau)^{\mathbb{N}}$ of $J(\tau)$ is a universal space for the class of metrizable spaces of wight $\leq \tau$ (cf. Corollary 2.3.7 of [50]). We define the subspace $J(\tau)_f^{\mathbb{N}}$ in $J(\tau)^{\mathbb{N}}$ as follows:

$$J(\tau)_f^{\mathbb{N}} = \{x = (x(n))_{n\in\mathbb{N}} \in J(\tau)^{\mathbb{N}} \mid x(n) = \mathbf{0} \text{ except for finitely many } n \in \mathbb{N}\}.$$

Applying the modified West's characterization Theorem 3.1.4 to the pair $(J(\tau)^{\mathbb{N}}, J(\tau)_f^{\mathbb{N}})$, we can also prove the following theorem:

Theorem 3.0.1. *Let τ be an infinite cardinal. The pair $(J(\tau)^{\mathbb{N}}, J(\tau)_f^{\mathbb{N}})$ is homeomorphic to $(\ell_2(\tau), \ell_2^f(\tau))$.*

3.1 West's characterization and the main result

Let $\mathcal{C}$ be a topological and closed hereditary class of spaces. We denote the collection of closed subspaces in a space X which belong to $\mathcal{C}$ by $\mathcal{C}(X)$. A subspace Y of X is said to be *weakly $\mathcal{C}(X)$-absorptive*[1] if the following condition hold:

(abs) For each $A \in \mathcal{C}(X)$, each closed subset B of A contained in Y and each open cover $\mathcal{U}$ of X, there exists an embedding $f : A \to Y$ such that f is $\mathcal{U}$-close to id_A and $f|_B = \mathrm{id}_B$.

A space Y has a *$\mathcal{C}$-complex structure* $\{\mathcal{A}_n\}_{n\in\omega}$ if each $\mathcal{A}_n$ is a subcollection of $\mathcal{C}(Y)$ with the following properties:

(1) $Y = \bigcup_{n\in\omega}(\bigcup \mathcal{A}_n)$;

[1]This notion is introduced in Theorem 6 of [61]. A subspace Y of X is $\mathfrak{C}(X)$-*absorptive* if for each $A \in \mathfrak{C}(X)$, each closed subset B of A contained in Y, and each open cover $\mathcal{U}$ of A in X, there exists a homeomorphism $f : X \to X$ such that $f(A) \subset Y$, $f|_{\bigcup\mathcal{U}}$ is $\mathcal{U}$-close to $\mathrm{id}_{\bigcup\mathcal{U}}$, and $f|_{(X\setminus\bigcup\mathcal{U})\cup B} = \mathrm{id}_{(X\setminus\bigcup\mathcal{U})\cup B}$. Moreover, if there exists an ambient isotopy h of f such that $\{h(\{x\} \times \mathbf{I}) \mid x \in A\} \prec \mathcal{U}$, then Y is called *strongly* $\mathfrak{C}(X)$-*absorptive*.

(2) $A_n = \bigcup_{i=0}^n(\bigcup \mathcal{A}_i)$ is closed in Y for each $n \in \omega$;

(3) For each $n \in \omega$, there exists a pairwise disjoint open cover $\mathcal{U}_n$ of $A_n \setminus A_{n-1}$ in Y such that $U \cap A_n \setminus A_{n-1} \in \{A \setminus A_{n-1} \mid A \in \mathcal{A}_n\}$ for each $U \in \mathcal{U}_n$, where $A_{-1} = \emptyset$.

J.E. West established the following characterization of $(\ell_2(\tau), \ell_2^f(\tau))$-manifold pairs in 1970, see Theorem 6 of [61].

Theorem 3.1.1. *Let τ be an infinite cardinal. For spaces $Y \subset X$, the pair (X, Y) is an $(\ell_2(\tau), \ell_2^f(\tau))$-manifold pair if and only if X is an $\ell_2(\tau)$-manifold, Y is weakly $\mathfrak{M}_0^f(X)$-absorptive and has an $\mathfrak{M}_0^f$-complex structure.*

Due to Theorem 6 of [32] (cf. Theorem C of [33]) and Theorem 1 of [61], we can classify $(\ell_2(\tau), \ell_2^f(\tau))$-manifold pairs according to homotopy types.

Theorem 3.1.2. *Let τ be an infinite cardinal. Suppose that (X, Y) and (X', Y') are $(\ell_2(\tau), \ell_2^f(\tau))$-manifold pairs. If X and X' (or Y and Y') have the same homotopy type, then (X, Y) and (X', Y') are homeomorphic.*

Remark 5. While it is not mentioned in [61], the similar characterization of $(\ell_2(\tau)\times\mathbf{Q}, \ell_2^f(\tau)\times\mathbf{Q})$-manifold pairs can be established as follows:

- A pair (X, Y) of spaces is an $(\ell_2(\tau) \times \mathbf{Q}, \ell_2^f(\tau) \times \mathbf{Q})$-manifold pair if and only if X is an $\ell_2(\tau)$-manifold[2], Y is weakly $\mathfrak{M}_0(X)$-absorptive and has an $\mathfrak{M}_0$-complex structure.

In addition, Theorem 3.1.2 is valid for $(\ell_2(\tau) \times \mathbf{Q}, \ell_2^f(\tau) \times \mathbf{Q})$-manifold pairs.

Although the complex structure is defined by imitating the simplicial complex structure, it is complicated. The following proposition is very useful for detecting a $\mathcal{C}$-complex structure with respect to a topological and closed hereditary class $\mathcal{C}$ in a metrizable space.

Proposition 3.1.3. *For a topological and closed hereditary class $\mathcal{C}$, a metrizable space X is a countable union of closed sets which are discrete unions of members of $\mathcal{C}$ if and only if X has a $\mathcal{C}$-complex structure.*

Proof. First, we show the "only if" part. Let $X = \bigcup_{n\in\omega}(\bigcup \mathcal{A}_n)$, where $\mathcal{A}_n$ is a discrete collection of X whose members are in $\mathcal{C}$ and the union $\bigcup \mathcal{A}_n$ is closed in X for each $n \in \omega$. Note that $\mathcal{A}_n \subset \mathcal{C}(X)$ for all $n \in \omega$. Then $A_n = \bigcup_{i=0}^n(\bigcup \mathcal{A}_i)$ is closed in X for every $n \in \omega$. Since each $\mathcal{A}_n$ is discrete in X, there exists a pairwise disjoint collection $\mathcal{U}_n = \{U(A) \mid A \in \mathcal{A}_n\}$ of open subsets of X such that $A \subset U(A)$ for each $A \in \mathcal{A}_n$. Observe that $U(A) \cap (A_n \setminus A_{n-1}) = A \setminus A_{n-1}$ for each $A \in \mathcal{A}_n$ and $n \in \omega$, where $A_{-1} = \emptyset$. Consequently, the collections $\{\mathcal{A}_n\}_{n\in\omega}$ is a $\mathcal{C}$-complex structure of X.

Next, we prove the "if" part. Let $\{\mathcal{A}_n\}_{n\in\omega}$ be a $\mathcal{C}$-complex structure of X. Then, for each $n \in \omega$ there exists a pairwise disjoint collection $\mathcal{U}_n$ of open subsets of X satisfying the following condition:

- Each $\mathcal{U}_n$ covers $A_n \setminus A_{n-1}$ so that $U \cap A_n \setminus A_{n-1} \in \{A \setminus A_{n-1} \mid A \in \mathcal{A}_n\}$ for every $U \in \mathcal{U}_n$, where $A_{-1} = \emptyset$.

For every $U \in \mathcal{U}_n$ and $n \in \omega$, we can choose $A \in \mathcal{A}_n$ so that $U \cap A_n \setminus A_{n-1} = A \setminus A_{n-1}$, which is open in A, so an F_σ set in A. Hence, we can write $U \cap A_n \setminus A_{n-1} = \bigcup_{m\in\omega} A^m_{(n,U)}$, where each $A^m_{(n,U)}$ is closed in A, so closed in X. It is easy to see that $\mathcal{A}_{(n,m)} = \{A^m_{(n,U)} \mid U \in \mathcal{U}_n\}$ is discrete in X and the union $\bigcup \mathcal{A}_{(n,m)}$ is closed in X for all $n, m \in \omega$. Moreover, $X = \bigcup_{n,m\in\omega}(\bigcup \mathcal{A}_{(n,m)})$. Indeed, for each $x \in X$, choose $n \in \omega$ such that $x \in A_n \setminus A_{n-1}$. Since $\mathcal{U}_n$ covers $A_n \setminus A_{n-1}$, there is $U \in \mathcal{U}_n$ such that $x \in U \cap A_n \setminus A_{n-1} = \bigcup_{m\in\omega} A^m_{(n,U)}$, which implies that $x \in A^m_{(n,U)} \subset \bigcup \mathcal{A}_{(n,m)}$ for some $m \in \omega$. Thus, X is a countable union of closed sets which are discrete unions of members of $\mathcal{C}$. □

[2]Remark that $\ell_2(\tau) \times \mathbf{Q}$ is homeomorphic to $\ell_2(\tau)$.

Combining Proposition 2.4.1 in Chapter 2 with the above, we can modify West's characterizations as follows:

Theorem 3.1.4. *Let $Y \subset X$ be spaces and τ an infinite cardinal. The pair (X, Y) is an $(\ell_2(\tau), \ell_2^f(\tau))$-manifold pair if and only if X is an $\ell_2(\tau)$-manifold, and Y is strongly countable-dimensional, σ-locally compact, and weakly $\mathfrak{M}_0^f(X)$-absorptive, and (X, Y) is an $(\ell_2(\tau) \times \mathbf{Q}, \ell_2^f(\tau) \times \mathbf{Q})$-manifold pair if and only if X is an $\ell_2(\tau)$-manifold, and Y is σ-locally compact and weakly $\mathfrak{M}_0(X)$-absorptive.*

Proposition 3.1.5. *Let $\mathcal{C}$ be a topological and closed hereditary subclass of $\mathfrak{M}$. Suppose that a homotopy dense subset Y of a metrizable space X satisfies the following conditions:*

- $(*)$ *Y is strongly universal for $\mathcal{C}$;*
- $(**)$ *Every closed subset $C \in \mathcal{C}(Y)$ is a Z-set in Y.*

Then Y is weakly $\mathcal{C}(X)$-absorptive.

Proof. Fix $A \in \mathcal{C}(X)$, a closed subset B of A contained in Y and an open cover $\mathcal{U}$ of X. Take an open cover $\mathcal{V}$ of X so that $\mathcal{V} \prec^{\star} \mathcal{U}$. Since Y is homotopy dense in X, we can find a homotopy $h : X \times \mathbf{I} \to X$ such that $h_0 = \mathrm{id}_X$ and $h(X \times (0, 1]) \subset Y$. Then, we have a map $k : A \to \mathbf{I}$ such that $k^{-1}(0) = B$ and $\{\{x\} \times [0, k(x)] \mid x \in A\} \prec h^{-1}(\mathcal{V})$. Define a map $f : A \to Y \subset X$ by $f(x) = h(x, k(x))$ for each $x \in A$, so f is $\mathcal{V}$-close to id_A and $f|_B = h_0|_B = \mathrm{id}_B$. On the other hand, since $\mathcal{C}$ is closed hereditary, it follows from $(**)$ that B is a Z-set in Y, hence the restriction $f|_B$ is a Z-embedding into Y. Then, applying the strong universality of Y to f, we can obtain a Z-embedding $g : A \to Y$ such that g is $\mathcal{V}|_Y$-close to f and $g|_B = f|_B = \mathrm{id}_B$, where $\mathcal{V}|_Y = \{V \cap Y \mid V \in \mathcal{V}\}$. Observe that g is $\mathcal{U}$-close to id_A. Consequently, Y is weakly $\mathcal{C}(X)$-absorptive. $\square$

A subset $A \subset X$ is said to be *locally homotopy negligible* in a space X if for each $n \in \omega$, $x \in X$ and open neighborhood U of x, there exists a neighborhood V of x such that given a map $f : (\mathbf{I}^n, \mathrm{bd}\,\mathbf{I}^n) \to (V, V \setminus A)$, there is a homotopy $h : (\mathbf{I}^n, \mathrm{bd}\,\mathbf{I}^n) \times \mathbf{I} \to (U, U \setminus A)$ with $h_0 = f$ and $h_1(\mathbf{I}^n) \subset U \setminus A$, where $\mathrm{bd}\,\mathbf{I}^n$ is the boundary of $\mathbf{I}^n$. It is easy to see that a subset $A \subset X$ is locally homotopy negligible in a space X if and only if each point of X has a neighborhood U such that $U \cap A$ is locally homotopy negligible in U. For every infinite cardinal τ, the subset $\ell_2(\tau) \setminus \ell_2^f(\tau)$ is locally homotopy negligible in $\ell_2(\tau)$. Now, we shall demonstrate Main Theorem.

Proof of Main Theorem. First, we prove the "only if" part. Since $\ell_2(\tau) \setminus \ell_2^f(\tau)$ is locally homotopy negligible in $\ell_2(\tau)$, it follows from Remark 2.2 of [57] that $U \setminus \ell_2^f(\tau)$ is locally homotopy negligible in U for every open subset $U \subset \ell_2(\tau)$. This means that $X \setminus Y$ is locally homotopy negligible in X, recall that (X, Y) is an $(\ell_2(\tau), \ell_2^f(\tau))$-manifold pair. Thus, Y is homotopy dense in X by Theorem 2.4 of [57].

Next, we show the "if" part. Since Y is an $\ell_2^f(\tau)$-manifold, it follows from the conditions (3) and (4) of the main theorem in Chapter 2 that Y satisfies the conditions $(*)$ and $(**)$ in Proposition 3.1.5 for the class $\mathfrak{M}_0^f$. Moreover, because Y is homotopy dense in X, we have that Y is weakly $\mathfrak{M}_0^f(X)$-absorptive by Proposition 3.1.5. Then, we can apply Theorem 3.1.4 to the pair (X, Y), so (X, Y) is an $(\ell_2(\tau), \ell_2^f(\tau))$-manifold pair. $\square$

Remark 6. Combining Theorems 2.4.3 and 3.1.4 with Proposition 3.1.5, we can obtain another characterization of $(\ell_2(\tau) \times \mathbf{Q}, \ell_2^f(\tau) \times \mathbf{Q})$-manifold pairs as follows:

- A pair (X, Y) of spaces is an $(\ell_2(\tau) \times \mathbf{Q}, \ell_2^f(\tau) \times \mathbf{Q})$-manifold pair if and only if X is an $\ell_2(\tau)$-manifold, Y is an $(\ell_2^f(\tau) \times \mathbf{Q})$-manifold and Y is homotopy dense in X.

Remark 7. The main theorem does not hold for other infinite-dimensional manifolds. For example, consider the pair $(\mathbf{Q} \times \ell_2, \mathbf{s} \times \ell_2^f)$. Recall that $\mathbf{s}$ is homeomorphic to the separable Hilbert space ℓ_2, see Section 1.3 in Chapter 1. Then we have $\mathbf{Q} \times \ell_2$ is homeomorphic to ℓ_2, $\mathbf{s} \times \ell_2^f$ is homeomorphic to $\ell_2 \times \ell_2^f$ and $\mathbf{s} \times \ell_2^f$ is homotopy dense in $\mathbf{Q} \times \ell_2$. However, $(\mathbf{Q} \times \ell_2, \mathbf{s} \times \ell_2^f)$ is not homeomorphic to $(\ell_2 \times \ell_2, \ell_2 \times \ell_2^f)$ because $\ell_2 \times \ell_2^f$ is an F_σ set in $\ell_2 \times \ell_2$ while $\mathbf{s} \times \ell_2^f$ is not an F_σ set but a $G_{\delta\sigma}$ set in $\mathbf{Q} \times \ell_2$.

3.2 An application

This section is devoted to proving Theorem 3.0.1. Throughout the section, we consider τ an infinite cardinal. We use an admissible metric d on $J(\tau)^{\mathbb{N}}$ as follows:

$$d(x, y) = \sum_{i \in \mathbb{N}} 2^{-i} \|x(i) - y(i)\|_1 \text{ for every } x = (x(i))_{i \in \mathbb{N}}, y = (y(i))_{i \in \mathbb{N}} \in J(\tau)^{\mathbb{N}}.$$

Let $\mathrm{pr}_i : J(\tau)^{\mathbb{N}} \to J(\tau)$ be the projection onto the ith coordinate. Define the vector $\mathbf{e}_\gamma \in \ell_1(\tau)$ for each $\gamma < \tau$ as follows:

$$\mathbf{e}_\gamma(\gamma') = \begin{cases} \mathbf{e}_\gamma(\gamma') = 1 & \text{if } \gamma' = \gamma, \\ \mathbf{e}_\gamma(\gamma') = 0 & \text{if } \gamma' \neq \gamma, \end{cases}$$

that is, $\mathbf{e}_\gamma$ is an unit vector of $\ell_1(\tau)$. Moreover, for $x, y \in \ell_1(\tau)$, the line segment between x and y is denoted by $\langle x, y \rangle$, that is,

$$\langle x, y \rangle = \{(1-t)x + ty \mid t \in \mathbf{I}\}.$$

First, we shall show the following:

Theorem 3.2.1. *The space $J(\tau)^{\mathbb{N}}$ is homeomorphic to $\ell_2(\tau)$.*

Proof. Since the hedgehog $J(\tau)$ is closed in $\ell_1(\tau)$, it is completely metrizable. As is easily observed, $J(\tau)$ is a metric polyhedron of a simplicial complex, and hence it is a contractible ANR (cf. Theorem 6.2.6 of [50]). Therefore $J(\tau)$ is an AR. According to Theorem 1.3.6, the countable product $J(\tau)^{\mathbb{N}}$ is homeomorphic to $\ell_2(\tau)$. □

Proposition 3.2.2. *The space $J(\tau)_f^{\mathbb{N}}$ is strongly countable-dimensional and σ-locally compact.*

Proof. According to Proposition 2.4.1 in Chapter 2, we need only to show that $J(\tau)_f^{\mathbb{N}}$ can be written as a countable union of closed subsets which are discrete unions of finite-dimensional compact subsets. Let $\mathrm{Fin}(\mathbb{N})$ be the all non-empty finite subsets of $\mathbb{N}$. For each $M \in \mathrm{Fin}(\mathbb{N})$, each $n \in \omega$ and each function $\psi_M : M \to \tau$, we define the finite-dimensional compact subset of $J(\tau)_f^{\mathbb{N}}$ as follows:

$$A_{(M,n)}^{\psi_M} = \left\{ x \in J(\tau)^{\mathbb{N}} \;\middle|\; \begin{array}{lcll} x(i) & \in & \langle 2^{-n}\mathbf{e}_{\psi_M(i)}, \mathbf{e}_{\psi_M(i)} \rangle, & \text{if } i \in M, \text{ and} \\ x(i) & = & \mathbf{0}, & \text{otherwise} \end{array} \right\},$$

which is homeomorphic to the cube $\mathbf{I}^{\mathrm{card}(M)}$. Let

$$\mathcal{A}_{(M,n)} = \{A_{(M,n)}^{\psi_M} \mid \psi_M : M \to \tau\} \text{ for each } M \in \mathrm{Fin}(\mathbb{N}) \text{ and } n \in \omega.$$

Fix a point $x \in J(\tau)_f^{\mathbb{N}} \setminus \{\mathbf{0}\}$, so we have the set $M = \{i \in \mathbb{N} \mid x(i) \neq \mathbf{0}\} \in \mathrm{Fin}(\mathbb{N})$. Define the function $\psi_M : M \to \tau$ as follows:

$$\psi_M(i) = \gamma < \tau \text{ if } x(i)(\gamma) > 0 \text{ for each } i \in M.$$

Taking $n \in \omega$ so that $2^{-n} \leq \min_{i \in M} \|x(i)\|_1$, we can easily see that $x \in A_{(M,n)}^{\psi_M}$. It follows that

$$J(\tau)_f^{\mathbb{N}} = \{\mathbf{0}\} \cup \left(\bigcup_{M \in \mathrm{Fin}(\mathbb{N}), n \in \omega} \left(\bigcup \mathcal{A}_{(M,n)}\right) \right).$$

Moreover, $\mathcal{A}_{(M,n)}$ is discrete in $J(\tau)^{\mathbb{N}}_f$ for each $M \in \mathrm{Fin}(\mathbb{N})$ and $n \in \omega$. Indeed, let $x \in J(\tau)^{\mathbb{N}}_f$. When $x(i) = \mathbf{0}$ for some $i \in M$, we have $B_d(x, 2^{-n}) \cap A^{\psi_M}_{(M,n)} = \emptyset$ for every $\psi_M : M \to \tau$. When $x(i) \neq \mathbf{0}$ for all $i \in M$, as is easily observed, we can take the unique function $\psi_M : M \to E$ such that $x(i) \in \langle \mathbf{0}, \mathbf{e}_{\psi_M(i)} \rangle \setminus \{\mathbf{0}\}$. Then, define $\delta = \min_{i\in M} \|x(i)\|_1$, so $B_d(x, \delta) \cap A^{\psi'_M}_{(M,n)} = \emptyset$ for every $\psi'_M : M \to \tau$ with $\psi'_M \neq \psi_M$. Thus, the proof is complete. □

Lemma 3.2.3. *The space $J(\tau)^{\mathbb{N}}_f$ is homotopy dense in $J(\tau)^{\mathbb{N}}$.*

Proof. We can take a contraction $\phi : J(\tau) \times \mathbf{I} \to J(\tau)$ such that $\phi_0 = \mathrm{id}_{J(\tau)}$ and $\phi_1(J(\tau)) = \{\mathbf{0}\}$. Then, the homotopy $h : J(\tau)^{\mathbb{N}} \times \mathbf{I} \to J(\tau)^{\mathbb{N}}$ is defined as follows: $h(x, 0) = x$ and

$$h(x,t) = (\mathrm{pr}_1(x), \cdots, \mathrm{pr}_{i-1}(x), \phi(\mathrm{pr}_i(x), 2^i t - 1), \mathbf{0}, \mathbf{0}, \cdots) \text{ for each } x \in J(\tau)^{\mathbb{N}} \text{ and } 2^{-i} \leq t \leq 2^{-i+1}.$$

It follows that $h_0 = \mathrm{id}_{J(\tau)}$ and $h(J(\tau)^{\mathbb{N}} \times (0,1]) \subset J(\tau)^{\mathbb{N}}_f$, hence $J(\tau)^{\mathbb{N}}_f$ is homotopy dense in $J(\tau)^{\mathbb{N}}$. □

Since $J(\tau)^{\mathbb{N}}$ is an AR, so $J(\tau)^{\mathbb{N}}_f$ is due to Proposition 1.2.6 and the above. Using the above lemma, we shall also show the following:

Proposition 3.2.4. *The space $J(\tau)^{\mathbb{N}}_f$ is $\mathfrak{M}^f_0(J(\tau)^{\mathbb{N}})$-absorptive.*

Proof. For the sake of convenience, let $X = J(\tau)^{\mathbb{N}}_f$, $\bar{X} = J(\tau)^{\mathbb{N}}$ and

$$X_m = \{x = (x(i))_{i\in\mathbb{N}} \in X \mid x(i) = \mathbf{0} \text{ for all } i > m\} \subset X \text{ for each } m \in \mathbb{N}.$$

Suppose that A is an finite-dimensional compact subset in $\bar{X}$, B is a closed subset of A contained in X, and $\mathcal{U}$ is an open cover of $\bar{X}$. It is sufficient to construct an embedding $\tilde{g} : A \to X$ such that $\tilde{g}$ is $\mathcal{U}$-close to id_A and $\tilde{g}|_B = \mathrm{id}_B$. We have $A \setminus B = \bigcup_{n\in\mathbb{N}} A_n$, where $A_1 \subset A_2 \subset \cdots$ are closed subsets of A, and an open cover $\mathcal{U}'$ of $\bar{X}$ such that $\mathcal{U} \succ^\star \mathcal{U}'$. Since X is homotopy dense in $\bar{X}$ due to Lemma 3.2.3, we can obtain a homotopy $\phi : \bar{X} \times \mathbf{I} \to \bar{X}$ so that $\phi_0 = \mathrm{id}_{\bar{X}}$ and $\phi(\bar{X} \times (0,1]) \subset X$. Let $k : A \to \mathbf{I}$ be a map such that $k^{-1}(0) = B$ and for each $x \in A \setminus B$, there exists $U \in \mathcal{U}'$ such that $\{x\} \times [0, k(x)] \subset \phi^{-1}(U \setminus B)$. We define the map $f : A \to X$ by $f(x) = \phi(x, k(x))$. Observe that f is $\mathcal{U}'$-close to id_A, $f|_B = \mathrm{id}_B$ and $f(A \setminus B) \subset X \setminus B$. Let $\lambda > 1$ be a Lebesgue number for $\mathcal{U}'$ with respect to $f(A)$. By the same argument of Lemma 2.1.4 in Chapter 2, we can find an open cover $\mathcal{V}$ of $X \setminus B$ of $\mathrm{mesh}\,\mathcal{V} < \lambda$ so as to satisfy the following conditions (cf. Lemma 3 of [4]):

($*$) For a map $h : f^{-1}(X \setminus B) = A \setminus B \to X \setminus B$, if $h \sim_{\mathcal{V}} f|_{A\setminus B}$, then h extends to the map $\tilde{h} : A \to X$ by $\tilde{h}|_B = \mathrm{id}_B$.

Take a sequence of open covers $\mathcal{V} \succ^\star \mathcal{V}_0 \succ^\star \mathcal{V}_1 \succ^\star \cdots$ of $X \setminus B$ of $\mathrm{mesh}\,\mathcal{V}_n < 2^{-n}$ for every $n \in \omega$. Since $X \setminus B$ is an ANR, by Proposition 1.2.5, we can choose an open cover $\mathcal{V}'_n$ of $X \setminus B$ for each $n \in \omega$ so that $\mathcal{V}_n \succ \mathcal{V}'_n$ and it has the following property:

($**$) Given a space Y and maps $h_1, h_2 : Y \to X \setminus B$, if $h_1 \sim_{\mathcal{V}'_n} h_2$, then $h_1 \simeq_{\mathcal{V}_n} h_2$.

By induction, we shall construct maps $g_n : A \setminus B \to X \setminus B$, $n \in \omega$, and a sequence of natural numbers $1 = m(0) < m(1) < \cdots$ such that

(1) $g_n|_{A_n}$ is an embedding into $X_{m(n)} \setminus B$,

(2) $g_{n+1}|_{A_n} = g_n|_{A_n}$ and

(3) $g_{n+1} \simeq_{\mathcal{V}_n} g_n$,

where $g_0 = f|_{A \setminus B}$ and $A_0 = \emptyset$. After completing the inductive construction, the sequence $\{g_n\}_{n\in\omega}$ converges to the injection $g : A \setminus B \to X \setminus B$ such that $g \sim_{\mathcal{V}} f|_{A\setminus B}$ and $g|_{A_n} = g_n|_{A_n}$ for all $n \in \omega$. Due to $(*)$, the map g is extended to the desired embedding $\tilde{g} : A \to X$ by $\tilde{g}|_B = \mathrm{id}_B$. Therefore, it remains to complete the induction.

Assume that g_j and $m(j)$ have been obtained for all $j < n$. Let $\lambda_n < 1$ be a Lebesgue number for $\mathcal{V}'_n$ with respect to $g_{n-1}(A_n)$. Then, there is a number $m(n)' \geq m(n-1)$ such that $\sum_{i>m(n)'} 2^{-i+1} < \lambda_n$. Let $m(n) = m(n)' + 2\dim(A) + 2$. Fix an unit vector $\mathbf{e}$ of $\ell_1(\tau)$. Remark that the segment $\langle \mathbf{e}/2, \mathbf{e}\rangle$ is contained in $J(\tau)$. By the finite dimensionality of A_n, there exists an embedding $q_n : A_n \to \langle \mathbf{e}/2, \mathbf{e}\rangle^{2\dim(A)+1}$. Taking a map $k_n : A_n \to \mathbf{I}$ with $k_n^{-1}(0) = A_{n-1}$, we can define the map $g'_n : A_n \to X_{m(n)} \setminus B$ as follows:

$$\mathrm{pr}_i\, g'_n(x) = \begin{cases} \mathrm{pr}_i\, g_{n-1}(x) & \text{if } i \leq m(n)', \\ k_n(x) p_{i-m'_{(n)}} q_n(x) & \text{if } m(n)' < i < m(n), \\ k_n(x)\mathbf{e} & \text{if } i = m(n), \\ \mathbf{0} & \text{if } m(n) < i, \end{cases}$$

where $p_j : \langle \mathbf{e}/2, \mathbf{e}\rangle^{2\dim(A)+1} \to \langle \mathbf{e}/2, \mathbf{e}\rangle$ is the projection onto the jth coordinate, $j = 1, \cdots, 2\dim(A)+1$. Then g'_n is an embedding. Indeed, take two distinct points $x, y \in A_n$ arbitrarily. In case $x, y \in A_{n-1}$, we have $k_n(x) = k_n(y) = 0$, so

$$g'_n(x) = g_{n-1}(x) \neq g_{n-1}(y) = g'_n(y)$$

since $g_{n-1}|_{A_{n-1}}$ is an embedding. In case $x \in A_n \setminus A_{n-1}$ and $y \in A_{n-1}$, we get $k_n(x) > 0 = k_n(y)$, hence

$$\mathrm{pr}_{m(n)}\, g'_n(x) = k_n(x)\mathbf{e} \neq \mathbf{0} = \mathrm{pr}_{m(n)}\, g'_n(y),$$

that is, $g'_n(x) \neq g'_n(y)$. In case $x, y \in A_n \setminus A_{n-1}$, we have $k_n(x), k_n(y) > 0$. When $k_n(x) \neq k_n(y)$, we see

$$\mathrm{pr}_{m(n)}\, g'_n(x) = k_n(x)\mathbf{e} \neq k_n(y)\mathbf{e} = \mathrm{pr}_{m(n)}\, g'_n(y),$$

so $g'_n(x) \neq g'_n(y)$. When $k_n(x) = k_n(y)$, there is $m(n)' < i < m(n)$ such that

$$\mathrm{pr}_i\, g'_n(x) = k_n(x)\,\mathrm{pr}_i\, q_n(x) \neq k_n(y)\,\mathrm{pr}_i\, q_n(y) = \mathrm{pr}_i\, g'_n(y)$$

because q_n is an embedding. Therefore $g'_n(x) \neq g'_n(y)$. Moreover, $g'_n|_{A_{n-1}} = g_{n-1}|_{A_{n-1}}$ because $g_{n-1}(A_{n-1}) \subset X_{m_{(n-1)}}$ and $k_n(A_{n-1}) = 0$. For every $x \in A_n$, we have

$$\begin{aligned} d(g'_n(x), g_{n-1}(x)) &= \sum_{i\in\mathbb{N}} 2^{-i} \|\mathrm{pr}_i\, g'_n(x) - \mathrm{pr}_i\, g_{n-1}(x)\|_1 \\ &\leq \sum_{i\leq m(n)'} 2^{-i} \|\mathrm{pr}_i\, g'_n(x) - \mathrm{pr}_i\, g_{n-1}(x)\|_1 + \sum_{i>m(n)'} 2^{-i+1} \\ &= \sum_{i>m(n)'} 2^{-i+1} < \lambda_n, \end{aligned}$$

hence $g'_n \sim_{\mathcal{V}'_n} g_{n-1}|_{A_n}$. By $(**)$, $g'_n \simeq_{\mathcal{V}_n} g_{n-1}|_{A_n}$. Applying the Homotopy Extension Theorem 1.2.4 to g'_n, we can obtain an extension $g_n : A \setminus B \to X \setminus B$ of g'_n such that $g_n \simeq_{\mathcal{V}_n} g_{n-1}$, which is desired. □

Now we can prove Theorem 3.0.1.

