On the construction of wavelets and its application to numerical analysis of differential equations

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Contents

1 Preliminaries

In this thesis, space $L^p(\mathbb{X})$ $(1 \leq p < \infty)$ denotes

$$L^p(\mathbb{X}) := \left\{ f : f \text{ is a Lebesgue measurable function and } \int_{\mathbb{X}} |f(x)|^p dx < \infty \right\}.$$

The inner product and the norm of $L^2(\mathbb{X})$ are defined by $\langle f, g \rangle_{L^2(\mathbb{X})} := \int_{\mathbb{X}} f(x)\overline{g(x)}dx$ and $\|f\|_{L^2(\mathbb{X})} := \langle f, f \rangle_{L^2(\mathbb{X})}^{1/2}$, respectively. For the case $p = \infty$, we define $L^\infty(\mathbb{X})$ to be the set of essentially bounded measurable functions on $\mathbb{X}$ and $\|f\|_{L^\infty(\mathbb{X})} = \operatorname{ess\,sup}_{x\in\mathbb{X}} |f(x)|$. For $f \in L^2(\mathbb{R})$, the Fourier transform and the inverse Fourier transform are expressed by

$$\mathcal{F}[f](\xi) \left(= \hat{f}(\xi) \right) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx$$

and

$$\mathcal{F}^{-1}[g](x) = \frac{1}{2\pi} \int_{\mathbb{R}} g(\xi) e^{ix\xi} d\xi.$$

In this thesis, we construct basis functions for numerical analysis of differential equations using the wavelet theory. Firstly, let us give definitions and results related to the wavelet theory. For their proofs, we refer to [6, 12, 25], etc.

The orthogonal wavelet is a L^2 function which is defined by the following:

Definition 1.1 *A function $\psi \in L^2(\mathbb{R})$ is called an orthogonal wavelet if the set $\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)\}_{j,k\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$.*

Orthogonal wavelets is usually constructed through a multiresolution analysis (MRA).

Definition 1.2 *An MRA $\{V_j\}_{j\in\mathbb{Z}}$ is a sequence of closed subspaces of $L^2(\mathbb{R})$ which satisfies the followings:*

(1) $V_j \subset V_{j+1}$ *for all* $j \in \mathbb{Z}$.

(2) $f(x) \in V_j \Longleftrightarrow f(2x) \in V_{j+1}$.

(3) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$.

(4) $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$.

(5) *There exists a function $\varphi \in V_0$ such that $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ forms an orthonormal basis for V_0. This function φ is called the scaling function.*

Here we remark that, for $f \in L^2(\mathbb{R})$, we can easily check the orthonormality of $\{f(\cdot - n)\}_{n\in\mathbb{Z}}$ in the Fourier domain.

Lemma 1.3 *Let $f \in L^2(\mathbb{R})$. Then $\{f(\cdot - n)\}_{n\in\mathbb{Z}}$ is an orthonormal system if and only if*

$$\sum_{k\in\mathbb{Z}} \left| \hat{f}(\xi + 2k\pi) \right|^2 = 1 \quad a.e.\ \xi \in \mathbb{R}.$$

Since $\varphi(x/2) \in V_{-1} \subset V_0 = \overline{\mathrm{span}\{\varphi(\cdot - k)\}}_{k\in\mathbb{Z}}$, there exists a sequence $\{h_k\}_{k\in\mathbb{Z}}$ satisfying the two-scale relation

$$\frac{1}{2}\varphi\left(\frac{x}{2}\right) = \sum_{k\in\mathbb{Z}} h_k \varphi(x-k). \tag{1}$$

By the Fourier transform of (1), we obtain

$$\hat{\varphi}(2\xi) = \hat{\varphi}(\xi) \sum_{k\in\mathbb{Z}} h_k e^{-ik\xi} \equiv m_0(\xi)\hat{\varphi}(\xi),$$

where $m_0(\xi) = \sum_{k\in\mathbb{Z}} h_k e^{-ik\xi} \in L^2(-\pi,\pi)$ is called the low-pass filter associated with the scaling function φ. The low-pass filter has the following important property:

Proposition 1.4 *Let m_0 be a low-pass filter. Then, it holds that*

$$|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1 \quad a.e.\ \xi \in \mathbb{R}.$$

Let $\{V_j\}_{j\in\mathbb{Z}}$ be a multiresolution analysis. By the orthogonal decomposition, there exists W_j such that $W_j \oplus V_j = V_{j+1}$. Using the above Proposition and Lemma, we can characterize V_0, V_{-1}, W_0 and W_{-1} as follows:

Lemma 1.5 *Let φ be a scaling function of an MRA $\{V_j\}_{j\in\mathbf{Z}}$ and m_0 be a low-pass filter associated with φ. Then, we have*

$$\begin{aligned}
V_{-1} &= \{f \in L^2(\mathbb{R}) : \hat{f}(\xi) = \alpha(2\xi)m_0(\xi)\hat{\varphi}(\xi),\ \alpha \in L^2(-\pi,\pi)\},\\
V_0 &= \{f \in L^2(\mathbb{R}) : \hat{f}(\xi) = \beta(\xi)\hat{\varphi}(\xi),\ \beta \in L^2(-\pi,\pi)\},\\
W_{-1} &= \{f \in L^2(\mathbb{R}) : \hat{f}(\xi) = e^{i\xi}\gamma(2\xi)\overline{m_0(\xi+\pi)}\hat{\varphi}(\xi),\ \gamma \in L^2(-\pi,\pi)\},\\
W_0 &= \{f \in L^2(\mathbb{R}) : \hat{f}(2\xi) = e^{i\xi}\gamma(2\xi)\overline{m_0(\xi+\pi)}\hat{\varphi}(\xi),\ \gamma \in L^2(-\pi,\pi)\}.
\end{aligned}$$

In fact, to find an orthogonal wavelet, we only have to find a function $\psi \in W_0$ such that $\{\psi(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for W_0 :

Proposition 1.6 *Let $\psi \in W_0$. If $\{\psi(\cdot - k)\}_{k\in\mathbb{Z}}$ forms an orthonormal basis for W_0, then ψ is an orthogonal wavelet, i.e., $\{\psi_{j,k}\}_{j,k\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$.*

From the above arguments, the construction of an orthogonal wavelet from an MRA is summarized as follows:

Theorem 1.7 *Let φ be a scaling function for an MRA $\{V_j\}_{j\in\mathbb{Z}}$ and m_0 is the associated low-pass filter. Suppose that ν is a 2π-periodic function satisfying $|\nu(\xi)| = 1$. Then ψ defined by*

$$\hat{\psi}(\xi) = e^{i\xi/2}\nu(\xi)\overline{m_0\left(\frac{\xi}{2}+\pi\right)}\hat{\varphi}\left(\frac{\xi}{2}\right)$$

is an orthogonal wavelet.

2 Introduction

2.1 The Galerkin method

The Galerkin method is a powerful tool for calculating numerical solutions of differential equations. In particular, lower-degree polynomials are often used for the basis and test functions since the resulting coefficient matrices of the Galerkin equations have simpler structures. This method is called the finite element method (FEM). Let us consider the following problem as an example:

$$\begin{cases} -u'' + u = f, \quad 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{2}$$

A weak form of the problem is given by

$$a(u, v) = \langle f, v \rangle_{L^2(\mathbb{R})} \quad \text{for all } v \in H_0^1(0, 1) \tag{3}$$

with a bilinear form

$$a(u, v) = \int_0^1 u(x)v(x)dx + \int_0^1 u'(x)v'(x)dx.$$

Here we denote the Sobolev space $H^1(0, 1) = \{u \in L^2(0, 1) \ : \ u' \in L^2(0, 1)\}$, and $H_0^1(0, 1) = \{u \in H^1(0, 1) \ : \ u(0) = u(1) = 0\}$ is its subspace. A solution of (3) is called a weak solution.

The Galerkin method constructs an approximate solution as the weak solution. Let $V_n \subset H_0^1$ be an n-dimensional subspace, and let $\varphi_1, \cdots, \varphi_n$ be a basis of V_n. By substituting $u_n \in V_n$ for u and $v_n \in V_n$ for v, we obtain

$$a(u_n, v_n) = \langle f, v_n \rangle_{L^2(\mathbb{R})} \quad \text{for all } v_n \in V_n. \tag{4}$$

We consider the approximate solution $u_J \in V_J$ of the form

$$u_n(x) = \sum_{j=1}^{n} U_j \varphi_j(x).$$

Taking $v_n = \varphi_j$ $(j = 1, 2, \cdots, n)$ in (4) we obtain a Galerkin equation

$$MU = F,$$

where $M = \{a(\varphi_i, \varphi_j)\}_{i,j=1,\cdots,n}$ is a coefficient matrix, $F = {}^t\{\langle f, \varphi_j \rangle_{L^2(\mathbb{R})}\}_{j=1,\cdots,n}$ is a vector generated by the inner products of f and the test functions, and U is a unknown vector $U = {}^t\{U_1, \cdots, U_n\}$. The coefficients $\{U_j\}_j$ are thus obtained as the solution of the equation $U = M^{-1}F$.

Classical FEM employees the hat function $B_2(x) = \max\{1 - |x|, 0\}$ as the basis and test functions. If we put $\{\varphi_i(x) = v_i(x) = B_2(x/h - i)\}_{i=1}^{1/h-1} \subset H_0^1(0, 1)$, then we can easily see that the components of the stiffness and mass matrices are given, respectively, by

$$a_{i,j} = \langle \varphi_i', v_j' \rangle_{L^2(\mathbb{R})} = \frac{1}{h} \times \begin{cases} 2, & j = i, \\ -1, & j = i \pm 1, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$c_{i,j} = \langle \varphi_i, v_j \rangle_{L^2(\mathbb{R})} = \frac{h}{6} \times \begin{cases} 4, & j = i, \\ 1, & j = i \pm 1, \\ 0, & \text{otherwise}. \end{cases}$$

Thus the coefficient matrix M is a tridiagonal matrix, and its components are given by $M_{i,i} = 2/h + 2h/3$, $M_{i,i\pm1} = -1/h + h/6$, and $M_{i,j} = 0$ otherwise. The sparsity of this matrix results in decreased computing costs.

Wavelet theory has been developing rapidly in several fields since its inception in the 1980's, and many wavelets have been introduced. The application of wavelets to the Galerkin method is an interesting topic, and the flexibility of wavelet functions provides many options for approximation spaces. Especially, compactly supported orthogonal wavelets or scaling functions give sparse matrices, including the stiffness matrix, because of their locality and orthogonality. Among these, the Daubechies scaling function [12], which is well known as a compactly supported orthogonal function, is commonly used for numerical analysis. But the Daubechies wavelets and scaling functions do not have explicit expressions in the time domain. So, if we try to compute the inner product on a wavelet $\langle f, \psi\rangle_{L^2(\mathbb{R})}$ or a scaling function $\langle f, \varphi\rangle_{L^2(\mathbb{R})}$ with high-dimensional accuracy, it is computationally expensive. Therefore, in some cases inner products with scaling functions are simply approximated by its sampling, i.e., $\langle f, 2^{j/2}\varphi(2^j \cdot -k)\rangle_{L^2(\mathbb{R})} \approx f(2^{-j}k)$, but the accuracy of these approximations depends on the smoothness of f, and getting high-precision analysis results requires an evaluation of the integrals. To overcome this difficulty with integrations, many methods using wavelets and scaling functions have been introduced [3, 8, 10, 11, 35].

When we use the orthogonal functions as basis and test functions, resulting mass matrix becomes a diagonal matrix, but in almost all cases, the highest derivative of the original equation is a leading term. Thus, in the above case, the structure of the stiffness matrix plays an important role.

In this paper, our aim is to find suitable (non orthogonal) Riesz bases for higher order differential equations in the sense that stiffness matrices are more sparse.

2.2 Uniform approach to find suitable bases

According to differential equations, we expect certain smoothness (at least Lipschitz continuity) for the subspace. Let us put the B-splines of orders 1 and 2 as follows:

$$N_1(x) = \begin{cases} 1 & \text{if } \ 0 \le x \le 1, \\ 0 & \text{otherwise,} \end{cases} \qquad N_2(x) = \begin{cases} x & \text{if } \ 0 \le x < 1, \\ 2-x & \text{if } \ 1 \le x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

$N_1(x)$ is called the Haar scaling function. $\{N_2(x-k) : k \in \mathbb{Z}\}$ which is a Riesz basis for the space V_0 of piecewise linear continuous functions on the intervals $[k, k+1]$ for all $k \in \mathbb{Z}$, is used in the standard FEM. We remark that the Franklin scaling function and the Strömberg scaling function can be also orthogonal bases for V_0 (see [18, 25, 34]). The Lipschitz continuity of functions in the subspace comes from the property of these bases. Therefore, our task is to determine a base scaling function rather than a subspace.

From the point of view of the study of differential equations, the coefficient of the highest order derivative has much more influence on the behavior of the solution. After the translation of the continuous problem into the discrete one, if the matrix corresponding to the principal part becomes simpler, the approximate solution will be more stable as an appropriate numerical treatment. In this section we shall give a uniform approach to find suitable bases such that the matrix corresponding to the principal part has just a form of three-point formula.

Firstly, for the simplicity, let us consider the second order equation $-\frac{d^2}{dx^2}u + u = f$ and V_0 i.e., $j = 0$. We are concerned with the following matrix coming from the principal part:

$$a_{k,\ell} := -\left\langle \frac{d}{dx}\varphi_{0,k}, \frac{d}{dx}\varphi_{0,\ell} \right\rangle_{L^2(\mathbb{R})} \left(= \left\langle \frac{d^2}{dx^2}\varphi_{0,k}, \varphi_{0,\ell} \right\rangle_{L^2(\mathbb{R})} \text{if } \varphi \in C^2 \right).$$

Since $\varphi_{0,k}(x) = \varphi(x-k)$, by Parseval's theorem we see that

$$\left\langle \frac{d}{dx}\varphi_{0,k}, \frac{d}{dx}\varphi_{0,\ell} \right\rangle_{L^2(\mathbb{R})} = \frac{1}{2\pi}\left\langle i\xi e^{-ik\xi}\hat{\varphi}, i\xi e^{-i\ell\xi}\hat{\varphi} \right\rangle_{L^2(\mathbb{R})} = \mathcal{F}^{-1}\left[\left|\xi\hat{\varphi}(\xi)\right|^2\right](\ell - k).$$

On the other hand, in order to get three-point formula for second order derivative, we need the tridiagonal matrix

$$\{a_{k,\ell}\}_{1\le k,\ell\le N} = \begin{pmatrix} -2 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & \cdots & 0 & 0 & 1 & -2 \end{pmatrix}, \tag{5}$$

where N depended on the interval in which $-\frac{d^2}{dx^2}u + u = f$ is considered. Thus, φ must satisfy the condition

$$\mathcal{F}^{-1}\Big[\big|\xi\hat{\varphi}(\xi)\big|^2\Big](\ell - k) = \begin{cases} 2 & \text{if } k = \ell, \\ -1 & \text{if } k = \ell \pm 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

It would not be easy to find φ from (6). Therefore, we shall try to change the condition (6). Further computations yield

$$\begin{aligned} \mathcal{F}^{-1}\Big[\big|\xi\hat{\varphi}(\xi)\big|^2\Big](\ell - k) &= \frac{1}{2\pi}\sum_{q\in\mathbb{Z}}\int_{2q\pi}^{2(q+1)\pi} e^{i(\ell-k)\xi}\big|\xi\hat{\varphi}(\xi)\big|^2 d\xi \\ &= \frac{1}{2\pi}\sum_{q\in\mathbb{Z}}\int_0^{2\pi} e^{i(\ell-k)\xi}\big|(\xi+2q\pi)\hat{\varphi}(\xi+2q\pi)\big|^2 d\xi \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{i(\ell-k)\xi}\sum_{q\in\mathbb{Z}}\big|(\xi+2q\pi)\hat{\varphi}(\xi+2q\pi)\big|^2 d\xi. \end{aligned}$$

