Semi-infinite Lakshmibai–Seshadri path model for level-zero extremal weight modules over quantum affine algebras

Motohiro Ishii

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Motohiro Ishii
Doctoral Program in Mathematics

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Contents

1 Introduction

The quantized universal enveloping algebra (or, quantum group) $U_q(\mathfrak{g})$ associated with a (symmetrizable) Kac–Moody Lie algebra $\mathfrak{g}$ was introduced independently by Drinfeld ([Dri85]) and Jimbo ([Jim85]) in their study of solvable lattice models in statistic physics. This is a non-commutative and non-cocommutative Hopf algebra defined over the field $\mathbb{C}(q)$, where q is transcendental over $\mathbb{C}$ or $q \in \mathbb{C}^{\times}$, and is defined by deforming the defining relations of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$; the classical $U(\mathfrak{g})$ can be obtained from $U_q(\mathfrak{g})$ by taking the limit $q \to 1$. Since then, the quantized universal enveloping algebras have provided many applications to mathematics such as the representation theory of algebraic groups and Hecke algebras, quantum invariants of knots and 3-manifolds ([RT91]), and so forth. Also, it is known that the quantized universal enveloping algebras naturally arise in mathematics. For example, Ringel ([Rin90]) gave a realization of the negative part $U_q^-(\mathfrak{g})$ of $U_q(\mathfrak{g})$ as the Hall algebra of quiver representations (or equivalently, representations of a path algebra) over a finite field; recently, Bridgeland ([Bri13]) described the whole of $U_q(\mathfrak{g})$ via the Hall algebra of $(\mathbb{Z}/2\mathbb{Z})$-graded complexes of quiver representations. Also, Khovanov–Lauda ([KL09, KL11]) and Rouquier ([Rou08]) provided a categorification of $U_q^-(\mathfrak{g})$ by use of representations of certain generalizations of affine Hecke algebras, called Khovanov–Lauda–Rouquier (or, quiver Hecke) algebras.

Motivated by Ringel's description of quantized universal enveloping algebras, Lusztig ([Lus90, Lus91]; see also [Lus93]) gave a geometric realization of $U_q^-(\mathfrak{g})$ by using the theory of perverse sheaves ([BBD82]) on moduli spaces of quiver representations, and then he discovered remarkable bases of $U_q^-(\mathfrak{g})$ and the integrable (irreducible) highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$ of highest weight λ, called canonical bases. Also, in an algebraic viewpoint, Kashiwara ([Kas90, Kas91]) independently developed the theory of (global) crystal bases of $U_q^-(\mathfrak{g})$ and $V(\lambda)$; in fact, it is known from [GL93] that the global crystal bases are identical to the canonical bases. The crystal basis $\mathcal{B}(\infty)$ (resp., $\mathcal{B}(\lambda)$) of $U_q^-(\mathfrak{g})$ (resp., $V(\lambda)$) can be regarded as a basis at the crystal limit $q = 0$, which has a combinatorial structure, called a crystal ([Kas93]; see also [HK02, Kas95, Kas02a, Lus93]). At the crystal limit, many of the problems in representation theory of $U_q(\mathfrak{g})$ can be reduced to combinatorial ones. So, in order to obtain further applications of crystal bases to the study of representation theory of $U_q(\mathfrak{g})$ and other areas of mathematics, it is a fundamental problem to give explicit realizations of crystal bases. For integrable highest weight $U_q(\mathfrak{g})$-modules, there are many useful realizations of their crystal bases. For example, Littelmann ([Lit94, Lit95]) introduced the Lakshmibai–Seshadri (LS for short) path model for $V(\lambda)$ in his study of the standard monomial theory ([LMS79, LS86]); soon after, Kashiwara ([Kas96]) and Joseph ([Jos95]) independently proved that the crystal $\mathbb{B}(\lambda)$ of LS paths of shape λ is isomorphic to the crystal basis $\mathcal{B}(\lambda)$. In addition, we know the following realizations of the crystal basis $\mathcal{B}(\lambda)$ of the integrable highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$:

- polyhedral realization, due to Nakashima–Zelevinsky ([NZ97]);
- Lagrangian construction via quiver varieties, due to Kashiwara–Saito ([KS97, Sai02]);
- monomial realization, due to Nakajima ([Nak03]; see also [HN06]);
- categorification via Khovanov–Lauda–Rouquier algebras, due to Lauda–Vazirani ([LV11]);

for $\mathfrak{g}$ of affine type,

- Kyoto path realization via perfect crystals, due to Kang–Kashiwara–Misra–Miwa–Nakashima–Nakayashiki ([KKMMNN92]);
- Young wall realization, due to Kang ([Kan03]);

and for $\mathfrak{g}$ of finite type,

- tableaux realization, due to Kashiwara–Nakashima ([KN94]);
- realization in terms of Mirković–Vilonen cycles and polytopes, due to Braverman–Finkelberg–Gaitsgory, Anderson, and Kamnitzer ([And03, BG01, BFG06, Kam07, Kam10]).

In his study of the canonical basis of the tensor product of an integrable highest weight $U_q(\mathfrak{g})$-module and an integrable lowest weight $U_q(\mathfrak{g})$-module, Lusztig ([Lus92]) introduced the modified version of the quantized universal enveloping algebra, and showed that this algebra has the canonical basis (or global crystal basis). Also, Kashiwara ([Kas94]) studied the crystal basis of the modified quantized universal enveloping algebra, and proved that the structure of it is essentially reduced to that of the crystal bases of extremal weight modules over $U_q(\mathfrak{g})$. Here, for each (not necessarily dominant) integral weight λ of $\mathfrak{g}$, the extremal weight $U_q(\mathfrak{g})$-module $V(\lambda)$ of extremal weight λ is a natural generalization of an integrable highest or lowest weight $U_q(\mathfrak{g})$-module; in fact, if λ is a dominant (resp., anti-dominant) integral weight for $\mathfrak{g}$, then $V(\lambda)$ is isomorphic to the integrable highest (resp., lowest) weight $U_q(\mathfrak{g})$-module of highest (resp., lowest) weight λ. However, the structures of a general extremal weight module and its crystal basis are more complicated than those of integrable highest (or, lowest) weight modules and their crystal bases; for example, general extremal weight modules are not necessarily semisimple, and there are no explicit realizations of crystal bases of extremal weight modules except in some very special cases.

Extremal weight modules over the quantized universal enveloping algebra associated with an affine Lie algebra are especially important, and we will give an explanation about them in detail. Let $\mathfrak{g}_{\mathrm{af}}$ be an (untwisted) affine Lie algebra (see [Kac90]); we call the quantized universal enveloping algebra $U_q(\mathfrak{g}_{\mathrm{af}})$ associated with $\mathfrak{g}_{\mathrm{af}}$ the quantum affine algebra for short. Let I_{af} be the set of indices of simple roots of $\mathfrak{g}_{\mathrm{af}}$ with a special index $0 \in I_{\mathrm{af}}$; we set $I := I_{\mathrm{af}} \setminus \{0\}$ and let $\mathfrak{g}$ be the finite-dimensional semisimple Lie subalgebra of $\mathfrak{g}_{\mathrm{af}}$ corresponding to I. The level of an integral weight λ of $\mathfrak{g}_{\mathrm{af}}$ is defined as the pairing $\langle c, \lambda \rangle \in \mathbb{Z}$ with the canonical central element c of $\mathfrak{g}_{\mathrm{af}}$. It is known that if the level of λ is positive (resp., negative), then the extremal weight module $V(\lambda)$ of extremal weight λ is a highest (resp., lowest) weight module, and the structure of $V(\lambda)$ is well-known in this case. So, it is essential to consider the level-zero extremal weight modules. In [Kas02a] (see also [AK97, Kas05]), Kashiwara studied level-zero extremal weight modules over quantum affine algebras, and showed that any finite-dimensional irreducible representation of the subalgebra $U_q'(\mathfrak{g}_{\mathrm{af}}) := U_q([\mathfrak{g}_{\mathrm{af}}, \mathfrak{g}_{\mathrm{af}}])$ of $U_q(\mathfrak{g}_{\mathrm{af}})$ can be obtained as a quotient of tensor product of level-zero extremal weight modules; finite-dimensional representation theory of $U_q'(\mathfrak{g}_{\mathrm{af}})$ is studied by many people in connection with integrable systems, geometric, categorical, and combinatorial representation theory including Kirillov–Reshetikhin modules and crystals, cluster algebras, Khovanov–Lauda–Rouquier algebras, and so forth (see e.g., [KR90, HKOTY99, HKOTT02, FZ02, FZ03, BFZ05, FZ07, KKK13a, KKK13b, KKK13c]). Also, it is known from [Nak04, Remark 2.15] (see also [CP01, Proposition 4.5]) that the extremal weight module $V(\lambda)$

of level-zero extremal weight λ is isomorphic to the quantum (or global) Weyl module $W_q(\lambda)$, introduced by Chari–Pressley ([CP01]). Notice that the level-zero extremal weight module is also equipped with an action of the quantum loop algebra $U_q(\mathcal{L}\mathfrak{g})$, where $\mathcal{L}\mathfrak{g} := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$ is the loop algebra associated with $\mathfrak{g}$; remark that $U_q(\mathcal{L}\mathfrak{g})$ (resp., $\mathcal{L}\mathfrak{g}$) is not a subalgebra of $U_q(\mathfrak{g}_{\mathrm{af}})$ (resp., $\mathfrak{g}_{\mathrm{af}}$). Then, when $\mathfrak{g}$ is of type A, D, E, level-zero $V(\lambda)$ is isomorphic to the universal standard module $M(\lambda)$, introduced by Nakajima ([Nak01]), which is defined as the Grothendieck group of equivariant coherent sheaves on the (Lagrangian) Nakajima quiver variety $\mathcal{L}(\lambda)$, with a level-zero dominant integral weight λ, equipped with the geometrically defined action of the quantum loop algebra $U_q(\mathcal{L}\mathfrak{g})$.

In [Kas02a], Kashiwara stated some conjectural structures on level-zero extremal weight modules over $U_q(\mathfrak{g}_{\mathrm{af}})$ and their crystal bases. His conjecture was proved by Beck–Nakajima ([BN04]; see also [Bec02, Nak04]), and in particular, it was shown that

$$\mathcal{B}(\lambda) \cong \bigotimes_{i \in I} \mathcal{B}(m_i \varpi_i) \tag{1.0.1}$$

with $\lambda = \sum_{i \in I} m_i \varpi_i$, $m_i \in \mathbb{Z}_{\geq 0}$, where ϖ_i, $i \in I$, denotes an i-th level-zero fundamental weight of $\mathfrak{g}_{\mathrm{af}}$. Also, Naito–Sagaki ([NS03, NS06]) gave an explicit realization of the crystal basis $\mathcal{B}(m_i \varpi_i)$ for each $i \in I$ in terms of LS paths of shape $m_i \varpi_i$. Namely, they proved that the set $\mathbb{B}(m_i \varpi_i)$ of LS paths of shape $m_i \varpi_i$ is isomorphic as a crystal to $\mathcal{B}(m_i \varpi_i)$, though neither of (the crystal graphs of) these two crystals is connected if $m_i > 1$. However, it turned out in [NS08] that the crystals $\mathbb{B}(\lambda)$ and $\mathcal{B}(\lambda)$ are not necessarily isomorphic for a general level-zero dominant integral weight λ; for example, if λ is of the form $\sum_{i \in K} \varpi_i$ for $K \subset I$ with $\#K \geq 2$, then the crystals $\mathbb{B}(\lambda)$ and $\mathcal{B}(\lambda)$ are never isomorphic, though both of these crystals are connected (for details, see [NS08, Appendix]). This is mainly because each connected component of the crystal $\mathbb{B}(\lambda)$ has fewer extremal elements than that of the crystal $\mathcal{B}(\lambda)$.

Let us explain the situation above more precisely. As an easy consequence of the isomorphism (1.0.1), we see that the extremal elements in the connected component $\mathcal{B}_0(\lambda)$ of $\mathcal{B}(\lambda)$ containing the extremal element u_λ of extremal weight λ is given as $\{u_x := S_x u_\lambda \mid x \in W_{\mathrm{af}}\} \cong W_{\mathrm{af}}/(W_J)_{\mathrm{af}}$, where $W_{\mathrm{af}} = W \ltimes \{t_\xi \mid \xi \in Q^\vee\}$ is the (affine) Weyl group of $\mathfrak{g}_{\mathrm{af}}$, and $(W_J)_{\mathrm{af}} := W_J \ltimes \{t_\xi \mid \xi \in Q_J^\vee\}$, with $J := \{i \in I \mid m_i = 0\}$, is equal to the stabilizer $\{x \in W_{\mathrm{af}} \mid S_x u_\lambda = u_\lambda\}$ of u_λ in W_{af} (see Proposition 5.1.1); here S_x, $x \in W_{\mathrm{af}}$, denote the action of W_{af} on $\mathcal{B}(\lambda)$. In contrast, the extremal elements in the connected component $\mathbb{B}_0(\lambda)$ of $\mathbb{B}(\lambda)$ containing the straight line path $\pi_\lambda := (\lambda\,; 0, 1)$ of weight λ is easily seen to be the straight lines $\{\pi_{x\lambda} := (x\lambda\,; 0, 1) \mid x \in W_{\mathrm{af}}\} \cong W_{\mathrm{af}}/(W_{\mathrm{af}})_\lambda$, where $(W_{\mathrm{af}})_\lambda$ is the stabilizer of λ in W_{af}, and equals $W_J \ltimes \{t_\xi \mid \langle \xi, \lambda \rangle = 0\}$. Here we have $(W_J)_{\mathrm{af}} \subset (W_{\mathrm{af}})_\lambda$ in general, with equality if and only if λ is a nonnegative integer multiple of ϖ_i for some $i \in I$ modulo the null root δ.

In order to overcome the dufficulty above, we introduce the notion of semi-infinite Lakshmibai–Seshadri ($\frac{\infty}{2}$-LS for short) paths of shape λ. A $\frac{\infty}{2}$-LS path of shape λ is, by definition, a pair $(\boldsymbol{x}\,; \boldsymbol{a})$ of a decreasing sequence $\boldsymbol{x} : x_1 >_{\frac{\infty}{2}} x_2 >_{\frac{\infty}{2}} \cdots >_{\frac{\infty}{2}} x_s$ in the set $(W^J)_{\mathrm{af}}$ of Peterson's coset representatives for the cosets in $W_{\mathrm{af}}/(W_J)_{\mathrm{af}}$, equipped with the semi-infinite Bruhat order $\geq_{\frac{\infty}{2}}$, and an increasing sequence $\boldsymbol{a} : 0 = a_0 < a_1 < \cdots < a_s = 1$ of rational numbers, while a (usual) LS path of shape λ is a pair $(\boldsymbol{\lambda}\,; \boldsymbol{a})$ of a decreasing sequence $\boldsymbol{\lambda} : \lambda_1 > \lambda_2 > \cdots > \lambda_s$ of elements in the affine Weyl group orbit $W_{\mathrm{af}}\lambda$, equipped with the partial order $\geq$ that Littelmann defined in [Lit95], and an increasing sequence $\boldsymbol{a}$ of rational numbers as above.

The coset representatives $(W^J)_{\mathrm{af}}$ were originally introduced by Peterson ([Pet97]; see also [LS10]) in his study of the relationship between the T-equivariant homology (ring) $H_\bullet^T(\mathcal{G}_G)$ of the affine Grassmannian $\mathcal{G}_G := G(\mathbb{C}(\!(t)\!))/G(\mathbb{C}[\![t]\!])$ and the T-equivariant (small) quantum cohomology ring $QH_T^\bullet(G/P)$ of the partial flag variety G/P, where G denotes a simply-connected simple algebraic group over $\mathbb{C}$, $P \subset G$ a parabolic subgroup, and $T \subset G$ a maximal torus; notice that $\mathcal{G}_G$ is weakly homotopy equivalent to the space ΩK of based loops into the maximal compact subgroup $K \subset G$ ([GR75, Mit88]), and the ring structure of the T-equivariant homology $H_\bullet^T(\mathcal{G}_G) \cong H_\bullet^T(\Omega K)$ comes from the group structure of ΩK. Also, we see from [Soe97, Claim 4.14] that the semi-infinite Bruhat order on the affine Weyl group W_{af} is essentially the same one as Lusztig's generic Bruhat order on W_{af} (for details, see Appendix A.2); the generic Bruhat order was originally introduced by Lusztig ([Lus80]) in his study of the conjectural character formula for the irreducible quotient of the Weyl module of a simply-connected almost simple algebraic group over an algebraically closed field of (sufficiently large) positive characteristic. As for the geometric meaning of the semi-infinite Bruhat order, it is known ([FFKM99, §5]; see also [FF90, §4]) that the semi-infinite Bruhat order describes the closure relation among the fine Schubert strata, parametrized by W_{af}, of the Drinfeld compactification $\mathcal{QM}^\alpha$, called the space of quasi-maps, of the variety of algebraic maps of a fixed degree $\alpha \in \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$ from the complex projective line $\mathbb{P}^1$ to the flag variety G/B. In addition, we remark that in Peterson's lecture note ([Pet97]), the semi-infinite Bruhat order (or, stable Bruhat order in his terminology) plays an important role, and that some of our arguments in the study of $\frac{\infty}{2}$-LS paths use (parabolic) quantum Bruhat graphs ([BFP99, LNSSS13a]), which appear in the equivariant quantum Chevalley formula for $QH_T^\bullet(G/P)$ ([Mih07]).

Now we are ready to state our main result. First, we can show that the natural surjection from the poset $(W^J)_{\mathrm{af}}$ (equipped with the semi-infinite Bruhat order) onto the poset $W_{\mathrm{af}}\lambda$ (equipped with Littelmann's partial order) is order-preserving, and hence that there exists a surjection from the set $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ of $\frac{\infty}{2}$-LS paths of shape λ onto the set $\mathbb{B}(\lambda)$ of LS paths of shape λ. We define a crystal structure on $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ for $U_q(\mathfrak{g}_{\mathrm{af}})$ in such a way that this surjection becomes a morphism of crystals. Then we obtain the following theorem.

Theorem 1.0.1. *Let $\mathfrak{g}_{\mathrm{af}}$ be an untwisted affine Lie algebra, and $\lambda = \sum_{i\in I} m_i\varpi_i$ a level-zero dominant integral weight, with $m_i \in \mathbb{Z}_{\geq 0}$ for all $i \in I$. Let $\mathcal{B}(\lambda)$ denote the crystal basis of the extremal weight module $V(\lambda)$ over $U_q(\mathfrak{g}_{\mathrm{af}})$, and let $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ denote the set of $\frac{\infty}{2}$-LS paths of shape λ, equipped with the $U_q(\mathfrak{g}_{\mathrm{af}})$-crystal structure as above. Then, we have an isomorphism of crystals*

$$\mathcal{B}(\lambda) \cong \mathbb{B}^{\frac{\infty}{2}}(\lambda). \tag{1.0.2}$$

We remark that for each $i \in I$, we have a natural identification $\mathbb{B}^{\frac{\infty}{2}}(m_i\varpi_i) = \mathbb{B}(m_i\varpi_i)$, since the equality $(W_{I\setminus\{i\}})_{\mathrm{af}} = (W_{\mathrm{af}})_{m_i\varpi_i}$ holds. Hence we recover the results obtained in [NS05, NS06] by Naito–Sagaki.

Also, we should mention that Hernandez and Nakajima ([HN06]) gave a monomial realization of a connected component of $\mathcal{B}(\lambda)$ for a general Kac–Moody Lie algebra; however, their realization is given in a recursive way, and hence it is difficult to determine all the elements in $\mathcal{B}(\lambda)$ explicitly in this realization.

This paper is organized as follows. In §2, we fix our notation for affine root data and review Peterson's coset representatives and semi-infinite Bruhat order on them. In §3, we introduce the notion of $\frac{\infty}{2}$-LS paths and define a crystal structure on $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$, deferring until §4 the proof of the stability property of $\frac{\infty}{2}$-LS paths under Kashiwara operators. Also, we state our main result, i.e., the isomorphism theorem above between $\mathcal{B}(\lambda)$ and $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$. In §4, we study the relation among quantum Bruhat graphs, Littelmann's level-zero weight posets, and semi-infinite Bruhat graphs, and then show that there exists a surjective morphism of crystals from $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ to $\mathbb{B}(\lambda)$. Also, we give the deferred proof of the fact that the set $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ is stable under the Kashiwara operators e_i, f_i, $i \in I_{\mathrm{af}}$, defined in §3. In §5, we show that the connected component $\mathcal{B}_0(\lambda)$ of $\mathcal{B}(\lambda)$ containing u_λ is isomorphic, as a crystal, to the connected component $\mathbb{B}_0^{\frac{\infty}{2}}(\lambda)$ of $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ containing the element $\eta_e := (e\,;0,1)$. In §6, we first study certain directed paths in (parabolic) quantum Bruhat graphs in order to give a bijection between all the connected components of $\mathcal{B}(\lambda)$ and those of $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$. By combining this result with the results in §4 and §5, we finally obtain the desired isomorphism $\mathcal{B}(\lambda) \cong \mathbb{B}^{\frac{\infty}{2}}(\lambda)$. In Appendix A, we give a new description of the semi-infinite Bruhat order on Peterson's coset representatives. Also, we mention the relation between the semi-infinite Bruhat order and Lusztig's generic Bruhat order.