Proof of Theorem 3.0.1. Combining Theorems 3.1.4, 3.2.1, Propositions 3.2.2 and 3.2.4, we have that $(J(\tau)^{\mathbb{N}}, J(\tau)^{\mathbb{N}}_f)$ is an $(\ell_2(\tau), \ell_2^f(\tau))$-manifold pair. According to Theorem 3.1.2, $(J(\tau)^{\mathbb{N}}, J(\tau)^{\mathbb{N}}_f)$ is homeomorphic to $(\ell_2(\tau), \ell_2^f(\tau))$. □

Chapter 4

Topological types of sigma-locally compact convex sets

The topological classification of convex sets in linear spaces has been an important problem of infinite-dimensional topology. By virtue of the efforts due to V. Klee [35], T. Dobrowolski [23] and H. Toruńczyk [25, 26], the following theorem can be established, see Corollary 5.2.3 of [10].

Theorem 4.0.1. *Let C be a separable completely metrizable closed convex set in a topological linear space. Suppose that C is an AR. Then, the convex set C is homeomorphic to $[0,1]^n \times [0,1)^m \times (0,1)^k$ for some cardinals $0 \leq n, k \leq \aleph_0$ and $0 \leq m \leq 1$. In particular, if C is not locally compact, then it is homeomorphic to the separable Hilbert space ℓ_2.*

Recall that a Fréchet space is a locally convex completely metrizable linear space. According to the Dugundji Extension Theorem [28] (cf. Theorem 6.1.1 of [50]), any convex subset of a locally convex topological linear space is an AE. It is well known that every infinite-dimensional Fréchet space is homeomorphic to a Hilbert space of the same weight (the Kadec [34] -Anderson [1] -Toruńczyk [59] Theorem). T. Banakh and R. Cauty [9] extended Theorem 4.0.1 to non-separable convex sets in Fréchet spaces as follows:

Theorem 4.0.2. *Let C be a closed convex set in a Fréchet space. Then, the convex set C is homeomorphic to $[0,1]^n \times [0,1)^m \times \ell_2(\tau)$ for some cardinals $0 \leq n \leq \aleph_0$, $0 \leq m \leq 1$ and $0 \leq \tau$. In particular, if C is not locally compact, then it is homeomorphic to a Hilbert space of the same weight.*

By ℓ_2^Q, we denote the linear span of $\prod_{n\in\mathbb{N}}[-2^{-n}, 2^{-n}]$ in ℓ_2. Remark that the pair (ℓ_2, ℓ_2^Q) is homeomorphic to $(\ell_2 \times \mathbf{Q}, \ell_2^f \times \mathbf{Q})$. D. Curtis, T. Dobrowolski, and J. Mogilski [22] studied on when σ-compact convex sets in a topological linear space is homeomorphic to the linear subspaces ℓ_2^f or ℓ_2^Q of the separable Hilbert space ℓ_2. They established the following theorem:

Theorem 4.0.3. *Let C be a σ-compact convex set in a completely metrizable linear space E. Suppose that the closure $\mathrm{cl}_E\, C$ is an AR and not locally compact. Then, the pair $(\mathrm{cl}_E\, C, C)$ is homeomorphic to (ℓ_2, ℓ_2^f) if C is strongly countable-dimensional, and $(\mathrm{cl}_E\, C, C)$ is homeomorphic to (ℓ_2, ℓ_2^Q) if C contains an infinite-dimensional locally compact convex set.*

Due to T. Dobrowolski [24] and T. Banakh [8], the above second assertion is strengthened as follows:

Theorem 4.0.4. *Suppose that C is a σ-compact convex set in a completely metrizable linear space E, whose closure $\mathrm{cl}_E\, C$ is an AR and not locally compact. If C contains a topological copy Q of the Hilbert cube having infinite codimension in C, then $(\mathrm{cl}_E\, C, C)$ is homeomorphic to (ℓ_2, ℓ_2^Q) .*

For two subsets $A \subset B$ of a linear space, we shall say that A has *infinite codimension* in B if the linear hull of A has infinite codimension in the linear hull of B.

Remark 8. In the second assertion of Theorem 4.0.3, the convex set C contains an infinite-dimensional compact convex set Q homeomorphic to the Hilbert cube, see Proposition 3.5 of [22] and Theorem 3.1 in Chapter III of [12]. Then it has infinite codimension in C, refer to Lemma 3.3 and Proposition 3.4 of [22].

The aim of this chapter is to extend the above theorems to non-separable convex sets in Fréchet spaces.

Main Theorem. *Let C be a σ-locally compact convex set of weight $\tau > \aleph_0$ in a Fréchet space F. Then the pair $(\mathrm{cl}_F C, C)$ is homeomorphic to $(\ell_2(\tau), \ell_2^f(\tau))$ if and only if C is strongly countable-dimensional, and $(\mathrm{cl}_F C, C)$ is homeomorphic to $(\ell_2(\tau) \times \mathbf{Q}, \ell_2^f(\tau) \times \mathbf{Q})$ if and only if C contains a topological copy of the Hilbert cube $\mathbf{Q}$.*

Remark 9. In the above theorem, we have $C \neq \mathrm{cl}_F C$. Indeed, by Proposition 2.4.1 in Chapter 2, we can write $C = \bigcup_{n<\aleph_0} C_n$, where each C_n is a closed locally compact set in C. According to Proposition 3.1 of [22], each compact subset of C is a Z-set in C. Since every Z-set is nowhere dense, for any $n < \aleph_0$, the closed subset C_n is nowhere dense in C. Therefore, the convex set C is of first category (, in fact, it is a Z_σ-set), which means that $C \neq \mathrm{cl}_F C$.

4.1 Proof of Main Theorem

This section is devoted to proving the main theorem. We shall use the modified West's Characterization Theorem 3.1.4 and the Classification Theorem 3.1.2 in Chapter 3.

Proof of Main Theorem. The "only if" part in the both statements are trivial. We shall show the "if" parts. According to Theorem 4.0.2, the closure $\mathrm{cl}_F C$ is homeomorphic to the Hilbert space $\ell_2(\tau)$. Now we consider two cases.

(1) First, assume that the convex set C is strongly countable-dimensional. By Theorems 3.1.4 and 3.1.2, the homeomorphism of the pairs $(\mathrm{cl}_F C, C)$ and $(\ell_2(\tau), \ell_2^f(\tau))$ will follow as soon as we check that C absorbs finite-dimensional compact subsets of $\mathrm{cl}_F C$. Fix a finite-dimensional compact subset $A \subset \mathrm{cl}_F C$, a closed subset B of A contained in C, and an open cover $\mathcal{U}$ of $\mathrm{cl}_F C$. By the density of C in $\mathrm{cl}_F C$ and the separability of A, there is a separable convex subset $D \subset C$ such that $B \subset D$ and $A \subset \mathrm{cl}_F D$. Moreover, using the fact that C is not separable, we can choose D so that the closure $\mathrm{cl}_F D$ is not locally compact. By Theorem 4.0.3, the pair $(\mathrm{cl}_F D, \mathrm{cl}_F D \cap C)$ is homeomorphic to (ℓ_2, ℓ_2^f), and hence by Theorem 3.1.4, the set $\mathrm{cl}_F D \cap C$ absorbs finite-dimensional compact subsets of $\mathrm{cl}_F D$. Consequently, for the finite-dimensional compact subset $A \subset \mathrm{cl}_F D \subset \mathrm{cl}_F C$, there is an embedding $f : A \to \mathrm{cl}_F D \cap C \subset C$ such that f is $\mathcal{U}$-close to id_A and $f|_B = \mathrm{id}_B$. This means that C absorbs finite-dimensional compact subsets of $\mathrm{cl}_F C$. Therefore the pair $(\mathrm{cl}_F C, C)$ is homeomorphic to $(\ell_2(\tau), \ell_2^f(\tau))$.

(2) Next, assume that C contains a subspace $Q \subset C$ homeomorphic to the Hilbert cube. Similarly, according to Theorem 3.1.4 and 3.1.2, the homeomorphism of the pairs $(\mathrm{cl}_F C, C)$ and $(\ell_2(\tau)\times\mathbf{Q}, \ell_2^f(\tau)\times\mathbf{Q})$ will follow as soon as we check that C absorbs compact subsets of $\mathrm{cl}_F C$. Take any compact subset $A \subset \mathrm{cl}_F C$, any closed subset B of A contained in C, and any open cover $\mathcal{U}$ of $\mathrm{cl}_F C$. Using the density of C in $\mathrm{cl}_F C$ and the separability of the compact set $A \cup Q$, we can find a separable convex subset $D \subset C$ such that $Q \cup B \subset D$ and $A \subset \mathrm{cl}_F D$. Then we may assume that $D = \mathrm{cl}_F D \cap C$. Since C is not separable, the compact set Q has infinite codimension in C. So we can choose D to be so large that Q has infinite codimension in D and $\mathrm{cl}_F D$ is not locally compact. Since C is σ-locally compact and D is separable, the convex set $D = \mathrm{cl}_F D \cap C$ is σ-compact. Since the topological copy Q of the Hilbert cube has infinite codimension in D, the pair $(\mathrm{cl}_F D, D)$ is homeomorphic to (ℓ_2, ℓ_2^Q) by Theorem 4.0.4. Due to Theorem 3.1.4, the convex set D absorbs compact subsets of $\mathrm{cl}_F D$. In particular, for the compact subset $A \subset \mathrm{cl}_F D$, there is an embedding $f : K \to D \subset C$ such that f is $\mathcal{U}$-close to id_A and $f|_B = \mathrm{id}_B$.

This implies that C absorbs compact subsets of $\mathrm{cl}_F C$. Consequently, $(\mathrm{cl}_F C, C)$ is homeomorphic to $(\ell_2(\tau) \times \mathbf{Q}, \ell_2^f(\tau) \times \mathbf{Q})$. This completes the proof. □

We do not know if the condition on Q to have infinite codimension in C in Theorem 4.0.4 can be omitted.

Probrem 2. Assume that a subset A of a Fréchet space is homeomorphic to the Hilbert cube $\mathbf{Q}$. Does A contain a subset B, which is homeomorphic to the Hilbert cube and has infinite codimension in A?

4.2 Infinite-dimensional convex sets in Fréchet spaces

In the proof of the main theorem, we show any strongly countable-dimensional, σ-locally compact convex set C in a Fréchet space F is weakly $\mathfrak{M}_0^f(\mathrm{cl}_F C)$-absorptive. In fact, each infinite-dimensional convex subset of a Fréchet space absorbs finite-dimensional compact subsets of its closure. For a subset K of a linear space, we denote the convex hull of K by $\mathrm{conv}(K)$ and the flat hull of K by $\mathrm{fl}(K)$. By the same argument of Lemma 3.2 in [22], we can show the following lemma:

Lemma 4.2.1. *Let F be a Fréchet space and D be an infinite-dimensional convex set in F. Suppose that A is a compact metrizable space, B is a closed subset of A, and $f : A \to \mathrm{cl}_F D$ is a map such that $f(B) \subset \mathrm{conv}(K)$ for some $K \subset D$. Then for each open cover $\mathcal{U}$ of $\mathrm{cl}_F D$, there exists a map $g : A \to D$ and a finite subset $L \subset D$ such that g is $\mathcal{U}$-close to f, $g|_B = f|_B$, and $g(A) \subset \mathrm{conv}(K \cup L)$.*

Proof. Fix an admissible F-norm $\|\cdot\|$ on F. Since A is a compact metrizable space, we can regard $A \subset \mathbf{I}^{\mathbb{N}}$. It follows from the Dugundji Extension Theorem that the convex set $\mathrm{cl}_F D$ is an AR, and hence the map f extends to a map $\tilde{f} : \mathbf{I}^{\mathbb{N}} \to \mathrm{cl}_F D$. We use an admissible metric d on $\mathbf{I}^{\mathbb{N}}$ defined as follows:

$$d(x,y) = \sum_{i \in \mathbb{N}} 2^{-i}|x(i) - y(i)| \text{ for each } x = (x(i))_{i\in\mathbb{N}}, y = (y(i))_{i\in\mathbb{N}} \in \mathbf{I}^{\mathbb{N}}.$$

Let $\epsilon > 0$ be a Lebesgue number of $\mathcal{U}$ with respect to $f(A)$. Take $\delta > 0$ so that for all $x, y \in \mathbf{I}^{\mathbb{N}}$, if $d(x,y) < \delta$, then $\|\tilde{f}(x) - \tilde{f}(y)\| < \epsilon/4$. Then we can choose $n \in \mathbb{N}$ such that the nth coordinate projection $p : \mathbf{I}^{\mathbb{N}} \to \mathbf{I}^n \times \{0\}$ is δ-close to $\mathrm{id}_{\mathbf{I}^{\mathbb{N}}}$, where $p(x) = (x(1), \cdots, x(n), 0, \cdots)$ for each $x \in \mathbf{I}^{\mathbb{N}}$. Note that $\tilde{f}p$ is $\epsilon/4$-close to $\tilde{f}$.

Since $p(A)$ is a finite-dimensional compact metric space of dimension $\leq n$, it has a finite open cover $\mathcal{V}$ of order $\leq n+1$ such that for all $x, y \in p(A)$, if some $V \in \mathcal{V}$ contains both x and y, then $\|\tilde{f}(x) - \tilde{f}(y)\| < \epsilon/(8(n+1))$. Take a nerve N of $\mathcal{V}$ and a canonical map $\phi : p(A) \to |N|$. Then we can choose $x_V \in V$ and $\psi(V) \in D$ for each $V \in N^{(0)} = \mathcal{V}$ so that $\|\psi(V) - \tilde{f}(x_V)\| < \epsilon/(8(n+1))$. Let $L = \{\psi(V) \in D \mid V \in N^{(0)}\}$, which is the desired finite subset. The choice ψ extends to the affine map $\tilde{\psi} : |N| \to \mathrm{conv}(L)$. Then $\tilde{\psi}\phi$ is $\epsilon/4$-close to $\tilde{f}|_{p(A)}$. Indeed, fix any $x \in p(A)$, so we can write $\phi(x) = \sum_{x \in V \in \mathcal{V}} t_V V \in |N|$, where $t_V \in \mathbf{I}$ and $\sum_{x\in V\in\mathcal{V}} t_V = 1$. Then we have

$$\begin{aligned}
\|\tilde{\psi}\phi(x) - \tilde{f}(x)\| &= \|\tilde{\psi}(\sum_{x\in V\in\mathcal{V}} t_V V) - \tilde{f}(x)\| = \|\sum_{x\in V\in\mathcal{V}} t_V \tilde{\psi}(V) - \tilde{f}(x)\| \\
&\leq \sum_{x\in V\in\mathcal{V}} \|t_V(\tilde{\psi}(V) - \tilde{f}(x))\| \leq \sum_{x\in V\in\mathcal{V}} \|\tilde{\psi}(V) - \tilde{f}(x)\| \\
&\leq \sum_{x\in V\in\mathcal{V}} (\|\tilde{\psi}(V) - \tilde{f}(x_V)\| + \|\tilde{f}(x_V) - \tilde{f}(x)\|) \\
&< (n+1)(\epsilon/(8(n+1)) + \epsilon/(8(n+1))) = \epsilon/4.
\end{aligned}$$

Hence $\tilde{\psi}\phi p|_A$ is $\epsilon/2$-close to f.

On the other hand, the restriction $f|_B$ extends to a map $\bar{f} : A \to \mathrm{conv}(K)$ because $\mathrm{conv}(K)$ is an AR. Taking a map $k : A \to \mathbf{I}$ such that $k(B) = \{0\}$ and $\{x \in A \mid \|\bar{f}(x) - f(x)\| \geq \epsilon/2\} \subset k^{-1}(1)$, we can define a map $g : A \to \mathrm{conv}(K \cup L)$ as follows:

$$g(x) = (1 - k(x))\bar{f}(x) + k(x)\tilde{\psi}\phi p(x).$$

Then g is the desired map. $\square$

The following proposition is the non-separable version of Propositions 2.2 and 3.4 in [22].

Proposition 4.2.2. *Let D be an infinite-dimensional convex set in a Fréchet space F. Then, D is weakly $\mathfrak{M}_0^f(\mathrm{cl}_F\, D)$-absorptive.*

Proof. We use an admissible metric d on F. For simplicity, denote $\mathrm{cl}_F\, D$ by $\overline{D}$. Let A be a finite-dimensional compact set in $\overline{D}$, B be a closed subset of A with $B \subset D$, and $\mathcal{U}$ be an open cover of $\overline{D}$. We shall construct an embedding $f : A \to D$ such that f is $\mathcal{U}$-close to id_A and $f|_B = \mathrm{id}_B$. According to Lemma 3 of [4], we can obtain an open refinement $\mathcal{V}$ of $\{U \cap \overline{D} \setminus B \mid U \in \mathcal{U}\}$ that has the following property:

(∗) For every map $h : A \setminus B \to \overline{D} \setminus B$, if h is $\mathcal{V}$-close to $\mathrm{id}_{A \setminus B}$, then h extends to the map $\tilde{h} : A \to \overline{D}$ by $\tilde{h}|_B = \mathrm{id}_B$.

Then, the space $\overline{D} \setminus B$ has a sequence of open covers $\mathcal{V} \succ^\star \mathcal{V}_0 \succ^\star \mathcal{V}_1 \succ^\star \cdots$ of $\mathrm{mesh}\, \mathcal{V}_n < 2^{-n}$ for each $n < \aleph_0$. It follows from the Dugundji Extension Theorem that $\overline{D} \setminus B$ is an ANR. Due to Proposition 1.2.5, we can choose open covers $\mathcal{V}'_n$ and $\mathcal{V}''_n$ of $\overline{D} \setminus B$ for each $n < \aleph_0$ so that $\mathcal{V}''_n \prec \mathcal{V}'_n \,{}^\star\!\!\prec \mathcal{V}_n$ and they satisfy the following condition:

(∗∗) Given a space Y and maps $h_1, h_2 : Y \to \overline{D} \setminus B$, if h_1 is $\mathcal{V}''_n$-close to h_2, then h_1 is $\mathcal{V}'_n$-homotopic to h_2.

We can write $A \setminus B = \bigcup_{n \in \mathbb{N}} A_n$ so that $A_1 \subset A_2 \subset \cdots$ are closed sets in A. Now, we shall inductively construct maps $f_n : A \setminus B \to \overline{D} \setminus B$, $n < \aleph_0$, and a tower of finite subsets $\emptyset = D_0 \subset D_1 \subset \cdots \subset D$ such that

(1) $f_n|_{A_n}$ is an embedding into $\mathrm{conv}(D_n) \setminus B$,

(2) $f_{n+1}|_{A_n} = f_n|_{A_n}$, and

(3) f_{n+1} is $\mathcal{V}_n$-close to f_n,

where $f_0 = \mathrm{id}_{A \setminus B}$ and $A_0 = \emptyset$. Assume that f_{n-1} and D_{n-1} have been obtained. Applying Lemma 4.2.1, we have a map $g : A_n \to D$ and a finite subset $L \subset D$ such that g is $\mathcal{V}''_{n-1}$-close to $f_{n-1}|_{A_n}$, $g|_{A_{n-1}} = f_{n-1}|_{A_{n-1}}$, and $g(A_n) \subset \mathrm{conv}(D_{n-1} \cup L)$. Note that $g(A_n) \cap B = \emptyset$ and g is $\mathcal{V}'_{n-1}$-homotopic to $f_{n-1}|_{A_n}$ by (∗∗). Moreover, we can find a map $k : A_n \to \mathbf{I}^{2\dim A_n + 2}$ such that $A_{n-1} = k^{-1}(\mathbf{0})$ and $k|_{A_n \setminus A_{n-1}}$ is an embedding (cf. Lemma 5.9.1 of [50]). Since D is infinite-dimensional, we can choose a subset $L' \subset D$ consisting of $2\dim A_n + 2$ points so that L' is affinely independent and $L' \cap \mathrm{fl}(D_{n-1} \cup L) = \emptyset$. Let $D_n = D_{n-1} \cup L \cup L'$. Then, there exists an embedding

$$i : \mathrm{conv}(D_{n-1} \cup L) \times \mathbf{I}^{2\dim A_n + 2} \to \mathrm{conv}(D_n)$$

such that i is $\mathcal{V}''_{n-1}$-close to the projection onto the first coordinate and $i(x, \mathbf{0}) = x$ for all $x \in \mathrm{conv}(D_{n-1} \cup L)$. Define $g' : A_n \to \mathrm{conv}(D_n)$ by $g'(x) = i(g(x), k(x))$ for each $x \in A_n$. Observe that g' is $\mathcal{V}''_{n-1}$-close to g, an extension of $f_{n-1}|_{A_{n-1}}$, and an embedding into $\mathrm{conv}(D_n) \setminus B$. By (∗∗), we have g' is $\mathcal{V}'_{n-1}$-homotopic to g, so g' is $\mathcal{V}_{n-1}$-homotopic to $f_{n-1}|_{A_n}$. Since $\overline{D} \setminus B$ is an ANR, due to the Homotopy Extension

Theorem 1.2.4, the embedding g' extends to a map $f_n : A \setminus B \to \overline{D} \setminus B$ such that f_n is $\mathcal{V}_{n-1}$-homotopic to f_{n-1}, which is the required map.

Due to conditions (2) and (3), and mesh $\mathcal{V}_n < 2^{-n}$ for each $n < \aleph_0$, the sequence $\{f_n\}_{n<\aleph_0}$ converges to a map $h : A \setminus B \to \overline{D} \setminus B$. Then, $h|_{A_n} = f_n|_{A_n}$ for all $n < \aleph_0$, so $h(A \setminus B) \subset D \setminus B$ and h is $\mathcal{V}$-close to $\mathrm{id}_{A\setminus B}$. It follows from $(*)$ that h extends to the map $f : A \to D$ by $f|_B = \mathrm{id}_B$. By condition (1), the map f is an embedding. It is clear that f is $\mathcal{U}$-close to id_A. Consequently, f is the desired embedding. $\square$

4.3 An application

A *full simplicial complex* K is a simplicial complex such that any finite vertices of K spans a simplex of K. We denote the full simplicial complex with the cardinality of vertices an infinite cardinal τ by $\Delta(\tau)$. The following assertion was proved by K. Sakai in 1987 (cf. Proposition 4.1 of [47]).

Proposition 4.3.1. *The metric polyhedron $|\Delta(\aleph_0)|_m$ is homeomorphic to ℓ_2^f.*

For each infinite cardinal τ, the metric polyhedron $|\Delta(\tau)|_m$ is a convex set in the Fréchet space $\ell_1(\tau)$ and it is strongly countable-dimensional and σ-locally compact due to the following proposition.

Proposition 4.3.2. *For any simplicial complex K, the metric polyhedron $|K|_m$ is a countable union of closed sets which are discrete unions of finite-dimensional compact metrizable spaces.*

Proof. For each simplex $\sigma \in K$, let $\hat{\sigma}$ and $\partial\sigma$ be the barycenter and the boundary of σ, respectively. Given $\sigma \in K \setminus K^{(0)}$ and $t \in \mathbf{I}$,

$$\sigma[t] = \{(1-s)\hat{\sigma} + sx \mid x \in \partial\sigma, 0 \leq s \leq t\}$$

is a closed subset of σ. Let $\mathcal{A}_0 = K^{(0)}$ and $\mathcal{A}_n = \{\sigma[1-2^{-n}] \mid \sigma \in K^{(n)} \setminus K^{(0)}\}$ for all $n \in \mathbb{N}$, so $\mathcal{A}_n$ is a discrete collection of finite-dimensional compact metrizable spaces in $|K|_m$. Then $|K|_m = \bigcup_{n\in\omega}(\bigcup \mathcal{A}_n)$. Consequently, the assertion holds. $\square$

Applying the main theorem, we can generalize Proposition 4.3.1 as follows:

Corollary 4.3.3. *For every infinite cardinal τ, the pair $(\mathrm{cl}_{\ell_1(\tau)} |\Delta(\tau)|_m, |\Delta(\tau)|_m)$ is homeomorphic to $(\ell_2(\tau), \ell_2^f(\tau))$.*

Chapter 5

A function space from a Peano space into a one-dimensional locally compact absolute retract and its compactification

Throughout this chapter, spaces are assumed to be regular. Given spaces X and Y, we denote by $\mathrm{C}(X,Y)$ the space of all maps from X to Y with the *compact-open topology*, that is, the topology of $\mathrm{C}(X,Y)$ is generated by the following set

$$\{f \in \mathrm{C}(X,Y) \mid K \text{ is a compact set in } X,\ U \text{ is an open set in } Y,\ f(K) \subset U\}.$$

When X is locally compact and σ-compact, and Y is metrizable, the space $\mathrm{C}(X,Y)$ is metrizable. In the paper [36], it was shown that if X is an infinite, locally compact, locally connected, separable metrizable space, then $\mathrm{C}(X,\mathbb{R})$ has a natural compactification $\overline{\mathrm{C}(X,\mathbb{R})}$ such that the pair $(\overline{\mathrm{C}(X,\mathbb{R})},\mathrm{C}(X,\mathbb{R}))$ is homeomorphic to $(\mathbf{Q},\mathbf{s})$ (cf. the compact case was proved in [51]). We shall generalize this result by replacing $\mathbb{R}$ with a 1-dimensional locally compact AR as follows:

Main Theorem. *Let X be an infinite, locally compact, locally connected, separable metrizable space, and let Y be a 1-dimensional locally compact AR. Suppose that X is non-discrete or Y is non-compact. Then the function space $\mathrm{C}(X,Y)$ has a natural compactification $\overline{\mathrm{C}(X,Y)}$ such that the pair $(\overline{\mathrm{C}(X,Y)},\mathrm{C}(X,Y))$ is homeomorphic to $(\mathbf{Q},\mathbf{s})$.*

Remark 10. In the main theorem, when X is discrete and Y is compact, the function space $\mathrm{C}(X,Y)$ is the product space Y^X, and hence it is homeomorphic to $\mathbf{Q}$ due to Toruńczyk's characterization of the Hilbert cube (Corollary 1.3.3 of Chapter 1, cf. [42, Corollary 8.1.2]).

The *Fell topology* on a hyperspace $\mathrm{Cld}^*(X)$ of closed sets in a space X is generated by the following collection

$$\{U^- \mid U \text{ is an open set in } X\} \cup \{(X \setminus K)^+ \mid K \text{ is a compact set in } X\},$$

and the space $\mathrm{Cld}^*(X)$ with the Fell topology is denoted by $\mathrm{Cld}^*_F(X)$. In the case X is compact, the Fell topology on $\mathrm{Cld}^*(X)$ coincides with the Vietoris topology and the empty set $\emptyset$ is an isolated point of $\mathrm{Cld}^*_F(X)$. It is known that $\mathrm{Cld}^*_F(X)$ is compact metrizable if and only if X is locally compact separable metrizable, see Theorem 5.1.5 of [11]. When X is a locally compact, locally connected space, and Y is a locally compact space, the function space $\mathrm{C}(X,Y)$ can be regarded as a subspace of the hyperspace $\mathrm{Cld}^*_F(X \times Y)$, where each $f \in \mathrm{C}(X,Y)$ is identified with the graph of f in $X \times Y$, refer to Lemma 2.1 of [36]. Thus, if X is locally compact, locally connected, separable metrizable, and Y is locally compact separable metrizable, then the closure $\mathrm{cl}_{\mathrm{Cld}^*_F(X\times Y)}\mathrm{C}(X,Y)$ of $\mathrm{C}(X,Y)$ in $\mathrm{Cld}^*_F(X \times Y)$ is a metrizable compactification of $\mathrm{C}(X,Y)$. In [36], the closure $\mathrm{cl}_{\mathrm{Cld}^*_F(X\times\overline{\mathbb{R}})}\mathrm{C}(X,\mathbb{R})$ was the desired compactification $\overline{\mathrm{C}(X,\mathbb{R})}$, where $\overline{\mathbb{R}} = [-\infty,\infty]$ is the extended real line:

Theorem 5.0.1. *Let X be an infinite, locally compact, locally connected, separable metrizable space. Then the pair* $(\mathrm{cl}_{\mathrm{Cld}^*_F(X\times\overline{\mathbb{R}})} \mathrm{C}(X,\mathbb{R}), \mathrm{C}(X,\mathbb{R}))$ *is homeomorphic to* $(\mathbf{Q},\mathbf{s})$.

We will prove that a space Y is a 1-dimensional locally compact AR if and only if Y has a dendrite compactification $\widetilde{Y}$ such that the remainder $\widetilde{Y}\setminus Y$ is closed and contained in the set of all end points of $\widetilde{Y}$ (Theorem 5.4.2). Taking a dendrite compactification $\widetilde{Y}$ of Y as above, we will adopt the closure $\mathrm{cl}_{\mathrm{Cld}^*_F(X\times\widetilde{Y})} \mathrm{C}(X,Y)$ as the compactification $\overline{\mathrm{C}(X,Y)}$ in the main theorem.

We denote the set consisting of all subsets of a space Y by $\mathrm{P}(Y)$. A set-valued function $\phi : X \to \mathrm{P}(Y)$ is said to be *upper semi-continuous* (briefly, u.s.c.) if $\phi^{-1}(U^+) = \{x\in X \mid \phi(x)\subset U\}$ is an open subset of X for every open subset U of Y. Let

$$\mathrm{USCC}(X,Y) = \{\phi : X \to \mathrm{Cld}(Y) \mid \phi \text{ is u.s.c. and } \phi(x) \text{ is connected for every } x\in X\}.$$

Due to Lemma 3.1 of [36], identifying each $\phi\in \mathrm{USCC}(X,Y)$ with the graph of ϕ, we can regard $\mathrm{USCC}(X,Y)$ as a subspace of $\mathrm{Cld}^*_F(X\times Y)$. Under our assumption, choosing a dendrite compactification $\widetilde{Y}$ of Y as above, we can show that if X is connected, then the closure $\mathrm{cl}_{\mathrm{Cld}^*_F(X\times\widetilde{Y})} \mathrm{C}(X,Y)$ coincides with $\mathrm{USCC}(X,\widetilde{Y})$ (Theorem 5.2.1). In Section 5.6, we will show that X is locally compact and locally connected if the above space $\mathrm{USCC}(X,\widetilde{Y})$ is homeomorphic to $\mathbf{Q}$, which is the converse of Main Theorem.

As mentioned in Proposition 6.3 of [36], the pair $(\mathrm{cl}_{\mathrm{Cld}^*_F(\mathbf{I}\times\mathbb{R})} \mathrm{C}(\mathbf{I},\mathbb{R}), \mathrm{C}(\mathbf{I},\mathbb{R}))$ is not homeomorphic to $(\mathbf{Q},\mathbf{s})$. In fact, the space $\mathrm{C}(\mathbf{I},\mathbb{R})$ is not homotopy dense in the closure $\mathrm{cl}_{\mathrm{Cld}^*_F(\mathbf{I}\times\mathbb{R})} \mathrm{C}(\mathbf{I},\mathbb{R})$. Even if we take the one-point compactification $\widetilde{Y}$, the above closure is not necessarily the desired compactification (Proposition 5.7.1). The n-dimensional Euclidean space $\mathbb{R}^n$ is a typical space that is a n-dimensional locally compact AR. It has a compactification $\overline{\mathbb{R}^n}$ that is homeomorphic to the n-dimensional closed unit ball. For each locally compact separable metrizable space X, the function space $\mathrm{C}(X,\mathbb{R}^n)$ is homeomorphic to Hilbert space ℓ_2 (Theorem 5.5.4). However, the pair $(\mathrm{cl}_{\mathrm{Cld}^*_F(X\times\overline{\mathbb{R}^n})} \mathrm{C}(X,\mathbb{R}^n), \mathrm{C}(X,\mathbb{R}^n))$ is not necessarily homeomorphic to $(\mathbf{Q},\mathbf{s})$ when $n\geq 2$. In fact, if X is the unit $(n-1)$-sphere, then $\mathrm{C}(X,\mathbb{R}^n)$ is not homotopy dense in $\mathrm{cl}_{\mathrm{Cld}^*_F(X\times\overline{\mathbb{R}^n})} \mathrm{C}(X,\mathbb{R}^n)$ (Proposition 5.7.2).