Hence, we find that (6) is equivalent to

$$\sum_{q\in\mathbb{Z}}\big|(\xi+2q\pi)\hat{\varphi}(\xi+2q\pi)\big|^2 \equiv -e^{i\xi} + 2e^{i0\xi} - e^{-i\xi}\Big(= 4\sin^2\frac{\xi}{2}\Big)$$

for almost everywhere $\xi \in \mathbb{R}$. Denoting the sinc function by $\operatorname{sinc}\xi = \frac{\sin\xi}{\xi}$, we see that the Haar scaling function $N_1(x)$ satisfies $\hat{N}_1(\xi) = e^{-i\xi/2}\operatorname{sinc}\frac{\xi}{2}$. We shall define $\Phi(x)$ by

$$\hat{\Phi}(\xi) = \frac{\hat{\varphi}(\xi)}{\hat{N}_1(\xi)}. \tag{7}$$

Then we also get

$$\sum_{q\in\mathbb{Z}}\big|\hat{\Phi}(\xi+2q\pi)\big|^2 = \sum_{q\in\mathbb{Z}}\frac{\big|(\xi+2q\pi)\hat{\varphi}(\xi+2q\pi)\big|^2}{4\sin^2\frac{\xi+2q\pi}{2}} = \frac{\sum_{q\in\mathbb{Z}}\big|(\xi+2q\pi)\hat{\varphi}(\xi+2q\pi)\big|^2}{-e^{i\xi} + 2e^{i0\xi} - e^{-i\xi}},$$

here we used

$$4\sin^2\frac{\xi+2q\pi}{2} = 4\sin^2\frac{\xi}{2} = -e^{i\xi} + 2e^{i0\xi} - e^{-i\xi}.$$

This means that

$$\sum_{q\in\mathbb{Z}}\big|\hat{\Phi}(\xi+2q\pi)\big|^2 \equiv 1 \quad \text{a.e. } \xi \in \mathbb{R}. \tag{8}$$

So, the condition (6) has been reduced to the conditions (7) and (8). Now we can easily find φ from (7) and (8), because the identity (8) is well-known as the orthonormal condition. The definition (7) yields

$$\varphi(x) = \mathcal{F}^{-1}\Big[\hat{N}_1(\xi)\hat{\Phi}(\xi)\Big](x) = N_1 * \Phi(x)\Big(= \int_{x-1}^{x} \Phi(y)dy\Big). \tag{9}$$

The new function φ is the *elevation* of Φ with N_1. Therefore N_1 is also called the *elevator* (see [32, 36]). More generally, let us represent the elevator by $\mathcal{E}$ and define

$$\varphi(x) = \mathcal{E} * \Phi(x).$$

Remark 2.1 The most typical example is the case when the elevator $\mathcal{E}(x)$ is $N_1(x)$ and $\Phi(x)$ is the Haar scaling function, i.e., $\mathcal{E}(x) = \Phi(x) = N_1(x)$. In this case, by (9) we obtain

$$\varphi(x) = N_1 * N_1(x) = N_2(x).$$

This case just coincides with the standard FEM. Choosing other scaling functions for $\Phi(x)$, we can obtain various types of bases.

2.3 Definition of elevator

We shall derive some properties for the case when the elevator $\mathcal{E}(x)$ is $N_1(x)$. By Taylor expansion we see that for $v \in C^4$

$$\sum_{\nu=-1}^{1} a_{k,k+\nu} v(x+\nu h) = h^2 \frac{d^2}{dx^2} v(x) + O(h^4) \quad \text{for all } k \in \mathbb{Z}. \tag{10}$$

Moreover, we assume that

$$\hat{\Phi}(0) = 1, \tag{11}$$

which allows scaling functions, but excludes wavelet functions. Hence, by (9) it follows that

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \varphi_{j,k}(x) &= \sum_{k\in\mathbb{Z}} \varphi(2^j x - k) = \sum_{k\in\mathbb{Z}} \int_{2^j x-k-1}^{2^j x-k} \Phi(y)dy \\ &= \int_{-\infty}^{\infty} \Phi(y)dy = \hat{\Phi}(0) = 1. \end{aligned} \tag{12}$$

This is just the partition of unity. Let us put $h = 2^{-j}$ and $w_j(x) = \sum_{\ell\in\mathbb{Z}} w_{j,\ell}\varphi_{j,\ell}(x)$. If w_j is sufficiently smooth and φ has compact support (or decays sufficiently fast), by (12) we have for $k \in \mathbb{Z}$

$$w_j(kh) = \sum_{\ell\in\mathbb{Z}} w_{j,\ell}\varphi_{j,\ell}(kh) \sim w_{j,k} \sum_{\ell\in\mathbb{Z}} \varphi_{j,\ell}(kh) = w_{j,k}. \tag{13}$$

Indeed, it holds that $w_j(kh) = w_{j,k}$ in the standard FEM.

Meanwhile we also get the following identity:

$$\sum_{q\in\mathbb{Z}} \big|\hat{\varphi}(\xi + 2q\pi)\big|^2 = \sum_{\nu\in\mathbb{Z}} c_{k,k+\nu} e^{i\nu\xi},$$

where $c_{k,\ell} := \langle \varphi_{0,k}, \varphi_{0,\ell}\rangle_{L^2(\mathbb{R})}$. In particular, taking $\xi = 0$, by (7) and (11) we find that for all $k \in \mathbb{Z}$

$$\sum_{\nu\in\mathbb{Z}} c_{k,k+\nu} = \sum_{q\in\mathbb{Z}} \big|\hat{\varphi}(2q\pi)\big|^2 = \sum_{q\in\mathbb{Z}} \big|\hat{N}_1(2q\pi)\hat{\Phi}(2q\pi)\big|^2 = \big|\hat{\Phi}(0)\big|^2 = 1,$$

here we used $\hat{\Phi}(2q\pi) = 0$ if $q \neq 0$, since $\sum_{q\neq 0} \left|\hat{\Phi}(2q\pi)\right|^2 - \left|\hat{\Phi}(0)\right|^2 = 0$ by (11) and (8) with $\xi = 0$. Noting that $c_{k,k+\nu} = c_{k,k-\nu}$, by Taylor expansion we see that for $v \in C_0^2$

$$\sum_{\nu\in\mathbb{Z}} c_{k,k+\nu}\ v(x+\nu h) = v(x) + O(h^2) \quad \text{for all } k \in \mathbb{Z}. \tag{14}$$

In our construction, to get the approximate solution $u_j(x) = \sum_{\ell\in\mathbb{Z}} u_{j,\ell}\varphi_{j,\ell}(x)$ in the interval $(0,1)$ for the equation $-\frac{d^2}{dx^2}u + u = f$, we solve the following system corresponding to the Galerkin equation:

$$\Big[-\big\{a_{k,\ell}h^{-2}\big\}_{1\le k,\ell\le N} + \big\{c_{k,\ell}h\big\}_{1\le k,\ell\le N}\Big]\ {}^t\big\{u_{j,\ell}\big\}_{1\le \ell\le N} = \big\{c_{k,\ell}\big\}_{1\le k,\ell\le N}\ {}^t\big\{f_{j,\ell}\big\}_{1\le \ell\le N}.$$

By (13) this can be regarded as

$$\Big[-\big\{a_{k,\ell}h^{-2}\big\}_{1\le k,\ell\le N} + \big\{c_{k,\ell}h\big\}_{1\le k,\ell\le N}\Big]\ {}^t\big\{u_j(\ell h)\big\}_{1\le \ell\le N} = \big\{c_{k,\ell}\big\}_{1\le k,\ell\le N}\ {}^t\big\{f_j(\ell h)\big\}_{1\le \ell\le N}.$$

Paying attention to each row, by (10) and (14) we find that

$$\begin{aligned}
-\sum_{1\le\ell\le 2^j} a_{k,\ell}h^{-2}u_j(\ell h) &= -\sum_{\nu} a_{k,k+\nu}h^{-2}u_j(kh+\nu h) = -\frac{d^2}{dx^2}u_j(kh) + O(h^2),\\
\sum_{1\le\ell\le 2^j} c_{k,\ell}u_j(\ell h) &= \sum_{\nu} c_{k,k+\nu}u_j(kh+\nu h) = u_j(kh) + O(h^2),\\
\sum_{1\le\ell\le 2^j} c_{k,\ell}f_j(\ell h) &= \sum_{\nu} c_{k,k+\nu}f_j(kh+\nu h) = f_j(kh) + O(h^2).
\end{aligned}$$

These give the numerical difference equation of the original differential equation $-\frac{d^2}{dx^2}u + u = f$ at the point $x = kh$. The accuracy of (13) depends on the case of application. We remark that (10) and (14) play an important role to guarantee the accuracy.

From the above observations for $\mathcal{E} = N_1$, we shall propose the following conditions to characterize qualitative elevators for the Galerkin method:

Definition 2.2 *Let Φ be a scaling function such that $\hat{\Phi}(0) = 1$ and $\hat{\Phi}(\xi) \neq 0$ for $-\pi \le \xi \le \pi$. Put $c_{k,\ell} := \langle \varphi_{0,k}, \varphi_{0,\ell}\rangle_{L^2(\mathbb{R})}$ and $a_{k,\ell} := -\langle \frac{d}{dx}\varphi_{0,k}, \frac{d}{dx}\varphi_{0,\ell}\rangle_{L^2(\mathbb{R})}$ for $\varphi(x) = \mathcal{E} * \Phi(x)$. The elevator $\mathcal{E}$ for the Galerkin method is a function satisfying*

(i) *$\hat{\mathcal{E}}(\xi) \neq 0$ for $-\pi \le \xi \le \pi$, in particular, $\hat{\mathcal{E}}(0) = 1$.*

(ii) *It holds that for $v \in C_0^4$*

$$\begin{aligned}
\sum_{\nu\in\mathbb{Z}} c_{k,k+\nu}\ v(x+\nu h) &= v(x) + O(h^2),\\
\sum_{\nu\in\mathbb{Z}} a_{k,k+\nu} v(x+\nu h) &= h^2\frac{d^2}{dx^2}v(x) + O(h^4).
\end{aligned}$$

(iii) *There exists a 2π-periodic function $m_{\mathcal{E}}(\xi)$ such that $\hat{\mathcal{E}}(2\xi) = m_{\mathcal{E}}(\xi)\hat{\mathcal{E}}(\xi)$.*

It is known that the exact frame is equivalent to the Riesz basis. The condition for the Riesz basis is given by

$$A \le \sum_{q\in\mathbb{Z}} \left|\hat{\varphi}(\xi + 2q\pi)\right|^2 \le B \tag{15}$$

for $0 < A \le B < \infty$ (see [4]). If $\hat{\Phi}(\xi) \neq 0$ for $-\pi \le \xi \le \pi$, by (i) we note that

$$\begin{aligned}\sum_{q\in\mathbb{Z}} \left|\hat{\varphi}(\xi+2q\pi)\right|^2 &= \sum_{q\in\mathbb{Z}} \left|\hat{\mathcal{E}}(\xi+2q\pi)\hat{\Phi}(\xi+2q\pi)\right|^2 \\ &\ge \left|\hat{\mathcal{E}}(\xi-2n\pi)\hat{\Phi}(\xi-2n\pi)\right|^2 \\ &\ge {}^{\exists}A > 0\end{aligned}$$

for $2n\pi - \pi \le \xi \le 2n\pi + \pi$ $(n \in \mathbb{Z})$, that is, $\xi \in \mathbb{R}$. Rewriting $\varphi(x) = N_1 * \Phi^\sharp(x)$ with $\hat{\Phi}^\sharp(\xi) = \frac{\hat{\mathcal{E}}(\xi)\hat{\Phi}(\xi)}{\hat{N}_1(\xi)}$, from (i) we can expect that the properties corresponding to (12), (13) and (14) still hold, since $\hat{\Phi}^\sharp(0) = 1$. In fact, we may omit $\sum_{\nu\in\mathbb{Z}} c_{k,k+\nu}\, v(x+\nu h) = v(x) + O(h^2)$ in (ii).

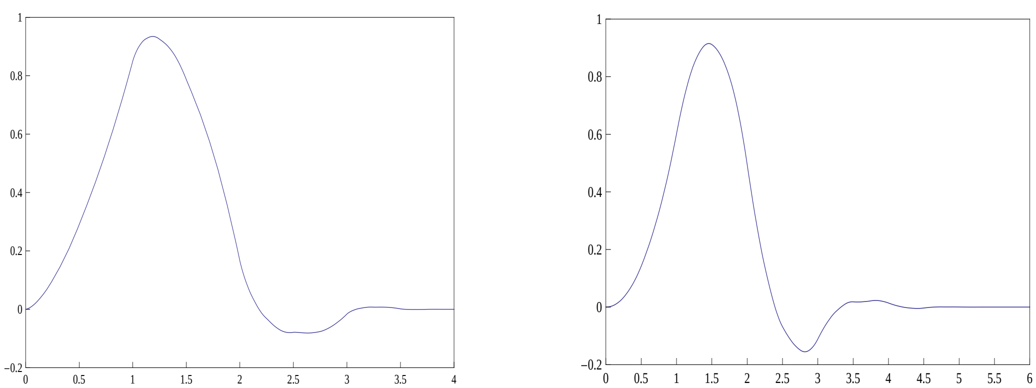
Replacing the definition $\varphi_{j,k}(x) = \varphi(2^j x - k)$ by $\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k)$, we could also get wavelet expansions. Thanks to the condition (iii) we obtain a semi-orthogonal wavelet $\hat{\psi}(\xi) = e^{i\xi/2}\overline{m(\xi/2+\pi)}\hat{\varphi}(\xi/2)$, where $m(\xi) = m_\varphi(\xi)\sum_{q\in\mathbb{Z}} |\hat{\varphi}(\xi+2q\pi)|^2 = m_{\mathcal{E}}(\xi) m_\Phi(\xi) \sum_{\nu\in\mathbb{Z}} c_{k,k+\nu} e^{i\nu\xi}$ (2π-periodic). A biorthogonal wavelet for the elevated φ can be also considered (see [16]).

3 Riesz basis of Daubechies type

3.1 Three-point formula for second order derivative

To get compactly-supported and also more smooth base than N_2, we may choose the Daubechies scaling function of order p for $\Phi \equiv \Phi_p^{\mathrm{D}}$ satisfying (11). Then by (9) we have

$$\varphi_p^{\mathrm{D}}(x) = N_1 * \Phi_p^{\mathrm{D}}(x). \tag{16}$$



(a) φ_2^{D} $\left(\text{supp } \varphi_2^{\mathrm{D}} \subset [0,4]\right)$ (b) φ_3^{D} $\left(\text{supp } \varphi_3^{\mathrm{D}} \subset [0,6]\right)$

Figure 1: Graphs of φ_2^{D} and φ_3^{D}.

The basis $\{\varphi_p^{\mathrm{D}}(x-k) : k \in \mathbb{Z}\}$ had been derived by [32] and [36]. Their approach is motivated from the observation that the integration of the Haar wavelet becomes N_2. Therefore, the pseudoframe was firstly considered by the integration of the Daubechies wavelet, and secondly it was arranged for the efficiency of the computation and arrived at φ_p^{D} (see also [27]).