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2 Preliminaries

2.1 Untwisted affine root data

Let $\mathfrak{g}_{\mathrm{af}}$ be an untwisted affine Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}_{\mathrm{af}}$. Let $\{\alpha_i\}_{i \in I_{\mathrm{af}}} \subset \mathfrak{h}_{\mathrm{af}}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_{\mathrm{af}}, \mathbb{C})$ and $\{\alpha_i^\vee\}_{i \in I_{\mathrm{af}}} \subset \mathfrak{h}_{\mathrm{af}}$ be the sets of simple roots and simple coroots, respectively. Let $\langle -, - \rangle : \mathfrak{h}_{\mathrm{af}} \times \mathfrak{h}_{\mathrm{af}}^* \to \mathbb{C}$ denote the canonical pairing. Throughout this paper, we take and fix a weight lattice $P_{\mathrm{af}} \subset \mathfrak{h}_{\mathrm{af}}^*$ satisfying

$$\begin{cases} \alpha_i \in P_{\mathrm{af}} \text{ and } \alpha_i^\vee \in \mathrm{Hom}_{\mathbb{Z}}(P_{\mathrm{af}}, \mathbb{Z}) \text{ for all } i \in I_{\mathrm{af}}, \\ \text{for each } i \in I_{\mathrm{af}}, \text{ there exists } \Lambda_i \in P_{\mathrm{af}} \text{ such that } \langle \alpha_j^\vee, \Lambda_i \rangle = \delta_{ij} \text{ for } j \in I_{\mathrm{af}}. \end{cases}$$

Let $\delta = \sum_{i \in I_{\mathrm{af}}} a_i \alpha_i$ and $c = \sum_{i \in I_{\mathrm{af}}} a_i^\vee \alpha_i^\vee$ be the null root and the canonical central element, respectively. We take and fix $0 \in I_{\mathrm{af}}$ such that $a_0 = a_0^\vee = 1$, and set $I := I_{\mathrm{af}} \setminus \{0\}$. For each $i \in I$, we define $\varpi_i := \Lambda_i - \langle c, \Lambda_i \rangle \Lambda_0$; note that $\langle c, \varpi_i \rangle = 0$ for all $i \in I$. Set

$$Q := \bigoplus_{i \in I} \mathbb{Z}\alpha_i, \qquad Q^\vee := \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee, \qquad P^+ := \sum_{i \in I} \mathbb{Z}_{\geq 0} \varpi_i. \tag{2.1.1}$$

Let $W_{\mathrm{af}} := \langle r_i \mid i \in I_{\mathrm{af}} \rangle$ be the (affine) Weyl group of $\mathfrak{g}_{\mathrm{af}}$, where r_i denotes the simple reflection with respect to α_i, and set $W := \langle r_i \mid i \in I \rangle \subset W_{\mathrm{af}}$. Let $e \in W_{\mathrm{af}}$ be the unit element, and $\ell : W_{\mathrm{af}} \to \mathbb{Z}_{\geq 0}$ the length function. Denote by $\leq$ the Bruhat order on W_{af}.

Denote by Δ_{af} the set of real roots of $\mathfrak{g}_{\mathrm{af}}$, and Δ_{af}^+ the set of positive real roots of $\mathfrak{g}_{\mathrm{af}}$; we know from [Kac90, Proposition 6.3] that

$$\begin{aligned} \Delta_{\mathrm{af}} &= \{\alpha + n\delta \mid \alpha \in \Delta,\ n \in \mathbb{Z}\}, \\ \Delta_{\mathrm{af}}^+ &= \Delta^+ \sqcup \{\alpha + n\delta \mid \alpha \in \Delta,\ n \in \mathbb{Z}_{>0}\}, \end{aligned} \tag{2.1.2}$$

where $\Delta := \Delta_{\mathrm{af}} \cap Q$ is the (finite) root system corresponding to I, and $\Delta^+ := \Delta \cap \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$. For each $x \in W_{\mathrm{af}}$, we set $\mathrm{Inv}(x) := \{\beta \in \Delta^+_{\mathrm{af}} \mid x\beta \in -\Delta^+_{\mathrm{af}}\}$; note that $\ell(x) = \#\mathrm{Inv}(x)$.

For $\beta \in \Delta_{\mathrm{af}}$, denote by $\beta^\vee$ the coroot of β, and $r_\beta \in W_{\mathrm{af}}$ the reflection with respect to β. For $\xi \in Q^\vee$, denote by $t_\xi \in W_{\mathrm{af}}$ the translation with respect to ξ. We know from [Kac90, Proposition 6.5] that $\{t_\xi \mid \xi \in Q^\vee\}$ is an abelian normal subgroup of W_{af}, with $t_\xi t_\zeta = t_{\xi+\zeta}$ for $\xi, \zeta \in Q^\vee$, and $W_{\mathrm{af}} = W \ltimes \{t_\xi \mid \xi \in Q^\vee\}$; remark that if $\beta \in \Delta_{\mathrm{af}}$ is of the form $\beta = \alpha + n\delta$ with $\alpha \in \Delta$ and $n \in \mathbb{Z}$, then

$$r_\beta = r_\alpha t_{n\alpha^\vee}. \tag{2.1.3}$$

For $w \in W$ and $\xi \in Q^\vee$, we have

$$wt_\xi\mu = w\mu - \langle \xi, \mu \rangle \delta \;\text{ if } \mu \text{ satisfies } \langle c, \mu \rangle = 0. \tag{2.1.4}$$

For a subset J of I, set

$$Q_J := \bigoplus_{j \in J} \mathbb{Z}\alpha_j, \qquad Q_J^\vee := \bigoplus_{j \in J} \mathbb{Z}\alpha_j^\vee, \qquad \Delta_J := \Delta \cap Q_J, \tag{2.1.5}$$

$$\Delta_J^+ := \Delta_J \cap \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i, \qquad W_J := \langle r_j \mid j \in J \rangle, \qquad \rho_J := \frac{1}{2} \sum_{\alpha \in \Delta_J^+} \alpha. \tag{2.1.6}$$

Set $\rho := \rho_I = (1/2)\sum_{\alpha \in \Delta^+} \alpha$. Since $\langle \alpha_j^\vee, \rho \rangle = \langle \alpha_j^\vee, \rho_J \rangle = 1$ for all $j \in J$, we have

$$\langle \xi, \rho - \rho_J \rangle = 0 \;\text{ for all } \xi \in Q_J^\vee. \tag{2.1.7}$$

Denote by W^J the set of minimal coset representatives of W/W_J; we see from [BB05, §2.4] that

$$W^J = \{w \in W \mid w\alpha \in \Delta^+ \text{ for all } \alpha \in \Delta_J^+\}. \tag{2.1.8}$$

For $w \in W$, we denote by $\lfloor w \rfloor \in W^J$ the minimal coset representative for the coset wW_J in W/W_J.

2.2 Peterson's coset representatives $(W^J)_{\mathrm{af}}$

Let J be a subset of I. Following [Pet97] (see also [LS10, §10]), we define

$$(\Delta_J)_{\mathrm{af}} := \{\alpha + n\delta \mid \alpha \in \Delta_J, n \in \mathbb{Z}\} \subset \Delta_{\mathrm{af}}, \tag{2.2.1}$$

$$(\Delta_J)^+_{\mathrm{af}} := (\Delta_J)_{\mathrm{af}} \cap \Delta^+_{\mathrm{af}} = \Delta_J^+ \sqcup \{\alpha + n\delta \mid \alpha \in \Delta_J, n \in \mathbb{Z}_{>0}\} \subset \Delta^+_{\mathrm{af}}, \tag{2.2.2}$$

$$(W_J)_{\mathrm{af}} := W_J \ltimes \{t_\xi \mid \xi \in Q_J^\vee\}, \tag{2.2.3}$$

$$(W^J)_{\mathrm{af}} := \{x \in W_{\mathrm{af}} \mid x\beta \in \Delta^+_{\mathrm{af}} \text{ for all } \beta \in (\Delta_J)^+_{\mathrm{af}}\}. \tag{2.2.4}$$

Remark 2.2.1. We see that $(W_J)_{\mathrm{af}} = \langle r_\beta \mid \beta \in (\Delta_J)^+_{\mathrm{af}} \rangle$. Indeed, we have $r_\beta = r_\alpha t_{n\alpha^\vee} \in (W_J)_{\mathrm{af}}$ for all $\beta = \alpha + n\delta \in (\Delta_J)^+_{\mathrm{af}}$, which implies that $(W_J)_{\mathrm{af}} \supset \langle r_\beta \mid \beta \in (\Delta_J)^+_{\mathrm{af}} \rangle$. Also, we have $W_J = \langle r_j \mid j \in J \rangle \subset \langle r_\beta \mid \beta \in (\Delta_J)^+_{\mathrm{af}} \rangle$, and $t_{\alpha_j^\vee} = r_{\alpha_j} r_{\alpha_j + \delta} \in \langle r_\beta \mid \beta \in (\Delta_J)^+_{\mathrm{af}} \rangle$ for $j \in J$, which implies that $t_\xi \in \langle r_\beta \mid \beta \in (\Delta_J)^+_{\mathrm{af}} \rangle$ for all $\xi \in Q_J^\vee$. Thus we have proved $(W_J)_{\mathrm{af}} \subset \langle r_\beta \mid \beta \in (\Delta_J)^+_{\mathrm{af}} \rangle$, and hence $(W_J)_{\mathrm{af}} = \langle r_\beta \mid \beta \in (\Delta_J)^+_{\mathrm{af}} \rangle$.

Lemma 2.2.2 ([Pet97]; see also [LS10, Lemma 10.6])**.** *For every $x \in W_{\mathrm{af}}$, there exists a unique factorization $x = x_1 x_2$ with $x_1 \in (W^J)_{\mathrm{af}}$ and $x_2 \in (W_J)_{\mathrm{af}}$.*

Definition 2.2.3. Define a (surjective) map $\Pi^J : W_{\mathrm{af}} \to (W^J)_{\mathrm{af}}$ by $\Pi^J(x) := x_1$ if $x = x_1 x_2$ with $x_1 \in (W^J)_{\mathrm{af}}$ and $x_2 \in (W_J)_{\mathrm{af}}$.

Lemma 2.2.4 ([Pet97]; see also [LS10, Proposition 10.10])**.** *(1)* $\Pi^J(w) = \lfloor w \rfloor$ *for* $w \in W$.

(2) $\Pi^J(xt_\xi) = \Pi^J(x)\Pi^J(t_\xi)$ *for* $x \in W_{\mathrm{af}}$ *and* $\xi \in Q^\vee$.

Definition 2.2.5 ([LNSSS13a, §3.4])**.** An element $\xi \in Q^\vee$ is said to be *J-adjusted* if $\langle \xi, \gamma \rangle \in \{-1, 0\}$ for all $\gamma \in \Delta_J^+$. Let $Q^\vee_{J\text{-adj}}$ denote the set of J-adjusted elements.

Lemma 2.2.6 ([LNSSS13a, (3.7) and Lemma 3.7])**.** *(1) For each* $\xi \in Q^\vee$*, there exists a unique* $\phi_J(\xi) \in Q_J^\vee$ *such that* $\xi + \phi_J(\xi) \in Q^\vee_{J\text{-adj}}$.

(2) For each $\xi \in Q^\vee$*, there exists a unique* $z_\xi \in W_J$ *such that* $\Pi^J(t_\xi) = z_\xi t_{\xi + \phi_J(\xi)}$.

(3) For $w \in W$ *and* $\xi \in Q^\vee$*, we have* $\Pi^J(wt_\xi) = \lfloor w \rfloor z_\xi t_{\xi+\phi_J(\xi)}$*. In particular,*

$$(W^J)_{\mathrm{af}} = \{ w z_\xi t_\xi \mid w \in W^J,\ \xi \in Q^\vee_{J\text{-adj}} \}. \tag{2.2.5}$$

Let us show some technical lemmas, which are needed later.

Lemma 2.2.7. *Let* $x \in (W^J)_{\mathrm{af}}$ *and* $i \in I_{\mathrm{af}}$*. If* $x^{-1}\alpha_i \notin (\Delta_J)_{\mathrm{af}}$*, then* $r_i x \in (W^J)_{\mathrm{af}}$.

Proof. Following the definition (2.2.4) of $(W^J)_{\mathrm{af}}$, we show that $r_i x \beta \in \Delta^+_{\mathrm{af}}$ for all $\beta \in (\Delta_J)^+_{\mathrm{af}}$. Let $\beta \in (\Delta_J)^+_{\mathrm{af}}$. Because $x \in (W^J)_{\mathrm{af}}$, we have $x\beta \in \Delta^+_{\mathrm{af}}$. Since $x^{-1}\alpha_i \notin (\Delta_J)_{\mathrm{af}}$ by assumption, it follows that $x^{-1}\alpha_i \neq \beta$, and hence $\alpha_i \neq x\beta$. Therefore, $r_i x\beta \in \Delta^+_{\mathrm{af}}$ since $\mathrm{Inv}(r_i) = \{\alpha_i\}$. Thus we have proved the lemma. □

Lemma 2.2.8. *Let* $x \in W_{\mathrm{af}}$*, and* $\xi \in Q^\vee_{J\text{-adj}}$*. Then,* $x z_\xi t_\xi \in (W^J)_{\mathrm{af}}$ *if and only if* $x \in (W^J)_{\mathrm{af}}$.

Proof. First we remark that

$$\begin{aligned} \Pi^J(x z_\xi t_\xi) &= \Pi^J(x z_\xi)\Pi^J(t_\xi) && \text{(by Lemma 2.2.4 (2))} \\ &= \Pi^J(x) z_\xi t_\xi && \text{(by Lemmas 2.2.4 (1) and 2.2.6 (2))}. \end{aligned} \tag{2.2.6}$$

Now, let us show the "only if" part. Assume that $x z_\xi t_\xi \in (W^J)_{\mathrm{af}}$; note that $\Pi^J(x z_\xi t_\xi) = x z_\xi t_\xi$. Combining this and (2.2.6), we obtain $\Pi^J(x) z_\xi t_\xi = x z_\xi t_\xi$, and hence $\Pi^J(x) = x$, which implies that $x \in (W^J)_{\mathrm{af}}$. Next, let us show the "if" part. Assume that $x \in (W^J)_{\mathrm{af}}$; note that $\Pi^J(x) = x$. Combining this and (2.2.6), we obtain $\Pi^J(x z_\xi t_\xi) = \Pi^J(x) z_\xi t_\xi = x z_\xi t_\xi$, which implies that $x z_\xi t_\xi \in (W^J)_{\mathrm{af}}$. Thus we have proved the lemma. □

2.3 Semi-infinite Bruhat order on $(W^J)_{\mathrm{af}}$

Definition 2.3.1 ([Pet97])**.** Let $x \in W_{\mathrm{af}}$, and write it as $x = vt_\zeta$ with $v \in W$ and $\zeta \in Q^\vee$. Then we define

$$\ell^{\frac{\infty}{2}}(x) := \ell(v) + 2\langle \zeta, \rho \rangle. \tag{2.3.1}$$

Definition 2.3.2. (1) Let J be a subset of I. Define the (parabolic) *semi-infinite Bruhat graph* SB^J to be the Δ^+_{af}-colored, oriented graph, with $(W^J)_{\mathrm{af}}$ the set of vertices, whose edges are drawn as follows: for $x \in (W^J)_{\mathrm{af}}$ and $\beta \in \Delta^+_{\mathrm{af}}$, we write $x \xrightarrow{\beta} r_\beta x$ if the following two conditions hold:

(i) $r_\beta x \in (W^J)_{\mathrm{af}}$,

(ii) $\ell^{\frac{\infty}{2}}(r_\beta x) = \ell^{\frac{\infty}{2}}(x) + 1$.

(2) Define a partial order $\le_{\frac{\infty}{2}}$ on $(W^J)_{\mathrm{af}}$, called the *semi-infinite Bruhat order*, as follows: for $x, y \in (W^J)_{\mathrm{af}}$, we write $x \le_{\frac{\infty}{2}} y$ if there exists a directed path from x to y in SB^J.

3 Main results

Throughout this section, we fix $\lambda \in P^+$, and set $J := \{i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0\} \subset I$.

3.1 Semi-infinite Lakshmibai–Seshadri paths

Definition 3.1.1. For a rational number $0 < a \le 1$, define $\mathrm{SB}(\lambda\,; a)$ to be the subgraph of SB^J consisting of the same vertices and only the edges

$$x \xrightarrow{\beta} y \text{ with } a\langle \beta^\vee, x\lambda \rangle \in \mathbb{Z}. \tag{3.1.1}$$

Note that $\mathrm{SB}(\lambda\,; 1) = \mathrm{SB}^J$.

Definition 3.1.2. A *semi-infinite Lakshmibai–Seshadri* ($\frac{\infty}{2}$*-LS* for short) *path* of shape λ is, by definition, a pair $(\boldsymbol{x}\,; \boldsymbol{a})$ of a decreasing sequence $\boldsymbol{x} : x_1 >_{\frac{\infty}{2}} \cdots >_{\frac{\infty}{2}} x_s$ in $(W^J)_{\mathrm{af}}$ and an increasing sequence $\boldsymbol{a} : 0 = a_0 < a_1 < \cdots < a_s = 1$ of rational numbers satisfying the condition that there exists a directed path from x_{u+1} to x_u in $\mathrm{SB}(\lambda\,; a_u)$ for each $u = 1, 2, \dots, s-1$. Denote by $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ the set of $\frac{\infty}{2}$-LS paths of shape λ.

For $\eta = (x_1, \dots, x_s\,; a_0, a_1, \cdots, a_s) \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$, we define a piecewise-linear, continuous map $\pi_\eta : [0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} P_{\mathrm{af}}$ by

$$\pi_\eta(t) := \sum_{u=1}^{k-1} (a_u - a_{u-1}) x_u \lambda + (t - a_{k-1}) x_k \lambda \qquad \text{for } t \in [a_{k-1}, a_k],\ 1 \le k \le s. \tag{3.1.2}$$

We will show the following proposition in §4.3; for the definition of a Lakshmibai–Seshadri path of shape λ, see Definition 4.1.3.

Proposition 3.1.3. *For every $\eta \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$, π_η is a Lakshmibai–Seshadri path of shape λ.*

Hence we see from [Lit95, §4] that $\pi_\eta(1) \in P_{\mathrm{af}}$ for all $\eta \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$. So, we define a map $\mathrm{wt} : \mathbb{B}^{\frac{\infty}{2}}(\lambda) \to P_{\mathrm{af}}$ by $\mathrm{wt}(\eta) := \pi_\eta(1)$.

Remark 3.1.4. We see from [Lit95, Lemma 4.5 d)] that for each $\eta \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ and $i \in I_{\mathrm{af}}$, all local minimal values of the function

$$H_i^{\pi_\eta}(t) := \langle \alpha_i^\vee, \pi_\eta(t) \rangle \quad \text{for } t \in [0,1], \tag{3.1.3}$$

are integers.

Now, let us define operators e_i, f_i, $i \in I_{\mathrm{af}}$, on $\mathbb{B}^{\frac{\infty}{2}}(\lambda) \sqcup \{\mathbf{0}\}$, where $\mathbf{0}$ is an additional element not contained in any crystal; we call these operators the root operators.

Definition 3.1.5 (cf. [Lit94, Proposition 4.2]). Let $\eta = (x_1, x_2, \dots, x_s\,; a_0, a_1, \dots, a_s) \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$, and $i \in I_{\mathrm{af}}$. Define $H_i^{\pi_\eta}(t)$, $t \in [0,1]$, as (3.1.3), and set

$$m_i^{\pi_\eta} := \min\{H_i^{\pi_\eta}(t) \mid t \in [0,1]\}; \tag{3.1.4}$$

note that $m_i^{\pi_\eta} \in \mathbb{Z}_{\le 0}$ and $H_i^{\pi_\eta}(1) - m_i^{\pi_\eta} \in \mathbb{Z}_{\ge 0}$ by Remark 3.1.4.

(1) If $m_i^{\pi_\eta} = 0$, then we define $e_i\eta = \mathbf{0}$. If $m_i^{\pi_\eta} \le -1$, then we set

$$\begin{aligned} t_1 &:= \min\{t \in [0,1] \mid H_i^{\pi_\eta}(t) = m_i^{\pi_\eta}\}, \\ t_0 &:= \max\{t \in [0,t_1] \mid H_i^{\pi_\eta}(t) = m_i^{\pi_\eta} + 1\}; \end{aligned} \tag{3.1.5}$$

we deduce from Remark 3.1.4 that $H_i^{\pi_\eta}(t)$ is strictly decreasing on $[t_0, t_1]$. Notice that there exists $1 \le q \le s$ such that $t_1 = a_q$. Let $1 \le p \le q$ be such that $a_{p-1} \le t_0 < a_p$. Then we define

$$\begin{aligned} e_i\eta := (&x_1, x_2, \dots, x_p, r_ix_p, r_ix_{p+1}, \dots, r_ix_q, x_{q+1}, x_{q+2}, \dots, x_s\,; \\ &a_0, a_1, a_2, \dots, a_{p-1}, t_0, a_p, a_{p+1}, \dots, a_{q-1}, t_1, a_{q+1}, a_{q+2}, \dots, a_{s-1}, a_s); \end{aligned}$$

if $t_0 = a_{p-1}$, then we drop x_p and a_{p-1}, and if $r_ix_q = x_{q+1}$, then we drop x_{q+1} and t_1.

(2) If $H_i^{\pi_\eta}(1) - m_i^{\pi_\eta} = 0$, then we define $f_i\eta := \mathbf{0}$. If $H_i^{\pi_\eta}(1) - m_i^{\pi_\eta} \ge 1$, then we set

$$\begin{aligned} t_0 &:= \max\{t \in [0,1] \mid H_i^{\pi_\eta}(t) = m_i^{\pi_\eta}\}, \\ t_1 &:= \min\{t \in [t_0,1] \mid H_i^{\pi_\eta}(t) = m_i^{\pi_\eta} + 1\}; \end{aligned} \tag{3.1.6}$$

we deduce from Remark 3.1.4 that $H_i^{\pi_\eta}(t)$ is strictly increasing on $[t_0, t_1]$. Notice that there exists $1 \le p \le s$ such that $t_0 = a_{p-1}$. Let $p \le q \le s$ be such that $a_{q-1} < t_1 \le a_q$. Then we define

$$\begin{aligned} f_i\eta := (&x_1, x_2, \dots, x_{p-1}, r_ix_p, r_ix_{p+1}, \dots, r_ix_q, x_q, x_{q+1}, \dots, x_s\,; \\ &a_0, a_1, a_2, \dots a_{p-2}, t_0, a_p, a_{p+1}, \dots, a_{q-1}, t_1, a_q, a_{q+1}, \dots, a_{s-1}, a_s); \end{aligned}$$

if $t_1 = a_q$, then we drop x_q and a_q, and if $x_{p-1} = r_ix_p$, then we drop x_{p-1} and $t_0 = a_{p-1}$.

(3) Define $e_i\mathbf{0} = f_i\mathbf{0} := \mathbf{0}$ for all $i \in I_{\mathrm{af}}$.