5.1 A convex structure on a dendrite

The standard unit simplex of dimension $n-1$ in $\mathbb{R}^n$ is denoted by P_n, namely

$$P_n = \left\{ t = (t(i))_{i=1}^n \in \mathbb{R}^n \;\middle|\; 0\leq t(i)\leq 1, \sum_{i=1}^n t(i) = 1 \right\}.$$

E. Michael [41] (cf. [46, Part B, Definitions 4.9 and 4.10]) introduced the convexity to subsets of metric spaces as follows:

Definition 1. A *convex structure* on a metric space $X=(X,d)$ is a sequence $(M_n,k_n)_{n\in\mathbb{N}}$ of pairs of subsets $M_n\subset X^n$ and functions $k_n : M_n\times P_n \to X$ such that the following conditions hold:

(1) If $x\in M_1$, then $k_1(x,1)=x$;

(2) If $x\in M_n$, $n\geq 2$, and $1\leq i\leq n$, then $\partial_i x\in M_{n-1}$ and $k_n(x,t) = k_{n-1}(\partial_i x, \partial_i t)$ for any $t\in P_n$ with $t(i)=0$, where ∂_i is the operator of forgetting the ith coordinate;

(3) If $x\in M_n$, $n\geq 2$, with $x(i)=x(i+1)$ for some $1\leq i<n$ and $t\in P_n$, then

$$k_n(x,t) = k_{n-1}(\partial_i x, (t(1),\cdots,t(i-1), t(i)+t(i+1), t(i+2),\cdots,t(n)));$$

(4) For each $n \in \mathbb{N}$ and each $x \in M_n$, the function $k_n(x, *) : P_n \ni t \mapsto k_n(x, t) \in X$ is continuous;

(5) For each $\epsilon > 0$, there exists a neighborhood U of the diagonal in $X \times X$ such that for every $n \in \mathbb{N}$ and every $x, y \in M_n$, if $(x(i), y(i)) \in U$ for all $1 \leq i \leq n$, then

$$d(k_n(x, t), k_n(y, t)) < \epsilon \text{ for all } t \in P_n.$$

Then a subset $C \subset X$ is said to be *convex* with respect to $(M_n, k_n)_{n \in \mathbb{N}}$ if $C^n \subset M_n$ and $k_n(C^n \times P_n) \subset C$ for every $n \in \mathbb{N}$.

It is said that a set-valued function $\phi : X \to \mathrm{P}(Y)$ is *lower semi-continuous* (briefly, l.s.c.) if $\phi^{-1}(U^-) = \{x \in X \mid \phi(x) \cap U \neq \emptyset\}$ is open in X for every open subset U of Y. A *continuous selection* for a set-valued function $\phi : X \to \mathrm{P}(Y)$ is a map (i.e., a continuous function) $f : X \to Y$ such that $f(x) \in \phi(x)$ for every $x \in X$. Michael [41] (cf. [46, Part B, Theorem 4.11]) established the selection theorem for metric spaces with convex structures as follows:

Theorem 5.1.1. *Let X be a paracompact space and $Y = (Y, d)$ a metric space with a convex structure $(M_n, k_n)_{n \in \mathbb{N}}$. For each l.s.c. set-valued function $\phi : X \to \mathrm{Cld}(Y)$, if each $\phi(x)$ is complete with respect to d and convex with respect to $(M_n, k_n)_{n \in \mathbb{N}}$, then ϕ has a continuous selection.* □

Michael [41] (cf. [46, Part B, Definition 4.12 and Theorem 4.13]) defined also geodesic structures on metric spaces, which can inductively generate convex structures.

Definition 2. A *geodesic structure* on a metric space $X = (X, d)$ is a pair (M, k) of a subset $M \subset X^2$ and a function $k : M \times \mathbf{I} \to X$ satisfying the following conditions:

(1) If $(x, x) \in M$, then $k((x, x), t) = x$ for all $t \in \mathbf{I}$;

(2) If $(x_1, x_2) \in M$, then $k((x_1, x_2), 0) = x_1$ and $k((x_1, x_2), 1) = x_2$;

(3) For each $(x_1, x_2) \in M$ and each $t \in \mathbf{I}$, if $((k((x_1, x_2), t), x_2) \in M$, then

$$k((k((x_1, x_2), t), x_2), s) = k((x_1, x_2), t + s(1 - t)) \text{ for all } s \in \mathbf{I};$$

(4) For each $x \in M$, the function $k(x, *) : \mathbf{I} \ni t \mapsto k(x, t) \in X$ is continuous;

(5) For each $\epsilon > 0$, there exist neighborhoods $V \subset U$ of the diagonal in $X \times X$ such that $(x, y) \in U$ implies that $d(x, y) < \epsilon$, and for every $(x_1, y_1), (x_2, y_2) \in M$, if $(x_1, x_2) \in U$ and $(y_1, y_2) \in V$, then

$$(k((x_1, y_1), t), k((x_2, y_2), t)) \in U \text{ for all } t \in \mathbf{I}.$$

Then it is said that a subset $G \subset X$ is *geodesic* with respect to (M, k) if $G^2 \subset M$ and $k(G^2 \times \mathbf{I}) \subset G$.

Proposition 5.1.2. *If a metric space has a geodesic structure, then it has a convex structure with respect to which every geodesic set is convex.* □

Remark 11. It is easy to see that the functions k_n, $n \in \mathbb{N}$, and k in Definitions 1 and 2 are continuous because of the conditions (4) and (5) of each definition.

We will prove that dendrites have convex structures.

Proposition 5.1.3. *Every dendrite $D = (D, d)$ with a convex metric has a convex structure (D^n, k_n) with respect to which any connected subset of it is convex.*

Proof. Due to Proposition 5.1.2, it is sufficient to show that D has a geodesic structure such that every connected subset is geodesic. Let $\gamma : D^2 \times \mathbf{I} \to D$ be the map as in Lemma 1.5.1 of Chapter 1. We shall first show that (D^2, γ) is a geodesic structure of D, that is, it satisfies the conditions (1), (2), (3), (4) and (5) of Definition 2. Clearly, the conditions (1), (2) and (4) are satisfied from the definition. By the property of d and the uniqueness of the arcs in D, the condition (3) holds. To check the condition (5), for each $\epsilon > 0$ choose a neighborhood $U = \{(x,y) \in D^2 \mid d(x,y) < \epsilon\}$ of the diagonal in D^2. Then the condition (†) of Lemma 1.5.1 implies the condition (5). Consequently, the pair (D^2, γ) is a geodesic structure.

It remains to show that if C is a connected subset in D, then C is geodesic. Indeed, for any $x, y \in C$ and $t \in \mathbf{I}$, we have $\gamma(x,y,t) = \gamma_{x,y}(t) \in C$ since C is arcwise connected from Fact 2(2) and $\gamma_{x,y}(\mathbf{I})$ is the unique arc between x and y from Fact 2(1). Hence C is geodesic. □

5.2 The closure of $\mathrm{C}(X, D)$ in $\mathrm{Cld}_F^*(X \times D)$

In this section, we shall show that the result in Theorem 4.1 of [36] (cf. [31, Theorem 1.10]) is valid for dendrites, that is,

Theorem 5.2.1. *For each locally compact, locally connected, paracompact space X with no isolated points and each dendrite D, the closure $\mathrm{cl}_{\mathrm{Cld}_F^*(X\times D)}\,\mathrm{C}(X,D)$ of $\mathrm{C}(X,D)$ coincides with $\mathrm{USCC}(X,D)$.*

For each $A \subset X \times Y$ and each $x \in X$, let $A(x) = \{y \in Y \mid (x,y) \in A\}$. When Y is compact, due to Proposition 3.1 of [36], A is closed in $X \times Y$ if and only if the set-valued function $A : X \ni x \mapsto A(x) \in \mathrm{P}(Y)$ is u.s.c. First, we shall extend Lemma 4.1 of [36] to the following lemma:

Lemma 5.2.2. *Let X be a locally compact, locally connected space, and let Y be a compact connected space. Then $\mathrm{USCC}(X,Y)$ is closed in $\mathrm{Cld}_F^*(X \times Y)$.*

Proof. Fix any $A \in \mathrm{Cld}_F^*(X \times Y) \setminus \mathrm{USCC}(X,Y)$. Then there exists $x \in X$ such that $A(x) = \emptyset$ or $A(x)$ is disconnected. When $A(x) = \emptyset$, we have an open neighborhood $W = (X \times Y \setminus \{x\} \times Y)^+$ of A in $\mathrm{Cld}_F^*(X \times Y)$. For each $B \in W$, we get $B(x) = \emptyset$, so $B \notin \mathrm{USCC}(X,Y)$. Therefore $W \cap \mathrm{USCC}(X,Y) = \emptyset$.

When $A(x)$ is disconnected, there exist disjoint open sets U, V in Y such that $A(x) \cap U \neq \emptyset$, $A(x) \cap V \neq \emptyset$ and $A(x) \subset U \cup V$. Then $C = Y \setminus (U \cup V)$ is a non-empty compact set because of the compactness and connectedness of Y. Since X is locally compact and locally connected, there are a compact connected neighborhood N_x of x in X and an open neighborhood N_C of C in Y such that $(N_x \times N_C) \cap A = \emptyset$. Then A has an open neighborhood

$$W = (\mathrm{int}\, N_x \times U)^- \cap (\mathrm{int}\, N_x \times V)^- \cap (X \times Y \setminus N_x \times C)^+$$

in $\mathrm{Cld}_F^*(X \times Y)$. To see $W \cap \mathrm{USCC}(X,Y) = \emptyset$, take any $B \in W$. If $B(y) = \emptyset$ for some $y \in X$, then $B \notin \mathrm{USCC}(X,Y)$. Otherwise, we have the u.s.c. set-valued function $B : X \ni z \mapsto B(z) \in \mathrm{Cld}(Y)$. Since $B \cap (N_x \times C) = \emptyset$ and $Y \setminus C = U \cup V$, we see that

$$N_U = \{z \in N_x \mid B(z) \cap U \neq \emptyset\} = N_x \setminus \{z \in X \mid B(z) \subset V\} \text{ and}$$
$$N_V = \{z \in N_x \mid B(z) \cap V \neq \emptyset\} = N_x \setminus \{z \in X \mid B(z) \subset U\}$$

are closed in N_x. Note that $N_x = N_U \cup N_V$. Since $B \in (\mathrm{int}\, N_x \times U)^- \cap (\mathrm{int}\, N_x \times V)^-$, there exist points $x_U, x_V \in N_x$ such that $B(x_U) \cap U \neq \emptyset$ and $B(x_V) \cap V \neq \emptyset$, that is, $N_U \neq \emptyset$ and $N_V \neq \emptyset$. By the connectedness of N_x, there exists $y \in N_U \cap N_V$. Then $B(y) \cap U \neq \emptyset$, $B(y) \cap V \neq \emptyset$ and $B(y) \subset Y \setminus C = U \cup V$, which means that $B(y)$ is disconnected. Hence $B \notin \mathrm{USCC}(X,Y)$. Thus, we have $W \cap \mathrm{USCC}(X,Y) = \emptyset$. Consequently, the space $\mathrm{USCC}(X,Y)$ is closed in $\mathrm{Cld}_F^*(X \times Y)$. □

Using Michael's Selection Theorem 5.1.1, we have the following:

Lemma 5.2.3. *Let X be a paracompact space with no isolated points and let D be a dendrite. Then* $\mathrm{C}(X,D)$ *is dense in* $\mathrm{USCC}(X,D)$.

Proof. Let $\phi \in \mathrm{USCC}(X,D)$ and W be a neighborhood of ϕ in $\mathrm{Cld}_F^*(X\times D)$. Then there exist open subsets $V_i \subset X\times D$, $i=1,\cdots,m$, and a compact subset $K\subset X\times D$ such that $\phi \in \bigcap_{i=1}^m V_i^- \cap (X\times D\setminus K)^+ \subset W$. For each $x\in X$, since D is locally compact and locally connected, $\phi(x)$ is a continuum, $K(x)$ is compact and $\phi(x)\cap K(x)=\emptyset$, we can find a continuum $A_x\subset D$ such that $\phi(x)\subset \mathrm{int}\, A_x$ and $A_x\cap K(x)=\emptyset$. Then each $x\in X$ has an open neighborhood U_x such that $(U_x\times A_x)\cap K=\emptyset$ and $\phi(x')\subset \mathrm{int}\, A_x$ for all $x'\in U_x$ because A_x is compact and ϕ is u.s.c. Since X is paracompact, the open cover $\{U_x \mid x\in X\}$ has a locally finite open refinement $\{U_\lambda \mid \lambda\in\Lambda\}$. For each $\lambda\in\Lambda$, choose $x(\lambda)\in X$ so that $U_\lambda\subset U_{x(\lambda)}$ and let $A_\lambda = A_{x(\lambda)}$. By the local finiteness of $\{U_\lambda \mid \lambda\in\Lambda\}$, we can define a set-valued function $\psi : X\to \mathrm{Cld}(D)$ by $\psi(x)=\bigcup\{A_\lambda \mid x\in U_\lambda\}$. Then $\psi(x)$ is a continuum for every $x\in X$. Indeed, for each $\lambda\in\Lambda$ with $x\in U_\lambda$, we have $\phi(x)\subset A_{x(\lambda)} = A_\lambda$ because $U_\lambda\subset U_{x(\lambda)}$. Hence $\psi(x)$ is continuum as a finite union of continua containing the continuum $\phi(x)$. Moreover, ψ is l.s.c. In fact, $\{x\in X \mid \psi(x)\cap V\neq\emptyset\} = \bigcup\{U_\lambda \mid A_\lambda\cap V\neq\emptyset\}$ for each open subset $V\subset D$.

Since X has no isolated points, we can choose $(x_i,y_i)\in\psi\cap V_i$ for each $i=1,\cdots,m$ so that $x_i\neq x_j$ if $i\neq j$. Take an admissible convex metric d on D (Fact 3). By virtue of Proposition 5.1.3, the dendrite D has a convex structure $(D^n,k_n)_{n\in\mathbb{N}}$ for d such that every continuum in D is convex with respect to it. Applying Theorem 5.1.1 to the l.s.c. convex-valued function ψ, we can obtain a continuous selection $f : X\to D$ for ψ such that $f(x_i)=y_i$ for each $i=1,\cdots,m$. It is easy to see that $f\in\bigcap_{i=1}^m V_i^-\cap(X\times D\setminus K)^+$. Consequently, $\mathrm{C}(X,D)$ is dense in $\mathrm{USCC}(X,D)$. □

Proof of Theorem 5.2.1. Combining Lemmas 5.2.2 and 5.2.3 implies Theorem 5.2.1. □

5.3 The homotopy denseness of $\mathrm{C}(X,D)$ in $\mathrm{USCC}(X,D)$

This section is devoted to proving the following theorem:

Theorem 5.3.1. *For each non-degenerate Peano continuum X and each dendrite D, the function space* $\mathrm{C}(X,D)$ *is homotopy dense in* $\mathrm{USCC}(X,D)$.

In the rest of this section, we assume that $X=(X,d_X)$ is a Peano continuum with a convex metric and $D=(D,d_D)$ is a dendrite with a convex metric. Moreover, we define an admissible metric ρ for the product $X\times D$ as follows:

$$\rho((x_1,y_1),(x_2,y_2)) = \max\{d_X(x_1,x_2), d_D(y_1,y_2)\}$$

and denote by ρ_H the Hausdorff metric on $\mathrm{Cld}(X\times D)$ induced from ρ. Here, the relative topology on $\mathrm{Cld}(X\times D)\subset \mathrm{Cld}_F^*(X\times D)$ is induced by the Hausdorff metric ρ_H. According to Theorem 5.2.1, the proof of Theorem 5.3.1 is reduced to showing that $(\mathrm{C}(X,D),\rho_H)$ satisfies the condition (hd) in Lemma 1.2.7. The following lemma can be proved by the same technique in the proof of Theorem 1.9 of [31].

Lemma 5.3.2. *For each map $f : X\to D$ and each point $x\in X$, the subset $A=N_\rho(f,\epsilon)(x)$ of D is connected.*

Proof. It suffices to prove that A is arcwise connected. For each $a_1,a_2\in A$, take the path γ_{a_1,a_2} as in Lemma 1.5.1. We shall show that $a=\gamma_{a_1,a_2}(t)\in A$ for each $t\in\mathbf{I}$. Since $(x,a_i)\in N_\rho(f,\epsilon)$ for $i=1,2$, we can take $x_i\in X$ so that $d((x,a_i),(x_i,f(x_i)))<\epsilon$. Then $d_X(x,x_i)<\epsilon$ and $d_D(a_i,f(x_i))<\epsilon$ for $i=1,2$. Let $b=\gamma_{f(x_1),f(x_2)}(t)$. It follows from Lemma 1.5.1 that

$$d_D(a,b) = d_D(\gamma_{a_1,a_2}(t),\gamma_{f(x_1),f(x_2)}(t)) \le \max_{i=1,2} d_D(a_i,f(x_i)) < \epsilon.$$

Since d_X is a convex metric, the ϵ-ball $B_{d_X}(x,\epsilon)$ of x in X is connected. It follows from the continuity of f that $f(B_{d_X}(x,\epsilon))$ is also connected, so it is arcwise connected due to Fact 2(2). The uniqueness of arcs in D implies that $b = \gamma_{f(x_1),f(x_2)}(t) \in f(B_{d_X}(x,\epsilon))$. Therefore, there exists $y \in B_{d_X}(x,\epsilon)$ such that $b = f(y)$. Note that

$$d((x,a),(y,b)) = \max\{d_X(x,y), d_D(a,b)\} < \epsilon,$$

that is, $(x,a) \in N_\rho(f,\epsilon)$. Consequently, $a \in A$. □

Using convex structures on dendrites, we can obtain the same result as Lemma 2 of [51] under our assumption.

Lemma 5.3.3. *Let K be a locally finite countable simplicial complex. If X has no isolated points, then any map $f : K^{(0)} \to \mathrm{C}(X,D)$ extends to a map $\tilde{f} : |K| \to \mathrm{C}(X,D)$ such that*

($*$) $\operatorname{diam}_{\rho_H} \tilde{f}(\sigma) \leq 4 \operatorname{diam}_{\rho_H} f(\sigma^{(0)})$ *for each* $\sigma \in K$.

Proof. By Proposition 5.1.3, the dendrite D has a convex structure $(D^n, k_n)_{n\in\mathbb{N}}$ such that every connected subset is convex. For each simplex $\sigma \in K \setminus K^{(0)}$, let $\epsilon_\sigma = 3\operatorname{diam}_{\rho_H} f(\sigma^{(0)})/2 \geq 0$. Moreover, for each vertex $v \in K^{(0)}$ with $\operatorname{diam}_{\rho_H} f(\mathrm{St}(v,K)^{(0)}) > 0$, let

$$\epsilon_v = \min\{\operatorname{diam}_{\rho_H} f(\sigma^{(0)}) \mid \sigma \in \mathrm{St}(v,K),\ \operatorname{diam}_{\rho_H} f(\sigma^{(0)}) > 0\} > 0,$$

where $\mathrm{St}(v,K)$ is the star at v in K.

Take the barycenter $\hat{\sigma}$ for each $\sigma \in K$ and the barycentric subdivision $\operatorname{Sd} K$ of K. For each $u \in K^{(0)}$, let $g(u) = f(u)$, and for each $\tau \in K \setminus K^{(0)}$ with $\operatorname{diam}_{\rho_H} f(\tau^{(0)}) = 0$, choose $w \in \tau^{(0)}$ and let $g(\hat{\tau}) = f(w)$. Since K is locally finite and X has no isolated points, for each $v \in K^{(0)}$ with $\epsilon_v > 0$, we can inductively take a finite subset $A_v \subset X$ and an open subset $U_v \subset X$ so that $f(v) \subset N_\rho(f(v)|_{A_v}, \epsilon_v)$, $A_v \subset U_v$ and $\operatorname{cl} U_v \cap \operatorname{cl} U_{v'} = \emptyset$ if $v \neq v' \in \sigma^{(0)}$ for some $\sigma \in K$ with $\operatorname{diam}_{\rho_H} f(\sigma^{(0)}) > 0$. Then we have a map $r_v : X \to \mathbf{I}$ such that $r_v(A_v) = 1$ and $r_v(X \setminus U_v) = 0$. Give $K^{(0)}$ a total order $\leqslant$. For each $\sigma \in K$, we can write $\sigma^{(0)} = \{v_1, \cdots, v_m\}$, where $v_1 \leqslant \cdots \leqslant v_m$. Now we define $g(\hat{\sigma}) \in \mathrm{C}(X,D)$ as follows:

$$g(\hat{\sigma})(x) = \begin{cases} k_m((f(v_1)(x), \cdots, f(v_m)(x)), (1/m, \cdots, 1/m)) & \text{if } x \notin \bigcup_{i=1}^m U_{v_i}, \\ k_m((f(v_1)(x), \cdots, f(v_m)(x)), \phi_j(x)) & \text{if } x \in \operatorname{cl} U_{v_j}, \end{cases}$$

where the m-tuple $\phi_j(x) \in P_m$ is defined by

$$\phi_j(x)(i) = \begin{cases} (1 - r_{v_j}(x))/m & \text{if } i \neq j, \\ (1 + (m-1) r_{v_j}(x))/m & \text{if } i = j. \end{cases}$$

Thus f has an extension $g : \operatorname{Sd} K^{(0)} \to \mathrm{C}(X,D)$.

It is easily observed that

($\star$) $g(\hat{\sigma})(x) = f(v)(x)$ for every $\sigma \in K$ with $\operatorname{diam}_{\rho_H} f(\sigma^{(0)}) > 0$, $v \in \sigma^{(0)}$ and $x \in A_v$.

For each $\sigma \in K$ such that $\operatorname{diam}_{\rho_H} f(\sigma^{(0)}) > 0$, since $\operatorname{diam}_{\rho_H} f(\sigma^{(0)}) < \epsilon_\sigma$, we have $f(u) \subset N_\rho(f(v), \epsilon_\sigma)$ for every $u, v \in \sigma^{(0)}$, which implies that $f(u)(x) \in N_\rho(f(v), \epsilon_\sigma)(x)$ for each $x \in X$. It follows from Lemma 5.3.2 that $N_\rho(f(v), \epsilon_\sigma)(x)$ is connected, so convex with respect to $(D^n, k_n)_{n\in\mathbb{N}}$, and hence $g(\hat{\sigma})(x) \in N_\rho(f(v), \epsilon_\sigma)(x)$ for each $x \in X$. Therefore, we have

($\star\star$) $g(\hat{\sigma}) \subset N_\rho(f(v), \epsilon_\sigma)$ for every $\sigma \in K$ with $\operatorname{diam}_{\rho_H} f(\sigma^{(0)}) > 0$ and $v \in \sigma^{(0)}$.

Next, extend g to a map $\tilde{f} : |K| = |\operatorname{Sd} K| \to \mathrm{C}(X, D)$ as follows:

$$\tilde{f}\left(\sum_{i=1}^{m} t_i \hat{\sigma_i}\right)(x) = k_m((g(\hat{\sigma_1})(x), \cdots, g(\hat{\sigma_m})(x)), (t_1, \cdots, t_m))$$

$$\text{for each } \sigma_1 \preccurlyeq \cdots \preccurlyeq \sigma_m \in K \text{ and } (t_1, \cdots, t_m) \in P_m.$$

Recall that the symbol $\tau' \preccurlyeq \tau$ means that τ' is a face of τ. Let $\sigma \in K$, $v \in \sigma^{(0)}$ and $y \in |\operatorname{St}(v, \operatorname{Sd} K)| \cap \sigma$. Then we can write $y = \sum_{i=1}^{m} t_i \hat{\sigma_i}$, where $v = \sigma_1 \preccurlyeq \cdots \preccurlyeq \sigma_m = \sigma \in K$ and $(t_1, \cdots, t_m) \in P_m$. In the case $\operatorname{diam}_{\rho_H} f(\sigma^{(0)}) = 0$, we have $\tilde{f}(y) = f(v)$ because $g(\hat{\sigma_i}) = f(v)$ for all $i = 1, \cdots, m$. Otherwise, we get $d(\tilde{f}(y), f(v)) < \epsilon_\sigma$. Indeed, when $x \in A_v$, it follows from $(\star)$ that $g(\hat{\sigma_i})(x) = f(v)(x)$ for every $i = 1, \cdots, m$, and hence

$$\begin{aligned}\tilde{f}(y)(x) = \tilde{f}\left(\sum_{i=1}^{m} t_i \hat{\sigma_i}\right)(x) &= k_m((g(\hat{\sigma_1})(x), \cdots, g(\hat{\sigma_m})(x)), (t_1, \cdots, t_m)) \\ &= k_m((f(v)(x), \cdots, f(v)(x)), (t_1, \cdots, t_m)) = f(v)(x).\end{aligned}$$

Therefore $f(v)|_{A_v} \subset \tilde{f}(y)$, which means that

$$f(v) \subset N_\rho(f(v)|_{A_v}, \epsilon_v) \subset N_\rho(\tilde{f}(y), \epsilon_\sigma).$$

On the other hand, $g(\hat{\sigma_i}) = f(v)$ if $\operatorname{diam}_{\rho_H} f(\sigma_i^{(0)}) = 0$, $i = 2, \cdots, m$, and it follows from $(\star\star)$ that

$$g(\hat{\sigma_i}) \subset N_\rho(f(v), \epsilon_{\sigma_i}) \subset N_\rho(f(v), \epsilon_\sigma)$$

if $\operatorname{diam}_{\rho_H} f(\sigma_i^{(0)}) > 0$, $i = 2, \cdots, m$. For every $z \in X$, since $N_\rho(f(v), \epsilon_\sigma)(z)$ is also convex with respect to $(D^n, k_n)_{n \in \mathbb{N}}$ by Lemma 5.3.2, we have

$$\tilde{f}(y)(z) = \tilde{f}\left(\sum_{i=1}^{m} t_i \hat{\sigma_i}\right)(z) = k_m((g(\hat{\sigma_1})(z), \cdots, g(\hat{\sigma_m})(z)), (t_1, \cdots, t_m)) \in N_\rho(f(v), \epsilon_\sigma)(z),$$

so $\tilde{f}(y) \subset N_\rho(f(v), \epsilon_\sigma)$. Hence $\rho_H(\tilde{f}(y), f(v)) < \epsilon_\sigma$.

To verify $(*)$, fix any $y, y' \in \sigma \in K$ and choose $v, v' \in \sigma^{(0)}$ so that $y \in |\operatorname{St}(v, \operatorname{Sd} K)|$ and $y' \in |\operatorname{St}(v', \operatorname{Sd} K)|$. Then we get

$$\begin{aligned}\rho_H(\tilde{f}(y), \tilde{f}(y')) &\leq \rho_H(\tilde{f}(y), f(v)) + \rho_H(f(v), f(v')) + \rho_H(\tilde{f}(y'), f(v')) \\ &< \epsilon_\sigma + \operatorname{diam}_{\rho_H} f(\sigma^{(0)}) + \epsilon_\sigma = 4 \operatorname{diam}_{\rho_H} f(\sigma^{(0)}).\end{aligned}$$

The proof is complete. □

Proof of Theorem 5.3.1. Combining Theorem 5.2.1 with Lemmas 1.2.7 and 5.3.3, we can establish Theorem 5.3.1. □

5.4 A dendrite compactification of a one-dimensional locally compact absolute retract

In this section, we show that every 1-dimensional locally compact AR has a dendrite compactification.

Lemma 5.4.1. *Let D be a dendrite with E the end points. Then $D \setminus E$ is homotopy dense in D. Consequently, the product space $(D \setminus E)^\Lambda$ is homotopy dense in D^Λ for any set Λ.*

Proof. Let $\gamma : D^2 \times \mathbf{I} \to D$ be the map obtained in Lemma 1.5.1. Fixing $x_0 \in D \setminus E$, we can define the desired homotopy $h : D \times \mathbf{I} \to D$ as $h(x,t) = \gamma(x, x_0, t)$ for each $x \in D$ and each $t \in \mathbf{I}$. □

D.W. Curtis showed in Proposition 2.4 and Lemma 3.2 of [20] that every locally compact, connected, locally connected, metrizable space Y has a Peano compactification $\widetilde{Y}$ such that the remainder $\widetilde{Y} \setminus Y$ is *locally non-separating*, that is, the following holds:

- For each non-empty connected open set U in $\widetilde{Y}$, the subset $U \cap Y$ is a non-empty connected set.

Using this result, we can characterize 1-dimensional locally compact ARs as follows:

Theorem 5.4.2. *A space Y is a 1-dimensional locally compact AR if and only if Y has a dendrite compactification $\widetilde{Y}$ such that the remainder $\widetilde{Y} \setminus Y$ is closed and contained in the set of end points of $\widetilde{Y}$.*

Proof. First, we will prove the "if" part. The space Y is locally compact and 1-dimensional because Y is open in the dendrite $\widetilde{Y}$. Moreover, it follows from Lemma 5.4.1 that Y is homotopy dense in $\widetilde{Y}$. Since the dendrite $\widetilde{Y}$ is an AR, so the homotopy dense subset Y is according to Proposition 1.2.6 in Chapter 1.

Next, we shall show the "only if" part. Due to Curtis' result mentioned in the above, since Y is locally compact, connected, locally connected and metrizable, we can obtain a Peano compactification $\widetilde{Y}$ of Y that has a locally non-separating remainder. Then $\widetilde{Y}$ has no simple closed curves, which means that it is a dendrite. Indeed, suppose that there exists an simple closed curve $C \subset \widetilde{Y}$. Since $\widetilde{Y}$ is locally connected and C is homeomorphic to a circle, we can find non-empty connected open sets $U_i \subset \widetilde{Y}$, $i = 1, 2, 3, 4$, so that $U_i \cap U_j = \emptyset$ if and only if $|i - j| = 2$, and $S \subset \bigcup_{i=1}^{4} U_i$. As the remainder $\widetilde{Y} \setminus Y$ is locally non-separating, each $V_i = U_i \cap Y$ is a non-empty connected open set in Y and $V_i \cap V_j = \emptyset$ if and only if $|i - j| = 2$. Then each V_i is arcwise connected because it is connected, locally connected, completely metrizable (cf. [50, Theorem 5.14.5]). Fix points $x_{i+1} \in V_i \cap V_{i+1}$, $i = 1, 2, 3$, and $x_1 \in V_1 \cap V_4$, and choose arcs $\gamma_i : \mathbf{I} \to V_i$ from x_i to x_{i+1}, $i = 1, 2, 3$, and $\gamma_4 : \mathbf{I} \to V_4$ from x_4 to x_1. It is easy to find a simple closed curve C' in the union $\bigcup_{i=1}^{4} \gamma_i(\mathbf{I}) \subset Y$. Then we have a retraction $r : Y \to C'$ because Y is 1-dimensional and C' is homeomorphic to a circle (cf. [50, Theorem 5.2.3]). Since Y is an AR, the curve C is also an AR, which is a contradiction. Thus $\widetilde{Y}$ is a dendrite.