In order to solve numerically the equation $-\frac{d^2}{dx^2}u + u = f$ with some base $\{\varphi(x-k) : k \in \mathbb{Z}\}$, we need to know the matrices $\left\{c_{k,\ell}\right\}_{1\le k,\ell\le N}$ and $\left\{a_{k,\ell}\right\}_{1\le k,\ell\le N}$. If one considers the orthogonal Daubechies scaling function, it holds that the matrix $\left\{c_{k,\ell}\right\}_{1\le k,\ell\le N} = I$. On the other hand, the matrix $\left\{a_{k,\ell}\right\}_{1\le k,\ell\le N}$ for the Daubechies scaling function is well studied in [1]. For all the bases constructed by the approach in §2.2, the matrix $\left\{a_{k,\ell}\right\}_{1\le k,\ell\le N}$ is just (5).

Remark 3.1 It would be preferable that bases are at least C^1 or Lipschitz continuous as N_2 in order that the weak form $-\langle \frac{d}{dx}\varphi_{0,k}, \frac{d}{dx}\varphi_{0,\ell}\rangle_{L^2}$ of $\langle \frac{d^2}{dx^2}\varphi_{0,k}, \varphi_{0,\ell}\rangle_{L^2}$ has a meaning. Especially for $p = 2$, the Daubechies scaling function $\Phi_2^{\mathrm{D}} \in C^{0.55}$ fails to satisfy the differentiability, but gives $\varphi_2^{\mathrm{D}} \in C^{1.55}$.

We shall also compute the exact value of $c_{k,\ell}$ for $\varphi_2^{\mathrm{D}}(x)$. Putting $\hat{\phi}(\xi) = \left|\hat{\varphi}_2^{\mathrm{D}}(\xi)\right|^2$, by Parseval's theorem we have

$$c_{k,\ell} = \langle \varphi_2^{\mathrm{D}}(x-k), \varphi_2^{\mathrm{D}}(x-\ell)\rangle_{L^2(\mathbb{R})} = \phi(\ell - k).$$

By (16) it holds that

$$\hat{\phi}(\xi) = \operatorname{sinc}^2\frac{\xi}{2}\hat{\Phi}_2^{\mathrm{D}}(\xi)^2 = \prod_{j=1}^{\infty}\cos^2\left(\frac{\xi}{2^{j+1}}\right)\left|m_2^{\mathrm{D}}\left(\frac{\xi}{2^j}\right)\right|^2 \equiv \prod_{j=1}^{\infty}\tilde{m}\left(\frac{\xi}{2^j}\right),$$

where m_2^{D} is the Daubechies low-pass filter. Since $m_2^{\mathrm{D}}(\xi) = \sum_{k=0}^{3}\eta_k e^{-ik\xi}$ with

$$\{\eta_0, \eta_1, \eta_2, \eta_3\} = \left\{\frac{1+\sqrt{3}}{8}, \frac{3+\sqrt{3}}{8}, \frac{3-\sqrt{3}}{8}, \frac{1-\sqrt{3}}{8}\right\}$$

and $\cos^2\xi = \frac{e^{2i\xi}+2+e^{-2i\xi}}{4}$, we find that $\tilde{m}(\xi) = \sum_{k=-4}^{4}\mu_k e^{-ik\xi}$ and its coefficients are given by

$$\{\mu_0, \mu_{\pm 1}, \mu_{\pm 2}, \mu_{\pm 3}, \mu_{\pm 4}\} = \left\{\frac{25}{64}, \frac{17}{64}, \frac{1}{16}, -\frac{1}{64}, -\frac{1}{128}\right\}.$$

The function ϕ satisfies

$$\phi(x) = 2\sum_{k=-4}^{4}\mu_k\phi(2x-k)$$

and $\operatorname{supp}\phi \subset [-4,4]$ since $\operatorname{supp}\varphi_2^D \subset [0,4]$ and $\phi(x) = \int_{\mathbb{R}}\varphi_2^D(t+x)\overline{\varphi_2^D(t)}dt$. Hence we have

$$M\ {}^t\{\phi(k)\}_{-3\le k\le 3} = \mathbf{0},$$

where

$$M = \begin{pmatrix} 1-2\mu_{-3} & -2\mu_{-4} & 0 & 0 & 0 & 0 & 0 \\ -2\mu_{-1} & 1-2\mu_{-2} & -2\mu_{-3} & -2\mu_{-4} & 0 & 0 & 0 \\ -2\mu_1 & -2\mu_0 & 1-2\mu_{-1} & -2\mu_{-2} & -2\mu_{-3} & -2\mu_{-4} & 0 \\ -2\mu_3 & -2\mu_2 & -2\mu_1 & 1-2\mu_0 & -2\mu_{-1} & -2\mu_{-2} & -2\mu_{-3} \\ 0 & -2\mu_4 & -2\mu_3 & -2\mu_2 & 1-2\mu_1 & -2\mu_0 & -2\mu_{-1} \\ 0 & 0 & 0 & -2\mu_4 & -2\mu_3 & 1-2\mu_2 & -2\mu_1 \\ 0 & 0 & 0 & 0 & 0 & -2\mu_4 & 1-2\mu_3 \end{pmatrix}.$$

We also remark that $\sum_{k=-3}^{3}\phi(k) = \sum_{k\in\mathbb{Z}}\phi(k) = \int_{\mathbb{R}}\frac{1}{2\pi}\sum_{k\in\mathbb{Z}}e^{ik\xi}\hat{\phi}(\xi)d\xi = \hat{\phi}(0) = 1$. Deriving the eigenvector with 0 eigenvalue such that $\sum_{k=-3}^{3}\phi(k) = 1$, we find that

$$\{\phi(0), \phi(\pm 1), \phi(\pm 2), \phi(\pm 3)\} = \left\{\frac{131}{180}, \frac{37}{240}, -\frac{11}{600}, \frac{1}{3600}\right\}.$$

Thus we obtain

$$c_{k,\ell}\Big(= \phi(l-k)\Big) = \begin{cases} 131/180 & \text{if } k = \ell, \\ 37/240 & \text{if } k = \ell \pm 1, \\ -11/600 & \text{if } k = \ell \pm 2, \\ 1/3600 & \text{if } k = \ell \pm 3, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, we get the following theorem:

Theorem 3.2 ([21]) *For $\varphi_2^D(x)$ defined by (16) with $\Phi = \Phi_2^D$ we have*

$$c_{k,\ell} = \begin{cases} 131/180 & \text{if } k = \ell, \\ 37/240 & \text{if } k = \ell \pm 1, \\ -11/600 & \text{if } k = \ell \pm 2, \\ 1/3600 & \text{if } k = \ell \pm 3. \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad a_{k,\ell} = \begin{cases} -2 & \text{if } k = \ell, \\ 1 & \text{if } k = \ell \pm 1, \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Moreover, it holds that for $v \in C_0^4$

$$\sum_{\nu \in \mathbb{Z}} c_{k,k+\nu}\ v(x + \nu h) = v(x) + O(h^2),$$
$$\sum_{\nu=-1}^{1} a_{k,k+\nu} v(x + \nu h) = h^2 \frac{d^2}{dx^2} v(x) + O(h^4).$$

3.2 Five-point formula for second order derivative

With small changes of the approach in §1.2 we can also consider the 5-point formula for 2nd order derivative. For this purpose, we replace (6) by

$$\mathcal{F}^{-1}\left[\left|\xi\hat{\varphi}(\xi)\right|^2\right](\ell - k) = \begin{cases} \frac{5}{2} & \text{if } k = \ell, \\ -\frac{4}{3} & \text{if } k = \ell \pm 1, \\ \frac{1}{12} & \text{if } k = \ell \pm 2, \\ 0 & \text{otherwise,} \end{cases}$$

which is equivalent to

$$\sum_{q \in \mathbb{Z}} \left|(\xi + 2q\pi)\hat{\varphi}(\xi + 2q\pi)\right|^2 \equiv \frac{1}{12}e^{2i\xi} - \frac{4}{3}e^{i\xi} + \frac{5}{2}e^{i0\xi} - \frac{4}{3}e^{-i\xi} + \frac{1}{12}e^{-2i\xi} \left(= \frac{4}{3}\sin^2\frac{\xi}{2}\left(\sin^2\frac{\xi}{2} + 3\right)\right)$$

for almost everywhere $\xi \in \mathbb{R}$. We shall define $\Phi(x)$ by

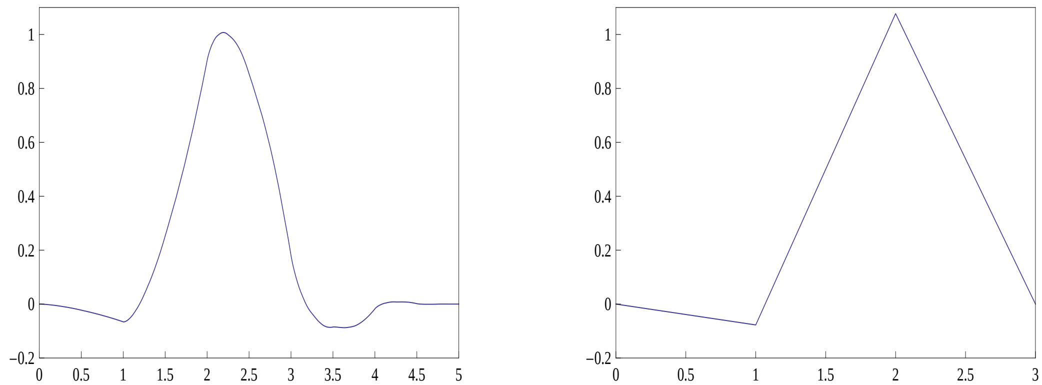
$$\hat{\Phi}(\xi) = \frac{\hat{\varphi}(\xi)}{(\gamma^- + \gamma^+ e^{-i\xi})\hat{N}_1(\xi)}, \tag{18}$$

where $\gamma^\pm = \frac{1}{2} \pm \frac{1}{\sqrt{3}}$. Then we also get

$$\sum_{q \in \mathbb{Z}} \left|\hat{\Phi}(\xi + 2q\pi)\right|^2 = \sum_{q \in \mathbb{Z}} \frac{\left|(\xi + 2q\pi)\hat{\varphi}(\xi + 2q\pi)\right|^2}{4\sin^2\frac{\xi+2q\pi}{2}\left|\gamma^- + \gamma^+ e^{-i(\xi+2q\pi)}\right|^2}$$
$$= \frac{\sum_{p \in \mathbb{Z}}\left|(\xi + 2q\pi)\hat{\varphi}(\xi + 2q\pi)\right|^2}{\frac{1}{12}e^{2i\xi} - \frac{4}{3}e^{i\xi} + \frac{5}{2}e^{i0\xi} - \frac{4}{3}e^{-i\xi} + \frac{1}{12}e^{-2i\xi}},$$

here we used

$$4\sin^2\frac{\xi + 2q\pi}{2}\left|\gamma^- + \gamma^+ e^{-i(\xi+2q\pi)}\right|^2 = 4\sin^2\frac{\xi}{2}\left|\gamma^- + \gamma^+ e^{-i\xi}\right|^2 = \frac{4}{3}\sin^2\frac{\xi}{2}\left(\sin^2\frac{\xi}{2} + 3\right)$$
$$= \frac{1}{12}e^{2i\xi} - \frac{4}{3}e^{i\xi} + \frac{5}{2}e^{i0\xi} - \frac{4}{3}e^{-i\xi} + \frac{1}{12}e^{-2i\xi}.$$



(a) $\Phi = \Phi_2^{\mathrm{D}}$ (supp $\tilde{\varphi} \subset [0,5]$). (b) $\Phi = N_1$ (supp $\tilde{\varphi} \subset [0,3]$).

Figure 2: Graphs of $\tilde{\varphi} = \big\{\gamma^- N_1(x) + \gamma^+ N_1(x-1)\big\} * \Phi(x)$.

Hence, the identity (8) still holds. Thus, the definition (18) yields

$$\tilde{\varphi}(x) = \big\{\gamma^- N_1(\cdot) + \gamma^+ N_1(\cdot - 1)\big\} * \Phi(x). \tag{19}$$

Let us put $\tilde{c}_{k,\ell} := \langle \tilde{\varphi}_{0,k}, \tilde{\varphi}_{0,\ell}\rangle_{L^2(\mathbb{R})}$ and $\tilde{a}_{k,\ell} := -\langle \frac{d}{dx}\tilde{\varphi}_{0,k}, \frac{d}{dx}\tilde{\varphi}_{0,\ell}\rangle_{L^2(\mathbb{R})}$. Similarly as §1.2, by (19) we also find that for all $\ell \in \mathbb{Z}$

$$\begin{aligned}\sum_{\nu\in\mathbb{Z}} \tilde{c}_{k,k+\nu} &= \sum_{q\in\mathbb{Z}} \left|\big\{\gamma^- \hat{N}_1(2q\pi) + \gamma^+ e^{-2q\pi i}\hat{N}_1(2q\pi)\big\}\hat{\Phi}(2q\pi)\right|^2 \\ &= \left|(\gamma^- + \gamma^+)\hat{\Phi}(0)\right|^2 = 1.\end{aligned}$$

We remark that $E(x) = \gamma^+ N_1(x-1) + \gamma^- N_1(x)$ satisfies (i) in Definition 2.2, since $|\hat{E}(\xi)| = \left|\gamma^- + \gamma^+ e^{i\xi}\right| \left|\mathrm{sinc}\ \frac{\xi}{2}\right| = \sqrt{\frac{1}{3}\sin^2\frac{\xi}{2}}\left|\mathrm{sinc}\ \frac{\xi}{2}\right|$.

It remains to compute the precise value of $\tilde{c}_{k,\ell}$ for $\tilde{\varphi}(x)$. Put $\varphi = N_1 * \Phi$ and $c_{k,\ell} = \langle \varphi_{0,k}, \varphi_{0,\ell}\rangle_{L^2}$ as in §3.1. Since $\tilde{\varphi}(x) = \gamma^-\varphi(x) + \gamma^+\varphi(x-1)$, we get

$$\begin{aligned}\tilde{c}_{k,\ell} &= \langle \gamma^-\varphi_{0,k} + \gamma^+\varphi_{0,k+1}, \gamma^-\varphi_{0,\ell} + \gamma^+\varphi_{0,\ell+1}\rangle_{L^2} \\ &= \left({\gamma^+}^2 + {\gamma^-}^2\right)\langle \varphi_{0,k}, \varphi_{0,\ell}\rangle_{L^2} + \gamma^+\gamma^-\big(\langle \varphi_{0,k}, \varphi_{0,\ell+1}\rangle_{L^2} + \langle \varphi_{0,k+1}, \varphi_{0,\ell}\rangle_{L^2}\big) \\ &= \frac{7}{6}c_{k,\ell} - \frac{1}{12}\left(c_{k,\ell+1} + c_{k+1,\ell}\right).\end{aligned}$$

In the case of $\Phi = \Phi_2^{\mathrm{D}}$, each $c_{k,\ell}$ is already given by (17). In the case of $\Phi = N_1$, since $\varphi = N_2$ we can easily see that

$$c_{k,\ell} = \begin{cases} 2/3 & \text{if } k = \ell, \\ 1/6 & \text{if } k = \ell \pm 1, \\ 0 & \text{otherwise.}\end{cases}$$

Consequently, we get the following theorem corresponding to Theorem 3.2:

Theorem 3.3 ([21]) *For $\tilde{\varphi}_2^D(x)$ defined by (19) with $\Phi = \Phi_2^D$ (resp. N_1) we have*

$$\tilde{c}_{k,\ell} = \begin{cases} 3557/4320 & \text{if } k=\ell, \\ 163/1350 & \text{if } k=\ell\pm 1, \\ -37/1080 & \text{if } k=\ell\pm 2, \\ 1/540 & \text{if } k=\ell\pm 3, \\ -1/43200 & \text{if } k=\ell\pm 4, \\ 0 & \text{otherwise,} \end{cases} \left(\textit{resp. } \tilde{c}_{k,\ell} = \begin{cases} 3/4 & \text{if } k=\ell, \\ 5/36 & \text{if } k=\ell\pm 1, \\ -1/72 & \text{if } k=\ell\pm 2, \\ 0 & \text{otherwise,} \end{cases} \right) \tag{20}$$

and

$$\tilde{a}_{k,\ell} = \begin{cases} -5/2 & \text{if } k=\ell, \\ 4/3 & \text{if } k=\ell\pm 1, \\ -1/12 & \text{if } k=\ell\pm 2, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, it holds that for $v \in C_0^6$

$$\begin{aligned} \sum_{\nu\in\mathbb{Z}} \tilde{c}_{k,k+\nu}\, v(x+\nu h) &= v(x) + O(h^2), \\ \sum_{\nu=-2}^{2} \tilde{a}_{k,k+\nu} v(x+\nu h) &= h^2 \frac{d^2}{dx^2} v(x) + O(h^6). \end{aligned}$$

3.3 Numerical results

Let us introduce some examples and numerical results in this section.