We will prove the following theorem in §4.4; for the definition of crystals, see [Kas95, §7.2] and [HK02, Definition 4.5.1] for example.

Theorem 3.1.6. *(1) The set $\mathbb{B}^{\frac{\infty}{2}}(\lambda) \sqcup \{\mathbf{0}\}$ is stable under the action of the root operators e_i and f_i, $i \in I_{\mathrm{af}}$.*

(2) For each $\eta \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ and $i \in I_{\mathrm{af}}$, we set

$$\begin{aligned} \varepsilon_i(\eta) &:= \max\{k \ge 0 \mid e_i^k\eta \ne \mathbf{0}\}, \\ \varphi_i(\eta) &:= \max\{k \ge 0 \mid f_i^k\eta \ne \mathbf{0}\}. \end{aligned} \tag{3.1.7}$$

The set $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ together with the maps wt, *e_i, f_i, $i \in I_{\mathrm{af}}$, and ε_i, φ_i, $i \in I_{\mathrm{af}}$, is a crystal with weights in P_{af}.*

3.2 Isomorphism theorem between $\mathcal{B}(\lambda)$ and $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$

Denote by $V(\lambda)$ the extremal weight module of extremal weight $\lambda \in P_{\mathrm{af}}$ over the quantized universal enveloping algebra $U_q(\mathfrak{g}_{\mathrm{af}})$ associated with $\mathfrak{g}_{\mathrm{af}}$, which is an integrable $U_q(\mathfrak{g}_{\mathrm{af}})$-module generated by a single element v_λ with the defining relation that "v_λ is an extremal weight vector of weight λ" (see [Kas94, §8] and [Kas02b, §3]). We know from [Kas94, §8] that $V(\lambda)$ has a crystal basis $\mathcal{B}(\lambda)$. The main result of this paper is the following theorem.

Theorem 3.2.1. *Let $\lambda \in P^+$. The crystal basis $\mathcal{B}(\lambda)$ of the extremal weight module $V(\lambda)$ of extremal weight λ is isomorphic, as a crystal, to the crystal $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ of $\frac{\infty}{2}$-LS paths of shape λ.*

Let us give a sketch of the proof of Theorem 3.2.1. Let $\mathbb{B}_0^{\frac{\infty}{2}}(\lambda)$ be the connected component of $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ containing $\eta_e := (e\,; 0, 1) \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$. Also, let u_λ be the element of $\mathcal{B}(\lambda)$ corresponding to the generator v_λ of $V(\lambda)$, and let $\mathcal{B}_0(\lambda)$ be the connected component of $\mathcal{B}(\lambda)$ containing $u_\lambda \in \mathcal{B}(\lambda)$. We will prove the following proposition in §5.

Proposition 3.2.2. *For $\lambda \in P^+$, there exists a unique isomorphism $\mathcal{B}_0(\lambda) \xrightarrow{\cong} \mathbb{B}_0^{\frac{\infty}{2}}(\lambda)$ of crystals that maps u_λ to η_e.*

Write $\lambda \in P^+$ as $\lambda = \sum_{i \in I} m_i \varpi_i$ with $m_i \in \mathbb{Z}_{\geq 0}$, $i \in I$, and set

$$\mathrm{Par}(\lambda) := \{\boldsymbol{\rho} = (\rho^{(i)})_{i \in I} \mid \rho^{(i)} \text{ is a partition of length less than } m_i \text{ for } i \in I\}; \tag{3.2.1}$$

we understand that $\rho^{(i)}$ is the empty partition if $m_i = 0$. We give $\mathrm{Par}(\lambda)$ a crystal structure as follows: for each $\boldsymbol{\rho} = (\rho^{(i)})_{i \in I} \in \mathrm{Par}(\lambda)$, we set

$$\begin{cases} e_i\boldsymbol{\rho} = f_i\boldsymbol{\rho} := \mathbf{0},\ \varepsilon_i(\boldsymbol{\rho}) = \varphi_i(\boldsymbol{\rho}) := -\infty & \text{for } i \in I_{\mathrm{af}}, \\ \mathrm{wt}(\boldsymbol{\rho}) := -\sum_{i \in I} |\rho^{(i)}|\delta, \end{cases} \tag{3.2.2}$$

where $|\rho^{(i)}| := \sum_{u=1}^{m_i-1} \rho_u^{(i)}$ if $\rho^{(i)} = (\rho_1^{(i)} \geq \rho_2^{(i)} \geq \cdots \geq \rho_{m_i-1}^{(i)} \geq 0)$. By Proposition 3.2.2, we have an isomorphism

$$\mathrm{Par}(\lambda) \otimes \mathcal{B}_0(\lambda) \cong \mathrm{Par}(\lambda) \otimes \mathbb{B}_0^{\frac{\infty}{2}}(\lambda) \tag{3.2.3}$$

of crystals. So, let $\mathcal{B}$ be either $\mathcal{B}_0(\lambda)$ or $\mathbb{B}_0^{\frac{\infty}{2}}(\lambda)$. For each $\boldsymbol{\rho} \in \mathrm{Par}(\lambda)$, we set $\{\boldsymbol{\rho}\} \otimes \mathcal{B} := \{\boldsymbol{\rho} \otimes b \mid b \in \mathcal{B}\} \subset \mathrm{Par}(\lambda) \otimes \mathcal{B}$; we see from the tensor product rule of crystals (see [Kas95, §7.2]) that $e_i(\boldsymbol{\rho} \otimes b) = \boldsymbol{\rho} \otimes e_i b$ and $f_i(\boldsymbol{\rho} \otimes b) = \boldsymbol{\rho} \otimes f_i b$ for all $b \in \mathcal{B}$ and $i \in I_{\mathrm{af}}$. Therefore, $\{\boldsymbol{\rho}\} \otimes \mathcal{B}$ is a connected subcrystal of $\mathrm{Par}(\lambda) \otimes \mathcal{B}$, anad

$$\mathrm{Par}(\lambda) \otimes \mathcal{B} = \bigsqcup_{\boldsymbol{\rho} \in \mathrm{Par}(\lambda)} \{\boldsymbol{\rho}\} \otimes \mathcal{B}.$$

Moreover, the map $\mathcal{B} \to \{\boldsymbol{\rho}\} \otimes \mathcal{B}$ defined by $b \mapsto \boldsymbol{\rho} \otimes b$ is bijective and commutes with the Kashiwara operators.

Now, we know the following proposition from [BN04, Theorem 4.16 (i)].

Proposition 3.2.3. *For $\lambda \in P^+$, there exists an isomorphism $\mathcal{B}(\lambda) \xrightarrow{\cong} \mathrm{Par}(\lambda) \otimes \mathcal{B}_0(\lambda)$ of crystals.*

Also, we will show the following proposition in §6.

Proposition 3.2.4. *For $\lambda \in P^+$, there exists an isomorphism $\mathbb{B}^{\frac{\infty}{2}}(\lambda) \xrightarrow{\cong} \mathrm{Par}(\lambda) \otimes \mathbb{B}_0^{\frac{\infty}{2}}(\lambda)$ of crystals.*

Thus we obtain

$$\begin{aligned} \mathcal{B}(\lambda) &\cong \mathrm{Par}(\lambda) \otimes \mathcal{B}_0(\lambda) && \text{(by Proposition 3.2.3)} \\ &\cong \mathrm{Par}(\lambda) \otimes \mathbb{B}_0^{\frac{\infty}{2}}(\lambda) && \text{(by (3.2.3))} \\ &\cong \mathbb{B}^{\frac{\infty}{2}}(\lambda) && \text{(by Proposition 3.2.4).} \end{aligned}$$

4 Proofs of Proposition 3.1.3 and Theorem 3.1.6

4.1 Lakshmibai–Seshadri paths

Throughout this subsection, we fix $\lambda \in P^+$.

Definition 4.1.1 ([Lit95, §4])**.** We define a partial order $\leq$ on $W_{\mathrm{af}}\lambda$ as follows: for $\mu, \nu \in W_{\mathrm{af}}\lambda$, we write $\mu \leq \nu$ if there exist a sequence $\mu = \nu_0, \nu_1, \dots, \nu_s = \nu$ of elements in $W_{\mathrm{af}}\lambda$ and a sequence $\beta_1, \dots, \beta_s$ of elements in Δ_{af}^+ such that $\nu_u = r_{\beta_u}\nu_{u-1}$ and $\langle \beta_u^\vee, \nu_{u-1} \rangle \in \mathbb{Z}_{>0}$ for $u = 1, \dots, s$. We call $(W_{\mathrm{af}}\lambda, \leq)$ the *level-zero weight poset* of shape λ.

Definition 4.1.2. (1) Define $\mathrm{LP}(\lambda)$ to be the Δ_{af}^+-colored, oriented graph, with $W_{\mathrm{af}}\lambda$ the set of vertices, whose edges are drawn as follows: for $\mu, \nu \in W_{\mathrm{af}}\lambda$, we write $\mu \xrightarrow{\beta} \nu$ if ν covers μ in the poset $W_{\mathrm{af}}\lambda$, where the label β of the edge is a unique positive real root $\beta \in \Delta_{\mathrm{af}}^+$ such that $\nu = r_\beta \mu$ and $\langle \beta^\vee, \mu \rangle > 0$.

(2) Let $0 < a \leq 1$ be a rational number. Define $\mathrm{LP}(\lambda\,; a)$ to be the subgraph of $\mathrm{LP}(\lambda)$ consisting of the same vertices and only the edges

$$\mu \xrightarrow{\beta} \nu \text{ with } a\langle \beta^\vee, \mu \rangle \in \mathbb{Z}. \tag{4.1.1}$$

Note that $\mathrm{LP}(\lambda\,; 1) = \mathrm{LP}(\lambda)$.

Definition 4.1.3 ([Lit95, §4])**.** A *Lakshmibai–Seshadri* (*LS* for short) *path* of shape λ is, by definition, a pair $(\boldsymbol{\nu}\,; \boldsymbol{a})$ of a decreasing sequence $\boldsymbol{\nu} : \nu_1 > \cdots > \nu_s$ in $W_{\mathrm{af}}\lambda$ and an increasing sequence $\boldsymbol{a} : 0 = a_0 < a_1 < \cdots < a_s = 1$ of rational numbers satisfying the condition that there exists a directed path from ν_{u+1} to ν_u in $\mathrm{LP}(\lambda\,; a_u)$ for each $u = 1, 2, \dots, s$. Let $\mathbb{B}(\lambda)$ denote the set of LS paths of shape λ.

As in (3.1.2), we identify $\pi = (\nu_1, \dots, \nu_s\,; a_0, a_1, \dots, a_s) \in \mathbb{B}(\lambda)$ with the piecewise-linear, continuous map $\pi : [0, 1] \to \mathbb{R} \otimes_{\mathbb{Z}} P_{\mathrm{af}}$ defined by

$$\pi(t) := \sum_{u=1}^{k-1} (a_u - a_{u-1})\nu_u + (t - a_{k-1})\nu_k \qquad \text{for } t \in [a_{k-1}, a_k],\ 1 \leq k \leq s. \tag{4.1.2}$$

Now, we give $\mathbb{B}(\lambda)$ a crystal structure as follows. We define $\mathrm{wt} : \mathbb{B}(\lambda) \to P_{\mathrm{af}}$ by $\mathrm{wt}(\pi) := \pi(1) \in P_{\mathrm{af}}$ (see [Lit95, §4]). Following [Lit95, §1] (see also [NS06, §1]), we define the root operators e_i, f_i, $i \in I_{\mathrm{af}}$, on $\mathbb{B}(\lambda) \sqcup \{\mathbf{0}\}$.

Definition 4.1.4. Let $\pi = (\nu_1, \nu_2, \dots, \nu_s\,; a_0, a_1, \dots, a_s) \in \mathbb{B}(\lambda)$, and $i \in I_{\mathrm{af}}$. Define $H_i^\pi(t)$, $t \in [0, 1]$, and m_i^π as (3.1.3) and (3.1.4), with π_η replaced by π, respectively.

(1) If $m_i^\pi = 0$, then we define $e_i\pi := \mathbf{0}$. If $m_i^\pi \le -1$, then we define $t_0, t_1 \in [0,1]$ as (3.1.5), and set

$$(e_i\pi)(t) := \begin{cases} \pi(t) & \text{if } t \in [0, t_0], \\ \pi(t_0) + r_i(\pi(t) - \pi(t_0)) & \text{if } t \in [t_0, t_1], \\ \pi(t) + \alpha_i & \text{if } t \in [t_1, 1], \end{cases}$$

or equivalently,

$$e_i\pi := (\nu_1, \nu_2, \dots, \nu_p, r_i\nu_p, r_i\nu_{p+1}, \dots, r_i\nu_q, \nu_{q+1}, \nu_{q+2}, \dots, \nu_s\,; \\ a_0, a_1, a_2, \dots, a_{p-1}, t_0, a_p, a_{p+1}, \dots, a_{q-1}, t_1, a_{q+1}, a_{q+2}, \dots, a_{s-1}, a_s),$$

where $1 \le p \le q \le s$ are such that $a_{p-1} \le t_0 < a_p$ and $t_1 = a_q$; if $t_0 = a_{p-1}$, then we drop ν_p and a_{p-1}, and if $r_i\nu_q = \nu_{q+1}$, then we drop ν_{q+1} and t_1.

(2) If $H_i^\pi(1) - m_i^\pi = 0$, then we define $f_i\pi := \mathbf{0}$. If $H_i^\pi(1) - m_i^\pi \ge 1$, then we define $t_0, t_1 \in [0,1]$ as (3.1.6), and set

$$(f_i\pi)(t) := \begin{cases} \pi(t) & \text{if } t \in [0, t_0], \\ \pi(t_0) + r_i(\pi(t) - \pi(t_0)) & \text{if } t \in [t_0, t_1], \\ \pi(t) - \alpha_i & \text{if } t \in [t_1, 1], \end{cases}$$

or equivalently,

$$f_i\pi := (\nu_1, \nu_2, \dots, \nu_{p-1}, r_i\nu_p, r_i\nu_{p+1}, \dots, r_i\nu_q, \nu_q, \nu_{q+1}, \dots, \nu_s\,; \\ a_0, a_1, a_2, \dots a_{p-2}, t_0, a_p, a_{p+1}, \dots, a_{q-1}, t_1, a_q, a_{q+1}, \dots, a_{s-1}, a_s),$$

where $1 \le p \le q \le s$ are such that $t_0 = a_{p-1}$ and $a_{q-1} < t_1 \le a_q$; if $t_1 = a_q$, then we drop ν_q and a_q, and if $\nu_{p-1} = r_i\nu_p$, then we drop ν_{p-1} and t_0.

(3) Define $e_i\mathbf{0} = f_i\mathbf{0} := \mathbf{0}$ for all $i \in I_{\text{af}}$.

We know from [Lit95, Proposition 4.7] that the set $\mathbb{B}(\lambda) \sqcup \{\mathbf{0}\}$ is stable under the root operators e_i, f_i, $i \in I_{\text{af}}$. So, we define $\varepsilon_i(\pi) := \max\{k \in \mathbb{Z}_{\ge 0} \mid e_i^k\pi \ne \mathbf{0}\}$, and $\varphi_i(\pi) := \max\{k \in \mathbb{Z}_{\ge 0} \mid f_i^k\pi \ne \mathbf{0}\}$ for $\pi \in \mathbb{B}(\lambda)$ and $i \in I_{\text{af}}$. We know from [Lit95, §2 and §4] that the set $\mathbb{B}(\lambda)$ together with the maps wt, e_i, f_i, $i \in I_{\text{af}}$, and ε_i, φ_i, $i \in I$, is a crystal with weights in P_{af}.

4.2 Quantum Bruhat graphs

Definition 4.2.1 ([LNSSS13a, §4]; see also [BFP99, §6])**.** Let J be a subset of I. Define the (parabolic) *quantum Bruhat graph* QB^J to be the $(\Delta^+ \setminus \Delta_J^+)$-colored oriented graph, with W^J the set of vertices, whose edges are drawn as follows: for $w \in W^J$ and $\alpha \in \Delta^+ \setminus \Delta_J^+$, we write $w \xrightarrow{\alpha} \lfloor wr_\alpha \rfloor$ if either of the following holds:

(B) $\ell(\lfloor wr_\alpha \rfloor) = \ell(w) + 1$, or

(Q) $\ell(\lfloor wr_\alpha \rfloor) = \ell(w) + 1 - 2\langle \alpha^\vee, \rho - \rho_J \rangle$.

We call an edge $w \xrightarrow{\alpha} \lfloor wr_\alpha \rfloor$ satisfying condition (B) (resp., (Q)) a *Bruhat* (resp., *quantum*) edge, and write as $w \xrightarrow[\mathsf{B}]{\alpha} \lfloor wr_\alpha \rfloor$ (resp., $w \xrightarrow[\mathsf{Q}]{\alpha} \lfloor wr_\alpha \rfloor$).

Remark 4.2.2. (1) If $w \xrightarrow[\mathsf{B}]{\alpha} \lfloor wr_\alpha \rfloor$, then we have $\lfloor wr_\alpha \rfloor = wr_\alpha \in W^J$ (see [LNSSS13b, Remark 3.1.2]).

(2) Let $w \in W^J$ and $\alpha \in \Delta^+ \setminus \Delta_J^+$. We have $w \xrightarrow[\mathsf{Q}]{\alpha} \lfloor wr_\alpha \rfloor$ if and only if the following condition (Q') holds ([LNSSS13a, §4.3]; see also [LS10, §10]):

(Q') $wr_\alpha t_{\alpha^\vee} \in (W^J)_{\mathrm{af}}$ and $\ell(wr_\alpha) = \ell(w) + 1 - 2\langle \alpha^\vee, \rho \rangle$.

(3) If $w \xrightarrow[\mathsf{Q}]{\alpha} \lfloor wr_\alpha \rfloor$, then we have $\ell(\lfloor wr_\alpha \rfloor) < \ell(w)$. Indeed, if we write $\alpha^\vee$ as $\alpha^\vee = \xi_1 + \xi_2$ with $\xi_1 \in \sum_{i \in I \setminus J} \mathbb{Z}_{\geq 0} \alpha_i^\vee$ and $\xi_2 \in \sum_{j \in J} \mathbb{Z}_{\geq 0} \alpha_j$, then $\xi_1 \neq 0$ since $\alpha \in \Delta^+ \setminus \Delta_J^+$. Also, we have $\langle \alpha^\vee, \rho - \rho_J \rangle = \langle \xi_1, \rho - \rho_J \rangle$ by (2.1.7). Therefore, we see that $\langle \alpha^\vee, \rho - \rho_J \rangle > 0$, which implies that $\ell(\lfloor wr_\alpha \rfloor) = \ell(w) + 1 - 2\langle \alpha^\vee, \rho - \rho_J \rangle < \ell(w)$.

Let $\lambda \in P^+$, and set $J := \{ i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \} \subset I$.

Definition 4.2.3. For a rational number $0 < a \leq 1$, let $\mathrm{QB}(\lambda\,; a)$ denote the subgraph of QB^J consisting of the same vertices and only the edges

$$w \xrightarrow{\alpha} \lfloor wr_\alpha \rfloor \text{ with } a\langle \alpha^\vee, \lambda \rangle \in \mathbb{Z}. \tag{4.2.1}$$

Note that $\mathrm{QB}(\lambda\,; 1) = \mathrm{QB}^J$.

4.3 Proof of Proposition 3.1.3

Throughout this subsection, we fix $\lambda \in P^+$ and set $J := \{ i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \}$. Let $\eta = (x_1, x_2, \ldots, x_s\,; a_0, a_1, \ldots, a_s) \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$, and define $\pi_\eta : [0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} P_{\mathrm{af}}$ as (3.1.2). By (4.1.2), this π_η corresponds to

$$\pi_\eta = (x_1\lambda, x_2\lambda, \ldots, x_s\lambda\,; a_0, a_1, \ldots, a_s) \in \mathbb{B}(\lambda). \tag{4.3.1}$$

Thus it suffices to show that for each $1 \leq u \leq s-1$, there exists a directed path from $x_{u+1}\lambda$ to $x_u\lambda$ in $\mathrm{LP}(\lambda\,; a_u)$. This follows immediately from the next proposition, which gives a relation between $\mathrm{SB}(\lambda\,; a)$ and $\mathrm{LP}(\lambda\,; a)$.

Proposition 4.3.1. *Let $0 < a \leq 1$ be a rational number, $x \in (W^J)_{\mathrm{af}}$, and $\beta \in \Delta_{\mathrm{af}}^+$. Then, $x \xrightarrow{\beta} r_\beta x$ in $\mathrm{SB}(\lambda\,; a)$ if and only if $x\lambda \xrightarrow{\beta} r_\beta x\lambda$ in $\mathrm{LP}(\lambda\,; a)$.*

We will show Proposition 4.3.1 "via $\mathrm{QB}(\lambda\,; a)$"; we will give a relation between $\mathrm{QB}(\lambda\,; a)$ and $\mathrm{LP}(\lambda\,; a)$ in Proposition 4.3.3, and a relation between $\mathrm{QB}(\lambda\,; a)$ and $\mathrm{SB}(\lambda\,; a)$ in Proposition 4.3.7.

Lemma 4.3.2 ([NS08, Lemma 2.11])**.** *Let $\mu, \nu \in W_{\mathrm{af}}\lambda$ and $\beta \in \Delta_{\mathrm{af}}^+$. If $\mu \xrightarrow{\beta} r_\beta \mu$ in $\mathrm{LP}(\lambda)$, then $\beta \in \Delta^+ \sqcup \{ -\gamma + \delta \mid \gamma \in \Delta^+ \}$.*

Proposition 4.3.3. *Let $0 < a \leq 1$ be a rational number.*

(1) Let $w \in W^J$, and $\alpha \in \Delta^+$. Assume that $w \xrightarrow[\mathsf{B}]{\alpha} \lfloor wr_\alpha \rfloor$ (resp., $w \xrightarrow[\mathsf{Q}]{\alpha} \lfloor wr_\alpha \rfloor$) in $\mathrm{QB}(\lambda\,; a)$. If we set $\beta := w\alpha$ (resp., $\beta := w\alpha + \delta$), then $\beta \in \Delta_{\mathrm{af}}^+$, and $wz_\xi t_\xi \lambda \xrightarrow{\beta} r_\beta wz_\xi t_\xi \lambda$ in $\mathrm{LP}(\lambda\,; a)$ for every $\xi \in Q_{J\text{-adj}}^\vee$.