It remains to show that the remainder $\widetilde{Y} \setminus Y$ is closed and contained in the set of all end points of $\widetilde{Y}$. From the local compactness of Y, it easily follows that $\widetilde{Y} \setminus Y$ is closed. Moreover, assume that there exists a point $x \in \widetilde{Y} \setminus Y$ such that x is not an end point, that is, x is a cut point (cf. [62, Chapter V, (1.1)]). Then we can obtain disjoint non-empty open sets W_1 and W_2 so that $\widetilde{Y} \setminus \{x\} = W_1 \cup W_2$. Since Y is connected, it misses the one of W_1 or W_2, which contains a non-empty connected open set. This contradicts that the remainder $\widetilde{Y} \setminus Y$ is locally non-separating. Hence the set of all end points of $\widetilde{Y}$ contains $\widetilde{Y} \setminus Y$. Thus the proof is complete. □

5.5 Proof of Main Theorem

In this section, we shall prove the main theorem. From now on let X and Y be spaces under the assumption in the main theorem and fix a dendrite compactification $\widetilde{Y}$ of Y such that the remainder $\widetilde{Y} \setminus Y$ is closed in $\widetilde{Y}$ and contained in the set of all end points of $\widetilde{Y}$. Remark that Y is homotopy dense in $\widetilde{Y}$ by Lemma 5.4.1. Then we have the following:

Proposition 5.5.1. *The space $\mathrm{C}(X, Y)$ is homotopy dense in $\mathrm{C}(X, \widetilde{Y})$.* □

For simplicity, we write

$$\overline{\mathrm{C}(X,Y)} = \mathrm{cl}_{\mathrm{Cld}_F^*(X \times \widetilde{Y})} \mathrm{C}(X,Y) = \mathrm{cl}_{\mathrm{Cld}_F^*(X \times \widetilde{Y})} \mathrm{C}(X, \widetilde{Y}),$$

so it is a compactification of $\mathrm{C}(X,Y)$. Furthermore, if X is connected, it coincides with $\mathrm{USCC}(X,\widetilde{Y})$ by Theorem 5.2.1.

R.D. Anderson [2, 3] introduced the concept of cap sets for the Hilbert cube $\mathbf{Q}$ to characterize subsets $M \subset \mathbf{Q}$ such that the pairs $(\mathbf{Q}, M)$ are homeomorphic to $(\mathbf{Q}, \mathbf{Q} \setminus \mathbf{s})$ (cf. [18, Lemma 8.1]). A subset $M \subset \mathbf{Q}$ is a *cap set* for $\mathbf{Q}$ if M is a Z_σ-set and has the following property:

(cap) For each pair A, B of compact sets in $\mathbf{Q}$ with $B \subset A \cap M$ and each $\epsilon > 0$, there exists an embedding $h : A \to M$ such that $h|_B = \mathrm{id}_B$ and $d(h(a), a) < \epsilon$ for every $a \in A$, where d is an admissible metric for $\mathbf{Q}$.

According to the above, we only need to check that $\overline{\mathrm{C}(X,Y)}$ is homeomorphic to $\mathbf{Q}$ and the complement $\overline{\mathrm{C}(X,Y)} \setminus \mathrm{C}(X,Y)$ is a cap set for $\overline{\mathrm{C}(X,Y)}$.

5.5.1 The case X is discrete.

First, we consider the case that X is discrete. Then X is homeomorphic to $\mathbb{N}$ and Y must be non-compact.

Lemma 5.5.2. *For every discrete space W and every compact space Z, the function space $\mathrm{C}(W,Z)$ is closed in $\mathrm{Cld}_F^*(W \times Z)$.*

Proof. Remark that for each $A \in \mathrm{Cld}_F^*(W \times Z)$, if $A(x)$ is a singleton for every $x \in W$, then $A \in \mathrm{C}(W,Z)$ because W is discrete. Hence, for any $B \in \mathrm{Cld}_F^*(W \times Z) \setminus \mathrm{C}(W,Z)$, we have some $x \in W$ such that $B(x) = \emptyset$ or $B(x)$ is non-degenerate. In the case $B(x) = \emptyset$, we take an open neighborhood $(W \times Z \setminus \{x\} \times Z)^+$ of B in $\mathrm{Cld}_F^*(W \times Z)$, which misses $\mathrm{C}(W,Z)$. In the case $B(x)$ is non-degenerate, we can find disjoint non-empty open subsets U and V of Z the both of which meet $B(x)$. Then $(\{x\} \times U)^- \cap (\{x\} \times V)^-$ is an open neighborhood of B in $\mathrm{Cld}_F^*(W \times Z)$. For every $B' \in (\{x\} \times U)^- \cap (\{x\} \times V)^-$, it is clear that $B'(x)$ is non-degenerate, hence $B' \in \mathrm{Cld}_F^*(W \times Z) \setminus \mathrm{C}(W,Z)$. As a result, the space $\mathrm{C}(W,Z)$ is closed in $\mathrm{Cld}_F^*(W \times Z)$. □

Applying this lemma to our setting, we have $(\overline{\mathrm{C}(X,Y)}, \mathrm{C}(X,Y))$ coincides with $(\mathrm{C}(X,\widetilde{Y}), \mathrm{C}(X,Y))$, which is homeomorphic to $(\widetilde{Y}^{\mathbb{N}}, Y^{\mathbb{N}})$ because X and $\mathbb{N}$ are homeomorphic. Therefore, we can establish the main theorem in the case X is discrete as a corollary of the following theorem:

Theorem 5.5.3. *Let D be a dendrite and let E_0 be a non-empty closed set of D which consists of end points. Then the pair $(D^{\mathbb{N}}, (D \setminus E_0)^{\mathbb{N}})$ of the countable products is homeomorphic to the pair $(\mathbf{Q}, \mathbf{s})$.*

Proof. Let $Z = D \setminus E_0$ for simplicity. Since Z is a non-compact separable completely metrizable AR, the countable product $Z^{\mathbb{N}}$ is homeomorphic to ℓ_2 due to Theorem 1.3.6. Moreover, D is a non-degenerate compact AR. Using Toruńczyk's characterization (Corollary 1.3.3, cf. [42, Corollary 8.1.2]), we can show that $D^{\mathbb{N}}$ is homeomorphic to $\mathbf{Q}$. Let $M = D^{\mathbb{N}} \setminus Z^{\mathbb{N}}$. It is sufficient to prove that the pair $(D^{\mathbb{N}}, M)$ is homeomorphic to $(\mathbf{Q}, \mathbf{Q} \setminus \mathbf{s})$.

First, the product space $Z^{\mathbb{N}}$ is a homotopy dense G_δ set in $D^{\mathbb{N}}$ by Lemma 5.4.1. It follows from Proposition 1.3.1 that the complement M is a Z_σ-set in $D^{\mathbb{N}}$. The countable product $D^{\mathbb{N}}$ assigns a metric d defined by

$$d(x,y) = \sum_{i \in \mathbb{N}} 2^{-i} d_D(x(i), y(i)) \text{ for each } x, y \in D^{\mathbb{N}},$$

where d_D is an admissible convex metric on D. Then the rest of the proof is to show the following:

($*$) For any compact subsets A, B contained in $D^{\mathbb{N}}$ with $B \subset A \cap M$ and each $\epsilon > 0$, there exists an embedding $h : A \to M$ such that $h|_B = \mathrm{id}_B$ and $d(h(a), a) < \epsilon$ for every $a \in A$, Define a map $\alpha : A \to \mathbf{I}$ by $\alpha(a) = \min\{1, \epsilon, d_D(a, B)\}/3$. Since $Z^{\mathbb{N}}$ is homotopy dense in $D^{\mathbb{N}}$, we can obtain a map $f : A \to D^{\mathbb{N}}$ so that $f(A \setminus B) \subset Z^{\mathbb{N}}$, $f|_B = \mathrm{id}_B$ and $d_D(f(a), a) < \alpha(a)$ for every $a \in A$. Let Z_i be a copy of Z for each $i \in \mathbb{N}$. Then $\prod_{i\in\mathbb{N}} Z_{2i}$ and $\prod_{i\in\mathbb{N}} Z_{2i-1}$ are homeomorphic to $Z^{\mathbb{N}}$, so they are homeomorphic to ℓ_2. Here we can take admissible metrics d_e on $\prod_{i\in\mathbb{N}} Z_{2i}$ and d_o on $\prod_{i\in\mathbb{N}} Z_{2i-1}$ defined as follows:

$$d_e(x, y) = \sum_{i\in\mathbb{N}} 2^{-2i} d_D(x(2i), y(2i)) \text{ and } d_o(x, y) = \sum_{i\in\mathbb{N}} 2^{-2i+1} d_D(x(2i-1), y(2i-1)).$$

It is well known that Hilbert spaces are strongly universal for the class of completely metrizable spaces of the same weight (cf. [59, Proposition 2.1]). Since $A \setminus B$ is completely metrizable and $\prod_{i\in\mathbb{N}} Z_{2i}$ is homeomorphic to ℓ_2, we can find an embedding $g_e : A \setminus B \to \prod_{i\in\mathbb{N}} Z_{2i}$ so that $d_e(g_e(a), (\mathrm{pr}_{2i} f(a))_{i\in\mathbb{N}}) < \alpha(a)$ for each $a \in A \setminus B$, where $\mathrm{pr}_i : Z^{\mathbb{N}} \to Z_i$ is the ith coordinate projection. Fix $e_0 \in E_0$ and define a map $g_o : A \setminus B \to \prod_{i\in\mathbb{N}} Z_{2i-1}$ as follows:

$$g_o(a) = (\mathrm{pr}_1 f(a), \cdots, \mathrm{pr}_{2i-3} f(a), \gamma_{\mathrm{pr}_{2i-1} f(a), e_0}(2^{2i-2}\alpha(a)/\operatorname{diam}_{d_D} D - 1), e_0, \cdots)$$
$$\text{if } 2^{-2i+2} \operatorname{diam}_{d_D} D < \alpha(a) \leq 2^{-2i+4} \operatorname{diam}_{d_D} D,$$

where $\gamma_{x,y} : \mathbf{I} \to D$ is the unique path from x to y as in Lemma 1.5.1. For any $a \in A \setminus B$, if $2^{-2j+2} \operatorname{diam}_{d_D} D < \alpha(a)$, then

$$\begin{aligned} d_o(g_o(a), (\mathrm{pr}_{2i-1} f(a))_{i\in\mathbb{N}}) &= \sum_{i\in\mathbb{N}} 2^{-2i+1} d_D(\mathrm{pr}_{2i-1} g_o(a), \mathrm{pr}_{2i-1} f(a)) \\ &\leq \sum_{i\geq j} 2^{-2i+1} \operatorname{diam}_{d_D} D = 2^{-2j+2} \operatorname{diam}_{d_D} D < \alpha(a). \end{aligned}$$

Now we define a map $g : A \setminus B \to M$ as follows:

$$\mathrm{pr}_i\, g(a) = \begin{cases} \mathrm{pr}_i\, g_e(a) & \text{if } i = 2j, \\ \mathrm{pr}_i\, g_o(a) & \text{if } i = 2j-1. \end{cases}$$

It follows from the definition of g that

$$d(g(a), a) \leq d(g(a), f(a)) + d(f(a), a) < 3\alpha(a) = \min\{1, \epsilon, d(a, B)\}$$

for each $a \in A \setminus B$. Hence we can extend g to a map $h : A \to M$ by $h|_B = \mathrm{id}_B$. Then h is clearly ϵ-close to id_A. Since g is injective and

$$h(A \setminus B) \cap h(B) = g(A \setminus B) \cap B = \emptyset,$$

the map h is an embedding. Thus the condition $(*)$ is satisfied. □

5.5.2 The case X is non-discrete.

Next, we consider the case X is non-discrete. As a corollary of the following theorem, we conclude that the function space $\mathrm{C}(X, Y)$ is homeomorphic to Hilbert space ℓ_2 under our assumption.

Theorem 5.5.4. *For a non-discrete, locally compact, separable metrizable space W and a separable completely metrizable AR Z with no isolated points, the function space $\mathrm{C}(W, Z)$ is homeomorphic to ℓ_2.*

Remark 12. The above theorem was proved by K. Sakai [48] when W is compact. Moreover, J. Smrekar and A. Yamashita [53] showed the case W is a countable CW-complex of dimension ≥ 1. This theorem cannot be generalized to the case that Z is an ANR. In fact, the space $\mathrm{C}(W, Z)$ is not an ANR even if Z is the unit circle (cf. [53, Introduction]).

Proposition 5.5.5. *For a locally compact space W and an AE Z, the function space* $\mathrm{C}(W,Z)$ *is an AE.*

Proof. Let A be a metrizable space, B a closed subset of A and let $f : B \to \mathrm{C}(W,Z)$ be a map. Define a function $F : B \times W \to Z$ by $F(b,x) = f(b)(x)$, which is continuous due to the local compactness of W. Since Z is an AE, the map F extends to a map $\widetilde{F} : A \times W \to Z$. Then we can define a map $\tilde{f} : A \to \mathrm{C}(W,Z)$ by $\tilde{f}(a)(x) = \widetilde{F}(a,x)$. Note that for each $b \in B$ and $x \in W$

$$\tilde{f}(b)(x) = \widetilde{F}(b,x) = F(b,x) = f(b)(x),$$

that is, the map $\tilde{f}$ is an extension of f. Consequently, the function space $\mathrm{C}(W,Z)$ is an AE. □

Proposition 5.5.6. *Let $W = \bigcup_{n\in\mathbb{N}} W_n$ be a σ-compact space, where each W_n is compact and contained in* $\operatorname{int} W_{n+1}$*, and let Z be a completely metrizable space. Then the function space* $\mathrm{C}(W,Z)$ *is completely metrizable.*

Proof. Take an admissible complete bounded metric d for Z and define a metric d^* on $\mathrm{C}(W,Z)$ as follows:

$$d^*(f,g) = \sum_{n\in\mathbb{N}} 2^{-n} \sup_{x\in W_n} d(f(x),g(x)) \text{ for each } f,g \in \mathrm{C}(W,Z),$$

so d^* is an admissible complete metric on it. □

By the same argument of [53, Proof of Theorem 1.2], we have the following:

Proposition 5.5.7. *Let W be a non-discrete, locally compact, separable metrizable space and let Z be an ANR with no isolated points. If* $\mathrm{C}(W,Z)$ *is path-connected, then* $\mathrm{C}(W,Z)$ *has the discrete approximation property.*

Proof. By the assumption, we can write $W = \bigcup_{n\in\mathbb{N}} W_n$, where each W_n is compact and contained in the interior $\operatorname{int} W_{n+1}$ of W_{n+1}, and choose countable distinct points $x_1, x_2, \cdots, x_\infty \in \operatorname{int} W_1$ so that $x_i \to x_\infty$ as $i \to \infty$. Moreover, since Z is an ANR with no isolated points, it has an admissible bounded metric d such that

(1) for each $\epsilon > 0$ there exists $\delta > 0$ such that any two δ-close maps from any space to Z is ϵ-homotopic, and

(2) every component P of Z has the diameter $\operatorname{diam}_d P > 1$.

We shall use an admissible metric d^* on $\mathrm{C}(W,Z)$ defined as in Proposition 5.5.6. Let $C_i = \{f \in \mathrm{C}(W,Z) \mid f(x_\infty) = f(x_j) \text{ for all } j \geq i\}$ for each $i \in \mathbb{N}$. Clearly, $C_i \subset C_{i+1}$. According to [53, Lemma 3.2], we need only to show the following two conditions:

(i) For each $\epsilon > 0$ and $f : \mathbf{I}^n \to \mathrm{C}(W,Z)$, $n \in \omega$, there are $i \in \mathbb{N}$ and $g : \mathbf{I}^n \to C_i$ such that g is ϵ-homotopic to f;

(ii) For each $\epsilon > 0$, there is $\delta > 0$ such that for any $i \in \mathbb{N}$ and $f : \mathbf{I}^n \to C_i$, $n \in \omega$, there exist $j \geq i$ and $g : \mathbf{I}^n \to C_j$ that is ϵ-homotopic to f and satisfies $d^*(f(\mathbf{I}^n), g(\mathbf{I}^n)) \geq \delta$.

(i) Let $\epsilon > 0$ and $f : \mathbf{I}^n \to \mathrm{C}(W,Z)$, $n \in \omega$. Take $\delta > 0$ so as to satisfy the condition (1). From the compactness of $\mathbf{I}^n$, we can find $i \in \mathbb{N}$ such that for any $s \in \mathbf{I}^n$ and $j \geq i$, $d^*(f(s)(x_j), f(s)(x_\infty)) < \delta$. Define $F : \mathbf{I}^n \times W \to Z$ by $F(s,x) = f(s)(x)$. Then the restriction $F|_{\mathbf{I}^n \times \{x_j | j \geq i\}}$ is δ-close to the constant map $F' : \mathbf{I}^n \times \{x_j \mid j \geq i\} \ni (s,x_j) \mapsto f(s)(x_\infty) \in Z$, and hence $F|_{\mathbf{I}^n \times \{x_j | j \geq i\}}$ is ϵ-homotopic to F' by the definition of δ. Since Z is an ANR, by the Homotopy Extension Theorem 1.2.4, there is an ϵ-homotopy $H : \mathbf{I}^n \times W \times \mathbf{I} \to Z$ such that $H(s,x,0) = F(s,x) = f(s)(x)$ and $H(s,x_j,1) = F'(s,x_j,1) = f(s)(x_\infty)$ for every $s \in \mathbf{I}^n$ and $j \geq i$. Define $g : \mathbf{I}^n \to \mathrm{C}(W,Z)$ by $g(s)(x) = H(s,x,1)$. Note that for each $s \in \mathbf{I}^n$

and $j \geq i$, $g(s)(x_j) = H(s, x_j, 1) = f(s)(x_\infty)$. Therefore $g(\mathbf{I}^n) \subset C_i$. Let $h : \mathbf{I}^n \times \mathbf{I} \to \mathrm{C}(W, Z)$ be the map defined by $h(s,t)(x) = H(s,x,t)$, which is an ϵ-homotopy linking f and g. Indeed, we have for each $s \in \mathbf{I}^n$ and $t, t' \in \mathbf{I}$,

$$d^*(h(s,t), h(s,t')) = \sum_{n \in \mathbb{N}} 2^{-n} \sup_{x \in W_n} d(H(s,x,t), H(s,x,t')) < \sum_{n \in \mathbb{N}} 2^{-n}\epsilon = \epsilon.$$

(ii) Take any $\epsilon > 0$. Due to (1), we can choose $0 < \delta \leq 1/8$ so that any two 10δ-close maps into Z are ϵ-homotopic. Fix $i \in \mathbb{N}$ and $f : \mathbf{I}^n \to C_i$. Let

$$K = \{f(s)(x_\infty) \mid s \in \mathbf{I}^n\} = F(\mathbf{I}^n \times \{x_j \mid i \leq j \leq \infty\}).$$

Since K is compact, there are finite points $y_0, \cdots, y_n \in K$ such that $K \subset \bigcup_{k=0}^n B_d(y_k, 2\delta)$. Then we can find a point $z_k \in B_d(y_k, 6\delta) \setminus B_d(y_k, 4\delta)$ for each $k = 0, \cdots, n$, because each path component of Z has the diameter $> 8\delta$ by (2). It follows from the choice of δ and the Homotopy Extension Theorem 1.2.4 that there is an ϵ-homotopies $h^k : Z \times \mathbf{I} \to Z$, $k = 0, \cdots, n$, such that $h^k(y, 0) = y$, $h^k(y, 1) = z_k$ if $y \in B_d(y_k, 4\delta)$, and $h^k(y,t) = y$ if $y \notin B_d(y_k, 6\delta)$. Using the Homotopy Extension Theorem 1.2.4 again, we can obtain an ϵ-homotopy $H : W \times Z \times \mathbf{I} \to Z$ so that $H(x, y, 0) = y$, $H(x_{i+k}, y, t) = h^k(y,t)$ for each $k = 0, \cdots, n$, and $H(x_j, y, t) = y$ for each $i + n < j \leq \infty$. Define the desired map $g : \mathbf{I}^n \to \mathrm{C}(W, Z)$ by $g(s)(x) = H(x, f(s)(x), 1)$. It follows that for each $i + n < j \leq \infty$,

$$g(s)(x_j) = H(x_j, f(s)(x_j), 1) = f(s)(x_j) = f(s)(x_\infty),$$

which implies that $g(\mathbf{I}^n) \subset C_{i+n+1}$. Moreover, we have an ϵ-homotopy $h : \mathbf{I}^n \times \mathbf{I} \to \mathrm{C}(W, Z)$ linking f and g defined by $h(s,t)(x) = H(x, f(s)(x), t)$. It remains to show that $d^*(f(\mathbf{I}^n), g(\mathbf{I}^n)) \geq \delta$. Fix any $s, s' \in \mathbf{I}^n$. In the case that $d(f(s)(x_\infty), f(s')(x_\infty)) \geq 2\delta$, we have

$$d(f(s)(x_\infty), g(s')(x_\infty)) = d(f(s)(x_\infty), f(s')(x_\infty)) \geq 2\delta.$$

Since $x_\infty \in W_1$, it follows that

$$d^*(f(s), g(s')) = \sum_{n \in \mathbb{N}} 2^{-n} \sup_{x \in W_n} d(f(s)(x), g(s')(x)) \geq 2^{-1} d(f(s)(x_\infty), g(s')(x_\infty)) \geq \delta.$$

In the case that $d(f(s)(x_\infty), f(s')(x_\infty)) < 2\delta$, taking some $k = 0, \cdots, n$ such that $f(s)(x_\infty) \in B_d(y_k, 2\delta)$, we have $f(s')(x_\infty) \in B_d(y_k, 4\delta)$. Then

$$g(s')(x_{i+k}) = H(x_{i+k}, f(s')(x_{i+k}), 1) = h^k(f(s')(x_\infty), 1) = z_k \notin B_d(y_k, 4\delta).$$

On the other hand, we get $f(s)(x_{i+k}) = f(s)(x_\infty) \in B_d(y_k, 2\delta)$, and hence $d(f(s)(x_{i+k}), g(s')(x_{i+k})) \geq 2\delta$. Since $x_{i+k} \in W_1$, it follows that

$$d^*(f(s), g(s')) = \sum_{n \in \mathbb{N}} 2^{-n} \sup_{x \in W_n} d(f(s)(x), g(s')(x)) \geq 2^{-1} d(f(s)(x_{i+k}), g(s')(x_{i+k})) \geq \delta.$$

Thus the proof is complete. □

Proof of Theorem 5.5.4. Combining Propositions 5.5.5, 5.5.6 and 5.5.7, we get $\mathrm{C}(W, Z)$ is a completely metrizable space with the discrete approximation property. The separability of $\mathrm{C}(W, Z)$ follows from the ones of W and Z, and the local compactness of W (cf. [29, Chapter XII, Theorem 5.2]). According to Toruńczyk's characterization (Theorem 1.3.5), the function space $\mathrm{C}(W, Z)$ is homeomorphic to ℓ_2. □

The following two lemmas guarantee that we may assume X is connected in the proof of the main theorem.

Lemma 5.5.8. *Let $W = \bigoplus_{\lambda\in\Lambda} W_\lambda$ be a locally connected space, where each W_λ is a component of W. For any spaces $Z' \subset Z$, the quadruplet*

$$(\mathrm{Cld}^*_F(W \times Z), \mathrm{cl}_{\mathrm{Cld}^*_F(W\times Z)} \mathrm{C}(W, Z), \mathrm{C}(W, Z), \mathrm{C}(W, Z'))$$

is homeomorphic to the quadruplet

$$\left(\prod_{\lambda\in\Lambda} \mathrm{Cld}^*_F(W_\lambda \times Z), \prod_{\lambda\in\Lambda} \mathrm{cl}_{\mathrm{Cld}^*_F(W_\lambda\times Z)} \mathrm{C}(W_\lambda, Z), \prod_{\lambda\in\Lambda} \mathrm{C}(W_\lambda, Z), \prod_{\lambda\in\Lambda} \mathrm{C}(W_\lambda, Z')\right).$$

Proof. Define a map $h : \mathrm{Cld}^*_F(W \times Z) \to \prod_{\lambda\in\Lambda} \mathrm{Cld}^*_F(W_\lambda \times Z)$ as follows:

$$h(A) = (A \cap (W_\lambda \times Z))_{\lambda\in\Lambda} \text{ for each } A \in \mathrm{Cld}^*_F(W \times Z),$$

which is the desired homeomorphism. □

Lemma 5.5.9. *Let W_n be a compact AR and let Z_n be a homotopy dense G_δ subset of W_n, $n \in \mathbb{N}$. Then the pair $(\mathbf{Q} \times \prod_{n\in\mathbb{N}} W_n, \mathbf{s} \times \prod_{n\in\mathbb{N}} Z_n)$ is homeomorphic to $(\mathbf{Q}, \mathbf{s})$.*

Proof. We may assume that each W_n is non-degenerate. By Toruńczyk's characterization (Corollary 1.3.3, cf. [42, Corollary 8.1.2]), the product space $\mathbf{Q} \times \prod_{n\in\mathbb{N}} W_n$ is homeomorphic to $\mathbf{Q}$. We shall show that the complement $M = (\mathbf{Q} \times \prod_{n\in\mathbb{N}} W_n) \setminus (\mathbf{s} \times \prod_{n\in\mathbb{N}} Z_n)$ is a cap set in $\mathbf{Q} \times \prod_{n\in\mathbb{N}} W_n$. It is easy to see that $(\mathbf{Q} \setminus \mathbf{s}) \times \prod_{n\in\mathbb{N}} W_n$ is a cap set in $\mathbf{Q} \times \prod_{n\in\mathbb{N}} W_n$ because $\mathbf{Q} \setminus \mathbf{s}$ is a cap set in $\mathbf{Q}$. Moreover, since each Z_n is a homotopy dense G_δ subset of W_n, the complement $W_n \setminus Z_n$ is a countable union of compact Z-sets in W_n due to Proposition 1.3.1. Let $\mathrm{pr}_m : \prod_{n\in\mathbb{N}} W_n \to W_m$ be the projection for each $m \in \mathbb{N}$. Then, as is easily observed,

$$M = \left((\mathbf{Q} \setminus \mathbf{s}) \times \prod_{n\in\mathbb{N}} W_n\right) \cup \bigcup_{m\in\mathbb{N}} (\mathbf{Q} \times \mathrm{pr}_m^{-1}(W_m \setminus Z_m))$$

is also a countable union of compact Z-sets in $\mathbf{Q} \times \prod_{n\in\mathbb{N}} W_n$, which contains $(\mathbf{Q}\setminus\mathbf{s}) \times \prod_{n\in\mathbb{N}} W_n$. It follows from Theorem 6.6 of [18] that M is a cap set in $\mathbf{Q} \times \prod_{n\in\mathbb{N}} W_n$, hence the pair $(\mathbf{Q} \times \prod_{n\in\mathbb{N}} W_n, \mathbf{s} \times \prod_{n\in\mathbb{N}} Z_n)$ is homeomorphic to $(\mathbf{Q}, \mathbf{s})$. □

Proof of Main Theorem in the Case X is Non-Discrete. We may suppose that X is connected as mentioned in the above. We divide the proof into the two case, the case X is compact, and the case X is non-compact.

(The compact case) Combining Theorem 5.3.1 with Proposition 5.5.1, we conclude that $\mathrm{C}(X, Y)$ is homotopy dense in $\overline{\mathrm{C}(X, Y)} = \mathrm{USCC}(X, \widetilde{Y})$. Since $\mathrm{C}(X, Y)$ is homeomorphic to ℓ_2 according to Theorem 5.5.4 (c.f. [48]), it easily follows that $\mathrm{USCC}(X, \widetilde{Y})$ is a compact AR with the disjoint cells property. Hence $\mathrm{USCC}(X, \widetilde{Y})$ is homeomorphic to $\mathbf{Q}$ by virtue of Toruńczyk's characterization (Corollary 1.3.3). Moreover, the complement $M = \mathrm{USCC}(X, \widetilde{Y}) \setminus \mathrm{C}(X, Y)$ is a Z_σ-set. Take an admissible metric d_X and an admissible convex metric $d_{\widetilde{Y}}$ on X and $\widetilde{Y}$, respectively, and define an admissible metric ρ on $X \times \widetilde{Y}$ as follows:

$$\rho((x, y), (x', y')) = \max\{d_X(x, x'), d_{\widetilde{Y}}(y, y')\}.$$

It remains to verify that the following condition holds:

(∗) For any compact sets $A, B \subset \mathrm{USCC}(X, \widetilde{Y})$ with $B \subset A \cap M$ and each $\epsilon > 0$, there exists an embedding $\tilde{h} : A \to M$ such that $\tilde{h}|_B = \mathrm{id}_B$ and $\rho_H(\tilde{h}(a), a) < \epsilon$ for every $a \in A$, where ρ_H is the Hausdorff metric on $\mathrm{Cld}(X \times \widetilde{Y})$ induced by ρ.

Let $\alpha : A \to \mathbf{I}$ be a map defined by $\alpha(a) = \min\{1, \epsilon, \rho_H(a, B)\}/3$. Since $\mathrm{C}(X,Y)$ is homotopy dense in $\mathrm{USCC}(X, \widetilde{Y})$, we can construct a map $f : A \to \mathrm{USCC}(X, \widetilde{Y})$ such that $f|_B = \mathrm{id}_B$, $f(A \setminus B) \subset \mathrm{C}(X,Y)$ and $\rho_H(f(a), a) \leq \alpha(a)$ for every $a \in A$. In addition, we can find an embedding $g : A \setminus B \to \mathrm{C}(X,Y)$ so that $\rho_H(g(a), f(a)) < \alpha(a)$ for each $a \in A \setminus B$ because $\mathrm{C}(X,Y)$ is homeomorphic to ℓ_2 and $A \setminus B$ is completely metrizable. Fix a point $x_0 \in X$ and define a function $h : A \setminus B \to \mathrm{Cld}(X \times \widetilde{Y})$ by

$$h(a)(x) = \begin{cases} \overline{B}(g(a)(x_0), \alpha(a)) & \text{if } x = x_0, \\ g(a)(x) & \text{if } x \neq x_0, \end{cases}$$

where $\overline{B}(g(a)(x_0), \alpha(a))$ is the closed ball. Remark that each $h(a)$ is an u.s.c. set-valued function due to Proposition 3.1 of [36]. Because $d_{\widetilde{Y}}$ is a convex metric, the function h is continuous and the closed ball $\overline{B}(g(a)(x_0), \alpha(a))$ is a subcontinuum of $\widetilde{Y}$, hence $h(A \setminus B) \subset M$. Since x_0 is not isolated point and g is an injection, the map h is also an injection. It follows that

$$\rho_H(h(a), a) \leq \rho_H(h(a), g(a)) + \rho_H(g(a), f(a)) + \rho_H(f(a), a) < 3\alpha(a) \leq \min\{1, \epsilon, \rho_H(a, B)\}$$

for each $a \in A \setminus B$. Therefore, the map $h : A \setminus B \to M$ can be extended to the map $\tilde{h} : A \to M$ by $\tilde{h}|_B = \mathrm{id}_B$. Moreover, we have $h(A \setminus B) \cap B = \emptyset$, hence $\tilde{h}$ is the desired embedding because A is compact. Thus the pair $(\overline{\mathrm{C}(X,Y)}, \mathrm{C}(X,Y))$ is homeomorphic to $(\mathbf{Q}, \mathbf{s})$.