Riesz base φ	Choice of Φ	Elevator $\mathcal{E}$	Length of support	Regularity in x	Remainder
N_2	N_1	N_1	2	C^1 (Lip)	$O(h^2)$
φ_2^{D}	Φ_2^{D}	N_1	4	$C^{1.5}$	$O(h^2)$
$\tilde{N}_2$	N_1	$\gamma^+ N_1(x-1)$ $+\gamma^- N_1(x)$	3	C^2	$O(h^4)$
$\tilde{\varphi}_2^{\mathrm{D}}$	Φ_2^{D}	$\gamma^+ N_1(x-1)$ $+\gamma^- N_1(x)$	5	$C^{1.5}$	$O(h^4)$
N_3	N_1	N_2	3	C^2	$O(h^2)$
$\mathring{\varphi}_2^{\mathrm{D}}$	Φ_2^{D}	N_2	5	$C^{2.5}$	$O(h^2)$

The boundary valued problem for

$$-\varepsilon^2 \frac{d^2}{dx^2} u^{(\varepsilon)} + u^{(\varepsilon)} = f, \quad 0 < x < 1, \quad u^{(\varepsilon)}(0) = u^{(\varepsilon)}(1) = 0,$$

has a solution represented by

$$u^{(\varepsilon)}(x) = -\frac{\sinh(x/\varepsilon)}{\varepsilon \sinh(1/\varepsilon)} \int_0^1 \sinh \frac{y-1}{\varepsilon} f(y)dy + \frac{1}{\varepsilon} \int_0^x \sinh \frac{y-x}{\varepsilon} f(y)dy. \tag{21}$$

For $f(x) = \sin 10\pi x$, by (21) the exact solution is $u^{(\varepsilon)}(x) = \frac{\sin 10\pi x}{1+100\varepsilon^2\pi^2}$ and the errors with the Riesz bases N_2 and φ_2^{D} are given by the following:

Table 1: The case of $\varepsilon = 1$.

Mesh size 2^{-j}	$E_j^{N_2}$	Q^{N_2}	$E_j^{\varphi_2^{\mathrm{D}}}$	$Q^{\varphi_2^{\mathrm{D}}}$
$j = 6$	1.57×10^{-4}	2.67	2.02×10^{-2}	6.07
$j = 7$	5.67×10^{-5}	2.77	3.58×10^{-3}	5.64
$j = 8$	2.02×10^{-5}	2.81	8.47×10^{-4}	4.23

Table 2: The case of $\varepsilon = 10^{-6}$.

Mesh size 2^{-j}	$E_j^{N_2}$	Q^{N_2}	$E_j^{\varphi_2^{\mathrm{D}}}$	$Q^{\varphi_2^{\mathrm{D}}}$
$j = 6$	1.58×10^{-1}	2.82	6.21×10^{-4}	5.57
$j = 7$	5.62×10^{-2}	2.81	1.17×10^{-4}	5.32
$j = 8$	2.00×10^{-2}	2.82	6.21×10^{-4}	5.57

For $f(x) = -\varepsilon^2(9N_1(3x) - 18N_1(3x-1) + N_1(3x-2)) + N_3(3x)$, the exact solution $u(x) = N_3(3x)$ belongs to H^2, since $\int_{\mathbb{R}} \langle\xi\rangle^4 |\hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}} \langle\xi\rangle^4 \left(\frac{\sin\xi/6}{\xi/6}\right)^6 d\xi < \infty$. The errors are given by the following:

Table 3: The case of $\varepsilon = 1$.

Mesh size 2^{-j}	$E_j^{N_2}$	Q^{N_2}	$E_j^{\varphi_2^{\mathrm{D}}}$	$Q^{\varphi_2^{\mathrm{D}}}$
$j = 6$	1.43×10^{-4}	2.81	6.08×10^{-4}	4.00
$j = 7$	5.20×10^{-5}	2.75	1.52×10^{-4}	4.00
$j = 8$	1.84×10^{-5}	2.82	3.82×10^{-5}	3.98

Table 4: The case of $\varepsilon = 10^{-6}$.

Mesh size 2^{-j}	$E_j^{N_2}$	Q^{N_2}	$E_j^{\varphi_2^{\mathrm{D}}}$	$Q^{\varphi_2^{\mathrm{D}}}$
$j = 6$	4.68×10^{-3}	2.79	5.59×10^{-4}	4.00
$j = 7$	1.69×10^{-3}	2.77	1.40×10^{-4}	3.99
$j = 8$	6.00×10^{-4}	2.82	3.51×10^{-5}	3.99

Here E_j^{φ} is relative L^2-error between the exact solution $u^{(\varepsilon)}(x)$ and the approximation $\tilde{u}^{(\varepsilon)}(x) = \sum_{\ell=1}^{N} u_\ell \varphi_{j,\ell}(x)$ on $[0,1]$ defined by

$$E_j^{\varphi} = E_j^{\varphi}(\varepsilon) = \frac{1}{\|u^{(\varepsilon)}\|_{L^2}} \sqrt{\sum_{\ell=0}^{2^j} \left\{ u^{(\varepsilon)}\left(\frac{\ell}{2^j}\right) - \tilde{u}^{(\varepsilon)}\left(\frac{\ell}{2^j}\right) \right\}^2}$$

and the ratio Q^{φ} is defined by $Q^{\varphi} = E_{j-1}^{\varphi}/E_j^{\varphi}$.

Concluding Remarks The method with the elevated Riesz bases converts a continuous operator to a discrete problem by featuring the highest order derivatives. Therefore, we can consider the following advantages:

1. For $-\varepsilon^2 \frac{d^2}{dx^2} u^{(\varepsilon)} + u^{(\varepsilon)} = f$, if $\varepsilon > 0$ is smaller, $-\varepsilon^2 \frac{d^2}{dx^2}$ gives more perturbation to the solution. The influence of the parameter $\varepsilon > 0$ can be reduced to some extent. Actually, for φ_2^{D} the relative L^2-error $E_j^{\varphi_2^{\mathrm{D}}}$ is stable for a smaller $\varepsilon > 0$.

2. For $(-1)^m \frac{d^{2m}}{dx^{2m}} u + u = f$, if the solution has lower regularity, $(-1)^m \frac{d^{2m}}{dx^{2m}}$ plays a more role in the structure of the equation. When we consider the solution in H^{2m} or of less regularity in H^s $(s < 2m)$, for instance $m = 2$, the relative L^2-errors $E_j^{N_3}$ and $E_j^{\mathring{\varphi}_2^D}$ keep good results even in comparison with $m = 1$ (For detail, see [21]).

3.4 Theoretical error estimates

In the previous sections, we constructed some basis functions and give error estimations through numerical simulations. On the other hand, theoretical error estimates are also important. Let us consider a Riesz scaling function φ which satisfies the Strang–Fix condition of order L, i.e.,

$$\hat{\varphi}(0) \neq 0, \quad \text{and} \quad \hat{\varphi}^{(k)}(2n\pi) = 0, \text{ for } n \neq 0, \quad k = 0, 1, \cdots, L-1.$$

For the Riesz scaling function φ, its dual function $\tilde{\varphi}$ is defined by

$$\widehat{\tilde{\varphi}}(\xi) = \frac{\hat{\varphi}(\xi)}{\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2}.$$

It is well known that when we consider the approximation of φ of the form $\sum_{k\in\mathbb{Z}} c_k \varphi\left(\frac{x}{T} - k\right)$, the projection

$$P_T f(x) = \sum_k \left\{ \int_{\mathbb{R}} f(y) \tilde{\varphi}\left(\frac{y}{T} - k\right) \frac{dy}{T} \right\} \varphi\left(\frac{x}{T} - k\right)$$

minimizes the L^2-error. In this situation, Blu and Unser [2] have derived the following theorem:

Theorem 3.4 ([2]) *Let φ satisfy the Strang–Fix condition of order L. Then, for any $f \in H^L(\mathbb{R})$ the approximation error is given by*

$$\|f - P_T f\|_{L^2(\mathbb{R})} \leq \left[\sup_{|\xi|<\pi} \frac{E(\xi)}{\xi^{2L}} + \frac{\|E\|_{L^\infty} \zeta(2L)}{\pi^{2L}} \right]^{1/2} \left\| f^{(L)} \right\|_{L^2(\mathbb{R})} T^L.$$

Here ζ is the zeta function $\zeta(t) = \sum_{k=1}^{\infty} \frac{1}{k^t}$ and $E(\xi) = 1 - \frac{|\hat{\varphi}(\xi)|^2}{\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2}$

With this theorem, let us estimate the ability of approximation for Daubechies scaling functions and elevated Daubechies scaling functions. L-th order Daubechies scaling function Φ_L^{D} satisfies the Strang–Fix condition of order L. Additionally, elevated function $\varphi_{L-1}^{\mathrm{D}}$ also satisfies the same condition. This means that increasing the order of Daubechies family, and elevating with N_1 produce the same effect that they increase the order of Strang–Fix condition from L to $L+1$. Let $C_L(\varphi) = \left[\sup_{|\xi|<\pi} \frac{E(\xi)}{\xi^{2L}} + \frac{\|E\|_{L^\infty} \zeta(2L)}{\pi^L} \right]^{1/2}$. Then, for Φ_3^{D} and φ_2^D, one can obtain $C_3\left(\varphi_2^{\mathrm{D}}\right) = 0.04027$, and $C_3\left(\Phi_3^{\mathrm{D}}\right) = 0.05948$. This shows the efficiency of the elevation scheme for numerical analysis.

4 Wavelet-Galerkin method with biorthogonal functions

In section 2, we introduced a uniform approach that generates Riesz bases such that the associated stiffness matrices become tridiagonal. This method is highly accurate, but the difficulty with the integral remains unsolved. In this section,we further develop this method and use the properties of the biorthogonality of the wavelets to overcome the difficulty with the integrals of the test functions. In particular, The Deslauriers–Dubuc interpolating scaling functions [13, 15] are used as basis functions.

4.1 Interpolating schemes

4.1.1 Deslauriers–Dubuc interpolating wavelet

Deslauriers and Dubuc [13] and Dubuc [15] introduced an interpolation scheme [29] that constructs a function on $\mathbb{R}$ from an initial value $\{f(k)\}_{k\in\mathbb{Z}}$. The functions obtained from the initial value $\{\delta_{k,0}\}_{k\in\mathbb{Z}}$ are called the fundamental functions. We denote the Deslauriers–Dubuc fundamental functions of order $D = 2L+1$ $(L = 0, 1, \cdots)$ by F_D. F_D satisfies the refinement relation

$$F_D(x) = \sum_{k\in\mathbb{N}} F_D(k/2)F_D(2x-k)$$

and supp $F_D = [-D, D]$. The smoothness of F_D increases as D increases [13].

F_D is known as a scaling functions of the interpolating wavelet function. In general, an interpolating scaling function φ has some useful properties. First, $\varphi(k) = \delta_{k,0}$ for $k \in \mathbb{Z}$, which is useful in terms of the approximation. Second, the two scale equation is given by $\varphi(x) = \sum_{k\in\mathbb{Z}} \varphi(k/2)\varphi(2x-k)$, which means that the filter coefficients $\{h_k\}_k$ are equal to the half values $\varphi(k/2)$. Moreover, the associate wavelet function is simply $\psi(x) = \varphi(2x-1)$.

In the case of Deslauriers–Dubuc scaling functions, the filter coefficients $\{h_k\}_k$ are easily calculated from the Lagrange polynomial: If

$$L_k(x) = \prod_{\substack{i=-N\\ i\neq k}}^{N+1} \frac{x-i}{k-i}, \quad k = -N, -N+1, \cdots, N+1,$$

then

$$\begin{aligned} h_{2k} &= \delta_{k,0}, \\ h_{2k+1} &= \begin{cases} L_{-k}(1/2), & k = -N-1, \cdots, N, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

For example,

$$\{h_{-1}, h_0, h_1\} = \left\{\frac{1}{2}, 1, \frac{1}{2}\right\}$$

when $D = 1$, and

$$\{h_{-3}, h_{-2}, h_{-1}, h_0, h_1, h_2, h_3\} = \left\{-\frac{1}{16}, 0, \frac{9}{16}, 1, \frac{9}{16}, 0, -\frac{1}{16}\right\}$$

when $D = 3$.