(2) *Let* $w \in W^J$, *and* $\xi \in Q^\vee_{J\text{-adj}}$. *Assume that* $wz_\xi t_\xi \lambda \xrightarrow{\beta} r_\beta w z_\xi t_\xi \lambda$ *in* $\mathrm{LP}(\lambda\,;a)$; *recall from Lemma 4.3.2 that* $\beta \in \Delta^+ \sqcup \{-\gamma + \delta \mid \gamma \in \Delta^+\}$. *Set* $\alpha := w^{-1}\beta$ *(resp.,* $\alpha := w^{-1}(\beta - \delta)$*) if* $\beta \in \Delta^+$ *(resp.,* $\beta \in \{-\gamma + \delta \mid \gamma \in \Delta^+\}$*). Then,* $\alpha \in \Delta^+ \setminus \Delta_J^+$, *and* $w \xrightarrow[\mathsf{B}]{\alpha} \lfloor w r_\alpha \rfloor$ *(resp.,* $w \xrightarrow[\mathsf{Q}]{\alpha} \lfloor w r_\alpha \rfloor$*) in* $\mathrm{QB}(\lambda\,;a)$.

Proof. (1) By [LNSSS13a, Theorem 6.5], we see that $wz_\xi t_\xi \lambda \xrightarrow{\beta} r_\beta w z_\xi t_\xi \lambda$ in $\mathrm{LP}(\lambda)$ for every $\xi \in Q^\vee_{J\text{-adj}}$. Since $a\langle \alpha^\vee, \lambda\rangle \in \mathbb{Z}$ and $\langle \beta^\vee, wz_\xi t_\xi \lambda\rangle = \langle \alpha^\vee, \lambda\rangle$, we get $a\langle \beta^\vee, wz_\xi t_\xi\lambda\rangle \in \mathbb{Z}$, which implies that the edge $wz_\xi t_\xi \lambda \xrightarrow{\beta} r_\beta w z_\xi t_\xi \lambda$ is in $\mathrm{LP}(\lambda\,;a)$.

(2) By [LNSSS13a, Theorem 6.5], we see that $w \xrightarrow{\alpha} \lfloor w r_\alpha \rfloor$ in QB^J. Since $a\langle \beta^\vee, wz_\xi t_\xi\lambda\rangle \in \mathbb{Z}$ and $\langle \alpha^\vee, \lambda\rangle = \langle \beta^\vee, wz_\xi t_\xi \lambda\rangle$, we get $a\langle \alpha^\vee, \lambda\rangle \in \mathbb{Z}$, which implies that the edge $w \xrightarrow{\alpha} \lfloor w r_\alpha \rfloor$ is in $\mathrm{QB}(\lambda\,;a)$. □

We know the following lemma from [BFP99, Lemma 4.3].

Lemma 4.3.4. *We have* $\ell(r_\alpha) \le 2\langle \alpha^\vee, \rho\rangle - 1$ *for all* $\alpha \in \Delta^+$.

Lemma 4.3.5. *Let* $x = wz_\xi t_\xi \in (W^J)_{\mathrm{af}}$ *with* $w \in W^J$ *and* $\xi \in Q^\vee_{J\text{-adj}}$, *and let* $\beta \in \Delta^+_{\mathrm{af}}$ *be such that* $x \xrightarrow{\beta} r_\beta x$ *in* SB^J. *Write* β *as* $\beta = \gamma + \chi\delta$ *with* $\gamma \in \Delta$ *and* $\chi \in \mathbb{Z}_{\ge 0}$, *and set* $\alpha := w^{-1}\gamma$; *note that* $\beta = w\alpha + \chi\delta$. *Then the following hold.*

(1) $\alpha \in \Delta^+ \setminus \Delta^+_J$,

(2) $\ell(wr_\alpha z_\xi) = \ell(wz_\xi) + 1 - 2\chi\langle z_\xi^{-1}\alpha^\vee, \rho\rangle$,

(3) $\chi \in \{0, 1\}$.

Proof. (1) We first show that $\alpha \notin \Delta_J$. Suppose that $\alpha \in \Delta_J$. By simple computation, we see that

$$x^{-1}\beta = t_{-\xi} z_\xi^{-1} w^{-1}(w\alpha + \chi\delta) = t_{-\xi}(z_\xi^{-1}\alpha + \chi\delta) = z_\xi^{-1}\alpha + n\delta \quad \text{for some } n \in \mathbb{Z}.$$

Because $\alpha \in \Delta_J$ and $z_\xi \in W_J$, it follows that $x^{-1}\beta \in (\Delta_J)_{\mathrm{af}}$, and hence $r_{x^{-1}\beta} \in (W_J)_{\mathrm{af}}$. Therefore, we deduce from Lemma 2.2.2 that $r_\beta x = x r_{x^{-1}\beta}$ is not contained in $(W^J)_{\mathrm{af}}$, which contradicts the assumption that $x \xrightarrow{\beta} r_\beta x$ in SB^J (see condition (i) in Definition 2.3.2). Thus we get $\alpha \notin \Delta_J$.

We next show that $\alpha \in \Delta^+$. Suppose that $-\alpha \in \Delta^+$. Then, $-z_\xi^{-1}\alpha \in \Delta^+$ since $\alpha \notin \Delta_J$. We have

$$\begin{aligned}
\ell^{\frac{\infty}{2}}(r_\beta x) &= \ell^{\frac{\infty}{2}}(w r_\alpha z_\xi t_{\xi + \chi z_\xi^{-1}\alpha^\vee}) \\
&= \ell(w r_\alpha z_\xi) + 2\langle \xi + \chi z_\xi^{-1}\alpha^\vee, \rho\rangle \\
&= \ell(w z_\xi r_{z_\xi^{-1}\alpha}) + 2\langle \xi + \chi z_\xi^{-1}\alpha^\vee, \rho\rangle && (\text{since } w r_\alpha z_\xi = w z_\xi r_{z_\xi^{-1}\alpha}) \\
&\le \ell(w z_\xi) + \ell(r_{z_\xi^{-1}\alpha}) + 2\langle \xi + \chi z_\xi^{-1}\alpha^\vee, \rho\rangle && (\text{see [BB05, Proposition 1.4.2 (v)]}) \\
&\le \ell(w z_\xi) + \bigl(2\langle -z_\xi^{-1}\alpha, \rho\rangle - 1\bigr) + 2\langle \xi + \chi z_\xi^{-1}\alpha^\vee, \rho\rangle && (\text{by Lemma 4.3.4}) \\
&= \ell^{\frac{\infty}{2}}(x) - 1 + 2(\chi - 1)\langle z_\xi^{-1}\alpha^\vee, \rho\rangle.
\end{aligned}$$

Since $\ell^{\frac{\infty}{2}}(r_\beta x) = \ell^{\frac{\infty}{2}}(x) + 1$, we deduce that $(\chi - 1)\langle z_\xi^{-1}\alpha^\vee, \rho\rangle \geq 1$. Because $\langle z_\xi^{-1}\alpha^\vee, \rho\rangle < 0$, we obtain $\chi = 0$, and hence $\beta = w\alpha$. Since $\beta = w\alpha$ is a positive real root by assumption, we get $w\alpha \in \Delta^+$. Because $-\alpha \in \Delta^+$ and $w\alpha \in \Delta^+$, we see that $\ell(wr_\alpha) < \ell(w)$. However we have

$$\begin{aligned}
\ell^{\frac{\infty}{2}}(r_\beta x) &= \ell^{\frac{\infty}{2}}(wr_\alpha z_\xi t_\xi) && \text{(since } \chi = 0\text{)} \\
&= \ell(wr_\alpha z_\xi) + 2\langle \xi, \rho\rangle \\
&\leq \ell(wr_\alpha) + \ell(z_\xi) + 2\langle \xi, \rho\rangle && \text{(see [BB05, Proposition 1.4.2 (v)])} \\
&< \ell(w) + \ell(z_\xi) + 2\langle \xi, \rho\rangle = \ell^{\frac{\infty}{2}}(x) && \text{(since } \ell(wr_\alpha) < \ell(w)\text{)},
\end{aligned}$$

which contradicts $\ell^{\frac{\infty}{2}}(r_\beta x) = \ell^{\frac{\infty}{2}}(x) + 1$. Thus we get $\alpha \in \Delta^+$. This proves part (1).

(2) By simple computation, we have

$$\begin{aligned}
1 &= \ell^{\frac{\infty}{2}}(r_\beta x) - \ell^{\frac{\infty}{2}}(x) && \text{(by condition (2) in Definition 2.3.2)} \\
&= \ell(wr_\alpha z_\xi) + 2\langle \xi + \chi z_\xi^{-1}\alpha^\vee, \rho\rangle - \ell(wz_\xi) - 2\langle \xi, \rho\rangle && \text{(since } r_\beta x = wr_\alpha z_\xi t_{\xi + \chi z_\xi^{-1}\alpha^\vee}\text{)} \\
&= \ell(wr_\alpha z_\xi) - \ell(wz_\xi) + 2\chi\langle z_\xi^{-1}\alpha^\vee, \rho\rangle.
\end{aligned}$$

Thus we have proved part (2).

(3) First, we remark that $z_\xi^{-1}\alpha \in \Delta^+$ since $z_\xi \in W_J$ and $\alpha \in \Delta^+ \setminus \Delta_J^+$ by (1). It follows that

$$\begin{aligned}
1 &= \ell(wr_\alpha z_\xi) - \ell(wz_\xi) + 2\chi\langle z_\xi^{-1}\alpha^\vee, \rho\rangle && \text{(by (2))} \\
&= \ell(wz_\xi r_{z_\xi^{-1}\alpha}) - \ell(wz_\xi) + 2\chi\langle z_\xi^{-1}\alpha^\vee, \rho\rangle && \text{(since } wr_\alpha z_\xi = wz_\xi r_{z_\xi^{-1}\alpha}\text{)} \\
&\geq \big(\ell(wz_\xi) - \ell(r_{z_\xi^{-1}\alpha})\big) - \ell(wz_\xi) + 2\chi\langle z_\xi^{-1}\alpha^\vee, \rho\rangle && \text{(see [BB05, Proposition 1.4.2 (v)])} \\
&= -\ell(r_{z_\xi^{-1}\alpha}) + 2\chi\langle z_\xi^{-1}\alpha^\vee, \rho\rangle \\
&\geq 1 - 2\langle z_\xi^{-1}\alpha^\vee, \rho\rangle + 2\chi\langle z_\xi^{-1}\alpha^\vee, \rho\rangle && \text{(by Lemma 4.3.4)},
\end{aligned}$$

which implies that $\chi \in \{0, 1\}$ since $\langle z_\xi^{-1}\alpha^\vee, \rho\rangle > 0$. This completes the proof of Lemma 4.3.5. □

The next lemma follows from [LS10, Lemma 10.3] and [LNSSS13a, Lemma 3.10].

Lemma 4.3.6. *For $\xi \in Q^\vee_{J\text{-adj}}$, we have $\mathrm{Inv}(z_\xi) = \{\gamma \in \Delta_J^+ \mid \langle \xi, \gamma\rangle = -1\}$ and $\ell(z_\xi) = -2\langle \xi, \rho_J\rangle$.*

Proposition 4.3.7 (cf. [LNSSS13a, Theorem 5.2])**.** *Let $0 < a \leq 1$ be a rational number.*

(1) Let $x = wz_\xi t_\xi \in (W^J)_{\mathrm{af}}$ with $w \in W^J$ and $\xi \in Q^\vee_{J\text{-adj}}$. Assume that $x \xrightarrow{\beta} r_\beta x$ in $\mathrm{SB}(\lambda; a)$ for $\beta \in \Delta^+_{\mathrm{af}}$; by Lemma 4.3.5, $\beta = w\alpha + \chi\delta$ for some $\alpha \in \Delta^+ \setminus \Delta^+_J$ and $\chi \in \{0, 1\}$. If $\chi = 0$ (resp., $\chi = 1$), then $w \xrightarrow[\mathsf{B}]{\alpha} \lfloor wr_\alpha \rfloor$ (resp., $w \xrightarrow[\mathsf{Q}]{\alpha} \lfloor wr_\alpha \rfloor$) in $\mathrm{QB}(\lambda; a)$.

(2) Let $w \in W^J$, and $\alpha \in \Delta^+ \setminus \Delta^+_J$. Assume that $w \xrightarrow[\mathsf{B}]{\alpha} \lfloor wr_\alpha \rfloor$ (resp., $w \xrightarrow[\mathsf{Q}]{\alpha} \lfloor wr_\alpha \rfloor$) in $\mathrm{QB}(\lambda; a)$. Set $\beta := w\alpha$ (resp., $\beta := w\alpha + \delta$). Then, $\beta \in \Delta^+_{\mathrm{af}}$, $r_\beta wz_\xi t_\xi \in (W^J)_{\mathrm{af}}$, and $wz_\xi t_\xi \xrightarrow{\beta} r_\beta wz_\xi t_\xi$ in $\mathrm{SB}(\lambda; a)$ for every $\xi \in Q^\vee_{J\text{-adj}}$.

Proof. (1) We first assume that $\chi = 0$, and hence $\beta = w\alpha$. Because $(W^J)_{\mathrm{af}} \ni r_\beta x = wr_\alpha z_\xi t_\xi$, it follows from (2.2.5) that $wr_\alpha \in W^J$. Hence, $\ell(wr_\alpha z_\xi) = \ell(wr_\alpha) + \ell(z_\xi)$ since $z_\xi \in W_J$. Also, we see from Lemma 4.3.5 (2) that $\ell(wr_\alpha z_\xi) = \ell(wz_\xi) + 1 = \ell(w) + \ell(z_\xi) + 1$. Combining these, we get $\ell(wr_\alpha) = \ell(w) + 1$. Thus, $w \xrightarrow[\mathsf{B}]{\alpha} wr_\alpha$ in $\mathrm{QB}(\lambda; a)$; notice that $a\langle \alpha^\vee, \lambda\rangle = a\langle \beta^\vee, x\lambda\rangle \in \mathbb{Z}$.

We next assume that $\chi = 1$, and hence $\beta = w\alpha + \delta$. Since $(W^J)_{\mathrm{af}} \ni r_\beta x = r_\beta w z_\xi t_\xi = w r_\alpha z_\xi t_{\xi + z_\xi^{-1}\alpha^\vee}$, we get $\xi + z_\xi^{-1}\alpha^\vee \in Q^\vee_{J\text{-adj}}$ by (2.2.5). We have

$$\begin{aligned} w r_\alpha z_\xi t_{\xi + z_\xi^{-1}\alpha^\vee} &= \Pi^J(w r_\alpha z_\xi t_{\xi + z_\xi^{-1}\alpha^\vee}) \\ &= \lfloor w r_\alpha \rfloor z_{\xi + z_\xi^{-1}\alpha^\vee} t_{\xi + z_\xi^{-1}\alpha^\vee} \qquad \text{(by Lemmas 2.2.6 (3))}, \end{aligned}$$

which implies that $w r_\alpha z_\xi = \lfloor w r_\alpha \rfloor z_{\xi + z_\xi^{-1}\alpha^\vee}$, and hence $\ell(w r_\alpha z_\xi) = \ell(\lfloor w r_\alpha \rfloor) + \ell(z_{\xi + z_\xi^{-1}\alpha^\vee})$. Therefore,

$$\begin{aligned} \ell(\lfloor w r_\alpha \rfloor) &= \ell(w r_\alpha z_\xi) - \ell(z_{\xi + z_\xi^{-1}\alpha^\vee}) \\ &= \ell(w z_\xi) + 1 - 2\langle z_\xi^{-1}\alpha^\vee, \rho\rangle - \ell(z_{\xi + z_\xi^{-1}\alpha^\vee}) && \text{(by Lemma 4.3.5 (2))} \\ &= \ell(w) + \ell(z_\xi) + 1 - 2\langle z_\xi^{-1}\alpha^\vee, \rho\rangle - \ell(z_{\xi + z_\xi^{-1}\alpha^\vee}) \\ &= \ell(w) - 2\langle \xi, \rho_J\rangle + 1 - 2\langle z_\xi^{-1}\alpha^\vee, \rho\rangle + 2\langle \xi + z_\xi^{-1}\alpha^\vee, \rho_J\rangle && \text{(by Lemma 4.3.6)} \\ &= \ell(w) + 1 - 2\langle z_\xi^{-1}\alpha^\vee, \rho - \rho_J\rangle \\ &= \ell(w) + 1 - 2\langle \alpha^\vee, \rho - \rho_J\rangle && \text{(by (2.1.7))}, \end{aligned}$$

which implies that $w \xrightarrow[\mathsf{Q}]{\alpha} \lfloor w r_\alpha \rfloor$ in $\mathrm{QB}(\lambda\,;a)$; note that $a\langle \alpha^\vee, \lambda\rangle = a\langle \beta^\vee, x\lambda\rangle \in \mathbb{Z}$.

(2) Set $x := w z_\xi t_\xi$. We first assume that $w \xrightarrow[\mathsf{B}]{\alpha} \lfloor w r_\alpha \rfloor = w r_\alpha \in W^J$ in $\mathrm{QB}(\lambda\,;a)$ (see Remark 4.2.2 (1)); we see by condition (B) in Definition 4.2.1 that $\beta = w\alpha \in \Delta^+ \subset \Delta^+_{\mathrm{af}}$. Since $r_\beta x = w r_\alpha z_\xi t_\xi$ and $w r_\alpha \in W^J$, it follows immediately from 2.2.5 that $r_\beta x \in (W^J)_{\mathrm{af}}$. Also, we have

$$\begin{aligned} \ell^{\frac{\infty}{2}}(r_\beta x) &= \ell(w r_\alpha z_\xi) + 2\langle \xi, \rho\rangle && \text{(since } r_\beta x = w r_\alpha z_\xi t_\xi) \\ &= \ell(w r_\alpha) + \ell(z_\xi) + 2\langle \xi, \rho\rangle && \text{(since } w r_\alpha \in W^J \text{ and } z_\xi \in W_J) \\ &= \ell(w) + 1 + \ell(z_\xi) + 2\langle \xi, \rho\rangle && \text{(since } w \xrightarrow[\mathsf{B}]{\alpha} w r_\alpha) \\ &= \ell(w z_\xi) + 2\langle \xi, \rho\rangle + 1 && \text{(since } w \in W^J \text{ and } z_\xi \in W_J) \\ &= \ell^{\frac{\infty}{2}}(x) + 1. \end{aligned}$$

Therefore, we obtain $x \xrightarrow{\beta} r_\beta x$ in $\mathrm{SB}(\lambda\,;a)$; note that $a\langle \beta^\vee, x\lambda\rangle = a\langle \alpha^\vee, \lambda\rangle \in \mathbb{Z}$.

We next assume that $w \xrightarrow[\mathsf{Q}]{\alpha} \lfloor w r_\alpha \rfloor$ in $\mathrm{QB}(\lambda\,;a)$; note that $\beta = w\alpha + \delta \in \Delta^+_{\mathrm{af}}$. Since $w r_\alpha t_{\alpha^\vee} \in (W^J)_{\mathrm{af}}$ by Remark 4.2.2 (2), it follows from Lemma 2.2.8 that $r_\beta x = w r_\alpha t_{\alpha^\vee} z_\xi t_\xi \in (W^J)_{\mathrm{af}}$. Because

$$\begin{aligned} r_\beta x &= \Pi^J(r_\beta x) && \text{(since } r_\beta x \in (W^J)_{\mathrm{af}}) \\ &= \Pi^J(w r_\alpha z_\xi t_{\xi + z_\xi^{-1}\alpha^\vee}) && \text{(since } r_\beta x = w r_\alpha z_\xi t_{\xi + z_\xi^{-1}\alpha^\vee}) \\ &= \lfloor w r_\alpha \rfloor z_{\xi + z_\xi^{-1}\alpha^\vee} t_{\xi + z_\xi^{-1}\alpha^\vee} && \text{(by Lemma 2.2.6 (3))}, \end{aligned}$$

we deduce that

$$\begin{aligned} \ell^{\frac{\infty}{2}}(r_\beta x) &= \ell^{\frac{\infty}{2}}(\lfloor w r_\alpha \rfloor z_{\xi + z_\xi^{-1}\alpha^\vee} t_{\xi + z_\xi^{-1}\alpha^\vee}) \\ &= \ell(\lfloor w r_\alpha \rfloor z_{\xi + z_\xi^{-1}\alpha^\vee}) + 2\langle \xi + z_\xi^{-1}\alpha^\vee, \rho\rangle \\ &= \ell(\lfloor w r_\alpha \rfloor) + \ell(z_{\xi + z_\xi^{-1}\alpha^\vee}) + 2\langle \xi + z_\xi^{-1}\alpha^\vee, \rho\rangle \end{aligned}$$

$$
\begin{aligned}
&= \ell(w) + 1 - 2\langle \alpha^\vee, \rho - \rho_J \rangle + \ell(z_{\xi + z_\xi^{-1}\alpha^\vee}) + 2\langle \xi + z_\xi^{-1}\alpha^\vee, \rho \rangle && (\text{since } w \xrightarrow[\mathsf{Q}]{\alpha} \lfloor w r_\alpha \rfloor) \\
&= \ell(w) + 1 - 2\langle \alpha^\vee, \rho - \rho_J \rangle - 2\langle \xi + z_\xi^{-1}\alpha^\vee, \rho_J \rangle + 2\langle \xi + z_\xi^{-1}\alpha^\vee, \rho \rangle && (\text{by Lemma 4.3.6}) \\
&= \ell(w) + 1 - 2\langle z_\xi^{-1}\alpha^\vee, \rho - \rho_J \rangle + 2\langle \xi + z_\xi^{-1}\alpha^\vee, \rho - \rho_J \rangle && (\text{by (2.1.7)}) \\
&= \ell(w) + 1 + 2\langle \xi, \rho - \rho_J \rangle \\
&= \ell(w) + 1 + 2\langle \xi, \rho \rangle + \ell(z_\xi) && (\text{by Lemma 4.3.6}) \\
&= \ell^{\frac{\infty}{2}}(x) + 1.
\end{aligned}
$$

Therefore we obtain $x \xrightarrow{\beta} r_\beta x$ in $\mathrm{SB}(\lambda\,;a)$; note that $a\langle \beta^\vee, x\lambda \rangle = a\langle \alpha^\vee, \lambda \rangle \in \mathbb{Z}$. Thus we have proved Proposition 4.3.7. □

Proof of Proposition 4.3.1. Write $x \in (W^J)_{\mathrm{af}}$ as $x = w z_\xi t_\xi$ with $w \in W^J$ and $\xi \in Q^\vee_{J\text{-adj}}$.