(The non-compact case) Similar to the compact case, it suffices to prove that $\mathrm{C}(X,Y)$ is homotopy dense in $\overline{\mathrm{C}(X,Y)}$, and that $\overline{\mathrm{C}(X,Y)}$ is homeomorphic to $\mathbf{Q}$. Let $\alpha X = X \cup \{\infty\}$ be the one-point compactification of X. Then it is a Peano continuum, refer to [55]. According to the compact case, the pair $(\overline{\mathrm{C}(\alpha X,Y)}, \mathrm{C}(\alpha X,Y))$ is homeomorphic to $(\mathbf{Q}, \mathbf{s})$, where

$$\overline{\mathrm{C}(\alpha X, Y)} = \mathrm{cl}_{\mathrm{Cld}^*_F(\alpha X \times \widetilde{Y})} \mathrm{C}(\alpha X, Y) = \mathrm{USCC}(\alpha X, \widetilde{Y}).$$

Due to Proposition 3.2 of [36], we have the embedding $e : \mathrm{Cld}^*_F(X \times \widetilde{Y}) \to \mathrm{Cld}^*_F(\alpha X \times \widetilde{Y})$ and the retraction $r : \mathrm{Cld}^*_F(\alpha X \times \widetilde{Y}) \to e(\mathrm{Cld}^*_F(X \times \widetilde{Y}))$ defined by

$$e(A) = A \cup (\{\infty\} \times \widetilde{Y}) \text{ and } r(B) = B \cup (\{\infty\} \times \widetilde{Y}),$$

where $r(\mathrm{C}(\alpha X, Y)) \subset e(\mathrm{C}(X,Y))$ and $e(\overline{\mathrm{C}(X,Y)}) = r(\overline{\mathrm{C}(\alpha X,Y)})$.

First, we will show that $\mathrm{C}(X,Y)$ is homotopy dense in $\overline{\mathrm{C}(X,Y)}$. Since $\mathrm{C}(\alpha X,Y)$ is homotopy dense in $\overline{\mathrm{C}(\alpha X,Y)}$, we can find a homotopy $h : \overline{\mathrm{C}(\alpha X,Y)} \times \mathbf{I} \to \overline{\mathrm{C}(\alpha X,Y)}$ so that $h_0 = \mathrm{id}_{\overline{\mathrm{C}(\alpha X,Y)}}$ and $h(\overline{\mathrm{C}(\alpha X,Y)} \times (0,1]) \subset \mathrm{C}(\alpha X,Y)$. Taking a homotopy

$$h' = e^{-1} r h (e \times \mathrm{id}_{\mathbf{I}}) : \overline{\mathrm{C}(X,Y)} \times \mathbf{I} \to \overline{\mathrm{C}(X,Y)},$$

we have $h'_0 = \mathrm{id}_{\overline{\mathrm{C}(X,Y)}}$ because $e(\overline{\mathrm{C}(X,Y)}) = r(\overline{\mathrm{C}(\alpha X,Y)})$. In addition, since $r(\mathrm{C}(\alpha X,Y)) \subset e(\mathrm{C}(X,Y))$, we get $h'(\overline{\mathrm{C}(X,Y)} \times (0,1]) \subset \mathrm{C}(X,Y)$. Hence $\mathrm{C}(X,Y)$ is homotopy dense in $\overline{\mathrm{C}(X,Y)}$.

Next, we shall prove that $\overline{\mathrm{C}(X,Y)}$ is homeomorphic to $\mathbf{Q}$. Since $e(\overline{\mathrm{C}(X,Y)}) = r(\overline{\mathrm{C}(\alpha X,Y)})$, we can regard $\overline{\mathrm{C}(X,Y)}$ as a retract of $\overline{\mathrm{C}(\alpha X,Y)}$, which is homeomorphic to $\mathbf{Q}$. Hence $\overline{\mathrm{C}(X,Y)}$ is a compact AR. Furthermore, the space $\mathrm{C}(X,Y)$ is homeomorphic to ℓ_2 by Theorem 5.5.4, so $\overline{\mathrm{C}(X,Y)}$ has the disjoint cells property. Using the Toruńczyk characterization (Corollary 1.3.3), we have $\overline{\mathrm{C}(X,Y)}$ is homeomorphic to $\mathbf{Q}$. Thus the proof is complete. □

5.6 The converse of Main Theorem

In this section, we shall prove the converse of the main theorem.

Lemma 5.6.1. *Let X be a space and Y a non-degenerate connected space. If* $\mathrm{USCC}(X,Y)$ *is Hausdorff, then X is locally compact.*

Proof. We shall show that for each point $x \in X$ and each open neighborhood U of x in X, there exists a compact neighborhood of x contained in U. Fix $y_0 \in Y$. Since $\mathrm{USCC}(X,Y)$ is Hausdorff, we can separate the following two functions

$$\phi = X \times \{y_0\} \cup (X \setminus U) \times Y \text{ and } \psi = X \times \{y_0\} \cup (X \setminus U) \times Y \cup \{x\} \times Y$$

by disjoint open sets V and W in $\mathrm{USCC}(X,Y)$. Then we can write

$$V = (X \times Y \setminus C)^+ \cap (\bigcap_{i=1}^{n} V_i^-) \cap \mathrm{USCC}(X,Y) \text{ and } W = (X \times Y \setminus D)^+ \cap (\bigcap_{j=1}^{m} W_j^-) \cap \mathrm{USCC}(X,Y),$$

where C and D are compact sets in $X \times Y$, and V_i's and W_j's are open sets in $X \times Y$. Moreover, we may assume that $\mathrm{pr}_X(D) \cap \mathrm{pr}_X(W_j) = \emptyset$ for each $1 \leq j \leq m$, where $\mathrm{pr}_X : X \times Y \to X$ is the projection onto X.

Note that $x \in \mathrm{pr}_X(C)$ because $\psi \notin V$, and $\mathrm{pr}_X(C) \subset U$ because $\phi \in V$. We prove that $\mathrm{pr}_X(C)$ is the desired neighborhood. Since $\phi \notin W$ and $\psi \in W$, we get $\{1 \leq j \leq m \mid x \in \mathrm{pr}_X(W_j)\} \neq \emptyset$. Let $\{j_k \mid 1 \leq k \leq l\} = \{1 \leq j \leq m \mid x \in \mathrm{pr}_X(W_j)\}$. Then there exists $j_k \in \{j_k \mid 1 \leq k \leq l\}$ such that $\mathrm{pr}_X(W_{j_k}) \subset \mathrm{pr}_X(C)$. Supposing the contrary, we can choose $x_{j_k} \in \mathrm{pr}_X(W_{j_k}) \setminus \mathrm{pr}_X(C)$ for each $1 \leq k \leq l$. Define the function

$$\xi = X \times \{y_0\} \cup (X \setminus U) \times Y \cup \bigcup_{k=1}^{l} \{x_{j_k}\} \times Y \in \mathrm{USCC}(X,Y).$$

Observe that $\xi \in V \cap W$, which is a contradiction. Hence we have $x \in \mathrm{pr}_X(W_j) \subset \mathrm{pr}_X(C)$ for some $1 \leq j \leq m$. This means that $\mathrm{pr}_X(C)$ is a neighborhood of x. The proof is complete. □

Let Y be a non-degenerate connected space. Then we can regard a space X as a subspace of $\mathrm{USCC}(X,Y)$. Indeed, taking $y_0 \in Y$, we have an embedding $i : X \ni x \mapsto X \times \{y_0\} \cup \{x\} \times Y \in \mathrm{USCC}(X,Y)$. Thus X is metrizable when $\mathrm{USCC}(X,Y)$ is so.

Proposition 5.6.2. *Let X be a space and Y a non-degenerate connected space. If* $\mathrm{USCC}(X,Y)$ *is compact metrizable, then X is locally compact, locally connected metrizable.*

Proof. According to Lemma 5.6.1, X is locally compact metrizable. So it remains to prove that X is locally connected. Suppose the contrary, that is, there exists a point $x_0 \in X$ and an open neighborhood U of x_0 such that every neighborhood V of x_0 contained in U is disconnected. We will show that there exists $x \in U$, open and closed subsets V_n in U containing x and $w_n \in W_n = U \setminus V_n$, $n \in \mathbb{N}$, such that $\{w_n\}_{n \in \mathbb{N}}$ converges to x. Let

$$\mathcal{V} = \{V \subset U \mid V \text{ is an open and closed subset of } U \text{ containing } x_0\}.$$

Then $\bigcap \mathcal{V}$ is not open in U. Otherwise, since $x_0 \in \bigcap \mathcal{V}$, we have $\bigcap \mathcal{V}$ is disconnected. So we can find an open and closed subset V of $\bigcap \mathcal{V}$ such that $x_0 \in V \subsetneq \bigcap \mathcal{V}$. Then V is open and closed in U, which is a contradiction to the minimality of $\bigcap \mathcal{V}$. Hence $\bigcap \mathcal{V}$ is not open in U. Choose a point $x \in \bigcap \mathcal{V}$ and a sequence $\{w_n\}_{n \in \mathbb{N}} \subset U \setminus \bigcap \mathcal{V}$ converging to x, and take $V_n \in \mathcal{V}$ so that $w_n \in W_n = U \setminus V_n$.

Now, we define

$$\phi_n = \bigcap_{i=1}^{n} V_i \times \{y_1\} \cup \bigcup_{i=1}^{n} W_i \times \{y_2\} \cup (X \setminus U) \times Y,$$

where y_1 and y_2 are distinct points of Y. Observe that $\phi_n \in \mathrm{USCC}(X, Y)$. By the assumption, $\mathrm{USCC}(X, Y)$ is a compact metrizable space. Therefore we may suppose that the sequence $\{\phi_n\}_{n\in\mathbb{N}}$ converges to some $\phi \in \mathrm{USCC}(X, Y)$. Then for each $n \in \mathbb{N}$, $\phi_n \cap U \times Y \setminus \{y_1, y_2\} = \emptyset$, which implies that $\phi \cap U \times Y \setminus \{y_1, y_2\} = \emptyset$. Since every ϕ_n contains (x, y_1), we have $y_1 \in \phi(x)$. Assume that $y_2 \notin \phi(x)$, so $\phi^{-1}((Y \setminus \{y_2\})^+)$ is an open neighborhood of x because ϕ is u.s.c. Since X is locally compact, we can take a compact neighborhood $N \subset \phi^{-1}((Y \setminus \{y_2\})^+)$ of x. Then $\phi \cap N \times \{y_2\} = \emptyset$, and hence, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $\phi_n \cap N \times \{y_2\} = \emptyset$. On the other hand, we can find $n \geq n_0$ such that $w_n \in N$, which means that $(w_n, y_2) \in \phi_n \cap N \times \{y_2\}$. This is a contradiction. Therefore $y_2 \in \phi(x)$. It follows that $\phi(x) = \{y_1, y_2\}$ is disconnected, which contradicts that $\phi \in \mathrm{USCC}(X, Y)$. Consequently, X is localIy connected. □

We can derive the following corollary from the above proposition immediately.

Corollary 5.6.3. *Let X be a space and Y a non-degenerate connected space. If* $\mathrm{USCC}(X, Y)$ *is homeomorphic to* $\mathbf{Q}$*, then X is locally compact, locally connected metrizable.*

Consequently, we have the following:

Theorem 5.6.4. *Let X be a non-degenerate connected space and Y a 1-dimensional locally compact AR. Then the following conditions are equivalent:*

(1) *X is locally compact, locally connected metrizable;*

(2) $(\mathrm{USCC}(X, \widetilde{Y}), \mathrm{C}(X, Y))$ *is homeomorphic to* $(\mathbf{Q}, \mathbf{s})$*;*

(3) $\mathrm{USCC}(X, \widetilde{Y})$ *is homeomorphic to* $\mathbf{Q}$*,*

where $\widetilde{Y}$ is a dendrite compactification of Y such that the remainder is closed and contained in the set of end points of $\widetilde{Y}$.

5.7 Examples

Let $\alpha\mathbb{R}$ be the one-point compactification of $\mathbb{R}$. Then we have the following proposition.

Proposition 5.7.1. *The function space* $\mathrm{C}(\mathbf{I}, \mathbb{R})$ *is not homotopy dense in the closure* $\mathrm{cl}_{\mathrm{Cld}^*_F(\mathbf{I}\times\alpha\mathbb{R})} \mathrm{C}(\mathbf{I}, \mathbb{R})$.

Proof. Let $\mathbf{S}^1$ be the unit circle in $\mathbb{R}^2$, that is, $\mathbf{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Since the pair $(\alpha\mathbb{R}, \mathbb{R})$ is homeomorphic to $(\mathbf{S}^1, \mathbf{S}^1 \setminus \{(1, 0)\})$, we need to prove that $\mathrm{C}(\mathbf{I}, \mathbf{S}^1 \setminus \{(1, 0)\})$ is not homotopy dense in $\mathrm{cl}_{\mathrm{Cld}^*_F(\mathbf{I}\times\mathbf{S}^1)} \mathrm{C}(\mathbf{I}, \mathbf{S}^1 \setminus \{(1, 0)\})$. For simplicity, we denote $\mathrm{cl}_{\mathrm{Cld}^*_F(\mathbf{I}\times\mathbf{S}^1)} \mathrm{C}(\mathbf{I}, \mathbf{S}^1 \setminus \{(1, 0)\})$ by $\overline{\mathrm{C}(\mathbf{I}, \mathbf{S}^1 \setminus \{(1, 0)\})}$. Let $f, g : \mathbf{I} \to \mathbf{S}^1 \setminus \{(1, 0)\}$ be the constant maps such that $f(\mathbf{I}) = \{(0, 1)\}$ and $g(\mathbf{I}) = \{(0, -1)\}$. Then f and g miss $K = \{(0, -1, 0)\} \subset \mathbf{I} \times \mathbf{S}^1$, that is, they are contained in the open set

$$U = (\mathbf{I} \times \mathbf{S}^1 \setminus K)^+ \cap \overline{\mathrm{C}(\mathbf{I}, \mathbf{S}^1 \setminus \{(1, 0)\})} \subset \overline{\mathrm{C}(\mathbf{I}, \mathbf{S}^1 \setminus \{(1, 0)\})}.$$

It is sufficient to show that f and g are connected by a path in U but not connected by any path in $U \cap \mathrm{C}(\mathbf{I}, \mathbf{S}^1 \setminus \{(1, 0)\})$.

First, we shall construct a path from f to g in U. For each $t \in \mathbf{I}$, let $\phi(t) : \mathbf{I} \to \mathbf{S}^1$ be the constant map such that

$$\phi(t)(\mathbf{I}) = \{(\sin \pi(1 - 2t)/2, \cos \pi(1 - 2t)/2)\}.$$

Then we have the path $\phi : \mathbf{I} \to \mathrm{C}(\mathbf{I}, \mathbf{S}^1) \subset \mathrm{Cld}(\mathbf{I} \times \mathbf{S}^1)$ between f and g in U.

Next, we will show that any path $\gamma : \mathbf{I} \to \mathrm{C}(\mathbf{I}, \mathbf{S}^1 \setminus \{(1, 0)\})$ from f to g cannot be contained in U. Let $\beta : \mathrm{C}(\mathbf{I}, \mathbf{S}^1 \setminus \{(1, 0)\}) \to \mathbf{S}^1 \setminus \{(1, 0)\}$ be the map defined by $\beta(h) = h(0)$. Then for the composition $\beta\gamma : \mathbf{I} \to \mathbf{S}^1 \setminus \{(1,0)\}$, we have $\beta\gamma(0) = f(0) = (0,1)$ and $\beta\gamma(1) = g(0) = (0,-1)$. Since $\mathbf{S}^1 \setminus \{(1,0)\}$ is homeomorphic to $\mathbb{R}$, according to the Mean Value Theorem, we can find $t \in \mathbf{I}$ such that $\gamma(t)(0) = \beta\gamma(t) = (-1,0)$, which means that $\gamma(t) \notin (\mathbf{I} \times \mathbf{S}^1 \setminus K)^+ \subset U$. Thus f and g are not connected by any path in $U \cap \mathrm{C}(\mathbf{I}, \mathbf{S}^1 \setminus \{(1,0)\})$. □

Let $\mathbf{S}^{n-1}$ be the unit $(n-1)$-sphere in $\mathbb{R}^n$, that is, $\mathbf{S}^{n-1} = \{x = (x(i))_{i=1}^n \in \mathbb{R}^n \mid \sum_{i=1}^n x(i)^2 = 1\}$. Recall that $\overline{\mathbb{R}^n}$ is a compactification of $\mathbb{R}^n$ that is homeomorphic to the n-dimensional unit closed ball. Then we can establish the following:

Proposition 5.7.2. *For $n \geq 2$, the function space $\mathrm{C}(\mathbf{S}^{n-1}, \mathbb{R}^n)$ is not homotopy dense in the closure $\mathrm{cl}_{\mathrm{Cld}^*_F(\mathbf{S}^{n-1} \times \overline{\mathbb{R}^n})} \mathrm{C}(\mathbf{S}^{n-1}, \mathbb{R}^n)$.*

Proof. Let $B = \{x = (x(i))_{i=1}^n \in \mathbb{R}^n \mid \|x\| < 2\}$ and $\overline{B} = \{x = (x(i))_{i=1}^n \in \mathbb{R}^n \mid \|x\| \leq 2\}$, where $\|x\| = \max\{|x(i)| \mid i = 1, \cdots, n\}$. Then the pair $(\overline{B}, B)$ is homeomorphic to $(\overline{\mathbb{R}^n}, \mathbb{R}^n)$. So it suffices to prove that $\mathrm{C}(\mathbf{S}^{n-1}, B)$ is not homotopy dense in $\mathrm{cl}_{\mathrm{Cld}^*_F(\mathbf{S}^{n-1} \times \overline{B})} \mathrm{C}(\mathbf{S}^{n-1}, B)$. For simplicity, denote $\mathrm{cl}_{\mathrm{Cld}^*_F(\mathbf{S}^{n-1} \times \overline{B})} \mathrm{C}(\mathbf{S}^{n-1}, B)$ by $\overline{\mathrm{C}(\mathbf{S}^{n-1}, B)}$. Define two maps $f, g \in \mathrm{C}(\mathbf{S}^{n-1}, B)$ by

$$f(x) = (x(1), \cdots, x(n)) \text{ and } g(x) = (x(1), \cdots, x(n-1), -x(n)) \text{ for each } x = (x(1), \cdots, x(n)) \in \mathbf{S}^{n-1}.$$

Let $K = \mathbf{S}^{n-1} \times \{(0, \cdots, 0)\} \subset \mathbf{S}^{n-1} \times \overline{B}$. Then the maps f and g are contained in the open subset

$$U = ((\mathbf{S}^{n-1} \times \overline{B}) \setminus K)^+ \cap \overline{\mathrm{C}(\mathbf{S}^{n-1}, B)} \subset \overline{\mathrm{C}(\mathbf{S}^{n-1}, B)}.$$

Now, we shall show that f and g are connected by a path in U but not in $U \cap \mathrm{C}(\mathbf{S}^{n-1}, B)$, which implies that $\mathrm{C}(\mathbf{S}^{n-1}, B)$ is not homotopy dense in $\overline{\mathrm{C}(\mathbf{S}^{n-1}, B)}$.

(1) We prove that the maps f and g are connected by a path in U. Set

$$A = \{(1, 0, \cdots, 0)\} \times \mathbf{S}^{n-1} \cup \mathbf{S}^{n-1} \times \{(-1, 0, \cdots, 0)\} \subset \mathbf{S}^{n-1} \times \overline{B}.$$

We will construct a path linking f to A in U. Define a map $\phi : \mathbf{S}^{n-1} \times [0,1) \to B$ as follows: For $x = (x(1), \cdots, x(n)) \in \mathbf{S}^{n-1}$ and $t \in [0,1)$, let

$$\phi(x,t) = \begin{cases} (-1, 0, \cdots, 0) & \text{if } x(1) \leq 2t-1, \\ ((x(1)-t)/(1-t), \alpha x(2), \cdots, \alpha x(n)) & \text{if } 2t-1 < x(1) < 1, \\ (1, 0, \cdots, 0) & \text{if } x(1) = 1, \end{cases}$$

where $\alpha = (((1-t)^2 - (x(1)-t)^2)/((1-t)^2(1-x(1)^2)))^{1/2}$. So we can get the function $\Phi : \mathbf{I} \to \mathrm{Cld}(\mathbf{S}^{n-1} \times \overline{B})$ defined by

$$\Phi(t) = \begin{cases} \phi_t & \text{if } t \in [0,1), \\ A & \text{if } t = 1. \end{cases}$$

Then it follows from the continuity of ϕ that Φ is continuous on $[0,1)$. To verify the continuity of Φ at $t = 1$, take any neighborhood N of $\Phi(1) = A$ in $\mathrm{Cld}^*_F(\mathbf{S}^{n-1} \times \overline{B})$. Then we can choose open sets $V_j \subset \mathbf{S}^{n-1} \times \overline{B}$, $j = 1, \cdots, m$, and a compact set $L \subset \mathbf{S}^{n-1} \times \overline{B}$ so that

$$A \in \bigcap_{j=1}^m V_j^- \cap ((\mathbf{S}^{n-1} \times \overline{B}) \setminus L)^+ \subset N.$$

We use an admissible metric ρ on $\mathbf{S}^{n-1} \times \overline{B}$ defined as follows:

$$\rho((x,y),(x',y')) = \max\{\|x - x'\|, \|y - y'\|\}.$$

Since $A \in \bigcap_{j=1}^m V_j^-$, we can find $(x_j, y_j) \in A$ and $\epsilon_j > 0$ for each $j = 1, \cdots, m$ so that $\rho((x, y), (x_j, y_j)) < \epsilon_j$ implies that $(x, y) \in V_j$. Moreover, we have $\epsilon_L = \inf\{\rho((x, y), L) \mid (x, y) \in A\} > 0$ because $A \in ((\mathbf{S}^{n-1} \times \overline{B}) \setminus L)^+$ and L is compact. Let $\epsilon = \min\{1, \epsilon_j, \epsilon_L \mid j = 1, \cdots, m\}$ and take any $t' \in ((1 + (1 - \epsilon^2)^{1/2})/2, 1)$ (i.e., $2t' - 1 > (1 - \epsilon^2)^{1/2}$).

First, we show that $\Phi(t') \in V_j^-$ for every $j = 1, \cdots, m$. When $x_j = (1, 0, \cdots, 0)$, we can find $x'_j \in \mathbf{S}^{n-1}$ with $x'_j(1) \geq 2t' - 1$ so that $\Phi(t')(x'_j) = \phi_{t'}(x'_j) = y_j$. Then, note that

$$1 - x'_j(1) \leq 1 - (2t' - 1) < 1 - (1 - \epsilon^2)^{1/2} \leq \epsilon \text{ and}$$
$$|x'_j(i)| \leq (1 - x'_j(1)^2)^{1/2} \leq (1 - (2t' - 1)^2)^{1/2} < (1 - (1 - \epsilon^2))^{1/2} = \epsilon$$

for $i = 2, \cdots, n$. It follows that

$$\begin{aligned}\rho((x'_j, \Phi(t')(x'_j)), (x_j, y_j)) &= \rho((x'_j, y_j), ((1, 0, \cdots, 0), y_j)) = \|x'_j - (1, 0, \cdots, 0)\| \\ &= \max\{1 - x'_j(1), |x'_j(i)| \mid i = 2, \cdots, n\} < \epsilon,\end{aligned}$$

hence $\Phi(t') \in V_j^-$. When $x_j \neq (1, 0, \cdots, 0)$, we get $y_j = (-1, 0, \cdots, 0)$. Observe that there exists $x'_j \in \{x = (x(i))_{i=1}^n \in \mathbf{S}^{n-1} \mid x(1) \leq 2t' - 1\}$ such that for each $i = 2, \cdots, n$,

$$|x_j(i) - x'_j(i)| \leq (1 - (2t' - 1)^2)^{1/2} < (1 - (1 - \epsilon^2))^{1/2} = \epsilon.$$

Moreover, we have

$$|x_j(1) - x'_j(1)| < 1 - (2t' - 1) < 1 - (1 - \epsilon^2)^{1/2} < \epsilon,$$

hence $\|x'_j - x_j\| < \epsilon$. Since

$$\Phi(t') \cap \mathbf{S}^{n-1} \times \{(-1, 0, \cdots, 0)\} = \{x = (x(i))_{i=1}^n \in \mathbf{S}^{n-1} \mid x(1) \leq 2t' - 1\} \times \{(-1, 0, \cdots, 0)\},$$

it follows that

$$\rho((x'_j, \Phi(t')(x'_j)), (x_j, y_j)) = \rho((x'_j, (-1, 0, \cdots, 0)), (x_j, (-1, 0, \cdots, 0))) = \|x'_j - x_j\| < \epsilon,$$

which implies that $\Phi(t') \in V_j^-$. Therefore, $\Phi(t') \in \bigcap_{j=1}^m V_j^-$.

Next, we verify that $\Phi(t') \in ((\mathbf{S}^{n-1} \times \overline{B}) \setminus L)^+$. Fix any $(x, y) \in \Phi(t')$. When $y = (-1, 0, \cdots, 0)$, the point $(x, y) \in A$, which means that $(x, y) \notin L$. When $y \neq (-1, 0, \cdots, 0)$, we have $x(1) > 2t' - 1$. Then

$$\begin{aligned}\rho((x, y), L) &\geq \rho(((1, 0, \cdots, 0), y), L) - \rho((x, y), ((1, 0, \cdots, 0), y)) \\ &\geq \epsilon_L - \|x - (1, 0, \cdots, 0)\| > \epsilon_L - \epsilon \geq 0.\end{aligned}$$

Hence $(x, y) \notin L$. It follows that $\Phi(t') \in ((\mathbf{S}^{n-1} \times \overline{B}) \setminus L)^+$. Consequently, Φ is continuous at $t = 1$.

Observe that $\Phi(\mathbf{I}) \subset U$. Hence f and A are linked by the path Φ in U. Similarly, we can construct a path from A to g in U, so f is connected to g by a path in U.

(2) We show that the maps f and g are not connected by any path in $U \cap \mathrm{C}(\mathbf{S}^{n-1}, B)$. Assume that f and g are connected by a path $\Phi : \mathbf{I} \to U \cap \mathrm{C}(\mathbf{S}^{n-1}, B)$. Then Φ induces a homotopy $h : \mathbf{S}^{n-1} \times \mathbf{I} \to B \setminus \{(0, \cdots, 0)\}$ from f to g. Taking a retract $r : B \setminus \{(0, \cdots, 0)\} \to \mathbf{S}^{n-1}$, we have the homotopy $rh : \mathbf{S}^{n-1} \times \mathbf{I} \to \mathbf{S}^{n-1}$ from $rf = \mathrm{id}_{\mathbf{S}^{n-1}}$ to $rg = -\mathrm{id}_{\mathbf{S}^{n-1}}$, where $-\mathrm{id}_{\mathbf{S}^{n-1}}(x) = (x(1), \cdots, x(n-1), -x(n))$ for each $x = (x(1), \cdots, x(n)) \in \mathbf{S}^{n-1}$. This is a contradiction. Therefore, f and g are not connected by any path in $U \cap \mathrm{C}(\mathbf{S}^{n-1}, B)$. Thus the proof is complete. $\square$

Chapter 6

A space of hypo-graphs and its compactification

For each function $f : X \to Y$ from a space X into a dendrite Y and $v \in Y$, we can define the hypo-graph $\downarrow_v f$ of f with respect to v as follows:

$$\downarrow_v f = \bigcup_{x \in X} \{x\} \times [v, f(x)] \subset X \times Y.$$

Recall that the symbol $[x, y]$ means the unique arc of two points x, y in a dendrite Y, see Fact 2. When f is continuous, the hypo-graph $\downarrow_v f$ is closed in $X \times Y$. Hence we can regard

$$\downarrow_v \mathrm{C}(X, Y) = \{\downarrow_v f \mid f : X \to Y \text{ is continuous}\}$$

as the subspace of the hyperspace $\mathrm{Cld}_V(X \times Y)$ endowed with the Vietoris topology. Let $\overline{\downarrow_v \mathrm{C}(X, Y)}$ be the closure of $\downarrow_v \mathrm{C}(X, Y)$ in $\mathrm{Cld}_V(X \times Y)$. In the case that $Y = \mathbf{I}$ and $v = 0$, we can consider

$$\downarrow_0 \mathrm{USC}(X, \mathbf{I}) = \{\downarrow_0 f \mid f : X \to \mathbf{I} \text{ is upper semi-continuous}\}$$

as the subspace in $\mathrm{Cld}_V(X \times \mathbf{I})$. Z. Yang and X. Zhou [63, 64] showed the following theorem:

Theorem 6.0.1. *Let X be a compact metrizable space. If the set of isolated points is not dense in X, then $\downarrow_0 \mathrm{USC}(X, \mathbf{I}) = \overline{\downarrow_0 \mathrm{C}(X, \mathbf{I})}$ and the pair $(\downarrow_0 \mathrm{USC}(X, \mathbf{I}), \downarrow_0 \mathrm{C}(X, \mathbf{I}))$ is homeomorphic to $(\mathbf{Q}, \mathbf{c}_0)$.*

This result is a counterpart of the one of [27] (cf. Chapter 6 of [43]) concerning function spaces endowed with the pointwise convergence topology. The aim of this chapter is to generalize the above theorem as follows:

Main Theorem. *Let X be an infinite, locally connected, compact metrizable space, Y a dendrite and $v \in Y$ an end point of Y. Then the pair $(\overline{\downarrow_v \mathrm{C}(X, Y)}, \downarrow_v \mathrm{C}(X, Y))$ is homeomorphic to $(\mathbf{Q}, \mathbf{c}_0)$.*

In the above, we assume the stronger condition for a compact metrizable space X than the one of Z. Yang and X. Zhou's. In the last section, we will discuss this gap.

Remark 13. The space $\downarrow_v \mathrm{C}(X, Y)$ has a cluster point in $\mathrm{Cld}_V(X \times Y)$ which is not the hypo-graph of any map from X to Y. For example, let $X = \mathbf{I}$, $Y = \{0\} \times \mathbf{I} \cup [-1, 1] \times \{1\}$ a triod and $v = (0, 0) \in Y$. Define a closed set A in $X \times Y$ as follows:

$$A = \mathbf{I} \times \{0\} \times \mathbf{I} \cup \{0\} \times [-1, 1] \times \{1\} \cup \{(x, t \sin(\pi/x), 1) \mid x \in (0, 1], t \in \mathbf{I}\}.$$

For each $n \in \mathbb{N}$, let $f_n : X \to [-1,1] \times \{1\} \subset Y$ be the map defined by

$$f_n(x) = \begin{cases} (\sin(\pi/x), 1) & \text{if } x \geq 1/2n, \\ (0,1) & \text{if } x \leq 1/2n. \end{cases}$$

Then observe that

$$\downarrow_v f_n = \mathbf{I} \times \{0\} \times \mathbf{I} \cup \{(x, t\sin(\pi/x), 1) \mid x \in [1/2n, 1], t \in \mathbf{I}\}$$

and the sequence $(\downarrow_v f_n)_{n\in\mathbb{N}}$ converges to A in $\mathrm{Cld}_V(X \times \mathbf{I})$. However, the set A is not the hypo-graph of any map from X to Y.