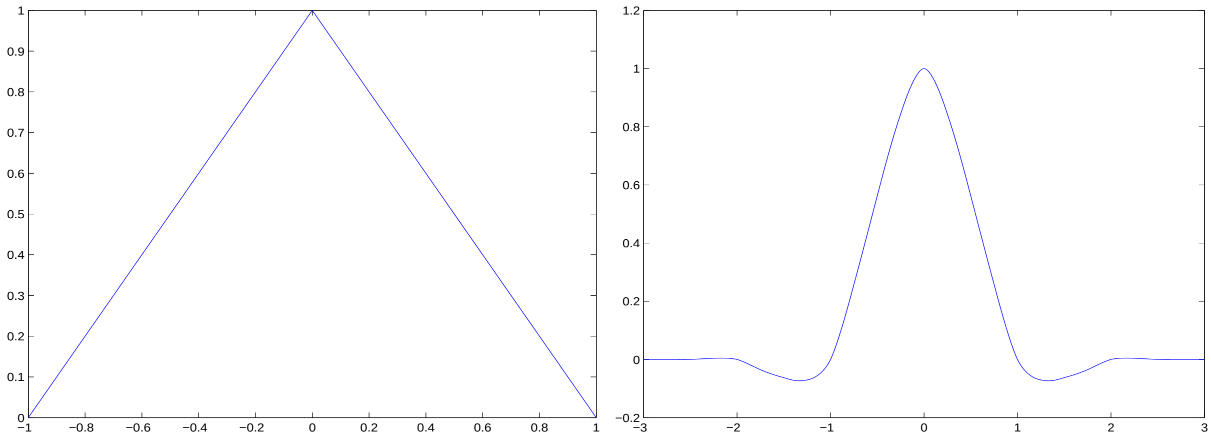
4.1.2 Average interpolation

Donoho [14] and Harten [24] generalized the Deslauriers–Dubuc interpolation scheme and also introduced a scheme called average interpolation. The fundamental functions of the average interpolation scheme A_D of order $D = 2L$ $(L = 1, 2, \cdots)$ still have compact supports supp $A_D \subset [-D, D+1]$ and satisfy the two scale equation

$$A_D(x) = \sum_{k\in\mathbb{Z}} c_k A_D(2x-k).$$

For example,

$$\{c_{-2}, c_{-1}, c_0, c_1, c_2, c_3\} = \left\{-\frac{1}{8}, \frac{1}{8}, 1, 1, \frac{1}{8}, -\frac{1}{8}\right\}$$



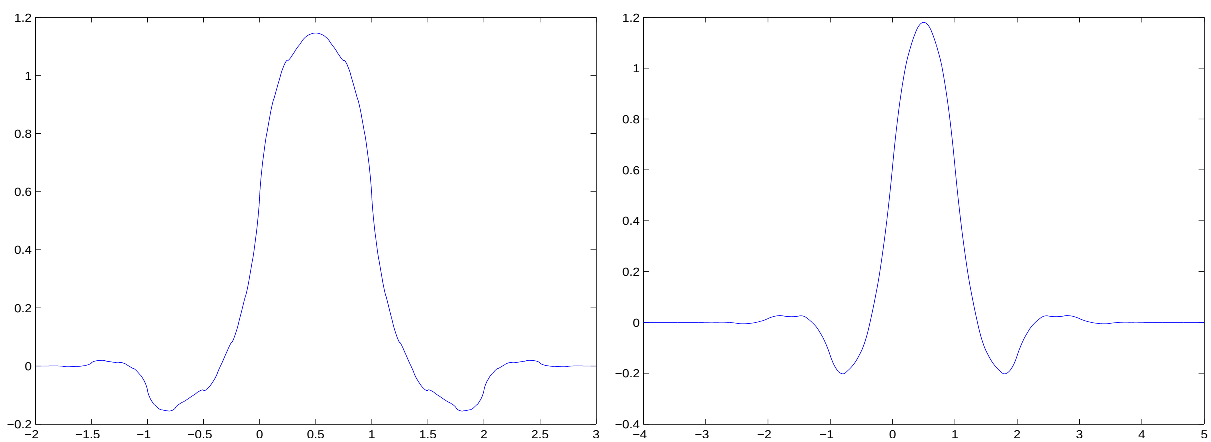
(a) F_1 (b) F_3

Figure 3: Deslauriers–Dubuc fundamental functions

when $D = 2$,

$$\{c_{-3}, c_{-2}, c_{-1}, c_0, c_1, c_2, c_3, c_4\} = \left\{ \frac{3}{128}, -\frac{3}{128}, -\frac{11}{64}, \frac{11}{64}, 1, 1, \frac{11}{64}, -\frac{11}{64}, -\frac{3}{128}, \frac{3}{128} \right\}$$

when $D = 4$.



(a) A_2 (b) A_4

Figure 4: Fundamental functions of the average interpolation scheme

The fundamental functions A_D and F_D have a strong relationship. If we set $\varphi = A_D$ and $\phi = F_{D+1}$, then it holds that

$$\phi'(x) = \varphi(x+1) - \varphi(x). \tag{22}$$

Since

$$\begin{aligned}\varphi(x+1)-\varphi(x) &= \frac{d}{dx}\int_x^{x+1}\varphi(y)dy \\ &= \frac{d}{dx}\int_{\mathbb{R}} N_1(y-x)\varphi(y)dy \\ &= \frac{d}{dx}\varphi * N_1^{\vee}(x),\end{aligned}$$

(22) is equivalent to

$$\phi = \varphi * N_1^{\vee},$$

where $f^{\vee}(x)=f(-x)$ and N_m is the m-th order B-spline, i.e., $N_1=\chi_{[0,1)}$ and $N_m=N_{m-1}*N_1$ $(m\geq 2)$. For the construction of Riesz bases, this means that ϕ is an elevation of φ with the elevator N_1 ([21]). In terms of the low-pass filters $m^{DD}(\xi)=\sum_k h_k^{DD}e^{-ik\xi}$ and $m^{A}(\xi)=\sum_k h_k^{A}e^{-ik\xi}$, it is denoted as $m^{DD}=m^A m$, where $m(\xi)=(1+e^{i\xi})/2$, or simply, $\{h^{DD}\}=\{h^A\}*\{1/2,1/2\}$.

Remark 4.1 *Deslauriers–Dubuc fundamental functions also have a special relationship to Daubechies scaling functions. Let Φ_N^D be a Daubechies scaling function of order N. Then Beylkin and Saito[31] proved the following equation:*

$$\int_R \Phi_N^D(x)\Phi_N^D(x-y)dx = F_{2N-1}(y). \tag{23}$$

Therefore F_{2N+1} is called the autocorrelation function of Φ_N^D.

Orthogonal wavelets lose several properties due to strong restrictions, but we can construct many wavelets by discarding the orthogonality. Cohen, Daubechies and Feauveau [7] constructed biorthogonal spline wavelets, whose primal and dual functions both have compact support.

Generally, the biorthogonal B-spline wavelets are specified with two parameters. Let φ_p and $\tilde{\varphi}_{p,\tilde{p}}$ be the primal and dual scaling functions of the biorthogonal B-spline wavelet, then the associated low-pass filters m_0 and $\tilde{m}_0$ are given by

$$m_0(\xi)=e^{-i\varepsilon\xi/2}\cos^p\left(\frac{\xi}{2}\right)$$

and

$$\tilde{m}_0(\xi)=e^{-i\varepsilon\xi/2}\cos^{\tilde{p}}\left(\frac{\xi}{2}\right)\sum_{k=0}^{(p+\tilde{p})/2-1}\binom{(p+\tilde{p})/2-1+k}{k}\sin^{2k}\left(\frac{\xi}{2}\right),$$

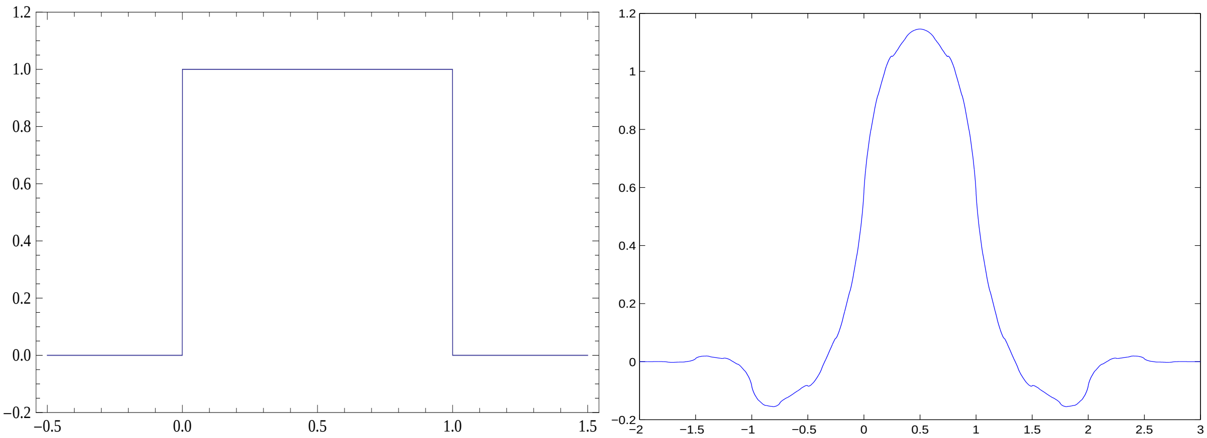
where $p+\tilde{p}$ is an even integer, $\varepsilon=0$ when p is even, and $\varepsilon=1$ when p is odd.

For $p=1$, we note that $m_0(\xi)=e^{-i\xi/2}\cos(\xi/2)$ is just the low-pass filter of the Haar wavelet. Thus, in this case, $\varphi_1=N_1(x)$. Moreover, Donoho [14] showed that the dual scaling functions are equal to the fundamental functions: more precisely, for $D=2,4,\cdots$, it holds that

$$\tilde{\varphi}_{1,D+1}=A_D. \tag{24}$$

4.2 Construction of coefficient matrices

We introduce a way to construct approximate solutions for certain differential equations by using Deslauriers–Dubuc fundamental functions. As mentioned above, these functions have compact support, are symmetric, and satisfy $F_D(k)=\delta_{k,0}$; the Daubechies functions do not have these properties.



(a) φ_1 (b) $\tilde{\varphi}_{1,3}$

Figure 5: Biorthogonal B-spline functions

With $\varphi = F_3$, $h = 1/(n+1)$ and $n \geq 5$, we seek a numerical solution

$$u_n(x) = \sum_{k=3}^{n-2} U_k \varphi(x/h - k)$$

for the Dirichlet boundary value problem (2). The standard Galerkin method leads to

$$a(u_n, \varphi_k) = \langle f, \varphi_k \rangle_{L^2(\mathbb{R})}, \quad k = 3, 4, \cdots, n-2. \tag{25}$$

From this, we obtain the Galerkin equation

$$MU = F, \tag{26}$$

where $M = \{\int_{\mathbb{R}} \varphi_i' \varphi_j' dx + \int_{\mathbb{R}} \varphi_i \varphi_j dx\}_{i,j}$ is a coefficient matrix; $F = {}^t\{\langle f, \varphi_j \rangle_{L^2}\}_{j=1,\cdots,n}$; and U is a unknown vector $U = {}^t\{U_1, \cdots, U_n\}$. This equation can be solved to obtain the coefficients U_k.

In this case, the stiffness matrix is a heptadiagonal matrix, which is relatively full compared with the one for classical FEM. Moreover, as in the case of the Daubechies function, Deslauriers–Dubuc fundamental function φ_k does not have an explicit formula; the difficulty of the integral on the right-hand side of (25) thus remains.

To deal with this problem, we replace φ_k by the hat functions $v_k = B_2(\cdot/h - k)$ and consider

$$a(u_n, v_k) = \langle f, v_k \rangle_{L^2(\mathbb{R})}, \quad k = 3, 4, \cdots, n-2.$$

This leads to a new Galerkin equation:

$$\tilde{M}U = \tilde{F}, \tag{27}$$

with $\tilde{M} = \{\int_{\mathbb{R}} \varphi_i'(x) v_j'(x) dx + \int_{\mathbb{R}} \varphi_i(x) v_j(x) dx\}_{i,j}$; $\tilde{F} = {}^t\{\langle f, v_j \rangle_{L^2(\mathbb{R})}\}_{j=1,\cdots,n}$; and $U = {}^t\{U_1, \cdots, U_n\}$. Equation (27) is more convenient and manageable than (26) for the following reasons:

(i) Both F_3 and B_2 are elevated functions of pair of biorthogonal functions with elevator $N_1^\vee$, thus the resulting stiffness matrix is a tridiagonal matrix, which is sparse compared with the one of (26).

(ii) Both F_3 and B_2 are refinement functions; therefore we can explicitly calculate the mass matrix.

(iii) Compared to (25), the integrals on the right-hand side of (27) are simpler, and they can be processed more quickly by computer. Thus our scheme quickly obtains the solution u once f has been set.

Let us more fully consider the advantages stated in (i), above. we have proved that if φ is orthogonal, i.e., $\langle\varphi, \varphi(\cdot - k)\rangle_{L^2} = \delta_{k,0}$, then the stiffness matrix generated by its elevated function $\Phi = \varphi * N_1$ is a tridiagonal matrix, i.e., $\langle\Phi', \Phi'(\cdot - k)\rangle_{L^2(\mathbb{R})} = 2\delta_{k,0} - \delta_{|k|,1}$. We can easily see that this is also true for a pair of biorthogonal functions, i.e., if $\langle\varphi_1, \varphi_2(\cdot - k)\rangle_{L^2(\mathbb{R})} = \delta_{k,0}$, then

$$\langle(\varphi_1 * N_1)', (\varphi_2 * N_1)'(\cdot - k)\rangle_{L^2(\mathbb{R})} = 2\delta_{k,0} - \delta_{|k|,1}. \tag{28}$$

Since F_3 and B_2 are elevated functions of the pair of biorthogonal functions $A_2 = \tilde{\varphi}_{1,3}$ and $N_1 = \varphi_1$ with elevator $N_1^\vee$ (see (22) and (24)), the resulting stiffness matrix is a tridiagonal matrix.

Remark 4.2 *One may expect that there exists an elevator $\mathcal{E}$ such that the stiffness matrix become a diagonal matrix, i.e., $\langle\varphi * \mathcal{E}, \varphi * \mathcal{E}(\cdot - k)\rangle = \delta_{k,0}$ with an orthogonal function φ. But this means that $\mathcal{E}$ is the sign function, and the resulting elevated function is thus non compactly supported. We therefore can not use this function for the Galerkin finite element method.*

Now let us consider (ii), above. Let f and g be compactly supported refinable functions. Then, $I_k = \int_{\mathbb{R}} f(x)g(x-k)dx = f * g^\vee(k)$. Here we remark that $p^\vee(x) = p(-x)$ is also refinable when p is refinable. Since the convolution of refinable functions is refinable [6], it can be given as a solution of an eigenvalue problem.

In the case $f = F_3$ and $g = B_2$, the above is summarized as follows:

Theorem 4.3 ([17]) *Set $M_k = \langle F_3, B_2(\cdot - k)\rangle_{L^2(\mathbb{R})}$ and $S_k = \langle F_3', B_2'(\cdot - k)\rangle_{L^2(\mathbb{R})}$. Then we obtain*

$$M_k = \begin{cases} 131/180 & \text{if } k = 0, \\ 37/240 & \text{if } k = \pm 1, \\ -11/600 & \text{if } k = \pm 2, \\ 1/3600 & \text{if } k = \pm 3, \\ 0 & \text{otherwise}, \end{cases} \tag{29}$$

and

$$S_k = \begin{cases} 2 & \text{if } k = 0, \\ -1 & \text{if } k = \pm 1, \\ 0 & \text{otherwise}. \end{cases} \tag{30}$$

Proof *Equation (30) is easily seen from (28), so let us prove (29). Set $f = F_3 * B_2^\vee = F_3 * B_2$. Then f is a refinable function with filter coefficients*

$$\begin{aligned} \{h_k\}_{k=-4}^{4} &= \frac{1}{2}\left\{-\frac{1}{16}, 0, \frac{9}{16}, 1, \frac{9}{16}, 0, -\frac{1}{16}\right\} * \left\{\frac{1}{2}, 1, \frac{1}{2}\right\} \\ &= \left\{-\frac{1}{64}, -\frac{1}{32}, \frac{1}{8}, \frac{17}{32}, \frac{25}{32}, \frac{17}{32}, \frac{1}{8}, -\frac{1}{32}, -\frac{1}{64}\right\}. \end{aligned}$$

From the two-scale equation, we get $M_k = f(k) = \sum_m h_m f(2k - m) = \sum_m h_{2k-m} f(m) = \sum_m h_{2k-m} M_m$, and we can obtain the M_k as the eigenvector of

$$\begin{pmatrix} M_{-3} \\ M_{-2} \\ M_{-1} \\ M_0 \\ M_1 \\ M_2 \\ M_3 \end{pmatrix} \begin{pmatrix} h_{-3} & h_{-4} & 0 & 0 & 0 & 0 & 0 \\ h_{-1} & h_{-2} & h_{-3} & h_{-4} & 0 & 0 & 0 \\ h_1 & h_0 & h_{-1} & h_{-2} & h_{-3} & h_{-4} & 0 \\ h_3 & h_2 & h_1 & h_0 & h_{-1} & h_{-2} & h_{-3} \\ 0 & h_4 & h_3 & h_2 & h_1 & h_0 & h_{-1} \\ 0 & 0 & 0 & h_4 & h_3 & h_2 & h_1 \\ 0 & 0 & 0 & 0 & 0 & h_4 & h_3 \end{pmatrix} = \begin{pmatrix} M_{-3} \\ M_{-2} \\ M_{-1} \\ M_0 \\ M_1 \\ M_2 \\ M_3 \end{pmatrix} \tag{31}$$

under the normalization $\sum_k M_k = 1$.