Assume first that $x \xrightarrow{\beta} r_\beta x$ in $\mathrm{SB}(\lambda\,;a)$. By Lemma 4.3.5, $\beta = w\alpha + \chi\delta$ for some $\alpha \in \Delta^+ \setminus \Delta_J^+$ and $\chi \in \{0, 1\}$. It follows from Proposition 4.3.7 (1) that $w \xrightarrow[\mathsf{B}]{\alpha} \lfloor w r_\alpha \rfloor$ (resp., $w \xrightarrow[\mathsf{Q}]{\alpha} \lfloor w r_\alpha \rfloor$) in $\mathrm{QB}(\lambda\,;a)$ if $\chi = 0$ (resp., $\chi = 1$). We see from Proposition 4.3.3 (1) that $x\lambda \xrightarrow{\beta} r_\beta x\lambda$ in $\mathrm{LP}(\lambda\,;a)$, as desired.

Assume next that $x\lambda \xrightarrow{\beta} r_\beta x\lambda$ in $\mathrm{LP}(\lambda\,;a)$. By Lemma 4.3.2, $\beta \in \Delta^+ \sqcup \{-\gamma + \delta \mid \gamma \in \Delta^+\}$. Set $\alpha := w^{-1}\beta$ (resp., $\alpha := w^{-1}(\beta - \delta)$) if $\beta \in \Delta^+$ (resp., $\beta \in \{-\gamma + \delta \mid \gamma \in \Delta^+\}$). It follows from Proposition 4.3.7 (2) that $\alpha \in \Delta^+ \setminus \Delta_J^+$, and $w \xrightarrow[\mathsf{B}]{\alpha} \lfloor w r_\alpha \rfloor$ (resp., $w \xrightarrow[\mathsf{Q}]{\alpha} \lfloor w r_\alpha \rfloor$) in $\mathrm{QB}(\lambda\,;a)$. Notice that $\beta = w\alpha$ (resp., $\beta = w\alpha + \delta$). Hence we deduce from Proposition 4.3.3 (2) that $x \xrightarrow{\beta} r_\beta x$ in $\mathrm{SB}(\lambda\,;a)$, as desired. Thus we have proved Proposition 4.3.1. □

4.4 Proof of Theorem 3.1.6

We show part (1) only for e_i; the proof for f_i is similar. Let $\eta = (x_1, \dots, x_s\,; a_0, a_1, \dots, a_s) \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ with $m_i^{\pi_\eta} \le -1$. Define $t_0, t_1 \in [0, 1]$ as (3.1.5), and let $1 \le p \le q \le s$ be such that $a_{p-1} \le t_0 < a_q$ and $t_1 = a_q$; then

$$
\begin{aligned}
e_i\eta = (&x_1, x_2, \dots, x_p, r_i x_p, r_i x_{p+1}, \dots, r_i x_q, x_{q+1}, x_{q+2}, \dots, x_s\,; \\
&a_0, a_1, a_2, \dots, a_{p-1}, t_0, a_p, a_{p+1}, \dots, a_{q-1}, t_1, a_{q+1}, a_{q+2}, \dots, a_{s-1}, a_s);
\end{aligned}
$$

if $t_0 = a_{p-1}$, then we drop x_p and a_{p-1}, and if $r_i x_q = x_{q+1}$, then we drop x_{q+1} and t_1. We need to prove that

(i) $r_i x_u \in (W^J)_{\mathrm{af}}$ for every $p \le u \le q$;

(ii) if $t_0 \ne a_{p-1}$ (resp., $t_0 = a_{p-1}$ and $p > 1$), then there exists a directed path from $r_i x_p$ to x_p (resp., x_{p-1}) in $\mathrm{SB}(\lambda\,;t_0)$;

(iii) for each $p \le u \le q - 1$, there exists a directed path from $r_i x_{u+1}$ to $r_i x_u$ in $\mathrm{SB}(\lambda\,;a_u)$;

(iv) if $r_i x_q \ne x_{q+1}$, then there exists a directed path from x_{q+1} to $r_i x_q$ in $\mathrm{SB}(\lambda\,;t_1) = \mathrm{SB}(\lambda\,;a_q)$.

First, let us show (i). As mentioned in Definition 3.1.5 (1), the function $H_i^{\pi_\eta}(t)$ is strictly decreasing on $[t_0, t_1]$. Thus we see that $\langle \alpha_i^\vee, x_u\lambda \rangle < 0$ for every $p \le u \le q$. In particular, $x_u^{-1}\alpha_i \notin (\Delta_J)_{\mathrm{af}}$. Hence it follows from Lemma 2.2.7 that $r_i x_u \in (W^J)_{\mathrm{af}}$.

Here recall from Proposition 3.1.3 (see also (4.3.1)) that

$$\pi_\eta = (x_1\lambda, \dots, x_s\lambda\,;a_0, a_1, \dots, a_s) \in \mathbb{B}(\lambda).$$

By Definition 4.1.4 (1), we see that

$$\begin{aligned} e_i\pi_\eta := (&x_1\lambda, x_2\lambda, \dots, x_p\lambda, r_ix_p\lambda, r_ix_{p+1}\lambda, \dots, r_ix_q\lambda, x_{q+1}\lambda, x_{q+2}\lambda, \dots, x_s\lambda\,; \\ &a_0, a_1, a_2, \dots, a_{p-1}, t_0, a_p, a_{p+1}, \dots, a_{q-1}, t_1, a_{q+1}, a_{q+2}, \dots, a_{s-1}, a_s) \in \mathbb{B}(\lambda); \end{aligned} \tag{4.4.1}$$

if $t_0 = a_{p-1}$, then we drop $x_p\lambda$ and a_{p-1}, and if $r_ix_q\lambda = x_{q+1}\lambda$, then we drop $x_{q+1}\lambda$ and t_1.

Now, let us show (ii). Since $\langle\alpha_i^\vee, x_p\lambda\rangle < 0$, we have $r_ix_p\lambda \xrightarrow{\alpha_i} x_p\lambda$ in $\mathrm{LP}(\lambda)$. Also, by applying [Lit95, Lemma 4.5 c)] to $\pi_\eta \in \mathbb{B}(\lambda)$ and $t_0 \in [0,1]$, we see that $t_0\langle\alpha_i^\vee, x_p\lambda\rangle \in \mathbb{Z}$, which implies that the edge $r_ix_p\lambda \xrightarrow{\alpha_i} x_p\lambda$ is in $\mathrm{LP}(\lambda\,;t_0)$. Hence it follows from Proposition 4.3.1 that $r_ix_p \xrightarrow{\alpha_i} x_p$ in $\mathrm{SB}(\lambda\,;t_0)$. Thus we have shown (ii) in the case that $t_0 \neq a_{p-1}$. Assume next that $t_0 = a_{p-1}$ and $p > 1$. By assumption, there exists a directed path from x_p to x_{p-1} in $\mathrm{SB}(\lambda\,;a_{p-1}) = \mathrm{SB}(\lambda\,;t_0)$. By concatenating this directed path and $r_ix_p \xrightarrow{\alpha_i} x_p$ obtained above, we get a directed path from r_ix_p to x_{p-1} in $\mathrm{SB}(\lambda\,;t_0)$. Thus we have proved (ii).

Next, let us show (iii). Fix $p \le u \le q-1$. Let

$$x_{u+1} = y_0 \xrightarrow{\beta_1} y_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_l} y_l = x_u$$

be a directed path from x_{u+1} to x_u in $\mathrm{SB}(\lambda, a_u)$. By using Proposition 4.3.1 repeatedly, we obtain a directed path

$$x_{u+1}\lambda = y_0\lambda \xrightarrow{\beta_1} y_1\lambda \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_l} y_l\lambda = x_u\lambda$$

from $x_{u+1}\lambda$ to $x_u\lambda$ in $\mathrm{LP}(\lambda\,;a_u)$. Then we deduce from [Lit95, Proof of Proposition 4.7] and (4.4.1) that $r_i\beta_m \in \Delta_{\mathrm{af}}^+$ for all $1 \le m \le l$, and

$$r_ix_{u+1}\lambda = r_iy_0\lambda \xrightarrow{r_i\beta_1} r_iy_1\lambda \xrightarrow{r_i\beta_2} \cdots \xrightarrow{r_i\beta_l} r_iy_l\lambda = r_ix_u\lambda$$

is a directed path from $r_ix_{u+1}\lambda$ to $r_ix_u\lambda$ in $\mathrm{LP}(\lambda\,;a_u)$. Again by using Proposition 4.3.1 repeatedly, we obtain a directed path

$$r_ix_{u+1} = r_iy_0 \xrightarrow{r_i\beta_1} r_iy_1 \xrightarrow{r_i\beta_2} \cdots \xrightarrow{r_i\beta_l} r_iy_l = r_ix_u$$

from r_ix_{u+1} to r_ix_u in $\mathrm{SB}(\lambda\,;a_u)$, as desired.

Finally we show (iv). As in the proof of (iii), let

$$x_{q+1} = y_0 \xrightarrow{\beta_1} y_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_l} y_l = x_q$$

be a directed path from x_{q+1} to x_q in $\mathrm{SB}(\lambda\,;a_q)$, and let

$$x_{q+1}\lambda = y_0\lambda \xrightarrow{\beta_1} y_1\lambda \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_l} y_l\lambda = x_q\lambda$$

be the corresponding directed path from $x_{q+1}\lambda$ to $x_q\lambda$ in $\mathrm{LP}(\lambda\,;a_q)$. We deduce from the definition of t_1 that $\langle\alpha_i, x_{q+1}\lambda\rangle \ge 0$ and $\langle\alpha_i^\vee, x_q\lambda\rangle < 0$. Set $m := \max\{0 \le k \le l \mid \langle\alpha_i^\vee, y_k\lambda\rangle \ge 0\}$. Then we see from [Lit95, Lemmas 4.1 and 4.3] and their proofs that $\beta_m = \alpha_i$ (and hence $y_{m-1} = r_iy_m$), $\beta_k \neq \alpha_i$ (and hence $r_i\beta_k \in \Delta_+$) for all $m+1 \le k \le l$, and

$$x_{q+1}\lambda = y_0\lambda \xrightarrow{\beta_1} y_1\lambda \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{m-1}} y_{m-1}\lambda = r_iy_m\lambda \xrightarrow{r_i\beta_{m+1}} \cdots \xrightarrow{r_i\beta_l} r_iy_l\lambda = r_ix_q\lambda.$$

is a directed path from $x_{q+1}\lambda$ to $r_i x_q \lambda$ in $\mathrm{LP}(\lambda\,;a_q)$. Again by using Proposition 4.3.1 repeatedly, we obtain a directed path

$$x_{q+1} = y_0 \xrightarrow{\beta_1} y_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{m-1}} y_{m-1} = r_i y_m \xrightarrow{r_i\beta_{m+1}} \cdots \xrightarrow{r_i\beta_l} r_i y_l = r_i x_q$$

from x_{q+1} to $r_i x_q$ in $\mathrm{SB}(\lambda\,;a_q)$, as desired. This proves part (1).

(2) We see from the definitions of the root operators that for each $\eta \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ and $i \in I_{\mathrm{af}}$, $e_i\eta \neq \mathbf{0}$ (resp., $f_i\eta \neq \mathbf{0}$) if and only if $e_i\pi_\eta \neq \mathbf{0}$ (resp., $f_i\pi_\eta \neq \mathbf{0}$). Hence,

$$\varepsilon_i(\eta) = \varepsilon_i(\pi_\eta),\ \varphi_i(\eta) = \varphi_i(\pi_\eta) \ \text{ for all } \eta \in \mathbb{B}^{\frac{\infty}{2}}(\lambda) \text{ and } i \in I_{\mathrm{af}}. \tag{4.4.2}$$

Also, it follows immediately from the definitions that

$$\mathrm{wt}(\eta) = \mathrm{wt}(\pi_\eta) \ \text{ for all } \eta \in \mathbb{B}^{\frac{\infty}{2}}(\lambda). \tag{4.4.3}$$

Thus we deduce that $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$, together with the maps wt, e_i, f_i, $i \in I_{\mathrm{af}}$, and ε_i, φ_i, $i \in I_{\mathrm{af}}$, satisfies the axioms of crystals except that

$$\text{for } \eta_1, \eta_2 \in \mathbb{B}^{\frac{\infty}{2}}(\lambda) \text{ and } i \in I_{\mathrm{af}},\ e_i\eta_1 = \eta_2 \text{ if and only if } \eta_1 = f_i\eta_2. \tag{4.4.4}$$

We give a proof only for the "only if" part; the proof for the "if" part is similar. Define $t_0, t_1 \in [0,1]$ as (3.1.5) for η_1 and $i \in I_{\mathrm{af}}$. Then we deduce that

$$\begin{aligned} t_0 &= \max\{t \in [0,1] \mid H_i^{\pi_{\eta_2}}(t) = m_i^{\pi_{\eta_2}}\}, \\ t_1 &= \min\{t \in [t_0,1] \mid H_i^{\pi_{\eta_2}}(t) = m_i^{\pi_{\eta_2}} + 1\}. \end{aligned}$$

Therefore, we see from the definition of the root operator f_i that $f_i\eta_2 = f_i e_i \eta_1 = \eta_1$. This proves part (2). Thus we have proved Theorem 3.1.6. □

Remark 4.4.1. (1) We see from the definition of the root operators (see also (4.1.2)), and (4.4.2), (4.4.3) that the map $\mathbb{B}^{\frac{\infty}{2}}(\lambda) \to \mathbb{B}(\lambda)$, $\eta \mapsto \pi_\eta$, is a strict morphism of crystals in the sense of [Kas94, §1.5]; in fact, this map is surjective.

(2) By (4.4.2) and [Lit95, Lemma 2.1 c)], we have $\varepsilon_i(\eta) = -m_i^{\pi_\eta}$ and $\varphi_i(\eta) = H_i^{\pi_\eta}(1) - m_i^{\pi_\eta}$ for all $\eta \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ and $i \in I_{\mathrm{af}}$.

5 Proof of Proposition 3.2.2

Throughout this section, we fix $\lambda \in P^+$ and set $J := \{i \in I \mid \langle \alpha_i^\vee, \lambda\rangle = 0\}$.

5.1 Extremal elements in $\mathcal{B}_0(\lambda)$ and $\mathbb{B}_0^{\frac{\infty}{2}}(\lambda)$

We know from [Kas94, §7] that the affine Weyl group W_{af} acts on $\mathcal{B}(\lambda)$ in such a way that

$$S_{r_i} b := \begin{cases} f_i^{\langle \alpha_i^\vee, \mathrm{wt}(b)\rangle} b & \text{if } \langle \alpha_i^\vee, \mathrm{wt}(b)\rangle \geq 0, \\ e_i^{-\langle \alpha_i^\vee, \mathrm{wt}(b)\rangle} b & \text{if } \langle \alpha_i^\vee, \mathrm{wt}(b)\rangle \leq 0 \end{cases} \tag{5.1.1}$$

for each $b \in \mathcal{B}(\lambda)$ and $i \in I_{\mathrm{af}}$.

Proposition 5.1.1 (cf. [Kas02b, Conjecture 5.11]; see also Remark 2.2.1)**.** *It holds that*

$$(W_J)_{\mathrm{af}} = \{x \in W_{\mathrm{af}} \mid S_x u_\lambda = u_\lambda\}.$$

Therefore, the correspondence $(W^J)_{\mathrm{af}} \ni x \mapsto S_x u_\lambda \in \{S_y u_\lambda \mid y \in W_{\mathrm{af}}\}$ *is bijective.*

Proof. Write $\lambda \in P^+$ as $\lambda = \sum_{i \in I} m_i \varpi_i$ with $m_i \in \mathbb{Z}_{\geq 0}$, $i \in I$. Then we know from [BN04, Remark 4.17] (see also [Kas02b, §13]) that there exists an embedding $\Psi : \mathcal{B}_0(\lambda) \hookrightarrow \bigotimes_{i \in I} \mathcal{B}(\varpi_i)^{\otimes m_i}$ of crystals such that $\Psi(u_\lambda) = \bigotimes_{i \in I} u_{\varpi_i}^{\otimes m_i}$. We can show by induction on $\ell(x)$ that $\Psi(S_x u_\lambda) = \bigotimes_{i \in I} (S_x u_{\varpi_i})^{\otimes m_i}$ for all $x \in W_{\mathrm{af}}$. Therefore,

$$\{x \in W_{\mathrm{af}} \mid S_x u_\lambda = u_\lambda\} = \bigcap_{i \in I \setminus J} \{x \in W_{\mathrm{af}} \mid S_x u_{\varpi_i} = u_{\varpi_i}\}. \tag{5.1.2}$$

We know from [Kas02b, Lemma 5.6] that

$$\{x \in W_{\mathrm{af}} \mid S_x u_{\varpi_i} = u_{\varpi_i}\} = \langle r_\beta \mid \beta \in \Delta_{\mathrm{af}}^+ \text{ with } \langle \beta^\vee, \varpi_i \rangle = 0 \rangle. \tag{5.1.3}$$

Notice that if $\beta \in (\Delta_J)_{\mathrm{af}}^+$, then $\langle \beta^\vee, \varpi_i \rangle = 0$ for every $i \in I \setminus J$, and hence $r_\beta \in \{x \in W_{\mathrm{af}} \mid S_x u_{\varpi_i} = u_{\varpi_i}\}$ for every $i \in I \setminus J$. Because $(W_J)_{\mathrm{af}} = \langle r_\beta \mid \beta \in (\Delta_J)_{\mathrm{af}}^+ \rangle$ by Remark 2.2.1, it follows immediately that $(W_J)_{\mathrm{af}} \subset \{x \in W_{\mathrm{af}} \mid S_x u_\lambda = u_\lambda\}$.

Let us prove the opposite inclusion. Let $x \in W_{\mathrm{af}}$ be such that $S_x u_\lambda = u_\lambda$. By (5.1.2), $S_x u_{\varpi_i} = u_{\varpi_i}$ for all $i \in I \setminus J$; in particular, $x\varpi_i = \varpi_i$ for all $i \in I \setminus J$ since the weight of $S_x u_{\varpi_i}$ is equal to $x\varpi_i$. Write x as $x = wt_\xi$ with $w \in W$ and $\xi \in Q^\vee$. Then, for every $i \in I \setminus J$, $\varpi_i = x\varpi_i = w\varpi_i - \langle \xi, \varpi_i \rangle \delta$ by (2.1.4). Hence, $w\varpi_i = \varpi_i$ and $\langle \xi, \varpi_i \rangle = 0$ for every $i \in I \setminus J$. Therefore, we see that $w \in W_J \subset (W_J)_{\mathrm{af}}$, and $\xi \in Q_J^\vee$, which implies that $t_\xi \in \langle t_{\alpha_j^\vee} \mid j \in J \rangle \subset (W_J)_{\mathrm{af}}$. Thus we obtain $x \in (W_J)_{\mathrm{af}}$, and hence $(W_J)_{\mathrm{af}} \supset \{x \in W_{\mathrm{af}} \mid S_x u_\lambda = u_\lambda\}$. This completes the proof of Proposition 5.1.1. □

For $x \in (W^J)_{\mathrm{af}}$, we set $u_x := S_x u_\lambda \in \mathcal{B}_0(\lambda)$; note that $S_y u_\lambda = u_{\Pi^J(y)}$ for $y \in W_{\mathrm{af}}$. We see from [Kas94, §8] that

$$\mathrm{wt}(u_x) = x\lambda, \qquad \varepsilon_i(u_x) = \max\{0, -\langle \alpha_i^\vee, x\lambda \rangle\}, \qquad \varphi_i(u_x) = \max\{0, \langle \alpha_i^\vee, x\lambda \rangle\} \tag{5.1.4}$$

for all $x \in (W^J)_{\mathrm{af}}$ and $i \in I_{\mathrm{af}}$.