6.1 Preliminaries

From now on, we proceed with our argument in the following assumption:

- $X = (X, d_X)$ is a compact metric space, and $Y = (Y, d_Y)$ is a dendrite with a convex metric d_Y and a distinguished end point $\mathbf{0} \in Y$.

Remark that any dendrite admits a convex metric, see Fact 3 in Chapter 1. For simplicity, we write $\downarrow\mathrm{C}(X,Y) = \downarrow_{\mathbf{0}} \mathrm{C}(X,Y)$. We use an admissible metric for the product space $X \times Y$ defined by

$$\rho((x,y),(x',y')) = \max\{d_X(x,x'), d_Y(y,y')\} \text{ for each } x,x' \in X \text{ and } y,y' \in Y.$$

Define $r : Y \times \mathbf{I} \to Y$ by $r(y,t) = \gamma(\mathbf{0}, y, t)$ for each $y \in Y$ and $t \in \mathbf{I}$, where γ is the map as in Lemma 1.5.1. Note that $r_0(Y) = \{\mathbf{0}\}$ and $r_1 = \mathrm{id}_Y$. Using this map r, we can define the homotopy $\overline{r} : \overline{\downarrow\mathrm{C}(X,Y)} \times \mathbf{I} \to \overline{\downarrow\mathrm{C}(X,Y)}$ as follows:

$$\overline{r}(A,t) = (\mathrm{id}_X \times r_t)(A) = \{(x, r_t(y)) \mid (x,y) \in A\}.$$

Then $\overline{r}_0(\overline{\downarrow\mathrm{C}(X,Y)}) = X \times \{\mathbf{0}\}$ and $\overline{r}_1 = \mathrm{id}_{\overline{\downarrow\mathrm{C}(X,Y)}}$. We shall verify the uniform continuity of $\overline{r}$. Take any $\epsilon > 0$. According to Lemma 1.5.1, the map r is uniform continuous. Hence we can choose $\epsilon > \delta > 0$ so that for each $y, y' \in Y$ and $t, t' \in \mathbf{I}$, if $d_Y(y,y') < \delta$ and $|t-t'| < \delta$, then $d_Y(r(y,t), r(y',t')) < \epsilon$. Now, let $A, A' \in \overline{\downarrow\mathrm{C}(X,Y)}$ and $t, t' \in \mathbf{I}$ such that $\rho_H(A,A') < \delta$ and $|t-t'| < \delta$. For each $(x,z) \in \overline{r}_t(A)$, there is a point $y \in A(x)$ such that $z = r_t(y)$. Since $\rho((x,y),A') < \delta$, we can find $(x',y') \in A'$ such that $\rho((x,y),(x',y')) < \delta$, which means that $d_X(x,x') < \delta$ and $d_Y(y,y') < \delta$. Let $z' = r_{t'}(y') \in A'(x')$. Then $(x',z') \in \overline{r}_{t'}(A')$ and $d_Y(z,z') = d_Y(r_t(y), r_{t'}(y')) < \epsilon$, and hence $\rho((x,z),(x',z')) = \max\{d_X(x,x'), d_Y(z,z')\} < \epsilon$. Thus we have $\rho((x,z), \overline{r}_{t'}(A')) < \epsilon$. By the same argument, we can show that $\rho((x',z'), \overline{r}_t(A)) < \epsilon$ for each $(x',z') \in r_{t'}(A')$. Therefore $\rho_H(\overline{r}_t(A), \overline{r}_{t'}(A)) < \epsilon$. Consequently, the map $\overline{r}$ is uniformly continuous. Then $\overline{r}$ is a contraction of $\overline{\downarrow\mathrm{C}(X,Y)}$.

The following lemma will often be used in this chapter, which can be easily proved.

Lemma 6.1.1. *Let A, A', B and B' be closed sets in a compact metric space $Z = (Z,d)$. Then*

$$d_H(A \cup B, A' \cup B') \leq \max\{d_H(A,A'), d_H(B,B')\}.$$

6.2 The closure of $\downarrow\mathrm{C}(X,Y)$ in $\mathrm{Cld}(X \times Y)$

This section is devoted to proving the following theorem:

Theorem 6.2.1. *If X has no isolated points, then $\overline{\downarrow\mathrm{C}(X,Y)}$ is an AR.*

For each $A \in \mathrm{Cld}(X \times Y)$, we define a set-valued function $A : X \to \mathrm{Cld}^*(Y)$ as follows:

$$A(x) = \{y \in Y \mid (x, y) \in A\} \in \mathrm{Cld}^*(Y).$$

For the sake of convenience, let $A(B) = \bigcup_{x \in B} A(x)$ for each $B \subset X$.

Lemma 6.2.2. *If X has no isolated points, then*

$$\overline{{\downarrow}\mathrm{C}(X, Y)} = \{A \in \mathrm{Cld}(X \times Y) \mid A(x) \neq \emptyset \textit{ for all } x \in X \textit{ and } y \in A(x) \Rightarrow [\mathbf{0}, y] \subset A(x)\}.$$

Proof. For convenience sake, let F be the set of the right side of the above equality. Then observe that ${\downarrow}\mathrm{C}(X, Y) \subset F$.

First, we prove that F is closed in $\mathrm{Cld}_V(X \times Y)$. Let A be the limit of a sequence $(A_n)_{n \in \mathbb{N}}$ in F. We shall show that $A(x) \neq \emptyset$ for every $x \in X$. For $n \in \mathbb{N}$, we can take $y_n \in A_n(x) \neq \emptyset$. Because of the compactness of Y, we can assume that $(y_n)_{n \in \mathbb{N}}$ converges to some $y \in Y$. Since $\rho_H(A_n, A) \to 0$ as $n \to \infty$ and

$$\rho((x, y), A_n) \leq \rho((x, y), (x, y_n)) = d_Y(y, y_n) \to 0 \text{ as } n \to \infty,$$

it follows that $(x, y) \in A$. Hence $A(x) \neq \emptyset$. To show that $[\mathbf{0}, y] \subset A(x)$ for each $y \in A(x)$, take any $z \in [\mathbf{0}, y]$. Since $(x, y) \in A$, we can choose $(x_n, y_n) \in A_n$, $n \in \mathbb{N}$, so that $(x_n, y_n) \to (x, y)$ as $n \to \infty$. According to Lemma 1.5.1, we can find $z_n \in [\mathbf{0}, y_n]$, $n \in \mathbb{N}$, such that $d_Y(z, z_n) \leq d_Y(y, y_n)$. Since $y_n \to y$ as $n \to \infty$, we have $z_n \to z$ as $n \to \infty$. Then $z_n \in [\mathbf{0}, y_n] \subset A_n(x_n)$, so $(x_n, z_n) \in A_n$ for every $n \in \mathbb{N}$. Because $(x_n, z_n) \to (x, z)$ as $n \to \infty$, it follows that $(x, z) \in A$, so $z \in A(x)$. Thus we have $[\mathbf{0}, y] \subset A(x)$. Consequently, $A \in F$, so F is closed in $\mathrm{Cld}_V(X \times Y)$.

Next, we will show that ${\downarrow}\mathrm{C}(X, Y)$ is dense in F. For each $\epsilon > 0$ and $A \in F$, because of the compactness of A, A has finite points (x_i, y_i), $i = 1, \cdots, n$, such that $A \subset \bigcup_{i=1}^n B_\rho((x_i, y_i), \epsilon/2)$, where we can take $x_i \neq x_j$ if $i \neq j$ because X has no isolated points. Let $A_0 = \bigcup_{i=1}^n \{x_i\} \times [\mathbf{0}, y_i] \subset A$. Then $A \subset N(A_0, \epsilon/2)$, which implies that $\rho_H(A_0, A) < \epsilon/2$. Let $\delta = \min\{\epsilon, d_X(x_i, x_j) \mid i \neq j\}/3 > 0$. Note that $\overline{B_{d_X}}(x_i, \delta) \cap \overline{B_{d_X}}(x_j, \delta) = \emptyset$ for every $i \neq j$. Using Urysohn maps, we can construct a map $f : X \to Y$ such that $f\big(X \setminus \bigcup_{i=1}^n B_{d_X}(x_i, \delta)\big) = \{0\}$, $f(B_{d_X}(x_i, \delta)) \subset [\mathbf{0}, y_i]$ and $f(x_i) = y_i$ for each $i = 1, \cdots, n$. Then $\rho_H({\downarrow}f, A_0) < \delta \leq \epsilon/3$. It follows that

$$\rho_H({\downarrow}f, A) \leq \rho_H({\downarrow}f, A_0) + \rho_H(A_0, A) \leq \epsilon/3 + \epsilon/2 < \epsilon.$$

Therefore ${\downarrow}\mathrm{C}(X \times Y)$ is dense in F. □

We show the uniformly local path-connectedness of $\overline{{\downarrow}\mathrm{C}(X, Y)}$ as follows:

Lemma 6.2.3. *If there are no isolated points in X, then $\overline{{\downarrow}\mathrm{C}(X, Y)}$ is uniformly locally path-connected with respect to ρ_H.*

Proof. Let $\epsilon > 0$ and $A, A' \in \overline{{\downarrow}\mathrm{C}(X, Y)}$ such that $\rho_H(A, A') < \epsilon/2$. We define a path $h : \mathbf{I} \to \overline{{\downarrow}\mathrm{C}(X, Y)}$ from A to $A \cup A'$ by $h(t) = A \cup \overline{r}_t(A')$, where Lemma 6.2.2 guarantees $h(\mathbf{I}) \subset \overline{{\downarrow}\mathrm{C}(X, Y)}$. The continuity of h follows from the one of $\overline{r}$ and Lemma 6.1.1. In fact,

$$\rho_H(h(t), h(t')) = \rho_H(A \cup \overline{r}_t(A'), A \cup \overline{r}_{t'}(A')) \leq \rho_H(\overline{r}_t(A'), \overline{r}_{t'}(A')).$$

Moreover, $A \subset h(t), h(t') \subset A \cup A'$, and hence

$$\rho_H(h(t), h(t')) \leq \rho_H(A, A \cup A') = \rho_H(A, A') < \epsilon/2.$$

It follows that $\mathrm{diam}_{\rho_H} h(\mathbf{I}) \leq \rho_H(A, A') < \epsilon/2$. Consequently, A is connected with $A \cup A'$ by an $\epsilon/2$-path. Similarly, A' is connected with $A \cup A'$ by an $\epsilon/2$-path. Therefore A and A' are connected by an ϵ-path. Thus the proof is complete. □

Now, we shall prove Theorem 6.2.1.

Proof of Theorem 6.2.1. By Lemma 6.2.3, $\overline{{\downarrow}\mathrm{C}(X,Y)}$ is a Peano continuum. Then, according to the Wojdysławski Theorem [65], refer to [42, Theorem 5.3.14], we have $\mathrm{Cld}_V(\overline{{\downarrow}\mathrm{C}(X,Y)})$ is an AR. Identifying $A \in \mathrm{Cld}_V(X \times Y)$ with $\{A\} \in \mathrm{Cld}_V(\mathrm{Cld}_V(X \times Y))$, we can regard $\mathrm{Cld}_V(X \times Y) \subset \mathrm{Cld}_V(\mathrm{Cld}_V(X \times Y))$. Then the union operator

$$\bigcup : \mathrm{Cld}_V(\mathrm{Cld}_V(X \times Y)) \ni \mathcal{A} \mapsto \bigcup \mathcal{A} \in \mathrm{Cld}_V(X \times Y)$$

is a retraction, see [42, Proposition 5.3.6]. As is easily observed due to Lemma 6.2.2, we have the image $\bigcup(\mathrm{Cld}_V(\overline{{\downarrow}\mathrm{C}(X,Y)})) = \overline{{\downarrow}\mathrm{C}(X,Y)}$. It follows that $\overline{{\downarrow}\mathrm{C}(X,Y)}$ is a retract of the AR $\mathrm{Cld}_V(\overline{{\downarrow}\mathrm{C}(X,Y)})$. Therefore $\overline{{\downarrow}\mathrm{C}(X,Y)}$ is an AR. $\square$

6.3 The homotopy denseness of ${\downarrow}\mathrm{C}(X,Y)$ in $\overline{{\downarrow}\mathrm{C}(X,Y)}$

In this section, we will prove the following theorem:

Theorem 6.3.1. *If X has no isolated points, then ${\downarrow}\mathrm{C}(X,Y)$ is homotopy dense in $\overline{{\downarrow}\mathrm{C}(X,Y)}$.*

Proof. We only need to verify condition (hd) with respect to $\alpha = 10$ in Lemma 1.2.7. Let K be a locally finite countable simplicial complex and $f : K^{(0)} \to {\downarrow}\mathrm{C}(X,Y)$. We shall construct a map $\overline{f} : |K| \to {\downarrow}\mathrm{C}(X,Y)$ such that the restriction $\overline{f}|_{K^{(0)}} = f$ and $\mathrm{diam}_{\rho_H} \overline{f}(\sigma) \leq 10 \,\mathrm{diam}_{\rho_H} f(\sigma^{(0)})$ for every $\sigma \in K$. For simplicity, let $\epsilon_\sigma = \mathrm{diam}_{\rho_H} f(\sigma^{(0)}) \geq 0$ for each $\sigma \in K \setminus K^{(0)}$. Let K_0 be the full subcomplex of K such that

$$K_0^{(0)} = \{v \in K^{(0)} \mid f(\mathrm{St}(v,K)^{(0)}) \text{ is a singleton}\},$$

where $\mathrm{St}(v,K)$ is the star at v in K. Note that $f(\sigma^{(0)})$ is a singleton if $\sigma \in K$ and $\sigma \cap |K_0| \neq \emptyset$. We define $K_1 = \{\sigma \in K \mid \sigma \cap |K_0| = \emptyset\}$. For every $v \in K_1^{(0)}$, since $\mathrm{diam}_{\rho_H} f(\mathrm{St}(v,K)^{(0)}) > 0$, we can define

$$\epsilon_v = \min\{\epsilon_\sigma \mid \sigma \in \mathrm{St}(v,K), \epsilon_\sigma > 0\} > 0.$$

Let $f_0 : |K_0| \to {\downarrow}\mathrm{C}(X,Y)$ be the map such that $f_0(\sigma) = f(\sigma^{(0)})$ for each $\sigma \in K_0$.

Since K is locally finite and X has no isolated points, we can choose a finite sets $A_v \subset X$ and $\delta_v > 0$, $v \in K_1^{(0)}$, so that

(1) $\rho_H(f(v)|_{A_v}, f(v)) < \epsilon_\sigma$,

(2) $B_{d_X}(a, \delta_v) \cap B_{d_X}(a', \delta_{v'}) = \emptyset$ if $v \neq v' \in K_1^{(0)}$, v and v' are contained in some $\sigma \in K$, $a \in A_v$, and $a' \in A_{v'}$,

(3) $B_{d_X}(a, \delta_v) \cap B_{d_X}(a', \delta_v) = \emptyset$ if $a \neq a' \in A_v$ and $v \in K_1^{(0)}$,

where $f(v)|_{A_v} = \bigcup_{a \in A_v} \{a\} \times [0, f(v)(a)]$. First, we will construct a map $f_1 : |K_1| \to {\downarrow}\mathrm{C}(X,Y)$ such that $\rho_H(f_1(v), f(v)) < \epsilon_v$ for each $v \in K_1^{(0)}$ and $\mathrm{diam}_{\rho_H} f_1(\sigma) < 7\epsilon_\sigma$ for each $\sigma \in K_1$. For every $v \in K_1^{(0)}$, we define $f_1(v) \in {\downarrow}\mathrm{C}(X,Y)$ as follows:

$$f_1(v)(x) = \begin{cases} r(f(v)(x) \times \{(\delta_v - d_X(x, A_v))/\delta_v\}) & \text{if } d_X(x, A_v) \leq \delta_v, \\ \{\mathbf{0}\} & \text{if } d_X(x, A_v) \geq \delta_v. \end{cases}$$

Since $f(v)|_{A_v} \subset f_1(v) \subset f(v)$, it follows that $\rho_H(f(v), f_1(v)) \leq \rho_H(f(v)|_{A_v}, f(v)) < \varepsilon_v$. Denote the barycenter of $\sigma \in K_1$ by $\hat{\sigma}$. For $\sigma \in K_1$, let

$$f_1(\hat{\sigma}) = \bigcup_{v \in \sigma^{(0)}} f_1(v) \in {\downarrow}\mathrm{C}(X,Y).$$

For each $z \in \sigma$, there exist faces $\sigma_0 \preccurlyeq \sigma_1 \preccurlyeq \cdots \preccurlyeq \sigma_n \preccurlyeq \sigma$ of σ such that $z = \sum_{i=0}^n t_i \hat{\sigma}_i$, where $\sum_{i=0}^n t_i = 1$ and $t_i > 0$. Then we can define

$$f_1(z) = \bigcup_{i=0}^n \overline{r}\Big(f_1(\hat{\sigma}_i), \sum_{i=j}^n t_j \Big) \in \downarrow\mathrm{C}(X,Y).$$

For each $\sigma \in K_1$ and $v \in \sigma^{(0)}$, the continuity of $f_1|_{\mathrm{St}(v,\mathrm{Sd}\,K)\cap\sigma}$ follows from the ones of both the map $\overline{r}$ and the union operator on $\mathrm{Cld}_V(X \times Y)$, where $\mathrm{Sd}\,K$ is the barycentric subdivision of K. Since K_1 is locally finite, it follows that f_1 is continuous. Thus we have a map $f_1 : |K_1| \to \downarrow\mathrm{C}(X,Y)$. For each $\sigma \in K_1$, let $v \in \sigma^{(0)}$ and $z \in |\operatorname{St}(v, \mathrm{Sd}\,K)| \cap \sigma$. By the definition of f_1, we have

$$f_1(v) \subset f_1(z) \subset f_1(\hat{\sigma}) = \bigcup_{v' \in \sigma^{(0)}} f(v').$$

Then it follows that

$$\begin{aligned}
\rho_H(f_1(z), f_1(v)) &\le \rho_H\Big(f_1(v), \bigcup_{v' \in \sigma^{(0)}} f(v') \Big) \le \rho_H(f_1(v), f(v)) + \rho_H\Big(f(v), \bigcup_{v' \in \sigma^{(0)}} f(v') \Big) \\
&\le \rho_H(f_1(v), f(v)) + \max\{\rho_H(f(v), f(v')) \mid v' \in \sigma^{(0)}\} \\
&\le \rho_H(f_1(v), f(v)) + \operatorname{diam}_{\rho_H} f(\sigma^{(0)}) \le \epsilon_v + \epsilon_\sigma \le 2\epsilon_\sigma.
\end{aligned}$$

For each $z, z' \in \sigma \in K_1$, we can choose vertices $v, v' \in \sigma^{(0)}$ such that $z \in |\operatorname{St}(v, \mathrm{Sd}\,K)|$ and $z' \in |\operatorname{St}(v'), \mathrm{Sd}\,K|$. Then we have

$$\begin{aligned}
\rho_H(f_1(z), f_1(z')) &\le \rho_H(f_1(z), f_1(v)) + \rho_H(f_1(v), f(v)) + \rho_H(f(v), f(v')) \\
&\qquad + \rho_H(f(v'), f_1(v')) + \rho_H(f_1(v'), f_1(z')) \\
&< 2\epsilon_\sigma + \epsilon_v + \epsilon_\sigma + \epsilon_{v'} + 2\epsilon_\sigma \le 7\epsilon_\sigma.
\end{aligned}$$

Consequently, $\operatorname{diam}_{\rho_H} f_1(\sigma) < 7\epsilon_\sigma$ for each $\sigma \in K_1$.

Next, we construct a map $f_* : |K| \cup K^{(0)} \times \mathbf{I} \to \downarrow\mathrm{C}(X,Y)$, where $|K|$ is identified with $|K| \times \{0\} \subset |K| \times \mathbf{I}$. Let $f_*|_{|K_0|} = f_0$ and $f_*|_{|K_1|} = f_1$. For each $z \in |K| \setminus |K_0 \cup K_1|$, there exits $\sigma_0 \in K_0$ and $\sigma_1 \in K_1$ such that z is contained in the join of σ_0 and σ_1, and hence z can be uniquely written as follows: $z = tz_0 + (1-t)z_1$ for some $z_0 \in \sigma_0$, $z_1 \in \sigma_1$ and $t \in \mathbf{I}$. Then we can define

$$f_*(z) = \overline{r}(f_0(z_0), t) \cup f_1(z_1) \in \downarrow\mathrm{C}(X,Y).$$

Observe that $f_*(z_0) = f_0(z_0)$ and $f_*(z_1) = f_1(z_1)$. For each $(v, t) \in K^{(0)} \times \mathbf{I}$, we define

$$f_*(v, t) = \overline{r}(f(v), t) \cup f_1(v),$$

where $f_*(v, 0) = f_1(v)$ and $f_*(v, 1) = f(v)$.

Thirdly, we can obtain a map $g : |K| \to |K| \cup K^{(0)} \times \mathbf{I}$ so that $g(v) = (v, 1)$ for each $v \in K^{(0)}$ and $g(\sigma) = \sigma \cup \sigma^{(0)} \times \mathbf{I}$ for each $\sigma \in K \setminus K^{(0)}$. In fact, let $v \in K^{(0)}$ and $z = \sum_{i=0}^n t_i \hat{\sigma}_i \in |\operatorname{St}(v, \mathrm{Sd}\,K)|$, where $\sigma_0 \preccurlyeq \sigma_1 \preccurlyeq \cdots \preccurlyeq \sigma_n \in K$, $\sum_{i=0}^n t_i = 1$ and $t_i \ge 0$. We define

$$g(z) = \begin{cases} (1 - 2t_0)z + 2t_0 v & \text{if } t_0 \le 1/2, \\ (v, 2t_0 - 1) & \text{if } t_0 \ge 1/2. \end{cases}$$

Now, the desired map $\overline{f} : |K| \to \downarrow\mathrm{C}(X,Y)$ can be defined by $\overline{f} = f_* g$. As is easily observed, $\overline{f}|_{K^{(0)}} = f$. We will show that $\operatorname{diam}_{\rho_H} \overline{f}(\sigma) \le 10\epsilon_\sigma$ for every $\sigma \in K$. When $\sigma \in K_0$, we have $\operatorname{diam}_{\rho_H} \overline{f}(\sigma) =$

$\operatorname{diam}_{\rho_H} f(\sigma^{(0)}) = 0$. For each $\sigma \in K_1$, since $\overline{f}(\sigma) = f_1(\sigma) \cup f_*(\sigma^{(0)} \times \mathbf{I})$, it follows that

$$\begin{aligned}\operatorname{diam}_{\rho_H} \overline{f}(\sigma) &\leq \operatorname{diam}_{\rho_H} f_1(\sigma) + \operatorname{diam}_{\rho_H} f_*(\sigma^{(0)} \times \mathbf{I}) \\ &\leq \operatorname{diam}_{\rho_H} f_1(\sigma) + \operatorname{diam}_{\rho_H} f(\sigma^{(0)}) + 2\max\{\rho_H(f_1(v), f(v)) \mid v \in \sigma^{(0)}\} \\ &< 7\epsilon_\sigma + \epsilon_\sigma + 2\epsilon_\sigma = 10\epsilon_\sigma.\end{aligned}$$

When $\sigma \in K \setminus (K_0 \cup K_1)$, we can take $\sigma_0 \in K_0$ and $\sigma_1 \in K_1$ so that σ is the join of σ_0 and σ_1. Since $\sigma \in \operatorname{St}(v_0, K)$ for any $v_0 \in \sigma_0^{(0)} \subset K_0^{(0)}$, $f(\sigma^{(0)})$ is a singleton. For each $z = tz_0 + (1-t)z_1 \in \sigma$, where $z_0 \in \sigma_0$, $z_1 \in \sigma_1$ and $0 \leq t \leq 1$, choose $v \in \sigma_1^{(0)}$ such that $z_1 \in |\operatorname{St}(v, \operatorname{Sd} K)|$. Then $f(\sigma^{(0)}) = \{f(v)\}$, $f_1(v) \subset f_1(z_1) \subset f(v)$ and $f_*(z) = \overline{r}(f_0(z_0), t) \cup f_1(z_1) \subset f(v)$. Hence we get

$$\rho_H(f_*(z), f(\sigma^{(0)})) = \rho_H(f_*(z), f(v)) \leq \rho_H(f_1(v), f(v)) < \epsilon_v \leq \epsilon_\sigma.$$

Therefore for each $z, z' \in \sigma$,

$$\rho_H(f_*(z), f_*(z')) \leq \rho_H(f_*(z), f(\sigma^{(0)})) + \rho_H(f(\sigma^{(0)}), f_*(z')) + \operatorname{diam}_{\rho_H} f(\sigma^{(0)}) < \epsilon_\sigma + \epsilon_\sigma = 2\epsilon_\sigma.$$

Consequently, $\operatorname{diam}_{\rho_H} f_*(\sigma) \leq 2\epsilon_\sigma$. Since

$$\operatorname{diam}_{\rho_H} f_*(\sigma^{(0)} \times \mathbf{I}) \leq \operatorname{diam}_{\rho_H} f(\sigma^{(0)}) + \max\{\rho_H(f(v), f_1(v)) \mid v \in \sigma_1^{(0)}\} \leq \epsilon_{\sigma_1} \leq \epsilon_\sigma,$$

it follows that

$$\operatorname{diam}_{\rho_H} \overline{f}(\sigma) \leq \operatorname{diam}_{\rho_H} f_*(\sigma) + \operatorname{diam}_{\rho_H} f_*(\sigma^{(0)} \times \mathbf{I}) \leq 2\epsilon_\sigma + \epsilon_\sigma = 3\epsilon_\sigma.$$

Thus the proof is complete. □

6.4 The space $\downarrow\mathrm{C}(X, Y)$ is an $F_{\sigma\delta}$ set in $\overline{\downarrow\mathrm{C}(X, Y)}$

A dendrite Y has an order $\leq$ defined as follows: $x \leq y$ if $x \in [\mathbf{0}, y]$. For each $\delta, \epsilon > 0$, let $\mathcal{A}(\delta, \epsilon)$ be the set which consists of $A \in \overline{\downarrow\mathrm{C}(X, Y)}$ such that the following condition is satisfied:

- For all $x, x' \in X$, if $d_X(x, x') < \delta$ and $y, y' \in Y$ are maximal points of $A(x), A(x')$, respectively, then $d_Y(y, y') \leq \epsilon$.

To prove that $\downarrow\mathrm{C}(X, Y)$ is an $F_{\sigma\delta}$ set in $\overline{\downarrow\mathrm{C}(X, Y)}$, we need the following lemma.

Lemma 6.4.1. *For each $\delta, \epsilon > 0$, the set $\mathcal{A}(\delta, \epsilon)$ is closed in $\overline{\downarrow\mathrm{C}(X, Y)}$.*

Proof. Take any sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}(\delta, \epsilon)$ that converges to A in $\overline{\downarrow\mathrm{C}(X, Y)}$. To show that $A \in \mathcal{A}(\delta, \epsilon)$, let $(x, y), (x', y') \in A$ such that $d_X(x, x') < \delta$ and y, y' are maximal in $A(x), A(x')$, respectively. Since $A_n \to A$, there exist $(x_n, y_n), (x'_n, y'_n) \in A_n$ such that $(x_n, y_n) \to (x, y)$ and $(x'_n, y'_n) \to (x', y')$, see [42, Lemma 5.3.1]. Without loss of generality, we may assume that $d_X(x_n, x'_n) < \delta$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, there exist maximal points $z_n \in A_n(x_n)$ and $z'_n \in A_n(x'_n)$ such that $z_n \geq y_n$ and $z'_n \geq y'_n$. Because Y is compact, replacing $(z_n)_{n\in\mathbb{N}}$ and $(z'_n)_{n\in\mathbb{N}}$ with subsequences, we can assume that $z_n \to z \in Y$ and $z'_n \to z' \in Y$. Using Lemma 5.3.1 of [42] again, we have $z \in A(x)$ and $z' \in A(x')$. Then y is contained in the arc $[\mathbf{0}, z]$ from $\mathbf{0}$ to z. Indeed, if not, we have $d_Y(y, [\mathbf{0}, z]) > 0$. Since $y_n \to y$ and $z_n \to z$, we can choose $m \in \mathbb{N}$ so that $d_Y(y, y_m), d_Y(z, z_m) < d_Y(y, [\mathbf{0}, z])/2$. Note that $y_m \in [\mathbf{0}, z_m]$. Then there exists a point $p \in [\mathbf{0}, z]$ such that $d_Y(y_m, p) \leq d_Y(z, z_m) < d_Y(y, [\mathbf{0}, z])/2$ by Lemma 1.5.1. It follows that

$$d_Y(y, p) \leq d_Y(y, y_m) + d_Y(y_m, p) < d_Y(y, [\mathbf{0}, z])/2 + d_Y(y, [\mathbf{0}, z])/2 = d_Y(y, [\mathbf{0}, z]),$$

which is a contradiction. Hence $y \in [\mathbf{0}, z]$. By the maximality of y in $A(x)$, we have $y = z$. Similarly, $y' = z'$.

Since each $A_n \in \mathcal{A}(\delta, \epsilon)$, $d_X(x_n, x'_n) < \delta$ and z_n, z'_n are maximal in $A(x_n), A(x'_n)$, respectively, it follows that $d_Y(z_n, z'_n) \leq \epsilon$. Recall that $z_n \to z = y$ and $z'_n \to z' = y'$, so $d_Y(y, y') \leq \epsilon$. Consequently, we have $A \in \mathcal{A}(\delta, \epsilon)$. Thus the proof is complete. □

Now, we show the following:

Proposition 6.4.2. *The space* $\downarrow\mathrm{C}(X, Y)$ *is an* $F_{\sigma\delta}$ *set in* $\overline{\downarrow\mathrm{C}(X, Y)}$.

Proof. By virtue of Lemma 6.4.1, it suffices to show that

$$\downarrow\mathrm{C}(X, Y) = \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \mathcal{A}(1/m, 1/n).$$

From the definition, we need only to prove that $A(x)$ has the unique maximal point in Y for every $A \in \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \mathcal{A}(1/m, 1/n)$ and $x \in X$. Let $y, y' \in Y$ be maximal points in $A(x)$. For each $n \in \mathbb{N}$, we can choose $m \in \mathbb{N}$ such that $A \in \mathcal{A}(1/m, 1/n)$, which implies that $d_Y(y, y') < 1/n$. It follows that $d_Y(y, y') = 0$, that is, $y = y'$. Therefore the maximal point of $A(x)$ is unique, and hence A is the hypo-graph of some continuous function. This completes the proof. □

6.5 The Digging Lemma

The following lemma will play an important role for the rest of this chapter.

Lemma 6.5.1 (The Digging Lemma)**.** *Suppose that* Z *is a paracompact space,* $\phi : Z \to \downarrow\mathrm{C}(X, Y)$ *is a map, and* $a \in X$ *is a non-isolated point. Then for each map* $\epsilon : Z \to (0, 1)$*, there exist maps* $\psi : Z \to \downarrow\mathrm{C}(X, Y)$ *and* $\delta : Z \to (0, 1)$ *such that for each* $z \in Z$*,*

(a) $\rho_H(\phi(z), \psi(z)) < \epsilon(z)$,

(b) $\psi(z)(B_{d_X}(a, \delta(z))) = \{\mathbf{0}\}$.