Remark 4.4 *In [21] and [36], with Φ_2^D and elevator N_1, a Riesz basis $\varphi_2^D = \Phi_2^D * N_1$ was constructed (see also (16)). Since Φ_2^D is orthogonal, $\langle \varphi_2^{D\prime}, \varphi_2^{D\prime}(\cdot - k)\rangle_{L^2(\mathbb{R})}$ corresponds to S_k in Theorem 4.3. Moreover, $\langle \varphi_2^D, \varphi_2^D(\cdot - k)\rangle_{L^2(\mathbb{R})}$ also corresponds to M_k in the theorem. Although this may seem strange, it is justified by the autocorrelation property (23); from $\hat{F}_3 = \mathcal{F}[\Phi_2^D * \Phi_2^{D\vee}] = |\hat{\Phi}_2^D|^2$, we have*

$$\begin{aligned}\langle F_3, B_2(\cdot - k)\rangle_{L^2(\mathbb{R})} &= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{F}_3(\xi)\hat{B}_2(\xi)e^{ik\xi}d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{\Phi}_2^D(\xi)|^2\hat{N}_1(\xi)\overline{\hat{N}_1(\xi)}e^{ik\xi}d\xi \\ &= \langle \varphi_2^D, \varphi_2^D(\cdot - k)\rangle_{L^2(\mathbb{R})}.\end{aligned}$$

4.3 Numerical results

In this section we present some numerical results to show the efficacy of our method. Let us illustrate some numerical examples. All computations were carried out with a Mac OS X, Intel Core i7, 3.4GHz, and by using Mathematica ver. 8.0.1.0.

We consider the following Dirichlet boundary value problem:

$$\begin{cases} -u'' + u = f, & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$

In classical FEM, the hat function B_2 is used to represent an approximate solution, and in [21], an elevated Daubechies scaling function $\varphi_2^D = \Phi_2^D * N_1$ was used. To compare these two, we calculated the approximate solutions using the Galerkin method:

$$\tilde{u}(x) = \sum_{n=1}^{2^j-1} u_n B_2(2^j x - k),$$

$$\tilde{u}(x) = \sum_{n=0}^{2^j-4} u_n \varphi_2^D(2^j x - k),$$

$$\tilde{u}(x) = \sum_{n=3}^{2^j-3} u_n F_3(2^j x - k),$$

with the test functions $B_2(2^jx-k)$ $(k = 1, \cdots, 2^j-1)$; $\varphi_2^D(2^jx-k)$ $(k = 0, 1, \cdots, 2^j-4)$; and $B_2(2^jx-k)$ $(k = 3, 3, \cdots, 2^j-3)$, respectively. The error was estimated by the relative ℓ^2-error:

$$e_j = \frac{\sqrt{\sum_{k=0}^{2^j}(u(k/2^j) - \tilde{u}(k/2^j))^2}}{\|u\|_{L^2(\mathbb{R})}}. \tag{32}$$

The results with various choices of u are presented as follows:

Table 5: The case of $u(x) = x^5(1-x)^5$

	B_2-B_2		φ_2^D-φ_2^D		F_3-B_2	
2^j	e_j	e_{j-1}/e_j	e_j	e_{j-1}/e_j	e_j	e_{j-1}/e_j
6	1.50×10^{-4}	—	2.87×10^{-4}	—	3.65×10^{-4}	—
7	5.30×10^{-5}	2.83	4.66×10^{-5}	6.16	1.76×10^{-5}	20.7
8	1.88×10^{-5}	2.83	9.49×10^{-6}	4.91	8.13×10^{-7}	21.7
9	6.63×10^{-6}	2.83	3.33×10^{-6}	2.85	3.68×10^{-8}	22.1
10	2.35×10^{-6}	2.83	3.15×10^{-6}	1.06	1.20×10^{-9}	30.7

Table 6: The case of $u(x) = N_5(5x)$

	B_2-B_2		φ_2^D-φ_2^D		F_3-B_2	
2^j	e_j	e_{j-1}/e_j	e_j	e_{j-1}/e_j	e_j	e_{j-1}/e_j
6	1.51×10^{-4}	—	4.31×10^{-4}	—	6.18×10^{-4}	—
7	5.33×10^{-5}	2.83	6.76×10^{-5}	6.38	5.50×10^{-5}	11.2
8	1.88×10^{-5}	2.83	1.28×10^{-5}	5.29	4.88×10^{-6}	11.3
9	6.66×10^{-6}	2.83	3.80×10^{-6}	3.36	4.32×10^{-7}	11.3
10	2.36×10^{-6}	2.83	3.14×10^{-6}	1.21	3.77×10^{-8}	11.4

Table 7: The case of $u(x) = N_3(3x)$

	B_2-B_2		φ_2^D-φ_2^D		F_3-B_2	
2^j	e_j	e_{j-1}/e_j	e_j	e_{j-1}/e_j	e_j	e_{j-1}/e_j
6	1.51×10^{-4}	—	3.64×10^{-2}	—	7.50×10^{-2}	—
7	5.22×10^{-5}	2.90	1.31×10^{-2}	2.78	2.67×10^{-2}	2.81
8	1.87×10^{-5}	2.79	4.66×10^{-3}	2.81	9.46×10^{-3}	2.82
9	6.57×10^{-6}	2.85	1.65×10^{-3}	2.82	3.35×10^{-3}	2.82
10	2.33×10^{-6}	2.82	5.84×10^{-4}	2.83	1.19×10^{-3}	2.83

Table 8: The case of $u(x) = N_3(10x/3 - 1/6)$ ($\operatorname{supp} u = [1/20, 19/20]$)

	B_2-B_2		φ_2^D-φ_2^D		F_3-B_2	
2^j	e_j	e_{j-1}/e_j	e_j	e_{j-1}/e_j	e_j	e_{j-1}/e_j
6	1.51×10^{-4}	—	8.24×10^{-4}	—	2.01×10^{-6}	—
7	5.21×10^{-5}	2.91	2.13×10^{-4}	3.87	7.80×10^{-7}	2.55
8	1.86×10^{-5}	2.80	5.17×10^{-5}	4.12	6.39×10^{-8}	12.4
9	6.63×10^{-6}	2.81	1.36×10^{-5}	3.79	2.61×10^{-8}	2.45
10	2.34×10^{-6}	2.84	4.47×10^{-6}	3.05	2.57×10^{-9}	10.1

Table 9: The case of $u(x) = \sin^2(2\pi x)$

	B_2-B_2		φ_2^D-φ_2^D		F_3-B_2	
2^j	e_j	e_{j-1}/e_j	e_j	e_{j-1}/e_j	e_j	e_{j-1}/e_j
6	1.53×10^{-4}	—	8.24×10^{-4}	—	2.01×10^{-6}	—
7	5.41×10^{-5}	2.82	2.13×10^{-4}	3.87	7.80×10^{-7}	2.55
8	1.91×10^{-5}	2.83	5.17×10^{-5}	4.12	6.39×10^{-8}	12.4
9	6.77×10^{-6}	2.83	1.36×10^{-5}	3.79	2.61×10^{-8}	2.45
10	2.39×10^{-6}	2.83	4.47×10^{-6}	3.05	2.57×10^{-9}	10.1

Figures 6-9 shows the CPU time required to calculate the integrals of F, i.e., the inner products of f and the test functions versus the error (32).

From these results, we can conclude that our method obtain smoother approximate solutions within the time required to perform classical FEM. In particular, we note that when an exact solution rapidly

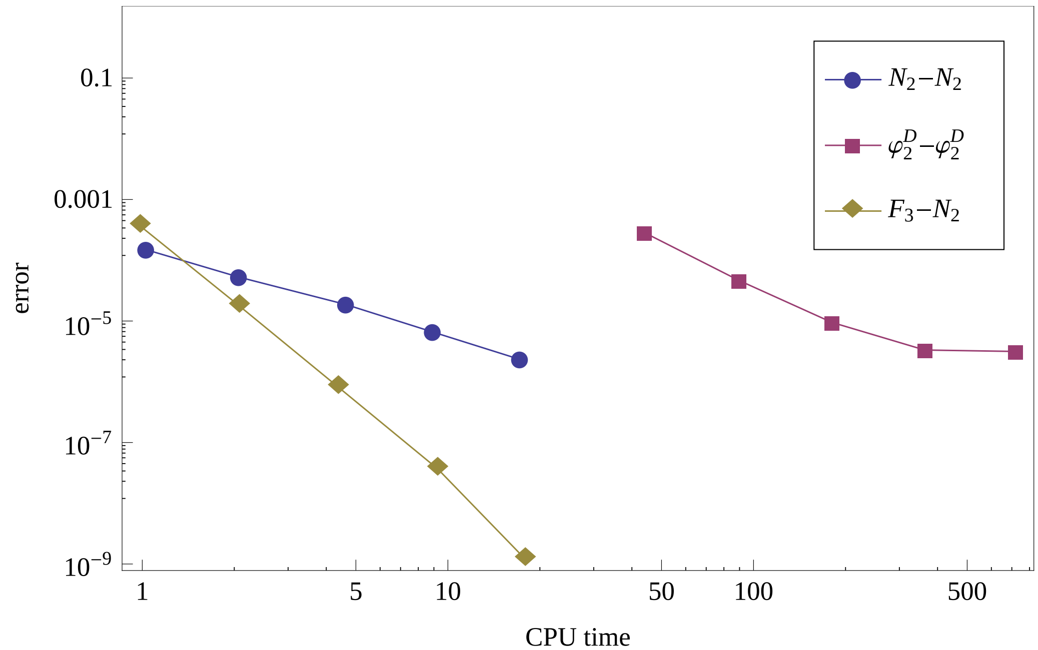


Figure 6: case 1; $u(x) = x^5(1-x)^5$.

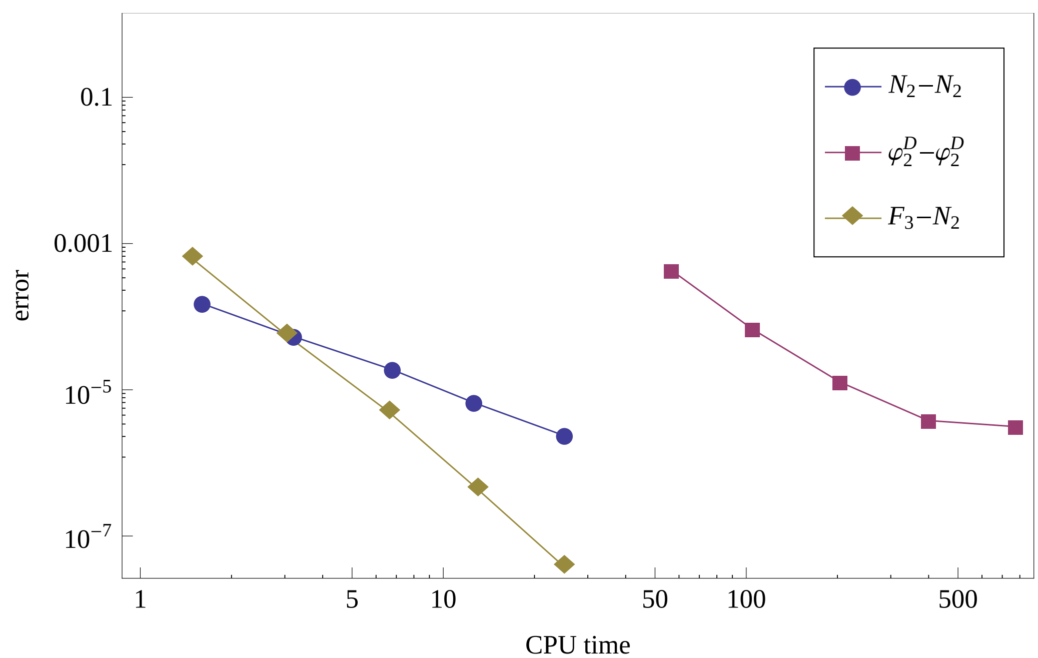


Figure 7: case 2; $u(x) = N_5(5x)$

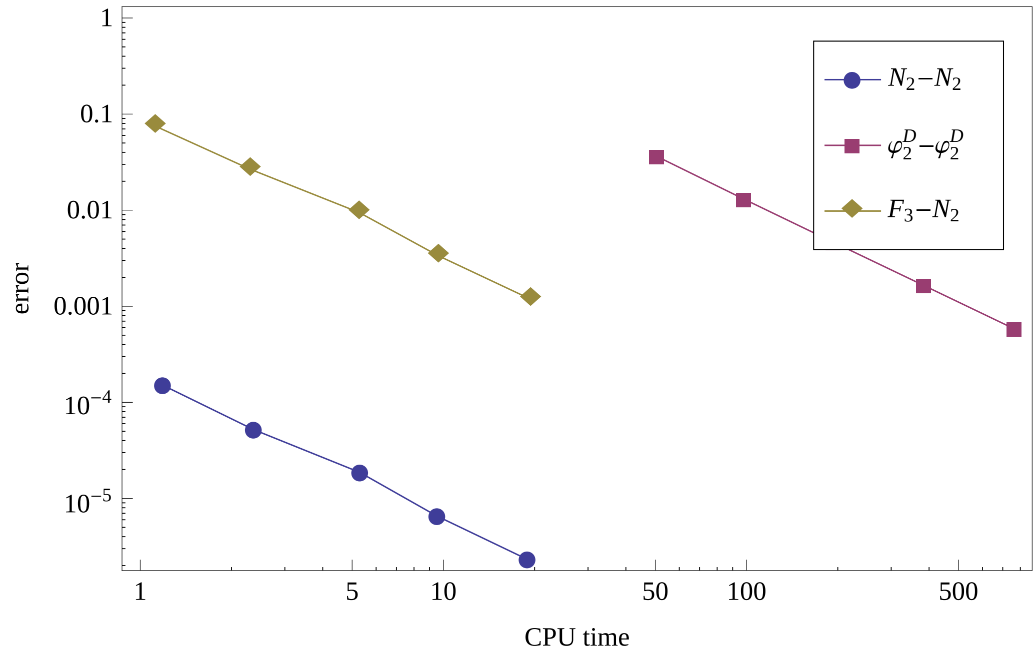


Figure 8: case 3; $u(x) = N_3(3x)$

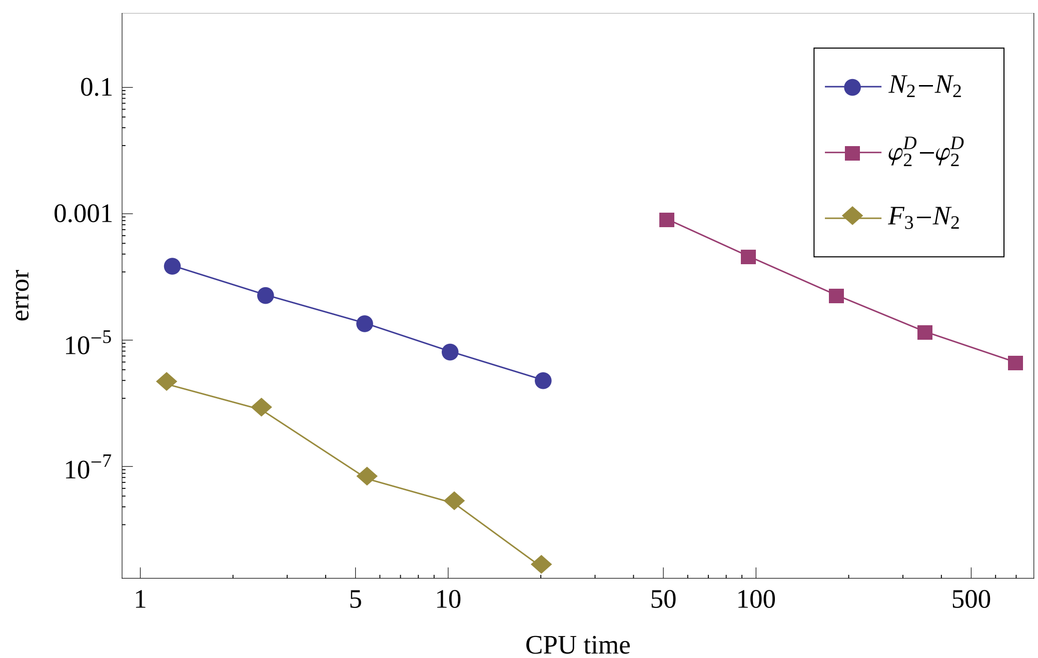


Figure 9: case 4; $u(x) = N_3(10x/3 - 1/6)$ ($\operatorname{supp} u = [1/20, 19/20]$)

decays to zero near the boundaries of the domain, our method is more effective. When the decay is not rapid, there is a slight loss of accuracy, which is presumably due to the shape of the basis F_3. Since F_3 is nearly zero at the endpoints of its support, non zero values of the exact solution cannot be represented well in this region. However, this weakness can be easily eliminated. Recall that our proposed method denotes an approximation solution using F_3 as

$$\tilde{u}(x) = \sum_{n=3}^{2^j-3} u_n F_3(2^j x - n). \tag{33}$$

To capture the behavior of u near the boundary of the domain, we denote the approximate solution using F_3 and B_2 as

$$\tilde{u}(x) = \sum_{n=3}^{2^j-3} u_n F_3(2^j x - n) + \sum_{n\in\{1,2,2^j-2,2^j-1\}} u_n B_2(2^j x - n). \tag{34}$$