Now, $\eta_x := (x\,; 0, 1)$ is an element of $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ for all $x \in (W^J)_{\mathrm{af}}$. By Remark 4.4.1 (2),

$$\mathrm{wt}(\eta_x) = x\lambda, \qquad \varepsilon_i(\eta_x) = \max\{0, -\langle \alpha_i^\vee, x\lambda \rangle\}, \qquad \varphi_i(\eta_x) = \max\{0, \langle \alpha_i^\vee, x\lambda \rangle\} \tag{5.1.5}$$

for all $x \in (W^J)_{\mathrm{af}}$ and $i \in I_{\mathrm{af}}$. For $x \in (W^J)$ and $i \in I_{\mathrm{af}}$, we defiine $S_{r_i}\eta_x \neq \mathbf{0}$ by

$$S_{r_i}\eta_x := \begin{cases} f_i^{\langle \alpha_i^\vee, x\lambda \rangle} \eta_x & \text{if } \langle \alpha_i^\vee, x\lambda \rangle \geq 0, \\ e_i^{-\langle \alpha_i^\vee, x\lambda \rangle} \eta_x & \text{if } \langle \alpha_i^\vee, x\lambda \rangle \leq 0. \end{cases}$$

Then,

$$S_{r_i}\eta_x = \eta_{\Pi^J(r_i x)} = \begin{cases} \eta_{r_i x} & \text{if } \langle \alpha_i^\vee, x\lambda \rangle \neq 0, \\ \eta_x & \text{if } \langle \alpha_i^\vee, x\lambda \rangle = 0. \end{cases} \tag{5.1.6}$$

Indeed, if $\langle \alpha_i^\vee, x\lambda \rangle = 0$, then it is obvious that $S_{r_i}\eta_x = \eta_x$. Also, we see that $x^{-1}\alpha_i \in (\Delta_J)_{\mathrm{af}}$, and hence $r_{x^{-1}\alpha_i} \in (W_J)_{\mathrm{af}}$. Thus, $\Pi^J(r_i x) = \Pi^J(x r_{x^{-1}\alpha_i}) = x$ since $x \in (W^J)_{\mathrm{af}}$. Assume that

$n := \langle \alpha_i^\vee, x\lambda \rangle > 0$; we see that $x^{-1}\alpha_i \notin (\Delta_J)_{\mathrm{af}}$, and hence $r_i x \in (W^J)_{\mathrm{af}}$ by Lemma 2.2.7. It can be easily seen by induction on k that

$$f_i^k \eta_x = (r_i x, x\,; 0, k/n, 1) \quad \text{for } 0 \le k \le n;$$

in particular, we get $f_i^n \eta_x = \eta_{r_i x} = \eta_{\Pi^J(r_i x)}$, as desired. The proof for the case that $\langle \alpha_i^\vee, x\lambda \rangle < 0$ is similar. Since $\Pi^J(x_1 \Pi^J(x_2)) = \Pi^J(x_1 x_2)$ for all $x_1, x_2 \in W_{\mathrm{af}}$, we deduce, by using (5.1.6) repeatedly, that

$$S_{r_{i_1}} S_{r_{i_2}} \cdots S_{r_{i_l}} \eta_x = \eta_{\Pi^J(r_{i_1} r_{i_2} \cdots r_{i_l} x)} \tag{5.1.7}$$

for every $i_1, i_2, \ldots, i_l \in I_{\mathrm{af}}$ and $x \in (W^J)_{\mathrm{af}}$. For $y \in W_{\mathrm{af}}$, we define $S_y := S_{r_{i_1}} S_{r_{i_2}} \cdots S_{r_{i_l}}$ if $y = r_{i_1} r_{i_2} \cdots r_{i_l}$; we see by (5.1.7) that S_y does not depend on the choice of an expression $y = r_{i_1} r_{i_2} \cdots r_{i_l}$ of y. Thus we get an action of W_{af} on the set $\{\eta_x \mid x \in (W^J)_{\mathrm{af}}\}$.

5.2 N-multiple maps

Proposition 5.2.1. *Let $N \in \mathbb{Z}_{>0}$. There exists a unique injective map $\sigma_N : \mathcal{B}_0(\lambda) \hookrightarrow \mathcal{B}_0(\lambda)^{\otimes N}$ such that $\sigma_N(u_\lambda) = u_\lambda^{\otimes N}$, and*

$$\mathrm{wt}(\sigma_N(b)) = N\mathrm{wt}(b), \qquad \varepsilon_i(\sigma_N(b)) = N\varepsilon_i(b), \qquad \varphi_i(\sigma_N(b)) = N\varphi_i(b), \tag{5.2.1}$$

$$\sigma_N(e_i b) = e_i^N \sigma_N(b), \qquad \sigma_N(f_i b) = f_i^N \sigma_N(b). \tag{5.2.2}$$

for $b \in \mathcal{B}_0(\lambda)$ and $i \in I_{\mathrm{af}}$. Here, we understand that $\sigma_N(\mathbf{0}) = \mathbf{0}$.

Proof. Write λ as $\lambda = \sum_{i \in I} m_i \varpi_i$, and let $N \in \mathbb{Z}_{>0}$. By [NS03, Theorem 3.7], there exists an injective map $\iota_1 : \mathcal{B}_0(\lambda) \hookrightarrow \mathcal{B}_0(N\lambda)$ such that $\iota_1(u_\lambda) = u_{N\lambda}$, and

$$\mathrm{wt}(\iota_1(b)) = N\mathrm{wt}(b), \qquad \varepsilon_i(\iota_1(b)) = N\varepsilon_i(b), \qquad \varphi_i(\iota_1(b)) = N\varphi_i(b),$$

$$\iota_1(e_i b) = e_i^N \iota_1(b), \qquad \iota_1(f_i b) = f_i^N \iota_1(b).$$

for $b \in \mathcal{B}_0(\lambda)$ and $i \in I_{\mathrm{af}}$. Also, we deduce from the existence of combinatorial R-matrices (see [Kas02b, §10]) and [BN04, Remark 4.17] that there exist injective maps

$$\iota_2 : \mathcal{B}_0(N\lambda) \hookrightarrow \bigotimes_{i \in I} \mathcal{B}(\varpi_i)^{\otimes N m_i} \quad \text{and} \quad \iota_3 : \mathcal{B}_0(\lambda)^{\otimes N} \hookrightarrow \bigotimes_{i \in I} \mathcal{B}(\varpi_i)^{\otimes N m_i}$$

such that $\iota_2(u_{N\lambda}) = \iota_3(u_\lambda^{\otimes N}) = \bigotimes_{i \in I} u_{\varpi_i}^{\otimes N m_i}$. By the connectedness of $\mathcal{B}_0(N\lambda)$, we deduce that the map ι_2 factors through ι_3, namely, there exists an (injective) map $\iota_4 : \mathcal{B}_0(N\lambda) \to \mathcal{B}_0(\lambda)^{\otimes N}$ such that $\iota_4(u_{N\lambda}) = u_\lambda^{\otimes N}$ and $\iota_2 = \iota_3 \circ \iota_4$. Now, we can easily see that the map $\sigma_N := \iota_4 \circ \iota_1 : \mathcal{B}_0(\lambda) \to \mathcal{B}_0(\lambda)^{\otimes N}$ satisfies the desired conditions. Thus we have proved Proposition 5.2.1. □

We can prove the following proposition in exactly the same way as [NS03, Proposition 3.12] and [Kas02a, Proposition 8.3.2 (3)].

Proposition 5.2.2. *Let $b \in \mathcal{B}_0(\lambda)$. There exists $N_b \in \mathbb{Z}_{>0}$ such that for every multiple $N \in \mathbb{Z}_{>0}$ of N_b,*

$$\sigma_N(b) = u_{x_1} \otimes u_{x_2} \otimes \cdots \otimes u_{x_N} \in \mathcal{B}_0(\lambda)^{\otimes N}$$

for some $x_1, x_2, \ldots, x_N \in (W^J)_{\mathrm{af}}$.

Let $N \in \mathbb{Z}_{>0}$. Notice that the set $\{i \in I \mid \langle \alpha_i^\vee, N\lambda \rangle = 0\}$ is identical to $J = \{i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0\}$. Hence, for every rational number $0 < a \leq 1$, the set of vertices for $\mathrm{SB}(N\lambda\,;a)$ is identical to that for $\mathrm{SB}(\lambda\,;a)$, i.e., $(W^J)_{\mathrm{af}}$. If $x \xrightarrow{\beta} y$ in $\mathrm{SB}(\lambda\,;a)$ for $x, y \in (W^J)_{\mathrm{af}}$ and $\beta \in \Delta_{\mathrm{af}}^+$, then it can be easily seen that $x \xrightarrow{\beta} y$ in $\mathrm{SB}(N\lambda\,;a)$. Hence, $\mathrm{SB}(\lambda\,;a)$ is a subgraph of $\mathrm{SB}(N\lambda\,;a)$.

Now, let $\eta = (x_1, \ldots, x_s\,; a_0, \ldots, a_s) \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$. By the observation above, there exists a directed path from x_{u+1} to x_u in $\mathrm{SB}(N\lambda\,;a_u)$ for each $1 \leq u \leq s-1$, which implies that $\eta \in \mathbb{B}^{\frac{\infty}{2}}(N\lambda)$. Thus we obtain the canonical inclusion

$$\Phi_N : \mathbb{B}^{\frac{\infty}{2}}(\lambda) \hookrightarrow \mathbb{B}^{\frac{\infty}{2}}(N\lambda), \ \eta \mapsto \eta. \tag{5.2.3}$$

Lemma 5.2.3 (cf. [Lit95, Lemma 2.4]). *We have $\Phi_N(\eta_e) = \eta_e$, $\Phi_N(e_i\eta) = e_i^N\Phi_N(\eta)$, and $\Phi_N(f_i\eta) = f_i^N\Phi_N(\eta)$ for all $\eta \in \mathbb{B}_0^{\frac{\infty}{2}}(\lambda)$ and $i \in I_{\mathrm{af}}$. Here, we understand that $\Phi_N(\mathbf{0}) = \mathbf{0}$.*

Proof. It is obvious that $\Phi_N(\eta_e) = \eta_e$. Let us show $\Phi_N(e_i\eta) = e_i^N\Phi_N(\eta)$; the equality $\Phi_N(f_i\eta) = f_i^N\Phi_N(\eta)$ can be shown similarly. For simplicity of notation, we set $\pi = \pi_\eta \in \mathbb{B}(\lambda)$ and $\pi' = \pi_{\Phi_N(\eta)} \in \mathbb{B}(N\lambda)$. We see by definition that

$$\pi'(t) = N\pi(t) \ \text{ for } t \in [0,1],$$

which implies that $H_i^{\pi'}(t) = NH_i^\pi(t)$, and hence $m_i^{\pi'} = Nm_i^\pi$.

If $e_i\eta = \mathbf{0}$, i.e., $m_i^\pi = 0$, then $m_i^{\pi'} = 0$, which implies that $e_i\Phi_N(\eta) = \mathbf{0}$, and hence $e_i^N\Phi_N(\eta) = \mathbf{0}$. Conversely, assume that $e_i^N\Phi_N(\eta) = \mathbf{0}$. Since $\varepsilon_i(\Phi_N(\eta)) = -m_i^{\pi'}$ by Remark 4.4.1 (2), it follows that $-m_i^{\pi'} < N$. Combining this and $m_i^{\pi'} = Nm_i^\pi$, we get $m_i^\pi = 0$, which implies that $e_i\eta = \mathbf{0}$.

Assume that $e_i\eta \neq \mathbf{0}$, or equivalently, $e_i^N\Phi_N(\eta) \neq \mathbf{0}$. For $\pi = \pi_\eta$, define $t_0, t_1 \in [0,1]$ as in (3.1.5); recall that $H_i^\pi(t)$ is strictly decreasing on $[t_0, t_1]$. Also, by Remark 3.1.4, $H_i^\pi(t) \geq m_i^\pi + 1$ for $t \in [0, t_0]$. Since $H_i^{\pi'}(t) = NH_i^\pi(t)$ for $t \in [0,1]$, it follows immediately that

$$t_0 = \min\{t \in [0,1] \mid H_i^{\pi'}(t) = m_i^{\pi'} = Nm_i^\pi\},$$
$$t_1 = \max\{t \in [0,t_0] \mid H_i^{\pi'}(t) = Nm_i^\pi + N\},$$

and that $H_i^{\pi'}(t)$ is strictly decreasing on $[t_0, t_1]$, and $H_i^{\pi'}(t) \geq Nm_i^\pi + N$ for $t \in [0, t_0]$. Hence we deduce from the definition of the root operator e_i that $\Phi_N(e_i\eta) = e_i^N\Phi_N(\eta)$, as desired. Thus we have proved the lemma. □

Let $N \in \mathbb{Z}_{\geq 2}$. We define an injective map $\psi_N : \mathbb{B}^{\frac{\infty}{2}}(N\lambda) \hookrightarrow \mathbb{B}^{\frac{\infty}{2}}(\lambda) \otimes \mathbb{B}^{\frac{\infty}{2}}((N-1)\lambda)$ as follows: let $\eta = (x_1, x_2, \ldots, x_a\,; a_0, a_1, \ldots, a_s) \in \mathbb{B}^{\frac{\infty}{2}}(N\lambda)$. Let $0 \leq p \leq s-1$ be such that $a_p \leq 1/N < a_{p+1}$, and set

$$\eta_1 := (x_1, x_2, \ldots, x_{p+1}\,; Na_0, Na_1, \ldots, Na_p, 1),$$
$$\eta_2 := \left(x_{p+1}, x_{p+2}, \ldots, x_s\,; 0, \frac{Na_{p+1}-1}{N-1}, \frac{Na_{p+2}-1}{N-1}, \ldots, \frac{Na_s - 1}{N-1} = 1\right);$$

if $a_p = 1/N$, i.e., $Na_p = 1$, then we drop x_{p+1} and 1 from η_1. It can be verified that $\eta_1 \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ and $\eta_2 \in \mathbb{B}^{\frac{\infty}{2}}((N-1)\lambda)$. So we define

$$\psi_N(\eta) := \eta_1 \otimes \eta_2 \in \mathbb{B}^{\frac{\infty}{2}}(\lambda) \otimes \mathbb{B}^{\frac{\infty}{2}}((N-1)\lambda).$$

By convention, we set $\psi_N(\mathbf{0}) := \mathbf{0}$.

Remark 5.2.4. Keep the notation above. Then we see that $\pi_\eta \in \mathbb{B}(N\lambda)$ is a "concatenation at $t = 1/N$" of $\pi_{\eta_1} \in \mathbb{B}(\lambda)$ and $\pi_{\eta_2} \in \mathbb{B}((N-1)\lambda)$, that is,

$$\pi_\eta(t) = \begin{cases} \pi_{\eta_1}(Nt) & \text{for } t \in [0, 1/N], \\ \pi_{\eta_1}(1) + \pi_{\eta_2}((N-1)^{-1}(Nt-1)) & \text{for } t \in [1/N, 1]. \end{cases}$$

Proposition 5.2.5. *The map $\psi_N : \mathbb{B}^{\frac{\infty}{2}}(\lambda) \hookrightarrow \mathbb{B}^{\frac{\infty}{2}}(\lambda) \otimes \mathbb{B}^{\frac{\infty}{2}}((N-1)\lambda)$ is a strict embedding of crystals in the sense of [Kas94, §1.5].*

Proof. We only show that $\psi_N(f_i\eta) = f_i\psi_N(\eta)$ for $\eta \in \mathbb{B}^{\frac{\infty}{2}}(N\lambda)$ and $i \in I_{\mathrm{af}}$; the proofs for the other conditions are similar or easier. Let $\eta \in \mathbb{B}^{\frac{\infty}{2}}(N\lambda)$, and assume that $\psi_N(\eta) = \eta_1 \otimes \eta_2$ with $\eta_1 \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ and $\eta_2 \in \mathbb{B}^{\frac{\infty}{2}}((N-1)\lambda)$. We deduce from the tensor product rule of crystals and Remark 4.4.1 (2) that

$$f_i(\eta_1 \otimes \eta_2) = \begin{cases} (f_i\eta_1) \otimes \eta_2 & \text{if } H_i^{\pi_{\eta_1}}(1) - m_i^{\pi_{\eta_1}} > -m_i^{\pi_{\eta_2}}, \\ \eta_1 \otimes (f_i\eta_2) & \text{otherwise.} \end{cases}$$

Assume that $H_i^{\pi_{\eta_1}}(1) - m_i^{\pi_{\eta_1}} > -m_i^{\pi_{\eta_2}}$. Since $-m_i^{\pi_{\eta_2}} \geq 0$, we see by definition that $f_i\eta_1 \neq \mathbf{0}$. Also, since $m_i^{\pi_{\eta_1}} < H_i^{\pi_{\eta_1}}(1) + m_i^{\pi_{\eta_2}}$, it follow immediately from Remark 5.2.4 that $H_i^{\pi_\eta}(t) > m_i^{\pi_{\eta_1}}$ for $t \in [1/N, 1]$. Hence, $m_i^{\pi_\eta} = m_i^{\pi_{\eta_1}}$, and

$$\begin{aligned} H_i^{\pi_\eta}(1) - m_i^{\pi_\eta} &\geq H_i^{\pi_{\eta_1}}(1) + H_i^{\pi_{\eta_2}}(1) - m_i^{\pi_\eta} && \text{(by Remark 5.2.4)} \\ &= H_i^{\pi_{\eta_1}}(1) + H_i^{\pi_{\eta_2}}(1) - m_i^{\pi_{\eta_1}} \\ &\geq H_i^{\pi_{\eta_1}}(1) + m_i^{\pi_{\eta_2}} - m_i^{\pi_{\eta_1}} > 0. \end{aligned}$$

Thus we get $f_i\eta \neq \mathbf{0}$ by definition. For the η, define $t_0, t_1 \in [0,1]$ as (3.1.6). Because $H_i^{\pi_\eta}(t) > m_i^{\pi_{\eta_1}}$ for $t \in [1/N, 1]$ as seen above, it follows immediately from Remark 5.2.4 that $H_i^{\pi_\eta}(t) \geq m_i^{\pi_{\eta_1}} + 1$ for $t \in [1/N, 1]$. Hence we get $t_0, t_1 \in [0, 1/N]$. We deduce from the definitions of the root operator f_i and the map ψ_N that $\psi_N(f_i\eta) = (f_i\eta_1) \otimes \eta_2 = f_i(\eta_1 \otimes \eta_2)$. Similarly, we can verify that if $H_i^{\pi_{\eta_1}}(1) - m_i^{\pi_{\eta_1}} \leq -m_i^{\pi_{\eta_2}}$, then $\psi_N(f_i\eta) = \eta_1 \otimes (f_i\eta_2) = f_i(\eta_1 \otimes \eta_2)$. Thus we have proved that $\psi_N(f_i\eta) = f_i\psi_N(\eta)$ for $\eta \in \mathbb{B}^{\frac{\infty}{2}}(N\lambda)$ and $i \in I_{\mathrm{af}}$, as desired. □

For each $N \in \mathbb{Z}_{>0}$, define a strict embedding $\Psi_N : \mathbb{B}^{\frac{\infty}{2}}(N\lambda) \hookrightarrow \mathbb{B}^{\frac{\infty}{2}}(\lambda)^{\otimes N}$ of crystals by

$$\Psi_1 := \mathrm{id}_{\mathbb{B}^{\frac{\infty}{2}}(\lambda)}, \quad \Psi_N := (\mathrm{id}_{\mathbb{B}^{\frac{\infty}{2}}(\lambda)} \otimes \Psi_{N-1}) \circ \psi_N \text{ for } N \geq 2,$$

and then define

$$\sigma_N := \Psi_N \circ \Phi_N : \mathbb{B}^{\frac{\infty}{2}}(\lambda) \hookrightarrow \mathbb{B}^{\frac{\infty}{2}}(\lambda)^{\otimes N}. \tag{5.2.4}$$

We see that this map σ_N has the following properties: $\sigma_N(\eta_e) = \eta_e^{\otimes N}$, and

$$\mathrm{wt}(\sigma_N(\eta)) = N\mathrm{wt}(\eta), \qquad \varepsilon_i(\sigma_N(\eta)) = N\varepsilon_i(\eta), \qquad \varphi_i(\sigma_N(\eta)) = N\varphi_i(\eta), \tag{5.2.5}$$

$$\sigma_N(e_i\eta) = e_i^N\sigma_N(\eta), \qquad \sigma_N(f_i\eta) = f_i^N\sigma_N(\eta) \tag{5.2.6}$$

for $\eta \in \mathbb{B}_0^{\frac{\infty}{2}}(\lambda)$ and $i \in I_{\mathrm{af}}$. Now, the following lemma can be easily shown by induction on $N \in \mathbb{Z}_{>0}$.

Lemma 5.2.6. *Let $\eta = (x_1, \dots, x_s\,; a_0, a_1, \dots, a_s) \in \mathbb{B}^{\frac{\infty}{2}}(N\lambda)$. If $k_u := Na_u \in \mathbb{Z}$ for all $0 \le u \le s$, then*

$$\Psi_N(\eta) = \underbrace{\eta_{x_1} \otimes \cdots \otimes \eta_{x_1}}_{(k_1 - k_0) \text{ times}} \otimes \underbrace{\eta_{x_2} \otimes \cdots \otimes \eta_{x_2}}_{(k_2 - k_1) \text{ times}} \otimes \cdots \otimes \underbrace{\eta_{x_s} \otimes \cdots \otimes \eta_{x_s}}_{(k_s - k_{s-1}) \text{ times}}.$$

Since $\langle c, \lambda \rangle = 0$, we see that

$$\{\langle \beta^\vee, \lambda \rangle \mid \beta \in \Delta_{\mathrm{af}}\} = \{\langle \alpha^\vee, \lambda \rangle \mid \alpha \in \Delta\}$$

is a finite set. Define $N_\lambda \in \mathbb{Z}_{>0}$ to be the least common multiple of the finite set $\{\langle \beta^\vee, \lambda \rangle \mid \beta \in \Delta_{\mathrm{af}}\} \setminus \{0\}$.

Lemma 5.2.7. *Let $N \in \mathbb{Z}_{>0}$ be a multiple of N_λ. For every $\eta = (x_1, \dots, x_s\,; a_0, a_1, \dots, a_s) \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$, it holds that $k_u := Na_u \in \mathbb{Z}$ for all $0 \le u \le s$, and*

$$\sigma_N(\eta) = \underbrace{\eta_{x_1} \otimes \cdots \otimes \eta_{x_1}}_{(k_1 - k_0) \text{ times}} \otimes \underbrace{\eta_{x_2} \otimes \cdots \otimes \eta_{x_2}}_{(k_2 - k_1) \text{ times}} \otimes \cdots \otimes \underbrace{\eta_{x_s} \otimes \cdots \otimes \eta_{x_s}}_{(k_s - k_{s-1}) \text{ times}}.$$

Proof. We show that $k_u = Na_u \in \mathbb{Z}$ for all $0 \le u \le s$. If $u = 0$ or s, then the assertion is obvious. Let $1 \le u \le s-1$; by the definition of a $\frac{\infty}{2}$-LS path, there exists a directed path from x_{u+1} to x_u in $\mathrm{SB}(\lambda\,; a_u)$. Let $x \xrightarrow{\beta} y$ be an edge in $\mathrm{SB}(\lambda\,; a_u)$. Then, $a_u\langle x^{-1}\beta^\vee, \lambda \rangle \in \mathbb{Z} \setminus \{0\}$. Indeed, since the edge is in $\mathrm{SB}(\lambda\,; a_u)$, we have $a_u\langle x^{-1}\beta^\vee, \lambda \rangle = a_u\langle \beta^\vee, x\lambda \rangle \in \mathbb{Z}$ by the definition. Suppose that $\langle x^{-1}\beta^\vee, \lambda \rangle = 0$. Then, $x^{-1}\beta \in (\Delta_J)_{\mathrm{af}}$, and hence $r_{x^{-1}\beta} \in (W_J)_{\mathrm{af}}$ by Remark 2.2.1. Since $x \in (W^J)_{\mathrm{af}}$, $y = r_\beta x = x r_{x^{-1}\beta} \notin (W^J)_{\mathrm{af}}$, which is a contradiction. Thus we have shown that $a_u\langle x^{-1}\beta^\vee, \lambda \rangle \in \mathbb{Z} \setminus \{0\}$. Therefore we see by the assumption on N that $k_u = Na_u \in \mathbb{Z}$. Since $\sigma_N(\eta) = \Psi_N(\Phi_N(\eta))$, the assertion of the lemma follows from Lemma 5.2.6 □

5.3 Proof of Proposition 3.2.2

Lemma 5.3.1. *Let $X = g_p g_{p-1} \cdots g_2 g_1$ be a monomial in the Kashiwara operators, where $g_q \in \{e_i, f_i \mid i \in I_{\mathrm{af}}\}$ for each $1 \le q \le p$.*

(1) Assume that $Xu_\lambda \ne \mathbf{0}$. Then, $X\eta_e \ne \mathbf{0}$. Further, take $N \in \mathbb{Z}_{>0}$ such that the element $\sigma_N(g_q g_{q-1} \cdots g_2 g_1 u_\lambda)$ is a tensor product of N elements in $\{u_x \mid x \in (W^J)_{\mathrm{af}}\}$ for each $0 \le q \le p$ (see Proposition 5.2.2). Write $\sigma_N(Xu_\lambda) = \sigma_N(g_p g_{p-1} \cdots g_2 g_1 u_\lambda)$ as $\sigma_N(Xu_\lambda) = u_{x_1} \otimes \cdots \otimes u_{x_N}$ with $x_1, \dots, x_N \in (W^J)_{\mathrm{af}}$. Then, $\sigma_N(X\eta_e) = \eta_{x_1} \otimes \cdots \otimes \eta_{x_N}$.