Proof. For each $z \in Z$, let $\xi(z) = \sup\{\eta > 0 \mid \rho_H(\phi(z), \phi(z)|_{X \setminus B_{d_X}(a, \eta)}) < \epsilon(z)\}$. Since a is not isolated and $\phi(z) \in \downarrow\mathrm{C}(X, Y)$, we have $\xi(z) > 0$. We shall prove $\xi : Z \to (0, \infty)$ is a lower semi-continuous function. Fix any $z \in Z$ and $\eta \in (0, \xi(z))$. From the definition of $\xi(z)$,

($\star$) $\rho_H(\phi(z), \phi(z)|_{X \setminus B_{d_X}(a, \xi(z) - \eta/2)}) < (n-1)\epsilon(z)/n$ for some $n \in \mathbb{N}$.

Let $t = \min\{\eta/2, \epsilon(z)/3n\}$. Since ϕ and ϵ are continuous, the point z has a neighborhood N in Z such that if $z' \in N$, then $\rho_H(\phi(z), \phi(z')) < t$ and $|\epsilon(z) - \epsilon(z')| < \epsilon(z)/3n$. We shall show that for every $z' \in N$, $\xi(z') \geq \xi(z) - \eta$. Take any $(x, y) \in \phi(z')|_{B_{d_X}(a, \xi(z) - \eta)}$. Since $\rho_H(\phi(z), \phi(z')) < t$, we can choose $(x', y') \in \phi(z)$ so that $\rho((x, y), (x', y')) < t \leq \eta/2$. Then $d_X(x, x') < \eta/2$, that is, $(x', y') \in \phi(z)|_{B_{d_X}(a, \xi(z) - \eta/2)}$. Due to ($\star$), there exists $(x'', y'') \in \phi(z)|_{X \setminus B_{d_X}(a, \xi(z) - \eta/2)}$ such that $\rho((x', y'), (x'', y'')) < (n-1)\epsilon(z)/n$. Since $\rho_H(\phi(z), \phi(z')) < t$, we can find a point $(x''', y''') \in \phi(z')$ such that $\rho((x'', y''), (x''', y''')) < t \leq \eta/2$, which implies that $x''' \in X \setminus B_{d_X}(a, \xi(z) - \eta)$. Then it follows that

$$\begin{aligned}
\rho((x, y), (x''', y''')) &\leq \rho((x, y), (x', y')) + \rho((x', y'), (x'', y'')) + \rho((x'', y''), (x''', y''')) \\
&< t + (n-1)\epsilon(z)/n + t \leq (2/3n + (n-1)/n)\epsilon(z) \\
&= \epsilon(z) - \epsilon(z)/3n < \epsilon(z').
\end{aligned}$$

Thus ξ is lower semi-continuous.

By Theorem 2.7.6 of [50], we can obtain a map $\delta : Z \to (0,1)$ so that $\delta(z) < \xi(z)/2$ for each $z \in Z$. Now, we can define the desired map $\psi : Z \to \downarrow\mathrm{C}(X,Y)$ as follows:

$$\begin{aligned}\psi(z) = \phi(z)|_{X \setminus B_{d_X}(a,2\delta(z))} \cup B_{d_X}(a,\delta(z)) \times \{\mathbf{0}\} \\ \cup \{(x,y) \in X \times Y \mid \delta(z) \leq d_X(x,a) \leq 2\delta(z), y \in [\mathbf{0}, r(\max\phi(z)(x), d_X(x,a)/\delta(z) - 1)]\}.\end{aligned}$$

Remark that $\phi(z) \in \downarrow\mathrm{C}(X,Y)$ is the hypo-graph of the map $X \ni x \mapsto \max\phi(z)(x) \in Y$. By the definition of ψ, it is easy to show that ψ satisfies conditions (a) and (b).

Claim. *The function ψ is continuous.*

For every $z \in Z$ and $\epsilon > 0$, by Lemma 1.5.1, there exists $\delta_1 > 0$ such that $\delta_1 < 1/2$ and

$$d_Y(y,y_1) < \delta_1 \text{ and } |t - t_1| < \delta_1 \Rightarrow d_Y(r(y,t), r(y_1,t_1)) < \epsilon.$$

Take $\delta_2 > 0$ such that $\delta_2 \leq \delta_1/2$ and $\delta_2 \operatorname{diam}_{d_Y} Y < \epsilon$. We can choose $\delta_3 > 0$ so that $\delta_3 < \delta(z)$ and

$$a, b \in [\delta(z)/2, 5\delta(z)/2] \text{ and } |a-b| < \delta_3 \Rightarrow |b/a - 1| < \delta_2.$$

Since ϕ and δ are continuous, there exists a neighborhood U of z such that for each $z' \in U$, $\rho_H(\phi(z),\phi(z')) < \min\{\epsilon, \delta(z)\delta_1/2, \delta_3/4\}$, $|1/\delta(z) - 1/\delta(z')| < 2\delta_1/9\delta(z)$ and $|\delta(z) - \delta(z')| < \delta_3/8$. We shall verify that $\rho_H(\psi(z),\psi(z')) < \epsilon$ for each $z' \in U$. Take any $(x,y) \in \psi(z)$. It is sufficient to show that $(x,y) \in N(\psi(z'),\epsilon)$.

Case I. $d_X(x,a) \leq \delta(z)$

Then we have $y = \mathbf{0}$. So $(x,y) = (x,\mathbf{0}) \in \psi(z')$.

Case II. $\delta(z) < d_X(x,a) < \delta(z) + \delta_3$

Then $|d_X(x,a)/\delta(z) - 1| < \delta_2$, so

$$\begin{aligned}d_Y(\mathbf{0},y) &\leq d_Y(\mathbf{0}, r(\max\phi(z)(x), d_X(x,a)/\delta(z) - 1)) = (d_X(x,a)/\delta(z) - 1) d_Y(\mathbf{0}, \max\phi(z)(x)) \\ &< \delta_2 \operatorname{diam}_{d_Y} Y < \epsilon.\end{aligned}$$

Therefore $\rho((x,y),(x,\mathbf{0})) = d_Y(\mathbf{0},y) < \epsilon$.

Case III. $d_X(x,a) \geq \delta(z) + \delta_3$

Since $\rho_H(\phi(z),\phi(z')) < \min\{\epsilon, \delta(z)\delta_1/2, \delta_3/4\}$, there exists a point $(x_1,y_1) \in \phi(z')$ such that

$$\rho((x, \max\phi(z)(x)), (x_1,y_1)) < \min\{\epsilon, \delta(z)\delta_1/2, \delta_3/4\}.$$

Then we have

$$d_X(x,x_1) \leq \rho((x,\max\phi(z)(x)),(x_1,y_1)) < \min\{\epsilon, \delta(z)\delta_1/2, \delta_3/4\}.$$

Moreover, $|\delta(z) - \delta(z')| < \delta_3/8$, and hence

$$d_X(x_1,a) \geq d_X(x,a) - d_X(x,x_1) > \delta(z) + \delta_3 - \delta_3/4 > \delta(z') - \delta_3/8 + \delta_3 - \delta_3/4 > \delta(z').$$

If $d_X(x_1,a) \geq 2\delta(z')$, we get $(x_1,y_1) \in \psi(z')$. Since $y \in [\mathbf{0}, \max\phi(z)(x)]$, by Lemma 1.5.1, we can find $y_2 \in [\mathbf{0}, y_1]$ such that $d_Y(y,y_2) \leq d_Y(\max\phi(z)(x), y_1) < \epsilon$. It follows that $(x_1,y_2) \in \psi(z')$ and

$$\rho((x,y),(x_1,y_2)) = \max\{d_X(x,x_1), d_Y(y,y_2)\} < \epsilon.$$

Now, we need only to consider the case that $\delta(z') < d_X(x_1,a) < 2\delta(z')$. Let $y_3 = r(y_1, d_X(x_1,a)/\delta(z') - 1)$. Then $y_3 \in [\mathbf{0}, r(\max\phi(z')(x_1), d_X(x_1,a)/\delta(z') - 1)]$, so $(x_1,y_3) \in \psi(z')$.

Case III-i. $\delta(z) + \delta_3 \le d_X(x,a) < 2\delta(z)$
Then we have

$$\begin{aligned}|d_X(x,a)/\delta(z) - 1 - (d_X(x_1,a)/\delta(z') - 1)| &\le |1/\delta(z) - 1/\delta(z')|d_X(x_1,a) + |d_X(x,a) - d_X(x_1,a)|/\delta(z)\\ &\le |1/\delta(z) - 1/\delta(z')|(d_X(x,x_1) + d_X(x,a)) + d_X(x,x_1)/\delta(z)\\ &< 2\delta_1(\delta(z)/4 + 2\delta(z))/9\delta(z) + \delta(z)\delta_1/2\delta(z)\\ &= \delta_1/2 + \delta_1/2 = \delta_1.\end{aligned}$$

On the other hand, we get

$$d_Y(\max\phi(z)(x), y_1) \le \rho((x, \max\phi(z)(x)), (x_1, y_1)) < \delta(z)\delta_1/2 < \delta_1.$$

It follows that

$$\begin{aligned}&d_Y(r(\max\phi(z)(x), d_X(x,a)/\delta(z) - 1), y_3)\\ &\qquad = d_Y(r(\max\phi(z)(x), d_X(x,a)/\delta(z) - 1), r(y_1, d_X(x_1,a)/\delta(z') - 1)) < \epsilon.\end{aligned}$$

Using Lemma 1.5.1, we can choose $y_4 \in [\mathbf{0}, y_3]$ so that

$$d_Y(y, y_4) \le d_Y(r(\max\phi(z)(x), d_X(x,a)/\delta(z) - 1), y_3) < \epsilon.$$

Then $(x_1, y_4) \in \psi(z')$ and $\rho((x,y),(x_1,y_4)) = \max\{d_X(x,x_1), d_Y(y,y_4)\} < \epsilon$.

Case III-ii. $2\delta(z) \le d_X(x,a) < 2\delta(z) + \delta_3/2$
It follows that

$$\begin{aligned}|2\delta(z') - d_X(x_1,a)| &\le |2\delta(z') - 2\delta(z)| + |2\delta(z) - d_X(x,a)| + |d_X(x,a) - d_X(x_1,a)|\\ &< \delta_3/4 + \delta_3/2 + \delta_3/4 = \delta_3.\end{aligned}$$

Therefore we have

$$|1 - (d_X(x_1,a)/\delta(z') - 1)| = |2 - d_X(x_1,a)/\delta(z')| < 2\delta_2 < \delta_1.$$

Observe that

$$d_Y(\max\psi(z)(x), y_3) = d_Y(\max\phi(z)(x), y_3) = d_Y(r(\max\phi(z)(x), 1), r(y_1, d_X(x_1,a)/\delta(z') - 1)) < \epsilon.$$

Due to Lemma 1.5.1, there exists $y_5 \in [\mathbf{0}, y_3]$ such that $d_Y(y, y_5) \le d_Y(\max\psi(z)(x), y_3) < \epsilon$. Then $(x_1, y_5) \in \psi(z')$ and $\rho((x,y),(x_1,y_5)) = \max\{d_X(x,x_1), d_Y(y,y_5)\} < \epsilon$.

Case III-iii. $d_X(x,a) \ge 2\delta(z) + \delta_3/2$
Note that

$$d_X(x_1,a) \ge d_X(x,a) - d_X(x,x_1) \ge 2\delta(z) + \delta_3/2 - \delta_3/4 > 2\delta(z') - \delta_3/4 + \delta_3/2 - \delta_3/4 = 2\delta(z'),$$

which is a contradiction.

Consequently, $(x,y) \in N(\psi(z'), \epsilon)$. Similarly, $\psi(z') \subset N(\psi(z), \epsilon)$. Thus $\rho_H(\psi(z), \psi(z')) < \epsilon$, and hence ψ is continuous. □

6.6 The disjoint cells property of $\overline{\downarrow\mathrm{C}(X,Y)}$

In this section, we shall show the following proposition:

Proposition 6.6.1. *If there are no isolated points in X, then $\overline{\downarrow\mathrm{C}(X,Y)}$ has the disjoint cells property.*

Proof. Let $f, g : \mathbf{Q} \to \overline{\downarrow\mathrm{C}(X,Y)}$ be maps and $0 < \epsilon < \operatorname{diam}_{d_Y} Y$. Since $\downarrow\mathrm{C}(X,Y)$ is homotopy dense in $\overline{\downarrow\mathrm{C}(X,Y)}$ by Theorem 6.3.1, we can obtain maps $f' : \mathbf{Q} \to \downarrow\mathrm{C}(X,Y)$ that is ϵ-close to f, and $g' : \mathbf{Q} \to \downarrow\mathrm{C}(X,Y)$ that is $\epsilon/3$-close to g. Take a non-isolated point $x_0 \in X$. Using the Digging Lemma 6.5.1, we can find a map $g'' : \mathbf{Q} \to \downarrow\mathrm{C}(X,Y)$ such that g'' is $\epsilon/3$-close to g' and $g''(z)(x_0) = \{\mathbf{0}\}$ for all $z \in \mathbf{Q}$. Define a map $g''' : \mathbf{Q} \to \overline{\downarrow\mathrm{C}(X,Y)}$ as follows:

$$g'''(z) = g''(z) \cup \{x_0\} \times \overline{B_{d_Y}}(0, \epsilon/3).$$

Then $\rho_H(g''(z), g'''(z)) < \epsilon/3$ for every $z \in \mathbf{Q}$, and hence g''' is $\epsilon/3$-close to g''. So it is ϵ-close to g. Take any $y \in Y$ with $d_Y(\mathbf{0}, y) = \epsilon/3$. Since $g''(z) \in \downarrow\mathrm{C}(X,Y)$ and $g''(z)(x_0) = \{\mathbf{0}\}$ for each $z \in \mathbf{Q}$, we can choose $\delta > 0$ so that $B_\rho((x_0, y), \delta) \cap g''(z) = \emptyset$. This implies that $g'''(z)$ is not the hypo-graph of any map because x_0 is a non-isolated point. Hence $g'''(z) \notin \downarrow\mathrm{C}(X,Y)$. Consequently, $f'(\mathbf{Q}) \cap g'''(\mathbf{Q}) = \emptyset$. Thus $\overline{\downarrow\mathrm{C}(X,Y)}$ has the disjoint cells property. □

Combining Theorem 6.2.1, Proposition 6.6.1, and Toruńczyk's characterization of the Hilbert cube, see Corollary 1.3.3 in Chapter 1, we can immediately obtain the following:

Corollary 6.6.2. *If X has no isolated points, then $\overline{\downarrow\mathrm{C}(X,Y)}$ is homeomorphic to the Hilbert cube $\mathbf{Q}$.*

Due to Proposition 6.4.2, $\downarrow\mathrm{C}(X,Y)$ is an $F_{\sigma\delta}$ set in $\overline{\downarrow\mathrm{C}(X,Y)}$ in the above. Hence we conclude as follows:

Corollary 6.6.3. *If X has no isolated points, then $\downarrow\mathrm{C}(X,Y)$ is an absolute $F_{\sigma\delta}$ set.*

6.7 Detecting a Z_σ-set in $\overline{\downarrow\mathrm{C}(X,Y)}$ containing $\downarrow\mathrm{C}(X,Y)$

In this section, we prove the following proposition:

Proposition 6.7.1. *If there are no isolated points in X, then $\downarrow\mathrm{C}(X,Y)$ is contained in some Z_σ-set in $\overline{\downarrow\mathrm{C}(X,Y)}$.*

We can easily prove the following:

Lemma 6.7.2. *Let Z be a Z-set in M that is homotopy dense in N. Then the closure $\overline{Z}$ of Z in N is a Z-set in N.*

Proof. Take any open cover $\mathcal{U}$ of N. Let $\mathcal{V}$ be an open cover of N such that $\mathcal{V} \stackrel{\star}{\prec} \mathcal{U}$. Since M is homotopy dense in N, we can find a map $f : N \to M$ such that f is $\mathcal{V}$-close to id_N. Moreover, since Z is a Z-set in M, there is a map $g : M \to M$ such that g is $\mathcal{V}|_M$-close to id_M and $g(M) \cap Z = \emptyset$, where $\mathcal{V}|_M = \{V \cap M \mid V \in \mathcal{V}\}$ is an open cover of M. Then the composition $gf : N \to M$ is $\mathcal{U}$-close to id_N and $gf(N) \cap \overline{Z} \subset g(M) \cap Z = \emptyset$. Consequently, $\overline{Z}$ is a Z-set in N. □

The next lemma is very useful for detecting Z-sets in $\overline{\downarrow\mathrm{C}(X,Y)}$.

Lemma 6.7.3. *Suppose that $F = E \cup Z$ is a closed set in $\overline{\downarrow\mathrm{C}(X,Y)}$ such that Z is a Z-set in $\overline{\downarrow\mathrm{C}(X,Y)}$, and for each $A \in E$, there exists a point $a \in X$ with $A(a) = \{\mathbf{0}\}$. Then F is a Z-set in $\overline{\downarrow\mathrm{C}(X,Y)}$.*

Proof. Let $\epsilon : \overline{\downarrow\mathrm{C}(X,Y)} \to (0,1)$. It suffices to construct a map $\phi : \overline{\downarrow\mathrm{C}(X,Y)} \to \overline{\downarrow\mathrm{C}(X,Y)}$ such that $\phi(\overline{\downarrow\mathrm{C}(X,Y)}) \cap F = \emptyset$ and $\rho_H(\phi(A), A) < \epsilon(A)$ for each $A \in \overline{\downarrow\mathrm{C}(X,Y)}$. Since Z is a Z-set, there exists a map $\psi : \overline{\downarrow\mathrm{C}(X,Y)} \to \overline{\downarrow\mathrm{C}(X,Y)} \setminus Z$ such that $\rho_H(\psi(A), A) < \epsilon(A)/2$ for each $A \in \overline{\downarrow\mathrm{C}(X,Y)}$. Fix a point $y_0 \in Y \setminus \{\mathbf{0}\}$. We define a map $\phi : \overline{\downarrow\mathrm{C}(X,Y)} \to \overline{\downarrow\mathrm{C}(X,Y)}$ by

$$\phi(A) = \psi(A) \cup \overline{r}([\mathbf{0}, y_0], t(A)),$$

where $t(A) = \min\{\epsilon(A), \rho_H(\psi(A), Z)\}/(2 \operatorname{diam}_{d_Y} Y) > 0$. Obviously, $\phi(A)(x) \neq \mathbf{0}$ for each $x \in X$, that is, $\phi(A) \notin E$. Observe that

$$\rho_H(\phi(A), \psi(A)) \leq t(A) d_Y(\mathbf{0}, y_0) \leq t(A) \operatorname{diam}_{d_Y} Y \leq \min\{\epsilon(A), \rho_H(\psi(A), Z)\}/2.$$

Hence $\phi(A) \notin Z$ and

$$\rho_H(\phi(A), A) \leq \rho_H(\phi(A), \psi(A)) + \rho_H(\psi(A), A) < \epsilon(A)/2 + \epsilon(A)/2 = \epsilon(A).$$

The continuity of ϕ follows from the ones of $\overline{r}$, ψ and t, and Lemma 6.1.1. This completes the proof. $\square$

Proof of Proposition 6.7.1. Take a countable dense set $D = \{d_n \mid n \in \mathbb{N}\}$ in X. For each $n, m \in \mathbb{N}$, let

$$F_{n,m} = \{\downarrow f \in \downarrow\mathrm{C}(X,Y) \mid d_Y(f(d_n), \mathbf{0}) \geq 1/m\}.$$

As is easily observed, $F_{n,m}$ is closed in $\downarrow\mathrm{C}(X,Y)$. For each map $\epsilon : \downarrow\mathrm{C}(X,Y) \to (0,1)$, by the Digging Lemma 6.5.1, we have $\phi : \downarrow\mathrm{C}(X,Y) \to \downarrow\mathrm{C}(X,Y)$ such that $\rho_H(\downarrow f, \phi(\downarrow f)) < \epsilon(\downarrow f)$ and $\phi(\downarrow f)(d_n) = \{\mathbf{0}\}$ for $\downarrow f \in \downarrow\mathrm{C}(X,Y)$. Obviously, $\phi(\downarrow\mathrm{C}(X,Y)) \cap F_{n,m} = \emptyset$. Thus each $F_{n,m}$ is a Z-set in $\downarrow\mathrm{C}(X,Y)$. It follows from Theorem 6.3.1 and Lemma 6.7.2 that the closure $\overline{F_{n,m}}$ is a Z-set in $\overline{\downarrow\mathrm{C}(X,Y)}$.

Let $F = \bigcap_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} (\downarrow\mathrm{C}(X,Y) \setminus F_{n,m})$. It remains to prove that the closure $\overline{F}$ of F in $\overline{\downarrow\mathrm{C}(X,Y)}$ is a Z-set. Observe that

$$F = \{\downarrow f \in \downarrow\mathrm{C}(X,Y) \mid f(d_n) = \mathbf{0} \text{ for each } n \in \mathbb{N}\} = \{\downarrow\mathbf{0}\},$$

where $\mathbf{0} : X \to \{\mathbf{0}\} \subset Y$ is the constant map. Hence $\overline{F} = \{\downarrow\mathbf{0}\} = \{X \times \{\mathbf{0}\}\}$. According to Lemma 6.7.3, $\overline{F}$ is a Z-set in $\overline{\downarrow\mathrm{C}(X,Y)}$. Consequently, $\downarrow\mathrm{C}(X,Y)$ is contained in the Z_σ-set $\overline{F} \cup \bigcup_{m,n\in\mathbb{N}} \overline{F_{n,m}}$. $\square$

6.8 The strong $(\mathfrak{M}_0, \mathcal{F}_{\sigma\delta})$-universality of $(\overline{\downarrow\mathrm{C}(X,Y)}, \downarrow\mathrm{C}(X,Y))$

In this section, we shall show the main theorem. Let (X_1, X_2) be a pair of spaces, and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes. We say that (X_1, X_2) is *strongly* $(\mathcal{C}_1, \mathcal{C}_2)$*-universal* if the following condition holds:

- Let $Z_1 \in \mathcal{C}_1$, $Z_2 \in \mathcal{C}_2$, K a closed subset of Z_1, and $f : Z_1 \to X_1$ a map such that $f|_K$ is a Z-embedding. Then for every open cover $\mathcal{U}$ of X_1, there exists a Z-embedding $g : Z_1 \to X_1$ such that g is $\mathcal{U}$-close to f, $g|_K = f|_K$ and $g^{-1}(X_2) \setminus K = Z_2 \setminus K$.

A pair (X_1, X_2) of spaces is $(\mathcal{C}_1, \mathcal{C}_2)$*-absorbing*[1] provided that the following conditions are satisfied:

(i) $X_1 \in \mathcal{C}_1$ and $X_2 \in \mathcal{C}_2$;

(ii) X_2 is contained in a Z_σ-set in X_1;

(iii) (X_1, X_2) is strongly $(\mathcal{C}_1, \mathcal{C}_2)$-universal.

[1]We modify the definition of [10].

Denote the class of compact metrizable spaces by $\mathfrak{M}_0$, and the one of separable metrizable absolute $F_{\sigma\delta}$ spaces by $\mathcal{F}_{\sigma\delta}$. According to Theorem 1.7.6 of [10], the following can be established.

Theorem 6.8.1. *Let X_1 and Z_1 be topological copies of the Hilbert cube $\mathbf{Q}$. If pairs (X_1, X_2) and (Z_1, Z_2) are $(\mathfrak{M}_0, \mathcal{F}_{\sigma\delta})$-absorbing, then there exists a homeomorphism $f : X_1 \to Z_1$ such that $f(X_2) = Z_2$.*

Let $\mathbf{c}_1 = \{(x_i)_{i\in\mathbb{N}} \in \mathbf{Q} \mid \lim_{i\to\infty} x_i = 1\}$. The following fact is well known.

Fact 4. The pairs $(\mathbf{Q}, \mathbf{c}_0)$ and $(\mathbf{Q}, \mathbf{c}_1)$ are $(\mathfrak{M}_0, \mathcal{F}_{\sigma\delta})$-absorbing, and hence $(\mathbf{Q}, \mathbf{c}_0)$ is homeomorphic to $(\mathbf{Q}, \mathbf{c}_1)$.

We needs the following lemma to verify the strong $(\mathfrak{M}_0, \mathcal{F}_{\sigma\delta})$-universality of $(\overline{\downarrow\mathrm{C}(X, Y)}, \downarrow\mathrm{C}(X, Y))$.

Lemma 6.8.2. *Let $x_m, x_\infty \in X$, $m \in \mathbb{N}$, such that $\{r_m = d_X(x_m, x_\infty)\}_{m\in\mathbb{N}}$ is a strictly decreasing sequence conversing to 0, and let $y_0 \in Y \setminus \{\mathbf{0}\}$ such that $d_Y(\mathbf{0}, y_0) \leq 1$. Suppose that $g : Z \to \mathbf{Q}$ is an injection from a space Z to the Hilbert cube $\mathbf{Q}$ and $\delta : Z \to (0, 1)$ is a map. Then there exists a map $\Phi : Z \to \overline{\downarrow\mathrm{C}(X, [\mathbf{0}, y_0])}$ satisfying the following conditions:*

1. *Φ is injective;*
2. *$\rho_H(\Phi(z), X \times \{\mathbf{0}\}) \leq \delta(z)$ for all $z \in Z$;*
3. *$\Phi(z)(X \setminus B_{d_X}(x_\infty, r_{2k})) = \{\mathbf{0}\}$ for all $z \in Z$ with $2^{-k} \leq \delta(z) \leq 2^{-k+1}$, $k \in \mathbb{N}$;*
4. *$z \in g^{-1}(\mathbf{c}_1)$ if and only if $\Phi(z) \in \downarrow\mathrm{C}(X, [\mathbf{0}, y_0])$;*
5. *$\Phi(z)(x_\infty) = [\mathbf{0}, r(y_0, \delta(z))]$ for all $z \in Z$.*

Proof. For each $k, m \in \mathbb{N}$, let $Z_k = \{z \in Z \mid 2^{-k} \leq \delta(z) \leq 2^{-k+1}\}$ and $S_m = \{x \in X \mid r_m \leq d_X(x, x_\infty) \leq r_{m-1}\}$. Note that $Z = \bigcup_{k\in\mathbb{N}} Z_k$, $x_{m-1}, x_m \in S_m$, $\bigcup_{m\in\mathbb{N}} S_m = X \setminus \{x_\infty\}$, and $S_m \cap S_{m'} \neq \emptyset$ if and only if $|m - m'| \leq 1$. We define maps $\phi_k : Z_k \to \mathbf{I}$ and $\psi_m : S_m \to \mathbf{I}$ for each $k, m \in \mathbb{N}$ by $\phi_k(z) = 2 - 2^k\delta(z)$ and $\psi_m(x) = (d_X(x, x_\infty) - r_m)/(r_{m-1} - r_m)$, respectively. Then $\psi_m(x_{m-1}) = 1$ and $\psi_m(x_m) = 0$. For each $i, k \in \mathbb{N}$, let $f_i^k : Z_k \to \mathbf{I}$ be a map defined by

$$f_i^k(z) = \begin{cases} 0 & \text{if } i = 1, \\ (1 - \phi_k(z))\delta(z) & \text{if } i = 2, \\ (1 - \phi_k(z))\delta(z)g(z)(1) & \text{if } i = 3, \\ \delta(z) & \text{if } i = 2j, j \geq 2, \\ \delta(z)((1 - \phi_k(z))g(z)((i-1)/2) + \phi_k(z)g(z)((i-3)/2)) & \text{if } i = 2j+1, j \geq 2. \end{cases}$$

Remark that $f_i^k(z) \leq \delta(z)$ for every $z \in Z$. We define a map $\Phi_k : Z_k \to \overline{\downarrow\mathrm{C}(X, [\mathbf{0}, y_0])}$, $k \in \mathbb{N}$, as follows:

$$\begin{aligned}\Phi_k(z) = \{x \in X \mid d_X(x, x_\infty) \geq r_{2k}\} \times \{\mathbf{0}\} \cup \{x_\infty\} \times [\mathbf{0}, r(y_0, \delta(z))] \\ \cup \bigcup_{i\in\mathbb{N}} \{(x, y) \in X \times Y \mid x \in S_{2k+i}, y \in [\mathbf{0}, r(y_0, \alpha_i^k(x, z))]\},\end{aligned}$$

where $\alpha_i^k(x, z) = \psi_{2k+i}(x) f_i^k(z) + (1 - \psi_{2k+i}(x)) f_{i+1}^k(z)$. Then $\Phi_k(z) = \Phi_{k+1}(z)$ for every $z \in Z_k \cap Z_{k+1}$. Indeed, take any $z \in Z_k \cap Z_{k+1}$. Since $\delta(z) = 2^{-k}$, we have $\phi_k(z) = 1$ and $\phi_{k+1}(z) = 0$. Observe that $f_1^k(z) = f_2^k(z) = f_3^k(z) = 0$. Hence for each $x \in X$,

$$\alpha_1^k(x, z) = \psi_{2k+1}(x) f_1^k(z) + (1 - \psi_{2k+1}(x)) f_2^k(z) = 0 \text{ and}$$
$$\alpha_2^k(x, z) = \psi_{2k+2}(x) f_2^k(z) + (1 - \psi_{2k+2}(x)) f_3^k(z) = 0.$$

It follows that

$$\Phi_k(z)(\{x \in X \mid d_X(x, x_\infty) \geq r_{2k+2}\}) = \{\mathbf{0}\} = \Phi_{k+1}(z)(\{x \in X \mid d_X(x, x_\infty) \geq r_{2k+2}\}).$$

We see $f_3^k(z) = 0 = f_1^{k+1}(z)$, $f_{2j+3}^k(z) = \delta(z)g(z)(j) = f_{2j+1}^{k+1}(z)$ and $f_{2j+2}^k(z) = \delta(z) = f_{2j}^{k+1}(z)$ for all $j \geq 1$, that is, $f_{i+2}^k(z) = f_i^{k+1}(z)$ for all $i \geq 1$. Therefore for each $x \in S_{2k+i+2}$, $i \geq 1$,

$$\Phi_k(z)(x) = [\mathbf{0}, r(y_0, \alpha_{i+2}^k(x,z))] = [\mathbf{0}, r(y_0, \alpha_i^{k+1}(x,z))] = \Phi_{k+1}(z)(x).$$

Moreover, $\Phi_k(z)(x_\infty) = [\mathbf{0}, r(y_0, \delta(z))] = \Phi_{k+1}(z)(x_\infty)$. Thus $\Phi_k(z) = \Phi_{k+1}(z)$.

Now, we can obtain the desired map $\Phi : Z \to \overline{{\downarrow}\mathrm{C}(X, [\mathbf{0}, y_0])}$ defined by $\Phi(z) = \Phi_k(z)$ if $z \in Z_k$. It follows from the definition that Φ satisfies conditions (2), (3) and (5). So it remains to verify that conditions (1) and (4) hold.