Figure 10 illustrates the basis and test functions of (33) and (34). This modification increases the size of the coefficient matrix from $2^j - 5$ to $2^j - 1$, but the form of the stiffness matrix does not change. In Figure 11 we show that the computational cost of the modification is comparable to the unmodified form and that the efficiency of the modification.

5 Two-dimensional cases

Thus far, we have considered the Galerkin method mainly for ordinary differential equations. For partial differential equations, some difficulties arise:

- **(Support)** For the general N-dimensional case, the number of nodes is $(1/h + 1)^N$. Therefore, we are forced to use only compactly supported bases, such as Daubechies scaling functions.
- **(Smoothness)** Some solutions become much smoother, according to the type of partial differential equation. Thus, smooth bases are preferable for representing the solutions.
- **(Symmetry)** For partial differential equations, the boundary is considered on general dimensions for partial differential equations. Then, larger asymmetries can occur with higher dimensions.

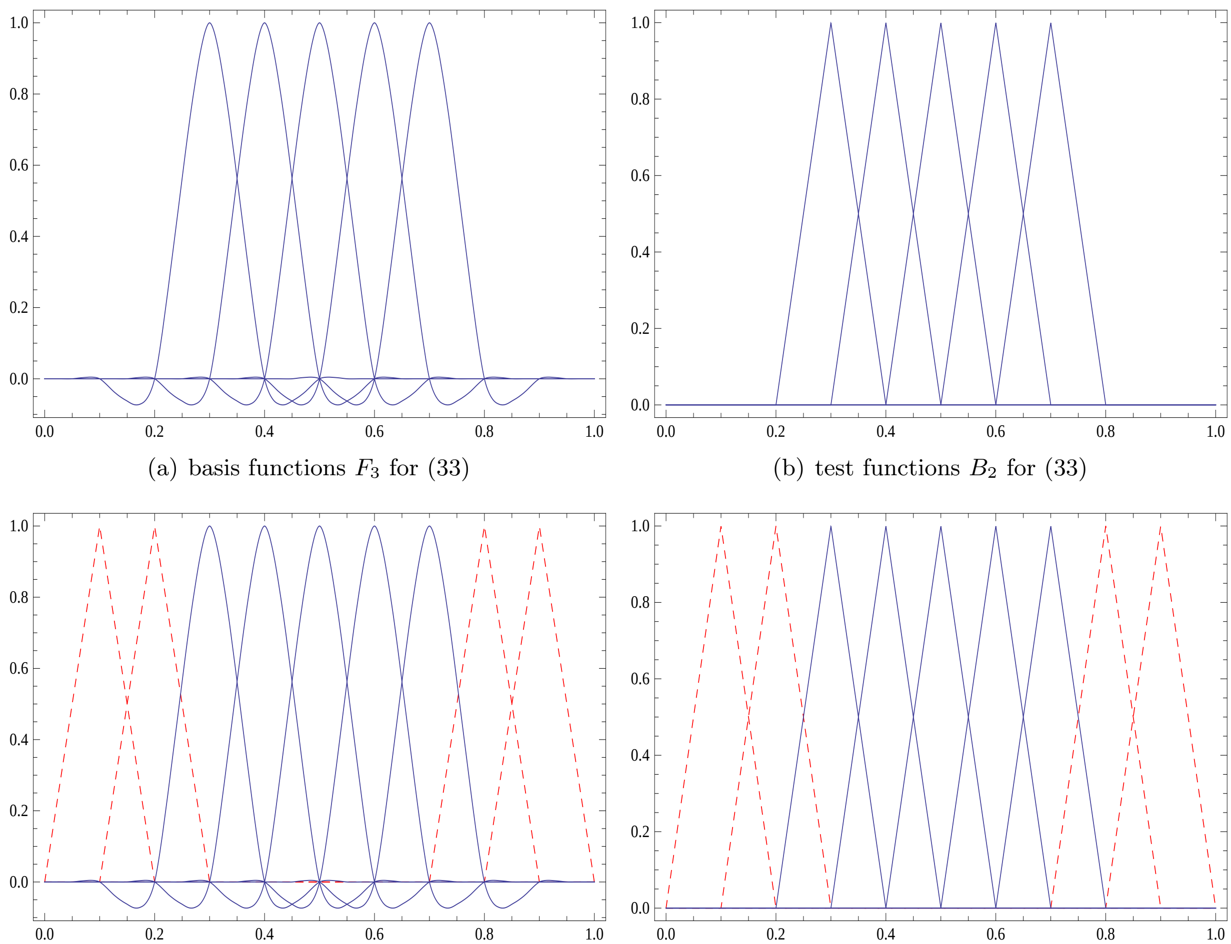
The purpose of this section is to overcome these difficulties and apply elevated basis functions to numerical solutions of boundary value problems for the two-dimensional Laplace equation. For Daubechies functions, there is a trade-off between the support size and the smoothness. We then construct new Riesz bases based on definite integrals of the scaling functions. The integrations extend the support of the scaling functions, but they improve the smoothness and the symmetry of the functions. In order to get better smoothness, the integrations are more efficient than increasing the order of Daubechies functions.

We consider the boundary value problem for the Poisson equation on the square domain $D = \{(x,y) \in \mathbb{R}^2 \;:\; 0 \le x, y \le 1\}$:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f, \\ u = 0 \quad \text{on } \partial D. \end{cases} \tag{35}$$

The exact solution is given by (see [30])

$$u(x,y) = -\int_0^1 \int_0^1 f(\xi,\eta) G(x,y,\xi,\eta) d\eta d\xi$$



(a) basis functions F_3 for (33)

(b) test functions B_2 for (33)

(c) basis functions F_3, B_2 for (34)

(d) test functions B_2 for (34)

Figure 10: basis and test functions for (33) and (34)

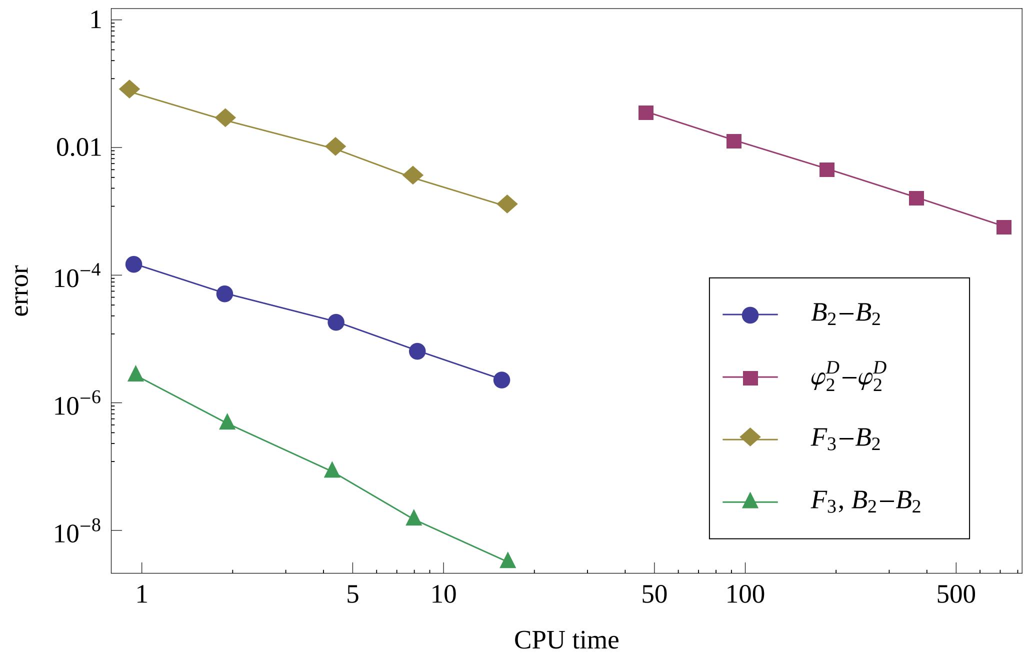


Figure 11: case 3; $u(x) = N_3(3x)$

with the Green's function

$$G(x,y,\xi,\eta) = \sum_{p=1}^{\infty} \frac{2\sin(\pi p x)\sin(\pi p \xi)}{\pi p \sinh(\pi p)} H_p(y,\eta),$$

where

$$H_p(y,\eta) = \begin{cases} \sinh(\pi p \eta)\sinh(\pi p(1-y)) & \text{if } 0 \le \eta < y \le 1, \\ \sinh(\pi p y)\sinh(\pi p(1-\eta)) & \text{if } 0 \le y < \eta \le 1. \end{cases}$$

Because of the infinity ($p = \infty$) in the double integrals, however, this solution is not practical. Therefore, the ability to represent an approximate solution with bases plays an important role.

5.1 Galerkin method

We now shall construct the approximate solutions to (35) in a manner similar to what we did for the one-dimensional case. The weak form of (35) is written as

$$-\left\langle \frac{\partial}{\partial x}u, \frac{\partial}{\partial x}v \right\rangle_{L^2(D)} - \left\langle \frac{\partial}{\partial y}u, \frac{\partial}{\partial y}v \right\rangle_{L^2(D)} = \langle f, v \rangle_{L^2(D)}. \tag{36}$$

To apply the one-dimensional case, we define the approximation space as $\mathrm{span}\{\phi_{j_1,j_2}(x,y) := \varphi_{j_1}(x)\varphi_{j_2}(y)\}_{j_1,j_2=1}^{n}$, and seek an approximate solution

$$\tilde{u}(x,y) = \sum_{j_1=1}^{n}\sum_{j_2=1}^{n} u_{j_1,j_2}\phi_{j_1,j_2}(x,y) \equiv \sum_{J=1}^{n^2} u_J \phi_J(x,y). \tag{37}$$

Substituting (37) into (36) and taking $v(x,y) = \phi_L(x,y)$ ($L = 1, 2, \cdots, n^2$) yields a linear system, written in matrix form as

$$MU = F,$$

where

$$M = -\begin{pmatrix} \langle \phi_1, \phi_1 \rangle & \langle \phi_2, \phi_1 \rangle & \cdots & \langle \phi_{n^2}, \phi_1 \rangle \\ \langle \phi_1, \phi_2 \rangle & \langle \phi_2, \phi_2 \rangle & \cdots & \langle \phi_{n^2}, \phi_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \phi_1, \phi_{n^2} \rangle & \langle \phi_2, \phi_{n^2} \rangle & \cdots & \langle \phi_{n^2}, \phi_{n^2} \rangle \end{pmatrix},$$

$U = {}^t\{u_k\}_{1 \le k \le n^2}$, and $F = {}^t\{f_\ell\}_{1 \le \ell \le n^2}$. Here we remark that $\langle \cdot, \cdot \rangle$ denotes $\langle \frac{\partial}{\partial x} \cdot, \frac{\partial}{\partial x} \cdot \rangle_{L^2(D)} + \langle \frac{\partial}{\partial y} \cdot, \frac{\partial}{\partial y} \cdot \rangle_{L^2(D)}$ and $f_n = \langle f, \phi_n \rangle_{L^2(D)}$.

We introduce the following notation in order to show the correspondence between the index J and the indexes j_1, j_2 of (37). We assume that $1 \le I, J \le n^2$ and $1 \le i_1, i_2, j_1, j_2 \le n$ are integers that satisfy

$$I = n(i_2 - 1) + i_1, \qquad J = n(j_2 - 1) + j_1, \tag{38}$$

and we set

$$R := j_1 - i_1, \qquad Q := j_2 - i_2. \tag{39}$$

We note that the correspondences (38) are one to one under $1 \le I, J \le n^2$ and $1 \le i_1, i_2, j_1, j_2 \le n$.

5.2 Choice of basis functions

We first calculate the (I, J)-th component of $M = \{M_{I,J}\}_{1 \le I, J \le n}$. We put

$$a_{i,j} := -\langle \varphi_i', \varphi_j' \rangle_{L^2(D)} \quad \text{and} \quad c_{i,j} := \langle \varphi_i, \varphi_j \rangle_{L^2(D)}.$$

Then, we get

$$\begin{aligned} M_{I,J} = &- \int_0^N \varphi'(x - i_1 + 1)\, \varphi'(x - j_1 + 1)\, dx \int_0^N \varphi(y - i_2 + 1)\, \varphi(y - j_2 + 1)\, dy \\ &- \int_0^N \varphi(x - i_1 + 1)\, \varphi(x - j_1 + 1)\, dx \int_0^N \varphi'(y - i_2 + 1)\, \varphi'(y - j_2 + 1)\, dy \\ \equiv\ & a_{i_1, j_1} c_{i_2, j_2} + c_{i_1, j_1} a_{i_2, j_2}, \end{aligned} \tag{40}$$

which depends on the choice of $\phi(x, y) := \varphi(x)\varphi(y)$.