(2) Assume that $X\eta_e \ne \mathbf{0}$. Then, $Xu_\lambda \ne \mathbf{0}$. Further, let $N \in \mathbb{Z}_{>0}$ be a multiple of N_λ (see Lemma 5.2.7), and write $\sigma_N(X\eta_e)$ as $\sigma_N(X\eta_e) = \eta_{x_1} \otimes \cdots \otimes \eta_{x_N}$ with some $x_1, \dots, x_N \in (W^J)_{\mathrm{af}}$. Then, $\sigma_N(Xu_\lambda) = u_{x_1} \otimes \cdots \otimes u_{x_N}$.

Proof. We give a proof only for part (1); the proof for part (2) is similar. Let us show part (1) by induction on p. If $p = 0$, then the assertion is obvious. Assume that $p > 0$. Set $Y := g_{p-1} \cdots g_2 g_1$; note that $Yu_\lambda \ne \mathbf{0}$. Write $\sigma_N(Yu_\lambda)$ as $\sigma_N(Yu_\lambda) = u_{y_1} \otimes \cdots \otimes u_{y_N}$ with some $y_1, \dots, y_N \in (W^J)_{\mathrm{af}}$. Then, by the induction hypothesis, $Y\eta_e \ne \mathbf{0}$, and $\sigma_N(Y\eta_e) = \eta_{y_1} \otimes \cdots \otimes \eta_{y_N}$. Here, we should recall from (5.1.4) and (5.1.5) that $\mathrm{wt}(u_x) = \mathrm{wt}(\eta_x)$, $\varepsilon_i(u_x) = \varepsilon_i(\eta_x)$, $\varphi_i(u_x) = \varphi_i(\eta_x)$ for all $x \in (W^J)_{\mathrm{af}}$ and $i \in I_{\mathrm{af}}$. Thus it follows from the tensor product rule of crystals that

$\mathrm{wt}(\sigma_N(Yu_\lambda)) = \mathrm{wt}(\sigma_N(Y\eta_e))$, $\varepsilon_i(\sigma_N(Yu_\lambda)) = \varepsilon_i(\sigma_N(Y\eta_e))$, $\varphi_i(\sigma_N(Yu_\lambda)) = \varphi_i(\sigma_N(Y\eta_e))$ for all $i \in I_{\mathrm{af}}$. Hence, by (5.2.1) and (5.2.5), we have $\varepsilon_i(Yu_\lambda) = \varepsilon_i(Y\eta_e)$ and $\varphi_i(Yu_\lambda) = \varphi_i(Y\eta_e)$ for all $i \in I_{\mathrm{af}}$. Thus, $Xu_\lambda = g_p Y u_\lambda \neq \mathbf{0}$ implies that $X\eta_e = g_p Y\eta_e \neq \mathbf{0}$. Moreover, $\sigma_N(Xu_\lambda)$ is of the form

$$\begin{aligned}\sigma_N(Xu_\lambda) &= \sigma_N(g_p Y u_\lambda) = g_p^N \sigma_N(Yu_\lambda) && \text{(by (5.2.2))}\\ &= g_p^N(u_{y_1} \otimes \cdots \otimes u_{y_N})\\ &= g_p^{n_1} u_{y_1} \otimes \cdots \otimes g_p^{n_N} u_{y_N}\end{aligned}$$

for some $n_1, \dots, n_N \in \mathbb{Z}_{\geq 0}$ with $n_1 + \cdots + n_N = N$. Then we deduce from (5.2.1), (5.2.5), and the tensor product rule of crystals, together with the equalities $\mathrm{wt}(u_x) = \mathrm{wt}(\eta_x)$, $\varepsilon_i(u_x) = \varepsilon_i(\eta_x)$, $\varphi_i(u_x) = \varphi_i(\eta_x)$ for all $x \in (W^J)_{\mathrm{af}}$ and $i \in I$, that

$$\begin{aligned}\sigma_N(X\eta_e) &= g_p^N(\eta_{y_1} \otimes \cdots \otimes \eta_{y_N}) && \text{(by (5.2.6))}\\ &= g_p^{n_1}\eta_{y_1} \otimes \cdots \otimes g_p^{n_N}\eta_{y_N}.\end{aligned}$$

Because $\sigma_N(Xu_\lambda) = g_p^{n_1} u_{y_1} \otimes \cdots \otimes g_p^{n_N} u_{y_N} = u_{x_1} \otimes \cdots \otimes u_{x_N}$ by assumption, we deduce that $g_p^{n_u}\eta_{y_u} = \eta_{x_u}$ for each $1 \leq u \leq N$. Thus we have proved part (1) □

Proof of Proposition 3.2.2. It suffices to show that the following hold for monomials X, Y in the Kashiwara operators (cf. [Kas96, Proof of Theorem 4.1] and [NS03, Proof of Theorem 5.1]):

(1) $Xu_\lambda \neq \mathbf{0}$ in $\mathcal{B}_0(\lambda)$ if and only if $X\eta_e \neq \mathbf{0}$ in $\mathbb{B}_0^{\frac{\infty}{2}}(\lambda)$,

(2) $Xu_\lambda = Yu_\lambda$ in $\mathcal{B}_0(\lambda)$ if and only if $X\eta_e = Y\eta_e$ in $\mathbb{B}_0^{\frac{\infty}{2}}(\lambda)$.

Assertion (1) has already been proved in Lemma 5.3.1. Let us show assertion (2). We first assume that $Xu_\lambda = Yu_\lambda \neq \mathbf{0}$. By Lemma 5.3.1 (1), we have $X\eta_e \neq \mathbf{0}$ and $Y\eta_e \neq \mathbf{0}$. Take $N \in \mathbb{Z}_{>0}$ such that the assumption of Lemma 5.3.1 (1) holds for both of Xu_λ and Yu_λ; write $\sigma_N(Xu_\lambda)$ and $\sigma_N(Yu_\lambda)$ as

$$\sigma_N(Xu_\lambda) = u_{x_1} \otimes u_{x_2} \otimes \cdots \otimes u_{x_N}, \qquad \sigma_N(Yu_\lambda) = u_{y_1} \otimes u_{y_2} \otimes \cdots \otimes u_{y_N}$$

with some $x_1, \dots, x_N \in (W^J)_{\mathrm{af}}$ and $y_1, \dots, y_N \in (W^J)_{\mathrm{af}}$. Then, by Lemma 5.3.1 (1),

$$\sigma_N(X\eta_e) = \eta_{x_1} \otimes \eta_{x_2} \otimes \cdots \otimes \eta_{x_N}, \qquad \sigma_N(Y\eta_e) = \eta_{y_1} \otimes \eta_{y_2} \otimes \cdots \otimes \eta_{y_N}.$$

Since $Xu_\lambda = Yu_\lambda$, we have $x_u = y_u$ for all $1 \leq u \leq N$. Therefore, we obtain $\sigma_N(X\eta_e) = \sigma_N(Y\eta_e)$, and hence $X\eta_e = Y\eta_e$ by the injectivity of σ_N. Thus we have proved the "only if" part of assertion (2). The "if" part of assertion (2) can be shown similarly; use Lemma 5.3.1 (2) instead of Lemma 5.3.1 (1). This completes the proof of Proposition 3.2.2. □

6 Proof of Proposition 3.2.4

Throughout this section, we fix $\lambda \in P^+$ and set $J := \{i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0\} \subset I$.

6.1 Directed paths from e in $\mathrm{QB}(\lambda\,;a)$

For a rational number $0 < a \le 1$, set

$$I(\lambda\,;a) := \{i \in I \mid a\langle \alpha_i^\vee, \lambda\rangle \in \mathbb{Z}\} \subset I, \tag{6.1.1}$$

$$\lfloor W_{I(\lambda\,;a)} \rfloor := \{\lfloor v \rfloor \mid v \in W_{I(\lambda\,;a)} = \langle r_i \mid i \in I(\lambda\,;a)\rangle\} \subset W^J; \tag{6.1.2}$$

note that $J \subset I(\lambda\,;a)$. Also, we set

$$W^J_{e\to}(\lambda\,;a) := \{w \in W^J \mid \text{there exists a directed path from } e \text{ to } w \text{ in } \mathrm{QB}(\lambda\,;a)\}. \tag{6.1.3}$$

Proposition 6.1.1. *We have* $\lfloor W_{I(\lambda\,;a)} \rfloor = W^J_{e\to}(\lambda\,;a)$.

In order to prove Proposition 6.1.1, we need some lemmas.

Lemma 6.1.2. *Let $w \in W^J$ be an arbitrary element. Every shortest directed path from e to w in QB^J consists only of Bruhat edges. In particular, the length of a shortest directed path from e to w in QB^J is equal to $\ell(w)$.*

Proof. Let $w \in W^J$. First, observe that there exists a directed path from e to w consisting only of Bruhat edges. Indeed, let $w = r_{i_1} r_{i_2} \cdots r_{i_{l-1}} r_{i_l}$ be a reduced expression of w. Then, $r_{i_k} r_{i_{k+1}} \cdots r_{i_{l-1}} r_{i_l} \in W^J$ for all $1 \le k \le l$, and

$$e \xrightarrow[\mathsf{B}]{\alpha_{i_l}} r_{i_l} \xrightarrow[\mathsf{B}]{r_{i_l}\alpha_{i_{l-1}}} r_{i_{l-1}} r_{i_l} \xrightarrow[\mathsf{B}]{r_{i_l} r_{i_{l-1}}\alpha_{i_{l-2}}} \cdots \xrightarrow[\mathsf{B}]{r_{i_l} r_{i_{l-1}}\cdots r_{i_2}\alpha_{i_1}} r_{i_1} \cdots r_{i_{l-1}} r_{i_l} = w \ \text{ in } \mathrm{QB}^J.$$

Thus the length p of a shortest directed path from e to w in QB^J is less than or equal to $l = \ell(w)$.

Let $e = w_0 \to w_1 \to \cdots \to w_p = w$ be a shortest directed path from e to w in QB^J. By the definition of QB^J (see also Remark 4.2.2 (3)), we have

$$\ell(w) = \sum_{q=1}^{p} (\underbrace{\ell(w_q) - \ell(w_{q-1})}_{=1 \text{ or } <0}) \le p. \tag{6.1.4}$$

Because $p \le \ell(w)$ as seen above, we obtain $\ell(w) = p$. The equality in (6.1.4) holds if and only if $\ell(w_q) - \ell(w_{q-1}) = 1$ for all $1 \le q \le p$, or equivalently, all the edges are Bruhat edges. Thus we have proved the lemma. □

We know the following from [LNSSS13b, Lemma 4.1.8 (1)].

Lemma 6.1.3. *Let $w, v \in W^J$, and assume that there exists a directed path*

$$v = v_0 \xrightarrow{\gamma_1} v_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_k} v_k = w \ \textit{in } \mathrm{QB}(\lambda\,;a).$$

Let $i \in I$. If there exists $0 \le m \le k-1$ such that $\langle \alpha_i^\vee, v_n\lambda\rangle < 0$ for all $m+1 \le n \le k$ and $\langle \alpha_i^\vee, v_m\lambda\rangle \ge 0$, then $\lfloor r_i v_{m+1} \rfloor = r_i v_{m+1} = v_m$ and there exists a directed path from v to $\lfloor r_i w \rfloor = r_i w$ of the form

$$v = v_0 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_m} v_m = r_i v_{m+1} \xrightarrow{\gamma_{m+2}} r_i v_{m+2} \xrightarrow{\gamma_{m+3}} \cdots \xrightarrow{\gamma_k} r_i v_k = r_i w \ \textit{in } \mathrm{QB}(\lambda\,;a).$$

Lemma 6.1.4. *If there exists a directed path from $w_1 \in W^J$ to $w_2 \in W^J$ in $\mathrm{QB}(\lambda\,;a)$, then $a(w_2\lambda - w_1\lambda) \in Q$.*

Proof. It suffices to show that if $w_1 \xrightarrow{\alpha} \lfloor w_1 r_\alpha \rfloor = w_2$ in $\mathrm{QB}(\lambda\,;a)$, then $a(w_2\lambda - w_1\lambda) \in Q$. Since $a\langle \alpha^\vee, \lambda \rangle \in \mathbb{Z}$, we have $a(w_2\lambda - w_1\lambda) = a(w_1 r_\alpha \lambda - w_1\lambda) = a\langle \alpha^\vee, \lambda \rangle w_1\alpha \in Q$, as desired. □

Lemma 6.1.5. *If there exists a directed path from e to $w \in W^J$ in $\mathrm{QB}(\lambda\,;a)$, then all shortest directed paths from e to w in QB^J are in $\mathrm{QB}(\lambda\,;a)$.*

Proof. We show the assertion by induction on $\ell(w)$. If $\ell(w) = 0$, i.e., $w = e$, then the assertion is obvious. Assume that $\ell(w) > 0$, and let

$$e = v_0 \xrightarrow{\gamma_1'} v_1 \xrightarrow{\gamma_2'} \cdots \xrightarrow{\gamma_k'} v_k = w$$

be a directed path from e to w in $\mathrm{QB}(\lambda\,;a)$. Take $i \in I$ such that $\langle \alpha_i^\vee, w\lambda \rangle < 0$; note that $r_i w \in W^J$ and $\ell(r_i w) = \ell(w) - 1$. Since $\langle \alpha_i^\vee, e\lambda \rangle = \langle \alpha_i^\vee, \lambda \rangle \geq 0$, it follows from Lemma 6.1.3 that there exists a directed path from e to $r_i w$ of the form:

$$e = v_0 \xrightarrow{\gamma_1'} v_1 \xrightarrow{\gamma_2'} \cdots \xrightarrow{\gamma_m'} v_m = r_i v_{m+1} \xrightarrow{\gamma_{m+2}'} \cdots \xrightarrow{\gamma_k'} r_i v_k = r_i w \;\text{ in } \mathrm{QB}(\lambda\,;a).$$

Therefore, by our induction hypothesis, all the shortest directed paths from e to $r_i w$ in QB^J are in $\mathrm{QB}(\lambda\,;a)$.

Now, let

$$e = w_0 \xrightarrow{\gamma_1} w_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_l} w_l = w \tag{6.1.5}$$

be an arbitrary shortest directed path from e to w in QB^J; note that $l = \ell(w)$ by Lemma 6.1.2. By the same argument as above, there exists a directed path from e to $r_i w$ of length $l-1$ of the form

$$e = w_0 \xrightarrow{\gamma_1} w_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_n} w_n = r_i w_{n+1} \xrightarrow{\gamma_{n+2}} \cdots \xrightarrow{\gamma_l} r_i w_l = r_i w \;\text{ in } \mathrm{QB}^J. \tag{6.1.6}$$

Notice that the directed path (6.1.6) is shortest by Lemma 6.1.2 since $\ell(r_i w) = \ell(w) - 1$. Hence the directed path (6.1.6) is in $\mathrm{QB}(\lambda\,;a)$ by our induction hypothesis. Therefore, $e = w_0 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_n} w_n$ and $w_{n+1} \xrightarrow{\gamma_{n+2}} \cdots \xrightarrow{\gamma_l} w_l = w$ are directed paths in $\mathrm{QB}(\lambda\,;a)$.

It remains to prove that $a\langle \gamma_{n+1}^\vee, \lambda \rangle \in \mathbb{Z}$. Since $w_{n+1} = \lfloor w_n r_{\gamma_{n+1}} \rfloor$, we have

$$\begin{aligned} -a\langle \gamma_{n+1}^\vee, \lambda \rangle w_n \gamma_{n+1} &= a(w_{n+1}\lambda - w_n\lambda) \\ &= a(w_l\lambda - e\lambda) - a(w_l\lambda - w_{n+1}\lambda) - a(w_n\lambda - e\lambda). \end{aligned}$$

Since there exists a directed path from e to $w_l = w$ in $\mathrm{QB}(\lambda\,;a)$ by assumption, it follows from Lemma 6.1.4 that $a(w\lambda - e\lambda) \in Q$. Similarly, since there exist directed paths from e to w_n and from w_{n+1} to $w_l = w$ in $\mathrm{QB}(\lambda\,;a)$ as seen above, we have $a(w_n\lambda - e\lambda) \in Q$ and $a(w_l\lambda - w_{n+1}\lambda) \in Q$ by Lemma 6.1.4. Thus we get $-a\langle \gamma_{n+1}^\vee, \lambda \rangle w_n \gamma_{n+1} \in Q$. Since $w_n\gamma_{n+1} \in \Delta$, it follows that $a\langle \gamma_{n+1}^\vee, \lambda \rangle$ is an integer. Thus we have proved Lemma 6.1.5. □

Proof of Proposition 6.1.1. We first show that $\lfloor W_{I(\lambda\,;a)} \rfloor \subset W^J_{e\rightarrow}(\lambda\,;a)$. Let $w \in \lfloor W_{I(\lambda\,;a)} \rfloor \subset W^J$, and let $w = r_{i_1} \cdots r_{i_l}$ be a reduced expression of w; note that $i_1, \dots, i_l \in I(\lambda\,;a)$. We see that

$$e \xrightarrow[\mathsf{B}]{\alpha_{i_l}} r_{i_l} \xrightarrow[\mathsf{B}]{r_{i_l}\alpha_{i_{l-1}}} r_{i_{l-1}} r_{i_l} \xrightarrow[\mathsf{B}]{r_{i_l} r_{i_{l-1}} \alpha_{i_{l-2}}} \cdots \xrightarrow[\mathsf{B}]{r_{i_l} r_{i_{l-1}} \cdots r_{i_2} \alpha_{i_1}} r_{i_1} \cdots r_{i_{l-1}} r_{i_l} = w \;\text{ in } \mathrm{QB}^J.$$

Since $i_1, \dots, i_l \in I(\lambda\,;a)$, we deduce that this directed path is in $\mathrm{QB}(\lambda\,;a)$, and hence $w \in W^J_{e\rightarrow}(\lambda\,;a)$.

We next show that $\lfloor W_{I(\lambda\,;a)} \rfloor \supset W^J_{e\rightarrow}(\lambda\,;a)$. Let $w \in W^J_{e\rightarrow}(\lambda\,;a)$. We show by induction on $l = \ell(w)$ that $w \in \lfloor W_{I(\lambda\,;a)} \rfloor$. If $l = \ell(w) = 0$, then it is obvious that $w = e \in \lfloor W_{I(\lambda\,;a)} \rfloor$. Assume that $\ell(w) > 0$. Recall from Lemma 6.1.2 that a shortest directed path from e to w in QB^J is of the form

$$e = w_0 \xrightarrow[\mathsf{B}]{\gamma_1} w_1 \xrightarrow[\mathsf{B}]{\gamma_2} \cdots \xrightarrow[\mathsf{B}]{\gamma_{l-1}} w_{l-1} \xrightarrow[\mathsf{B}]{\gamma_l} w_l = w. \tag{6.1.7}$$

Because $w \in W^J_{e\rightarrow}(\lambda\,;a)$, the directed path (6.1.7) is contained in $\mathrm{QB}(\lambda\,;a)$ by Lemma 6.1.5. In particular, $w_{l-1} \in W^J_{e\rightarrow}(\lambda\,;a)$, and hence $w_{l-1} \in \lfloor W_{I(\lambda\,;a)} \rfloor$ by induction hypothesis.