Condition (1) Φ is injective.
Let $z_1, z_2 \in Z$ such that $\Phi(z_1) = \Phi(z_2)$. Then

$$[\mathbf{0}, r(y_0, \delta(z_1))] = \Phi(z_1)(x_\infty) = \Phi(z_2)(x_\infty) = [\mathbf{0}, r(y_0, \delta(z_2))],$$

which implies that $\delta(z_1) = \delta(z_2)$. Hence both of z_1 and z_2 are contained in Z_k for some $k \in \mathbb{N}$ and

$$\phi_k(z_1) = 2 - 2^k\delta(z_1) = 2 - 2^k\delta(z_1) = \phi_k(z_2).$$

Since $\psi_{2k+i}(x_{2k+i}) = 0$ for all $i \in \mathbb{N}$, we have

$$[\mathbf{0}, r(y_0, f_{i+1}^k(z_1))] = \Phi_k(z_1)(x_{2k+i}) = \Phi_k(z_2)(x_{2k+i}) = [\mathbf{0}, r(y_0, f_{i+1}^k(z_2))],$$

which implies that $f_j^k(z_1) = f_j^k(z_2)$ for every $j \geq 2$. In the case $\phi_k(z_1) = 1$, for each $j \in \mathbb{N}$, we have

$$g(z_1)(j) = f_{2j+3}^k(z_1) = f_{2j+3}^k(z_2) = g(z_2)(j),$$

In the case $\phi_k(z_1) \neq 1$, we have

$$(1 - \phi_k(z_1))\delta(z_1)g(z_1)(1) = f_3^k(z_1) = f_3^k(z_2) = (1 - \phi_k(z_2))\delta(z_2)g(z_2)(1),$$

which implies that $g(z_1)(1) = g(z_2)(1)$. Assume that $g(z_1)(i) = g(z_2)(i)$ for $i \in \mathbb{N}$. Then

$$\begin{aligned}\delta(z_1)((1 - \phi_k(z_1))g(z_1)(i+1) + \phi_k(z_1)g(z_1)(i)) &= f_{2i+3}^k(z_1) = f_{2i+3}^k(z_2) \\ &= \delta(z_2)((1 - \phi_k(z_2))g(z_2)(i+1) + \phi_k(z_2)g(z_2)(i)),\end{aligned}$$

so $g(z_1)(i+1) = g(z_2)(i+1)$. By induction, for all $j \in \mathbb{N}$, we get $g(z_1)(j) = g(z_2)(j)$. It follows that $g(z_1) = g(z_2)$. Since g is injective, $z_1 = z_2$. Therefore Φ is injective.

Condition (4) $z \in g^{-1}(\mathbf{c}_1)$ if and only if $\Phi(z) \in {\downarrow}\mathrm{C}(X, [\mathbf{0}, y_0])$.
We define a function $h(z) : X \to [\mathbf{0}, y_0] \subset Y$ for each $z \in Z_k$ and $k \in \mathbb{N}$ as follows:

$$h(z)(x) = \begin{cases} 0 & \text{if } d_X(x, x_\infty) \geq r_{2k}, \\ r(y_0, \alpha_i^k(x,z)) & \text{if } x \in S_{2k+i}, i \in \mathbb{N}, \\ r(y_0, \delta(z)) & \text{if } x = x_\infty. \end{cases}$$

Observe that ${\downarrow}h(z) = \Phi(z)$ and $h(z)$ is continuous on $X \setminus \{x_\infty\}$. When $h(z)$ is continuous at the point x_∞, $\Phi(z) = {\downarrow}h(z) \in {\downarrow}\mathrm{C}(X, [\mathbf{0}, y_0])$. So we need only to show that $z \in g^{-1}(\mathbf{c}_1)$ if and only if $h(z)$ is continuous at x_∞.

First, we shall prove the only if part. Take any $\epsilon > 0$. We may assume that $\epsilon < \delta(z)$. Since $g(z) \in \mathbf{c}_1$, there exists $i_0 \in \mathbb{N}$ such that for every $i \geq i_0$, $g(z)(i) > 1 - \epsilon/\delta(z)$. Fix any point $x \neq x_\infty$ in the neighborhood $\{x_\infty\} \cup \bigcup_{i \geq 2i_0+3} S_{2k+i}$ of x_∞ in X, where $z \in Z_k$. Then $x \in S_{2k+i}$ for some $i \geq 2i_0 + 3$. When i is even, $f_i^k(z) = \delta(z)$. When i is odd,

$$\begin{aligned} f_i^k(z) &= \delta(z)((1 - \phi_k(z))g(z)((i-1)/2) + \phi_k(z)g(z)((i-3)/2)) \\ &> \delta(z)((1 - \phi_k(z))(1 - \epsilon/\delta(z)) + \phi_k(z)(1 - \epsilon/\delta(z))) > \delta(z) - \epsilon. \end{aligned}$$

Hence we have

$$\begin{aligned}\alpha_i^k(x,z) &= \psi_{2k+i}(x) f_i^k(z) + (1-\psi_{2k+i}(x)) f_{i+1}^k(z)\\ &> \psi_{2k+i}(x)(\delta(z)-\epsilon) + (1-\psi_{2k+i}(x))(\delta(z)-\epsilon) = \delta(z)-\epsilon.\end{aligned}$$

It follows that

$$\begin{aligned}d_Y(h(z)(x_\infty), h(z)(x)) = d_Y(r(y_0,\delta(z)), r(y_0,\alpha_i^k(z))) &= (\delta(z)-\alpha_i^k(z)) d_Y(\mathbf{0}, y_0)\\ &< \delta(z) - (\delta(z)-\epsilon) = \epsilon.\end{aligned}$$

Consequently, $h(z)$ is continuous.

Next, we shall show the if part. Let $\epsilon \in (0,1)$ and $\epsilon' = \epsilon\phi_k(z)\delta(z)$, where $z \in Z_k$ with $\phi_k(z) > 0$. Since $h(z)$ is continuous at x_∞, we can choose $i_0 \geq 5$ so that for any $x \in X$,

$$d_X(x, x_\infty) \leq r_{2k+i_0} \Rightarrow d_Y(h(z)(x), h(z)(x_\infty)) < \epsilon' d_Y(\mathbf{0}, y_0).$$

Recall that $\psi_m(x_m) = 0$ for all $m \in \mathbb{N}$. Therefore for every $i \geq i_0$,

$$\begin{aligned}d_Y(r(y_0, f_{i+1}^k(z)), r(y_0,\delta(z))) &= d_Y(r(y_0, \psi_{2k+i}(x_{2k+i}) f_i^k(z) + (1-\psi_{2k+i}(x_{2k+i})) f_{i+1}^k(z)), r(y_0,\delta(z)))\\ &= d_Y(r(y_0, \alpha_i^k(x_{2k+i}, z)), r(y_0, \delta(z)))\\ &= d_Y(h(z)(x_{2k+i}), h(z)(x_\infty)) < \epsilon' d_Y(\mathbf{0}, y_0).\end{aligned}$$

Note that for all $i \geq i_0 + 1$,

$$\delta(z) - f_i^k(z) = d_Y(r(y_0, f_i^k(z)), r(y_0, \delta(z)))/d_Y(\mathbf{0}, y_0) < \epsilon'.$$

It follows that for any $j \geq (i_o - 2)/2$,

$$\begin{aligned}g(z)(j) &= (f_{2j+3}^k(z)/\delta(z) - (1-\phi_k(z))g(z)(j+1))/\phi_k(z) \geq (f_{2j+3}^k(z)/\delta(z) - (1-\phi_k(z)))/\phi_k(z)\\ &> ((\delta(z)-\epsilon')/\delta(z) - (1-\phi_k(z)))/\phi_k(z) = ((\delta(z) - \epsilon\phi_k(z)\delta(z))/\delta(z) - (1-\phi_k(z)))/\phi_k(z)\\ &= 1-\epsilon.\end{aligned}$$

Hence $g(z) \in \mathbf{c}_1$. Thus the proof is complete. $\square$

Proposition 6.8.3. *If X has no isolated points, then the pair $(\overline{\downarrow\mathrm{C}(X,Y)}, \downarrow\mathrm{C}(X,Y))$ is strongly $(\mathfrak{M}_0, \mathcal{F}_{\sigma\delta})$-universal.*

Proof. Let $Z \in \mathfrak{M}_0$, $C \in \mathcal{F}_{\sigma\delta}$, K a closed subset of Z, $\epsilon > 0$ and $\Phi : Z \to \overline{\downarrow\mathrm{C}(X,Y)}$ a map such that the restriction $\Phi|_K$ is a Z-embedding. We shall construct a Z-embedding $\Psi : Z \to \overline{\downarrow\mathrm{C}(X,Y)}$ so that Ψ is ϵ-close to Φ, $\Psi|_K = \Phi|_K$ and $\Psi^{-1}(\downarrow\mathrm{C}(X,Y))\setminus K = C\setminus K$. Since $\Phi(K)$ is a Z-set in $\overline{\downarrow\mathrm{C}(X,Y)}$, we may assume that $\Phi(K) \cap \Phi(Z\setminus K) = \emptyset$. Define a map $\delta : Z \to [0,1)$ by $\delta(z) = \min\{\epsilon, \rho_H(\Phi(z), \Phi(K))\}/4$. Observe that $\delta(z) = 0$ if and only if $z \in K$. Since $\downarrow\mathrm{C}(X,Y)$ is homotopy dense in $\overline{\downarrow\mathrm{C}(X,Y)}$ by Theorem 6.3.1, there exists a homotopy $H : \overline{\downarrow\mathrm{C}(X,Y)} \times \mathbf{I} \to \overline{\downarrow\mathrm{C}(X,Y)}$ such that $H_0 = \mathrm{id}_{\overline{\downarrow\mathrm{C}(X,Y)}}$, $H_t(\overline{\downarrow\mathrm{C}(X,Y)}) \subset \downarrow\mathrm{C}(X,Y)$ for all $t \in (0,1]$ and $\rho_H(H_t(\downarrow A), \downarrow A) \leq t$ for each $\downarrow A \in \overline{\downarrow\mathrm{C}(X,Y)}$ and $t \in \mathbf{I}$. Let $h : Z \to \overline{\downarrow\mathrm{C}(X,Y)}$ be a map defined by $h(z) = H(\Phi(z), \delta(z))$. Remark that $\rho_H(h(z), \Phi(z)) = \rho_H(H(\Phi(z), \delta(z)), \Phi(z)) \leq \delta(z)$ for every $z \in Z$, in particular, $h(z) = \Phi(z)$ for all $z \in K$, and $h(Z\setminus K) \subset \downarrow\mathrm{C}(X,Y)$. Take a non-isolated point $x_\infty \in X$. According to the Digging Lemma 6.5.1, we can obtain maps $\psi : Z\setminus K \to \downarrow\mathrm{C}(X,Y)$ and $r : Z \setminus K \to (0,1)$ so that for each $z \in Z\setminus K$,

(a) $\rho_H(h(z), \psi(z)) \leq \delta(z)$,

(b) $\psi(z)(B_{d_X}(x_\infty, r(z))) = \{\mathbf{0}\}$.

Let $Z_k = \{z \in Z \mid 2^{-k} \leq \delta(z) \leq 2^{-k+1}\} \subset Z \setminus K$ for each $k \in \mathbb{N}$. Then each Z_k is compact and $Z \setminus K = \bigcup_{k \in \mathbb{N}} Z_k$. Since x_∞ is a non-isolated point, there exists a point $x_1 \in X \setminus \{x_\infty\}$ such that $d_X(x_1, x_\infty) < \min\{1, r(z) \mid z \in Z_1\}$. By induction, we can choose $x_m \in X \setminus \{x_\infty\}$ for each $m \geq 2$ so that $d_X(x_m, x_\infty) < \min\{1/m, d_X(x_{m-1}, x_\infty), r(z) \mid z \in Z_m\}$. Let $r_m = d_X(x_m, x_\infty)$ for each $m \in \mathbb{N}$, so r_m converges to 0 as m intends to ∞. Note that for every $z \in Z_k$ and $k \in \mathbb{N}$, $\psi(z)(B_{d_X}(x_\infty, r_k)) = \{\mathbf{0}\}$. Since the pair $(\mathbf{Q}, \mathbf{c}_1)$ is strongly $(\mathfrak{M}_0, \mathcal{F}_{\sigma\delta})$-universal due to Fact 4, we can take am embedding $g : Z \to \mathbf{Q}$ so that $g^{-1}(\mathbf{c}_1) = C$. Choose $y_0 \in Y \setminus \{\mathbf{0}\}$ with $d_Y(\mathbf{0}, y_0) \leq 1$.

Using Lemma 6.8.2, we can obtain a map $\psi' : Z \setminus K \to \overline{{\downarrow}\mathrm{C}(X, [\mathbf{0}, y_0])}$ satisfying the following conditions:

(1) ψ' is injective;

(2) $\rho_H(\psi'(z), X \times \{\mathbf{0}\}) \leq \delta(z)$ for all $z \in Z \setminus K$;

(3) $\psi'(z)(X \setminus B_{d_X}(x_\infty, r_{2k})) = \{\mathbf{0}\}$ for all $z \in Z_k$, $k \in \mathbb{N}$;

(4) $z \in C \setminus K$ if and only if $\psi'(z) \in {\downarrow}\mathrm{C}(X, [\mathbf{0}, y_0])$;

(5) $\psi'(z)(x_\infty) = [\mathbf{0}, r(y_0, \delta(z))]$ for all $z \in Z \setminus K$.

Define $\psi'' : Z \setminus K \to \overline{{\downarrow}\mathrm{C}(X, Y)}$ by $\psi''(z) = \psi(z) \cup \psi'(z)$. The continuity of ψ'' follows from the ones of ψ and ψ', and Lemma 6.1.1. By conditions (a) and (2), and Lemma 6.1.1, for each $z \in Z \setminus K$,

$$\begin{aligned}\rho_H(h(z), \psi''(z)) &= \rho_H(h(z) \cup X \times \{\mathbf{0}\}, \psi(z) \cup \psi'(z)) \\ &\leq \max\{\rho_H(h(z), \psi(z)), \rho_H(X \times \{\mathbf{0}\}, \psi'(z))\} \leq \delta(z).\end{aligned}$$

According to conditions (b), (3) and (4), we have $z \in C \setminus K$ if and only if $\psi''(z) \in {\downarrow}\mathrm{C}(X, Y)$. Moreover, ψ'' is injective. Indeed, take any $z_1, z_2 \in Z \setminus K$ with $\psi''(z_1) = \psi''(z_2)$. Then there exist $k_1, k_2 \in \mathbb{N}$ such that $z_1 \in Z_{k_1}$ and $z_2 \in Z_{k_2}$, respectively. It follows from (b) and (5) that

$$[\mathbf{0}, r(y_0, \delta(z_1))] = \psi'(z_1)(x_\infty) = \psi''(z_1)(x_\infty) = \psi''(z_2)(x_\infty) = \psi'(z_2)(x_\infty) = [\mathbf{0}, r(y_0, \delta(z_2))],$$

which implies that $\delta(z_1) = \delta(z_2)$. Hence $z_1, z_2 \in Z_k$, where $k = k_1 = k_2$. Since $\psi(z_1)(B_{d_X}(x_\infty, r_k)) = \{\mathbf{0}\} = \psi(z_2)(B_{d_X}(x_\infty, r_k))$ by (b), we have

$$\psi'(z_1)(x) = \psi''(z_1)(x) = \psi''(z_2)(x) = \psi'(z_2)(x) \text{ for every } x \in B_{d_X}(x_\infty, r_{2k}).$$

On the other hand, by (3), $\psi'(z_1)(X \setminus B_{d_X}(x_\infty, r_{2k})) = \{\mathbf{0}\} = \psi'(z_2)(X \setminus B_{d_X}(x_\infty, r_{2k}))$. Therefore $\psi'(z_1) = \psi'(z_2)$. Due to (1), we get $z_1 = z_2$, so ψ'' is injective.

We can extend ψ'' to the desired map $\Psi : Z \to \overline{{\downarrow}\mathrm{C}(X, Y)}$ by $\Psi|_K = \Phi|_K$. Then for each $z \in Z$,

$$\rho_H(\Phi(z), \Psi(z)) \leq \rho_H(\Phi(z), h(z)) + \rho_H(h(z), \Psi(z)) \leq 2\delta(z) \leq \min\{\epsilon, \rho_H(\Phi(z), \Phi(K))\}/2,$$

which means that Ψ is continuous. Moreover, it follows that $\rho_H(\Phi(z), \Psi(z)) \leq \epsilon$ for all $z \in Z$, and $\Psi(z) \in \overline{{\downarrow}\mathrm{C}(X, Y)} \setminus \Phi(K)$ for all $z \in Z \setminus K$. Since $z \in C \setminus K$ if and only if $\psi''(z) \in {\downarrow}\mathrm{C}(X, Y)$, we have $\Psi^{-1}({\downarrow}\mathrm{C}(X, Y)) \setminus K = C \setminus K$. It remains to show that Ψ is a Z-embedding. It is easy to see that Ψ is an embedding. Recall that $\Psi(K) = \Phi(K)$ is a Z-set in $\overline{{\downarrow}\mathrm{C}(X, Y)}$. Since $x_{2k} \in B_{d_X}(x_\infty, r_k) \setminus B_{d_X}(x_\infty, r_{2k})$ for every $k \in \mathbb{N}$, it follows from (b) and (3) that

$$\Psi(z)(x_{2k}) = \psi''(z)(x_{2k}) = \psi(z)(x_{2k}) \cup \psi'(z)(x_{2k}) = \{\mathbf{0}\} \text{ for each } z \in Z_k.$$

Applying Lemma 6.7.3, $\Psi(Z) = \Psi(Z \setminus K) \cup \Psi(K)$ is a Z-set in $\overline{{\downarrow}\mathrm{C}(X, Y)}$. Consequently, Ψ is a Z-embedding. □

Finally, we prove the main theorem.

Proof of Main Theorem. We can write $X = \bigoplus_{i=1}^n X_i$, where each X_i is a component of X. Note that the pair $(\overline{\downarrow_v \mathrm{C}(X,Y)}, \downarrow_v \mathrm{C}(X,Y))$ is homeomorphic to $\big(\prod_{i=1}^n \overline{\downarrow_v \mathrm{C}(X_i,Y)}, \prod_{i=1}^n \downarrow_v \mathrm{C}(X_i,Y)\big)$, refer to Lemma 6.8 of [39]. Since X is infinite, there exists at least one component that is non-degenerate. When X_i is a singleton, $(\overline{\downarrow_v \mathrm{C}(X_i,Y)}, \downarrow_v \mathrm{C}(X_i,Y))$ is homeomorphic to (Y,Y). When X_i is non-degenerate, it is compact and has no isolated points. Combining Corollary 6.6.2, Proposition 6.4.2, Proposition 6.7.1 and Proposition 6.8.3, we can obtain that $\overline{\downarrow_v \mathrm{C}(X_i,Y)}$ is homeomorphic to $\mathbf{Q}$ and that $(\overline{\downarrow_v \mathrm{C}(X_i,Y)}, \downarrow_v \mathrm{C}(X_i,Y))$ is $(\mathfrak{M}_0, \mathcal{F}_{\sigma\delta})$-absorbing. It follows from Theorem 6.8.1 and Fact 4 that $(\overline{\downarrow_v \mathrm{C}(X_i,Y)}, \downarrow_v \mathrm{C}(X_i,Y))$ is homeomorphic to $(\mathbf{Q}, \mathbf{c}_0)$. On the other hand, using Theorem 6.8.1, we can easily show that the pairs $(\mathbf{Q}\times\mathbf{Q}, \mathbf{c}_0\times\mathbf{c}_0)$ and $(\mathbf{Q}\times Y, \mathbf{c}_0\times Y)$ are homeomorphic to $(\mathbf{Q}, \mathbf{c}_0)$. This means that $\big(\prod_{i=1}^n \overline{\downarrow_v \mathrm{C}(X_i,Y)}, \prod_{i=1}^n \downarrow_v \mathrm{C}(X_i,Y)\big)$ is homeomorphic to $(\mathbf{Q}, \mathbf{c}_0)$. Thus the proof is complete. □

6.9 Remarks

In this section, we will give some remarks on the main theorem. Z. Yang and X. Zhou [64] proved the stronger result as follows:

Theorem 6.9.1. *The pair $(\downarrow\mathrm{USC}(X,\mathbf{I}), \downarrow\mathrm{C}(X,\mathbf{I}))$ is homeomorphic to $(\mathbf{Q}, \mathbf{c}_0)$ if and only if the set of isolated points of X is not dense.*

It is unknown whether the same result holds or not in the general case. However, we show the following theorem (cf. Z. Yang [63] proved the case that $Y = \mathbf{I}$).

Theorem 6.9.2. *The space $\downarrow\mathrm{C}(X,Y)$ is a Baire space if and only if the set of isolated points is dense in X.*

The following two assertions are counterparts to Lemma 6.7.3 and Proposition 6.7.1, respectively.

Lemma 6.9.3. *Suppose that $F = E \cup Z \subset \downarrow\mathrm{C}(X,Y)$ is a closed set such that Z is a Z-set in $\downarrow\mathrm{C}(X,Y)$, and there exists a point $x \in X$ such that for every $\downarrow f \in E$, $f(x) = \mathbf{0}$. Then F is a Z-set in $\downarrow\mathrm{C}(X,Y)$.*

Proof. Let $\epsilon : \downarrow\mathrm{C}(X,Y) \to (0,1)$. It suffices to construct a map $\phi : \downarrow\mathrm{C}(X,Y) \to \downarrow\mathrm{C}(X,Y)$ such that $\phi(\downarrow\mathrm{C}(X,Y)) \cap F = \emptyset$ and $\rho_H(\phi(\downarrow f), \downarrow f) < \epsilon(\downarrow f)$ for each $\downarrow f \in \downarrow\mathrm{C}(X,Y)$. Since Z is a Z-set, there exists a map $\psi : \downarrow\mathrm{C}(X,Y) \to \downarrow\mathrm{C}(X,Y) \setminus Z$ such that $\rho_H(\psi(\downarrow f), \downarrow f) < \epsilon(\downarrow f)/2$ for every $\downarrow f \in \downarrow\mathrm{C}(X,Y)$. Fix a point $y_0 \in Y \setminus \{\mathbf{0}\}$ with $d_Y(\mathbf{0}, y_0) \le 1$ and let $t(\downarrow f) = \min\{\epsilon(\downarrow f), \rho_H(\psi(\downarrow f), Z)\}/2 > 0$ for each $\downarrow f \in \downarrow\mathrm{C}(X,Y)$.

First, we consider the case that $x \in X$ is an isolated point. Define a map $\phi : \downarrow\mathrm{C}(X,Y) \to \downarrow\mathrm{C}(X,Y)$ by

$$\phi(\downarrow f) = \psi(\downarrow f)|_{X\setminus\{x\}} \cup [\mathbf{0}, \gamma(\max \psi(\downarrow f)(x), y_0, t(\downarrow f)/\operatorname{diam}_{d_Y} Y)] \text{ for each } \downarrow f \in \downarrow\mathrm{C}(X,Y),$$

where $\gamma : Y^2 \times \mathbf{I} \to Y$ is as in Lemma 1.5.1. Obviously, $\phi(\downarrow f)(x) \neq \mathbf{0}$, that is, $\phi(\downarrow f) \notin E$. Observe that

$$\rho_H(\psi(\downarrow f), \phi(\downarrow f)) \le t(\downarrow f) \le \rho_H(\psi(\downarrow f), Z)/2,$$

which implies that $\phi(\downarrow f) \notin Z$. Moreover,

$$\rho_H(\downarrow f, \phi(\downarrow f)) \le \rho_H(\downarrow f, \psi(\downarrow f)) + \rho_H(\psi(\downarrow f), \phi(\downarrow f)) < \epsilon(\downarrow f)/2 + t(\downarrow f) \le \epsilon(\downarrow f).$$

Next, we consider the case that $x \in X$ is a non-isolated point. Using the Digging Lemma 6.5.1, we can obtain maps $\xi : \downarrow\mathrm{C}(X,Y) \to \downarrow\mathrm{C}(X,Y)$ and $\delta : \downarrow\mathrm{C}(X,Y) \to (0,1)$ such that for each $\downarrow f \in \downarrow\mathrm{C}(X,Y)$,

(a) $\rho_H(\psi(\downarrow f), \xi(\downarrow f)) < t(\downarrow f)/2$,

(b) $\xi(\downarrow f)(B_{d_X}(x, \delta(\downarrow f))) = \{\mathbf{0}\}$.

For each $\downarrow f \in \downarrow \mathrm{C}(X,Y)$, let

$$\eta(\downarrow f) = \bigcup_{x' \in B_{d_X}(x,\delta(\downarrow f))} [\mathbf{0}, r(y_0, t(\downarrow f)(\delta(\downarrow f) - d_Y(x,x'))/(2\delta(\downarrow f))].$$

We define a map $\phi : \downarrow \mathrm{C}(X,Y) \to \downarrow \mathrm{C}(X,Y)$ as follows:

$$\phi(\downarrow f) = \xi(\downarrow f) \cup \eta(\downarrow f).$$

Note that $\phi(\downarrow f)(x) \neq \mathbf{0}$, and hence $\phi(\downarrow \mathrm{C}(X,Y)) \cap E = \emptyset$. For every $\downarrow f \in \downarrow \mathrm{C}(X,Y)$, we have

$$\begin{aligned}\rho_H(\psi(\downarrow f), \phi(\downarrow f)) &\leq \rho_H(\psi(\downarrow f), \xi(\downarrow f)) + \rho_H(\xi(\downarrow f), \phi(\downarrow f)) \\ &< t(\downarrow f)/2 + t(\downarrow f)/2 \leq \rho_H(\psi(\downarrow f), Z)/2.\end{aligned}$$

Therefore $\phi(\downarrow f) \notin Z$. It follows that

$$\rho_H(\downarrow f, \phi(\downarrow f)) \leq \rho_H(\downarrow f, \psi(\downarrow f)) + \rho_H(\psi(\downarrow f), \phi(\downarrow f)) < \epsilon(\downarrow f)/2 + t(\downarrow f) \leq \epsilon(\downarrow f).$$

This completes the proof. □

Proposition 6.9.4. *If the set of isolated points is not dense in X, then $\downarrow \mathrm{C}(X,Y)$ is a Z_σ-set in itself, and hence it is not a Baire space.*

Proof. Let X_0 be the set of isolated points in X and take a countable dense set $D = \{d_n \mid n \in \mathbb{N}\}$ in $X \setminus X_0$. For each $n, m \in \mathbb{N}$, let

$$F_{n,m} = \{\downarrow f \in \downarrow \mathrm{C}(X,Y) \mid d_Y(f(d_n), \mathbf{0}) \geq 1/m\}.$$

As is easily observed, $F_{n,m}$ is closed in $\downarrow \mathrm{C}(X,Y)$. For each map $\epsilon : \downarrow \mathrm{C}(X,Y) \to (0,1)$, by the Digging Lemma 6.5.1, we have $\phi : \downarrow \mathrm{C}(X,Y) \to \downarrow \mathrm{C}(X,Y)$ such that $\rho_H(\downarrow f, \phi(\downarrow f)) < \epsilon(\downarrow f)$ and $\phi(\downarrow f)(d_n) = \mathbf{0}$ for $\downarrow f \in \downarrow \mathrm{C}(X,Y)$. Obviously, $\phi(\downarrow \mathrm{C}(X,Y)) \cap F_{n,m} = \emptyset$. Thus each $F_{n,m}$ is a Z-set in $\downarrow \mathrm{C}(X,Y)$.

Let $F = \bigcap_{n \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} (\downarrow \mathrm{C}(X,Y) \setminus F_{n,m})$. It remains to prove that the closure $\overline{F}$ of F in $\downarrow \mathrm{C}(X,Y)$ is a Z-set. Observe that

$$F = \{\downarrow f \in \downarrow \mathrm{C}(X,Y) \mid f(d_n) = \mathbf{0} \text{ for each } n \in \mathbb{N}\},$$

which implies that $f(x) = \mathbf{0}$ for all $\downarrow f \in F$ and all $x \in X \setminus \overline{X_0}$. Fix $x \in X \setminus \overline{X_0}$ and $\delta > 0$ such that $B_{d_X}(x, \delta) \subset X \setminus \overline{X_0}$. For every $\downarrow f \in \overline{F}$, we have $f(x) = \mathbf{0}$. Indeed, for each $\epsilon \in (0, \delta)$, there exists $\downarrow g \in F$ such that $\rho_H(\downarrow f, \downarrow g) < \epsilon$. Then we can find $(a,b) \in \downarrow g$ such that $\rho((x, f(x)), (a,b)) < \epsilon$. Since $d_X(x,a) < \epsilon < \delta$, we get $g(a) = \mathbf{0}$, so $d_Y(f(x), \mathbf{0}) = d_Y(f(x), b) < \epsilon$. Hence $f(x) = \mathbf{0}$. According to Lemma 6.9.3, the closure $\overline{F}$ is a Z-set in $\downarrow \mathrm{C}(X,Y)$. Consequently, $\downarrow \mathrm{C}(X,Y) = \overline{F} \cup \bigcup_{m,n \in \mathbb{N}} F_{n,m}$ is a Z_σ-set in itself. □

We prove the "if" part of Theorem 6.9.2.

Proposition 6.9.5. *If the set of isolated points is dense in X, then $\downarrow \mathrm{C}(X,Y)$ is a Baire space.*

Proof. Let X_0 be the set of isolated points in X and $\mathcal{F}$ be the finite subsets of X_0. For each $F \in \mathcal{F}$ and $n \in \mathbb{N}$, we define

$$U_{F,n} = \{A \in \overline{\downarrow \mathrm{C}(X,Y)} \mid A(x) \subset B_{d_Y}(\mathbf{0}, 1/n) \text{ for all } x \in X \setminus F\}.$$

Since $F \subset X_0$, $U_{F,n}$ is open in $\overline{\downarrow \mathrm{C}(X,Y)}$. Let $U_n = \bigcup_{F \in \mathcal{F}} U_{F,n}$. We shall prove that each U_n is dense in $\overline{\downarrow \mathrm{C}(X,Y)}$. For each $\downarrow f \in \downarrow \mathrm{C}(X,Y)$ and $\epsilon > 0$, we can obtain $F \in \mathcal{F}$ so that $\rho_H(\downarrow f|_F, \downarrow f) < \epsilon$ because $\downarrow f$ is compact and X_0 is dense in X. Define a map $g : X \to Y$ as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x \in F, \\ \mathbf{0} & \text{if } x \in X \setminus F. \end{cases}$$

Then $\downarrow g \in U_{F,n} \subset U_n$ and $\rho_H(\downarrow g, \downarrow f) \leq \rho_H(\downarrow f|_F, \downarrow f) < \epsilon$. Hence U_n is dense in $\overline{\downarrow \mathrm{C}(X,Y)}$.

Next, we will show that $G = \bigcap_{n\in\mathbb{N}} U_n \subset \downarrow \mathrm{C}(X,Y)$. Let $A \in G$. Observe that for each $x \in X \setminus X_0$, $A(x) = \{\mathbf{0}\}$. Moreover, for each $n \in \mathbb{N}$, we can find $F \in \mathcal{F}$ such that $A \in U_{F,n}$. Then $A(y) \subset B_{d_Y}(\mathbf{0}, 1/n)$ for all $y \in X \setminus F$, which means that A is a hypo-graph of a function from X to Y that is continuous at x. Therefore $A \in \downarrow \mathrm{C}(X,Y)$. Since $\overline{\downarrow \mathrm{C}(X,Y)}$ is compact, the G_δ-set $G = \bigcap_{n\in\mathbb{N}} U_n$ is a Baire space and dense in $\downarrow \mathrm{C}(X,Y)$.Consequently, $\downarrow \mathrm{C}(X,Y)$ is a Baire space. □

Remark 14. In the above proof, if $A \in \overline{\downarrow \mathrm{C}(X,Y)}$ and $x \in X_0$, then $A(x)$ is an arc or the singleton $\{\mathbf{0}\}$. Hence the restriction $A|_{X_0}$ is a hypo-graph of a continuous function from X_0 to Y.

Combining Propositions 6.9.4 and 6.9.5, we can establish Theorem 6.9.2. The space $\mathbf{c}_0$ is not a Baire space. In fact, it is a Z_σ-set in it. Immediately, we have the following:

Corollary 6.9.6. *If $\downarrow \mathrm{C}(X,Y)$ is homeomorphic to $\mathbf{c}_0$, then the set of isolated points is not dense in X.*

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