5.2.1 Case of B-spline N_2

We begin with the consideration of the simplest case $\phi(x, y) := N_2(x) N_2(y)$ with $N_2 = N_1 * N_1$ ($\mathcal{E} = \Phi = N_1$). An easy calculation shows that

$$c_{i,j} = \begin{cases} 2/3 & \text{if } |j - i| = 0, \\ 1/6 & \text{if } |j - i| = 1, \\ 0 & \text{otherwise}, \end{cases} \quad a_{i,j} = \begin{cases} -2 & \text{if } |j - i| = 0, \\ 1 & \text{if } |j - i| = 1, \\ 0 & \text{otherwise}. \end{cases} \tag{41}$$

Combining (40) and (41), we obtain

$$M_{I,J} = \begin{cases} -8/3 & \text{if } (|R|, |Q|) = (0, 0), \\ 1/3 & \text{if } (|R|, |Q|) = (1, 0), (0, 1) \\ 1/3 & \text{if } (|R|, |Q|) = (1, 1), \\ 0 & \text{otherwise}, \end{cases}$$

where $R = R(I, J)$ and $Q = Q(I, J)$ are integers determined by (38) and (39). Therefore, the matrix M is the block tridiagonal matrix with the tridiagonal matrices A and B:

$$\mathbf{M} = \begin{pmatrix} A & B & & \text{O} \\ B & A & \ddots & \\ & \ddots & \ddots & B \\ \text{O} & & B & A \end{pmatrix}_{n^2 \times n^2}, \tag{42}$$

where

$$A = \begin{pmatrix} -\frac{8}{3} & \frac{1}{3} & & 0 \\ \frac{1}{3} & -\frac{8}{3} & \ddots & \\ & \ddots & \ddots & \frac{1}{3} \\ 0 & & \frac{1}{3} & -\frac{8}{3} \end{pmatrix}_{n \times n}, \quad B = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & & 0 \\ \frac{1}{3} & \frac{1}{3} & \ddots & \\ & \ddots & \ddots & \frac{1}{3} \\ 0 & & \frac{1}{3} & \frac{1}{3} \end{pmatrix}_{n \times n}.$$

5.2.2 Case of Riesz bases of Daubechies-type

We now turn to the case $\Phi = \Phi_2^D$. Let us put

$$\phi(x, y) := \varphi_2^D(x)\varphi_2^D(y) \quad \text{with} \quad \varphi_2^D = N_1 * \Phi_2^D.$$

In this case, $a_{i,j}$ provides a three-point formula for the second-order derivative (see (17)). In addition,

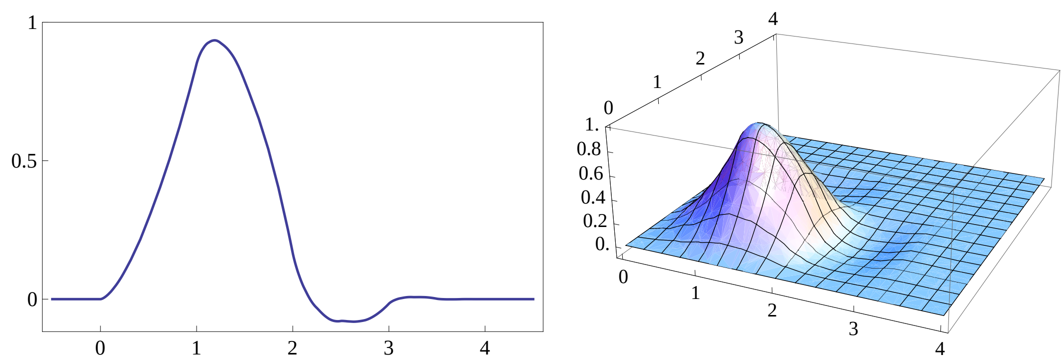


Figure 12: Graphs of $\varphi_2^D(x)$ (left) and $\varphi_2^D(x)\varphi_2^D(y)$ (right).

we derived $c_{i,j}$ in section 3 (see Theorem 3.2).

Using (17) we can calculate $M_{I,J}$, e.g.,

$$M_{1,1} = a_{1,1}c_{1,1} + c_{1,1}a_{1,1} = -131/45,$$
$$M_{1,2} = a_{1,2}c_{1,1} + c_{1,2}a_{1,1} = 151/360,$$

and so on. Here we set

$$\Lambda(r, q) = \begin{cases} -131/45 & \text{if } (|r|, |q|) = (0, 0), \\ 151/360 & \text{if } (|r|, |q|) = (1, 0), (0, 1), \\ 11/300 & \text{if } (|r|, |q|) = (2, 0), (0, 2), \\ -1/1800 & \text{if } (|r|, |q|) = (3, 0), (0, 3), \\ 37/120 & \text{if } (|r|, |q|) = (1, 1), \\ -11/600 & \text{if } (|r|, |q|) = (2, 1), (1, 2), \\ 1/3600 & \text{if } (|r|, |q|) = (3, 1), (1, 3), \\ 0 & \text{otherwise.} \end{cases} \tag{43}$$

Then, from straightforward computation, we see that

$$\sum_{(R,Q)\in\mathbb{Z}^2} \Lambda(R,Q)R^n Q^{n-k} = 0$$

for $n = 0, 1, 3$ and $0 \le k \le n$, and

$$\begin{aligned}\sum_{(R,Q)\in\mathbb{Z}^2} \Lambda(R,Q)R^2 = \sum_{(R,Q)\in\mathbb{Z}^2} \Lambda(R,Q)Q^2 = 2,\\ \sum_{(R,Q)\in\mathbb{Z}^2} \Lambda(R,Q)RQ = 0.\end{aligned}$$

Consequently, since

$$\sum_{(R,Q)\in\mathbb{Z}^2} \Lambda(R,Q)w(x+hR, y+hQ) = \sum_{k=0}^{3} \frac{h^k}{k!}\left(R\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}\right)^k w(x,y) + O(h^4)$$

for $w \in C_0^6(\mathbb{R}^2)$, we obtain the following theorem:

Theorem 5.1 *For $\phi(x,y) := \varphi_2^D(x)\varphi_2^D(y)$, we have*

$$M_{I,J} = \Lambda(R,Q), \tag{44}$$

where $R = R(I,J)$ and $Q = Q(I,J)$ are integers determined by (38) and (39). Moreover, it holds that for $w \in C_0^4(\mathbb{R}^2)$

$$\sum_{(R,Q)\in\mathbb{Z}^2} \Lambda(R,Q)\ w(x+hR, y+hQ) = \left(\frac{\partial^2}{\partial x^2}w(x,y) + \frac{\partial^2}{\partial y^2}w(x,y)\right)h^2 + O(h^4). \tag{45}$$

Here we introduce the following notation for simplicity. Let $STM(a_1, \cdots, a_k)$ (resp. $SBTM(A_1, \cdots, A_k)$) denote the symmetric diagonal Toeplitz matrix (resp. symmetric block diagonal Toeplitz matrix)

$$\begin{pmatrix} a_1 & \cdots & a_k & & \mathbf{0} \\ \vdots & \ddots & & \ddots & \\ a_k & & \ddots & & a_k \\ & \ddots & & & \vdots \\ \mathbf{0} & & a_k & \cdots & a_1 \end{pmatrix} \quad \left(\text{resp.} \begin{pmatrix} A_1 & \cdots & A_k & & \mathbf{0} \\ \vdots & \ddots & & \ddots & \\ A_k & & \ddots & & A_k \\ & \ddots & & & \vdots \\ \mathbf{0} & & A_k & \cdots & A_1 \end{pmatrix}\right).$$

Then, we can rewrite (44) as

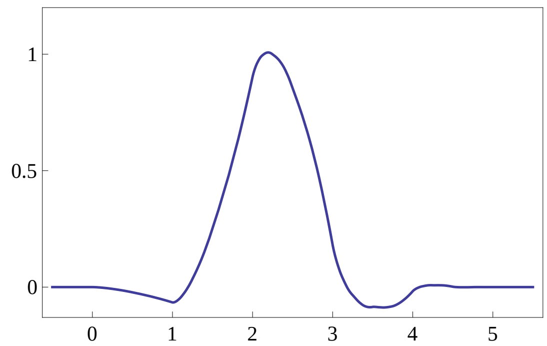
$$M = SBTM(A, B, C, D),$$

where

$$\begin{aligned} A &= STM\{-131/45, 151/360, 11/300, -1/1800\},\\ B &= STM\{151/360, 37/120, -11/600, 1/3600\},\\ C &= STM\{11/300, -11/600\},\\ D &= STM\{-1/1800, 1/3600\}.\end{aligned}$$

We next consider the case $\phi(x,y) := \tilde{\varphi}_2^D(x)\tilde{\varphi}_2^D(y)$ with

$$\tilde{\varphi}_2^D = \left\{\left(\frac{1}{2} + \frac{1}{\sqrt{3}}\right)N_1(\cdot - 1) + \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)N_1(\cdot)\right\} * \Phi_2^D.$$



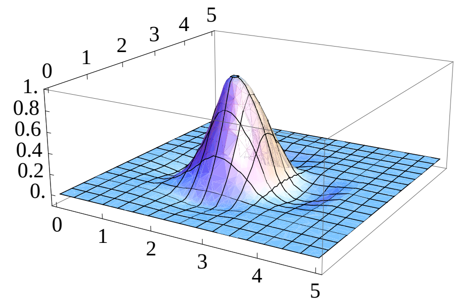


Figure 13: Graphs of $\tilde{\varphi}_2^D(x)$ (left) and $\tilde{\varphi}_2^D(x)\tilde{\varphi}_2^D(y)$ (right).

In this case, $\tilde{\varphi}_2^D$ gives a five-point formula for the second-order derivative. Using (20) we can calculate $M_{I,J}$, e.g.,

$$M_{1,1} = a_{1,1}c_{1,1} + c_{1,1}a_{1,1} = -\frac{-3557}{864},$$
$$M_{1,2} = a_{1,2}c_{1,1} + c_{1,2}a_{1,1} = \frac{2579}{3240},$$

and so on. Here we set

$$\tilde{\Lambda}(r,q) = \begin{cases} -3557/864 & (|r|,|q|) = (0,0), \\ 2579/3240 & (|r|,|q|) = (1,0),(0,1), \\ 883/51840 & (|r|,|q|) = (2,0),(0,2), \\ -1/216 & (|r|,|q|) = (3,0),(0,3), \\ 1/17280 & (|r|,|q|) = (4,0),(0,4), \\ 652/2025 & (|r|,|q|) = (1,1), \\ -301/5400 & (|r|,|q|) = (2,1),(1,2), \\ 1/405 & (|r|,|q|) = (3,1),(1,3), \\ -1/32400 & (|r|,|q|) = (4,1),(1,4), \\ 37/6480 & (|r|,|q|) = (2,2), \\ -1/6480 & (|r|,|q|) = (2,3),(3,2), \\ 1/518400 & (|r|,|q|) = (2,4),(4,2), \\ 0 & \text{otherwise.} \end{cases}$$

Then, from straightforward computation, we see that $\sum_{(R,Q)\in\mathbb{Z}^2} \tilde{\Lambda}(R,Q)R^nQ^{n-k} = 0$ for $n = 0,1,3,4,5$ and $0 \le k \le n$; $\sum_{(R,Q)\in\mathbb{Z}^2} \Lambda(R,Q)R^2 = \sum_{(R,Q)\in\mathbb{Z}^2} \Lambda(R,Q)Q^2 = 2$, and $\sum_{(R,Q)\in\mathbb{Z}^2} \Lambda(R,Q)RQ = 0$. Since $\sum_{(R,Q)\in\mathbb{Z}^2} \Lambda(R,Q)w(x+hR, y+hQ) = \sum_{k=0}^{5} \frac{h^k}{k!}\left(R\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}\right)^k w(x,y) + O(h^6)$ for $w \in C_0^6(\mathbb{R}^2)$, we obtain the following theorem:

Theorem 5.2 *For $\phi(x,y) := \tilde{\varphi}_2^D(x)\tilde{\varphi}_2^D(y)$, we have*

$$M_{I,J} = \tilde{\Lambda}(R,Q),$$

where $R = R(I,J)$ and $Q = Q(I,J)$ are integers determined by (38) and (39). Moreover, it holds that for $w \in C_0^6(\mathbb{R}^2)$

$$\sum_{(R,Q)\in\mathbb{Z}^2} \Lambda(R,Q)\ w(x+hR, y+hQ) = \left(\frac{\partial^2}{\partial x^2}w(x,y) + \frac{\partial^2}{\partial y^2}w(x,y)\right)h^2 + O(h^6).$$

From this theorem it follows that the coefficient matrix M is the block nonadiagonal matrix

$$\mathbf{M} = SBTM\{A, B, C, D, E\},$$

where

$$\begin{aligned}
A &= STM\Big\{-\frac{3557}{864}, \frac{2579}{3240}, \frac{883}{51840}, -\frac{1}{216}, \frac{1}{17280}\Big\},\\
B &= STM\Big\{\frac{2579}{3240}, \frac{652}{2025}, -\frac{301}{5400}, \frac{1}{405}, -\frac{1}{32400}\Big\},\\
C &= STM\Big\{\frac{883}{51840}, -\frac{301}{5400}, \frac{37}{6480}, -\frac{1}{6480}, \frac{1}{518400}\Big\},\\
D &= STM\Big\{-\frac{1}{216}, \frac{1}{405}, -\frac{1}{6480}\Big\},\\
E &= STM\Big\{\frac{1}{17280}, -\frac{1}{32400}, \frac{1}{518400}\Big\}.
\end{aligned}$$

5.3 Numerical results

We now present some examples and numerical results. We define the relative L^2-error E_j^φ between the exact solution $u(x, y)$ and the approximation

$$\tilde{u}(x, y) = \sum_{j_2=1}^{n} \sum_{j_1=1}^{n} u_{j_1,j_2} \varphi\left(\frac{x}{h} - j_1 + 1\right) \varphi\left(\frac{y}{h} - j_2 + 1\right)$$

by

$$E_n^\varphi = \frac{1}{\|u\|_{L^2(\mathbb{R})}} \sqrt{\sum_{\ell_1=1}^{\frac{1}{h}-1} \sum_{\ell_2=1}^{\frac{1}{h}-1} \big(u(h\ell_1, h\ell_2) - \tilde{u}(h\ell_1, h\ell_2)\big)^2},$$

and we define the ratio Q_n^φ by $Q_n^\varphi = E_{n-5}^\varphi / E_n^\varphi$. Here we remark that n depends to the step size h and the size of supp φ. For example, if $\varphi = N_2$, i.e., meas(supp φ) $= 2$, then, $n = 4$. Generally, it holds that $n = 1/h + 1 -$ meas(supp φ).

Table 10: The case of $u(x, y) = 2^8 x^2 (1 - x)^2 y^2 (1 - y)^2$.

n	$E_n^{N_2}$	$Q_n^{N_2}$	$E_n^{\varphi_2^D}$	$Q_n^{\varphi_2^D}$	$E_n^{\tilde{\varphi}_2^D}$	$Q_n^{\tilde{\varphi}_2^D}$
10	0.0794	1.77	3.18	1.31	3.63	1.28
15	0.0549	1.45	2.49	1.28	2.91	1.25
20	0.0419	1.31	2.03	1.22	2.41	1.20
25	0.0339	1.24	1.72	1.19	2.05	1.17
30	0.0284	1.19	1.48	1.16	1.79	1.15

Table 11: The case of $u(x, y) = 2^{20} x^5 (1 - x)^5 y^5 (1 - y)^5$.

n	$E_n^{N_2}$	$Q_n^{N_2}$	$E_n^{\varphi_2^D}$	$Q_n^{\varphi_2^D}$	$E_n^{\tilde{\varphi}_2^D}$	$Q_n^{\tilde{\varphi}_2^D}$
10	0.149	1.83	0.416	4.12	0.337	3.80
15	0.102	1.46	0.142	2.93	0.122	2.76
20	0.0775	1.32	0.0616	2.31	0.0551	2.22
25	0.0626	1.24	0.0317	1.94	0.0291	1.89
30	0.0524	1.19	0.0185	1.71	0.0174	1.68

The above results indicate that Daubechies-type Riesz bases produce a good approximation to the solution when the exact solution decays quickly at the boundaries of the region. Here, we constructed two-dimensional basis functions of Daubechies type. Obviously, one can also construct basis functions of Deslauriers–Dubuc type by using the results in section 4.

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