Now, let $w = r_{i_1} r_{i_2} \cdots r_{i_l}$ be a reduced expression of w. Since $w_{l-1} \xrightarrow[\mathsf{B}]{\gamma_l} w_l = w$, we have $\ell(w) = \ell(w_{l-1}) + 1$, and $w > w_{l-1}$ with respect to the Bruhat order on W. Thus, by the Subword Property (see [BB05, Theorem 2.2.2]), there exists $1 \leq p \leq l-1$ such that

$$w_{l-1} = r_{i_1} \cdots r_{i_{p-1}} r_{i_{p+1}} \cdots r_{i_l}$$

is a reduced expression of w_{l-1}. Because $w_{l-1} \in \lfloor W_{I(\lambda\,;a)} \rfloor$ as mentioned above, we have $i_1, \dots, i_{p-1}, i_{p+1}, \dots, i_l \in I(\lambda\,;a)$. Thus it remains to show that $i_p \in I(\lambda\,;a)$, that is, $a\langle \alpha_{i_p}^\vee, \lambda \rangle \in \mathbb{Z}$. Since $w_{l-1} = w r_{\gamma_l}$, we deduce that $\gamma_l = r_{i_l} \cdots r_{i_{p+1}} \alpha_{i_p}$, and hence $\gamma_l^\vee \in \alpha_{i_p}^\vee + \sum_{p+1 \leq q \leq l} \mathbb{Z} \alpha_{i_q}^\vee$; notice that $a\langle \alpha_{i_q}^\vee, \lambda \rangle \in \mathbb{Z}$ for every $p+1 \leq q \leq l$ since $i_q \in I(\lambda\,;a)$. Because the directed path (6.1.7) is contained in $\mathrm{QB}(\lambda\,;a)$, we have $a\langle \gamma_l^\vee, \lambda \rangle \in \mathbb{Z}$. Combining these, we see that $a\langle \alpha_{i_p}^\vee, \lambda \rangle \in \mathbb{Z}$. Thus we get $i_p \in I(\lambda\,;a)$, as desired. □

Corollary 6.1.6. *Let $w, v \in W^J_{e\rightarrow}(\lambda\,;a)$. If $w \xrightarrow{\alpha} v$ in $\mathrm{QB}(\lambda\,;a)$, then $\alpha \in \sum_{i \in I(\lambda\,;a)} \mathbb{Z}_{\geq 0} \alpha_i$.*

Proof. Since $v = \lfloor w r_\alpha \rfloor$, there exists $z \in W_J$ such that $r_\alpha = w^{-1} v z$. By Proposition 6.1.1, $w, v \in W_{I(\lambda\,;a)}$. Also, $z \in W_J \subset W_{I(\lambda\,;a)}$ since $J \subset I(\lambda\,;a)$. Therefore, $r_\alpha \in W_{I(\lambda\,;a)}$, and hence $\alpha \in \sum_{i \in I(\lambda\,;a)} \mathbb{Z}_{\geq 0} \alpha_i$. Thus we have proved the corollary. □

6.2 J-adjusted coroots

Let $p_{I \setminus J} : Q^\vee \to Q^\vee_{I \setminus J}$ be the projection from $Q^\vee = \bigoplus_{i \in I} \mathbb{Z} \alpha_i^\vee$ onto $Q^\vee_{I \setminus J} = \bigoplus_{i \in I \setminus J} \mathbb{Z} \alpha_i^\vee$.

Proposition 6.2.1. *For each $i \in I \setminus J$, there exists a unique positive root $\tilde{\alpha}_i \in \Delta^+$ satisfying the following conditions:*

(1) $p_{I \setminus J}(\tilde{\alpha}_i^\vee) = \alpha_i^\vee$;

(2) $\ell(r_{\tilde{\alpha}_i}) = 2\langle \tilde{\alpha}_i^\vee, \rho \rangle - 1$;

(3) $\tilde{\alpha}_i^\vee \in Q^\vee_{J\text{-adj}}$;

(4) $z_{\tilde{\alpha}_i^\vee} r_{\tilde{\alpha}_i} \in W^J$, and $z_{\tilde{\alpha}_i^\vee} r_{\tilde{\alpha}_i} \xrightarrow[\mathsf{Q}]{\tilde{\alpha}_i} e$ in QB^J.

For a subset $K \subset I$, let $\theta_K \in \Delta_K^+$ be the highest root of Δ_K; we write $\theta = \theta_I \in \Delta^+$.

Lemma 6.2.2. *For every $\gamma = w^{-1}\theta \in W\theta \cap (\Delta \setminus \Delta_J)$, the element $z\gamma^\vee \in Q^\vee$ is J-adjusted, where $z \in W_J$ is defined by $r_\theta w = \lfloor r_\theta w \rfloor z$.*

Proof. Let us show that $\langle z\gamma^\vee, \alpha\rangle \in \{-1, 0\}$ for all $\alpha \in \Delta_J^+$. Let $\alpha \in \Delta_J^+$. We have

$$\begin{aligned}\langle z\gamma^\vee, \alpha\rangle &= \langle zw^{-1}\theta^\vee, \alpha\rangle = -\langle zw^{-1}r_\theta\theta^\vee, \alpha\rangle \\ &= -\langle \theta^\vee, r_\theta wz^{-1}\alpha\rangle = -\langle \theta^\vee, \lfloor r_\theta w\rfloor\alpha\rangle;\end{aligned}$$

we need only to show that $\langle \theta^\vee, \lfloor r_\theta w\rfloor\alpha\rangle \in \{0, 1\}$. Notice that $\lfloor r_\theta w\rfloor\alpha \in \Delta^+$ by (2.1.8). Let us show that $\lfloor r_\theta w\rfloor\alpha \neq \theta$. Suppose that $\lfloor r_\theta w\rfloor\alpha = \theta$. Then we have

$$\gamma = w^{-1}\theta = -w^{-1}r_\theta\theta = -z^{-1}\lfloor r_\theta w\rfloor^{-1}\theta = -z^{-1}\alpha.$$

Because $z \in W_J$ and $\alpha \in \Delta_J$, we see that $\gamma = -z^{-1}\alpha \in \Delta_J$, which contradicts the assumption. Thus we get $\lfloor r_\theta w\rfloor\alpha \neq \theta$. Hence, it follows from [Bou68, Proposition 25 (iv)] that $\langle \theta^\vee, \lfloor r_\theta w\rfloor\alpha\rangle \in \{0, 1\}$, as desired. □

We know the following lemma from [BMO11, Lemma 7.2]; in a simply-laced root system, we understand that all roots are long.

Lemma 6.2.3. *Let $\alpha \in \Delta^+$, and write it as $\alpha = \sum_{i\in I} c_i\alpha_i$ with $c_i \in \mathbb{Z}_{\geq 0}$, $i \in I$. We have $\ell(r_\alpha) = 2\langle\alpha^\vee, \rho\rangle - 1$ if and only if α is a long root, or α is a short root, satisfying the condition that $c_i = 0$ for all $i \in I$ such that α_i is long.*

Proof of Proposition 6.2.1. We know from [Woo05, Lemma/Definition 1] (see also [LS10, Theorem 10.15]) that an element satisfying conditions (1) and (3) is unique.

Let $i \in I \setminus J$. We first show that there exists $\tilde{\alpha}_i^\vee \in \Delta^+$ satisfying conditions (1), (2), and (3).

Case 1. If α_i is a long root of Δ, then it follows that $\alpha_i \in W\theta \cap (\Delta \setminus \Delta_J)$. By Lemma 6.2.2, there exists $z \in W_J$ such that $z\alpha_i^\vee$ is J-adjusted. Set $\tilde{\alpha}_i := z\alpha_i \in \Delta^+$. Since $z \in W_J$, we get $p_{I\setminus J}(\tilde{\alpha}_i^\vee) = p_{I\setminus J}(z\alpha_i^\vee) = \alpha_i^\vee$. Because $\tilde{\alpha}_i = z\alpha_i$ is a long root of Δ, it follows from Lemma 6.2.3 that $\ell(r_{\tilde{\alpha}_i}) = 2\langle\tilde{\alpha}_i^\vee, \rho\rangle - 1$. Thus, $\tilde{\alpha}_i = z\alpha_i \in \Delta^+$ satisfies conditions (1), (2), and (3).

Case 2. Assume that α_i is a short root. Let $J \sqcup \{i\} = K \sqcup J_1 \sqcup \cdots \sqcup J_m$ be the decomposition of (the Dynkin subdiagram corresponding to) $J \sqcup \{i\}$ into its connected components, with $i \in K$; note that $\alpha_i \in \Delta_K$.

Subcase 2-1. If Δ_K contains a long root of Δ, then Δ_K is of type B, C, F, or G. We see from [Bou68] that $\sum_{k\in K}\alpha_k^\vee$ is a short coroot for Δ_K. Let $\gamma \in \Delta_K^+$ be the long root such that $\gamma^\vee = \sum_{k\in K}\alpha_k^\vee$; notice that $\gamma \in W\theta \cap (\Delta \setminus \Delta_J)$. Hence, by Lemma 6.2.2, there exists $z \in W_J$ such that $z\gamma^\vee$ is J-adjusted. Set $\tilde{\alpha}_i := z\gamma$. Since $z \in W_J$, we get $p_{I\setminus J}(\tilde{\alpha}_i^\vee) = p_{I\setminus J}(z\gamma^\vee) = p_{I\setminus J}(\gamma^\vee) = \alpha_i^\vee$. Because $\tilde{\alpha}_i = z\gamma$ is a long root of Δ, it follows from Lemma 6.2.3 that $\ell(r_{\tilde{\alpha}_i}) = 2\langle\tilde{\alpha}_i^\vee, \rho\rangle - 1$. Thus, $\tilde{\alpha}_i = z\gamma \in \Delta^+$ satisfies conditions (1), (2), and (3).

Subcase 2-2. Assume that Δ_K does not contain a long root of Δ; in this case, Δ_K is a simply-laced root system. Because $\alpha_i \in W_K\theta_K \cap (\Delta_K \setminus \Delta_{K\setminus\{i\}})$, we see from Lemma 6.2.2 that there exists $z \in W_{K\setminus\{i\}}$ such that $z\alpha_i^\vee$ is $(K \setminus \{i\})$-adjusted. We claim that $z\alpha_i^\vee$ is J-adjusted. Remark that $J = (K \setminus \{i\}) \sqcup J_1 \sqcup \cdots \sqcup J_m$, which implies that $\Delta_J^+ = \Delta_{K\setminus\{i\}}^+ \sqcup \Delta_{J_1}^+ \sqcup \cdots \sqcup \Delta_{J_m}^+$. Since $z\alpha_i^\vee$ is $(K \setminus \{i\})$-adjusted, we have $\langle z\alpha_i^\vee, \gamma\rangle \in \{-1, 0\}$ for all $\gamma \in \Delta_{K\setminus\{i\}}^+$. Because K is a connected component of $J \sqcup \{i\}$, it follows that $\langle z\alpha_i^\vee, \gamma\rangle = 0$ for all $\gamma \in \Delta_{J_1}^+ \sqcup \cdots \sqcup \Delta_{J_m}^+$. Combining these, we get $\langle z\alpha_i^\vee, \gamma\rangle \in \{-1, 0\}$ for all $\gamma \in \Delta_J^+$, which implies that $z\alpha_i^\vee$ is J-adjusted. Since $z \in W_{K\setminus\{i\}}$, we get $p_{I\setminus J}(z\alpha_i^\vee) = \alpha_i^\vee$. Because every root in Δ_K is a short root of Δ, we deduce from Lemma 6.2.3 that $\ell(r_{z\alpha_i}) = 2\langle z\alpha_i^\vee, \rho\rangle - 1$. Thus, $\tilde{\alpha}_i := z\alpha_i \in \Delta^+$ satisfies conditions (1), (2), and (3).

Next, we show that $\tilde{\alpha}_i \in \Delta^+$ above satisfies condition (4). Since $\tilde{\alpha}_i^\vee$ is J-adjusted by condition (3), we see from Lemma 4.3.6 that

$$\mathrm{Inv}(z_{\tilde{\alpha}_i^\vee}) = \{\gamma \in \Delta_J^+ \mid \langle \tilde{\alpha}_i^\vee, \gamma\rangle = -1\}. \tag{6.2.1}$$

Let $\gamma \in \Delta_J^+$; note that $\langle \tilde{\alpha}_i^\vee, \gamma\rangle \in \{-1, 0\}$. If $\langle \tilde{\alpha}_i^\vee, \gamma\rangle = -1$, then $r_{\tilde{\alpha}_i}(\gamma) = \gamma + \tilde{\alpha}_i \in \Delta^+ \setminus \Delta_J^+$ by condition (1), and hence $z_{\tilde{\alpha}_i^\vee} r_{\tilde{\alpha}_i}(\gamma) \in \Delta^+$ since $z_{\tilde{\alpha}_i^\vee} \in W_J$. If $\langle \tilde{\alpha}_i^\vee, \gamma\rangle = 0$, then $r_{\tilde{\alpha}_i}(\gamma) = \gamma \notin \mathrm{Inv}(z_{\tilde{\alpha}_i^\vee})$ by (6.2.1), and hence $z_{\tilde{\alpha}_i^\vee} r_{\tilde{\alpha}_i}(\gamma) \in \Delta^+$. Thus we get $z_{\tilde{\alpha}_i^\vee} r_{\tilde{\alpha}_i}(\gamma) \in \Delta^+$ for all $\gamma \in \Delta_J^+$, and hence $z_{\tilde{\alpha}_i^\vee} r_{\tilde{\alpha}_i} \in W^J$ by (2.1.8).

Now, we have $r_{\tilde{\alpha}_i}^{-1}(\mathrm{Inv}(z_{\tilde{\alpha}_i^\vee})) \subset \Delta^+$ by (6.2.1). Therefore, by [Mac03, (2.2.4)],

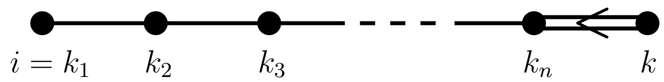
$$\ell(z_{\tilde{\alpha}_i^\vee} r_{\tilde{\alpha}_i}) = \ell(z_{\tilde{\alpha}_i^\vee}) + \ell(r_{\tilde{\alpha}_i}); \tag{6.2.2}$$

note that $\ell(z_{\tilde{\alpha}_i^\vee}) = -2\langle \tilde{\alpha}_i^\vee, \rho_J\rangle$ by Lemma 4.3.6 and condition (3), and that $\ell(r_{\tilde{\alpha}_i}) = 2\langle \tilde{\alpha}_i^\vee, \rho\rangle - 1$ by condition (2). Combining these, we have $\ell(z_{\tilde{\alpha}_i^\vee} r_{\tilde{\alpha}_i}) = 2\langle \tilde{\alpha}_i^\vee, \rho - \rho_J\rangle - 1$. Thus,

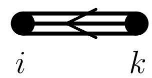
$$\ell(z_{\tilde{\alpha}_i^\vee} r_{\tilde{\alpha}_i}) + 1 - 2\langle \tilde{\alpha}_i^\vee, \rho - \rho_J\rangle = 0 = \ell(e),$$

which implies that $z_{\tilde{\alpha}_i^\vee} r_{\tilde{\alpha}_i} \xrightarrow[\mathsf{Q}]{\tilde{\alpha}_i} e$ in QB^J. Thus we have proved Proposition 6.2.1. □

Remark 6.2.4. The element $\tilde{\alpha}_i \in \Delta^+$, $i \in I \setminus J$, is explicitly described as follows: let K be the connected component of $J \sqcup \{i\}$ containing i, and let $k \in K$ be the (unique) most nearest node whose corresponding simple root α_k is a long root of Δ_K. If α_i is a long root of Δ_K, then $k = i$. If α_i is a short root of Δ_K, then $k \neq i$ and K contains either of the following Dynkin subdiagrams:

(i) 

$(n \in \mathbb{Z}_{\geq 1})$

(ii) 

If α_i is a long root of Δ_K (and hence $k = i$), then we have $\tilde{\alpha}_i = \alpha_i$; if K contains (i), then we have $\tilde{\alpha}_i = r_{k_1} r_{k_2} r_{k_3} \cdots r_{k_n} \alpha_k$; and if K contains (ii), then we have $\tilde{\alpha}_i = r_i \alpha_k$.

6.3 Directed paths from e to e in QB^J

For a directed path

$$\mathbf{d} : v = w_0 \xrightarrow{\gamma_1} w_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_l} w_l = w \text{ in } \mathrm{QB}^J,$$

we define the weight $\mathrm{wt}(\mathbf{d})$ of $\mathbf{d}$ by

$$\mathrm{wt}(\mathbf{d}) := \sum_{\substack{1 \leq u \leq l \text{ s.t.} \\ w_{u-1} \xrightarrow[\mathsf{Q}]{} w_u}} \gamma_u^\vee \in \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i^\vee. \tag{6.3.1}$$

Proposition 6.3.1. *Let $0 < a \le 1$ be a rational number such that $I(\lambda\,;a) \supsetneq J$. For each $i \in I(\lambda\,;a) \setminus J$, there exists a directed path $\mathbf{d}(i)$ from e to e in $\mathrm{QB}(\lambda\,;a)$ such that $\mathrm{wt}(\mathbf{d}(i)) = \tilde{\alpha}_i^\vee$, where $\tilde{\alpha}_i$ is the positive root given in Proposition 6.2.1.*

Proof. Let $i \in I(\lambda\,;a) \setminus J$. We see from condition (1) in Proposition 6.2.1 that $r_{\tilde{\alpha}_i} \in W_{J \sqcup \{i\}} \subset W_{I(\lambda\,;a)}$ since $i \in I(\lambda\,;a)$ and $J \subset I(\lambda\,;a)$. Also, $z_{\tilde{\alpha}_i^\vee} \in W_J \subset W_{I(\lambda\,;a)}$. Hence, $z_{\tilde{\alpha}_i^\vee} r_{\tilde{\alpha}_i} \in W_{I(\lambda\,;a)}$. Because $z_{\tilde{\alpha}_i^\vee} r_{\tilde{\alpha}_i} = \lfloor z_{\tilde{\alpha}_i^\vee} r_{\tilde{\alpha}_i} \rfloor \in W^J$ by condition (4) in Proposition 6.2.1, we have $z_{\tilde{\alpha}_i^\vee} r_{\tilde{\alpha}_i} \in \lfloor W_{I(\lambda\,;a)} \rfloor = W^J_{e\to}(\lambda\,;a)$ (see Proposition 6.1.1). It follows from Lemmas 6.1.2 and 6.1.5 that there exists a directed path of the form

$$e = w_0 \xrightarrow[\mathsf{B}]{\gamma_1} w_1 \xrightarrow[\mathsf{B}]{\gamma_2} \cdots \xrightarrow[\mathsf{B}]{\gamma_l} w_l = z_{\tilde{\alpha}_i^\vee} r_{\tilde{\alpha}_i} \text{ in } \mathrm{QB}(\lambda\,;a).$$

Also, by condition (4) in Proposition 6.2.1, we have $z_{\tilde{\alpha}_i^\vee} r_{\tilde{\alpha}_i} \xrightarrow[\mathsf{Q}]{\tilde{\alpha}_i} e$ in QB^J. We see that $\langle \tilde{\alpha}_i^\vee, \lambda \rangle = \langle \alpha_i^\vee, \lambda \rangle$ by condition (1) in Proposition 6.2.1. Since $i \in I(\lambda\,;a)$, it follows immediately that $a\langle \tilde{\alpha}_i^\vee, \lambda \rangle = a\langle \alpha_i^\vee, \lambda \rangle \in \mathbb{Z}$, and hence the edge $z_{\tilde{\alpha}_i^\vee} r_{\tilde{\alpha}_i} \xrightarrow[\mathsf{Q}]{\tilde{\alpha}_i} e$ is contained in $\mathrm{QB}(\lambda\,;a)$. Concatenating these, we obtain a directed path $\mathbf{d}(i)$ of the form

$$\mathbf{d}(i) : e = w_0 \xrightarrow[\mathsf{B}]{\gamma_1} w_1 \xrightarrow[\mathsf{B}]{\gamma_2} \cdots \xrightarrow[\mathsf{B}]{\gamma_l} w_l = z_{\tilde{\alpha}_i^\vee} r_{\tilde{\alpha}_i} \xrightarrow[\mathsf{Q}]{\tilde{\alpha}_i} e \text{ in } \mathrm{QB}(\lambda\,;a) \tag{6.3.2}$$

such that $\mathrm{wt}(\mathbf{d}(i)) = \tilde{\alpha}_i^\vee$. Thus we have proved Proposition 6.3.1. □

Corollary 6.3.2. *For a rational number $0 < a \le 1$,*

$$\left\{ p_{I\setminus J}(\mathrm{wt}(\mathbf{d})) \mid \mathbf{d} \textit{ is a directed path from } e \textit{ to } e \textit{ in } \mathrm{QB}(\lambda\,;a) \right\} = \sum_{i \in I(\lambda\,;a)\setminus J} \mathbb{Z}_{\ge 0} \alpha_i^\vee.$$

Proof. Let

$$\mathbf{d} : e = w_0 \xrightarrow{\gamma_1} w_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_p} w_p = e$$

be a directed path from e to e in $\mathrm{QB}(\lambda\,;a)$. Then, $w_0, w_1, \dots, w_p \in W^J_{e\to}(\lambda\,;a)$. Applying Corollary 6.1.6 to $w_{q-1} \xrightarrow{\gamma_q} w_q$ for each $1 \le q \le p$, we see that $\gamma_q \in \sum_{i \in I(\lambda\,;a)} \mathbb{Z}_{\ge 0} \alpha_i^\vee$ for every $1 \le q \le p$. Therefore, $\mathrm{wt}(\mathbf{d}) \in \sum_{i \in I(\lambda\,;a)} \mathbb{Z}_{\ge 0} \alpha_i^\vee$, and hence $p_{I\setminus J}(\mathrm{wt}(\mathbf{d})) \in \sum_{i \in I(\lambda\,;a)\setminus J} \mathbb{Z}_{\ge 0} \alpha_i^\vee$. Thus the inclusion (LHS) $\subset$ (RHS) holds.

Let us show the opposite inclusion. Remark that (LHS) contains 0 since the weight of the "trivial" directed path $e = w_0 = e$ in $\mathrm{QB}(\lambda\,;a)$ is equal to 0. Hence, if $I(\lambda\,;a) = J$, then the assertion is obvious. Assume that $I(\lambda\,;a) \supsetneq J$. For each $i \in I(\lambda\,;a)$, let $\mathbf{d}(i)$ be a directed path from e to e such that $\mathrm{wt}(\mathbf{d}(i)) = \tilde{\alpha}_i^\vee$ (see Proposition 6.3.1); note that $p_{I\setminus J}(\tilde{\alpha}_i^\vee) = \alpha_i$ by condition (1) in Proposition 6.2.1. Thus, for each $\xi \in$ (RHS), we obtain a directed path $\mathbf{d}$ from e to e in $\mathrm{QB}(\lambda\,;a)$ such that $p_{I\setminus J}(\mathrm{wt}(\mathbf{d})) = \xi$, by concatenating these $\mathbf{d}(i)$'s, $i \in I(\lambda\,;a) \setminus J$. Therefore the opposite inclusion (LHS) $\supset$ (RHS) holds. Thus we have proved Corollary 6.3.2. □

6.4 Existence condition for directed paths in $\mathrm{SB}(\lambda\,;a)$

Lemma 6.4.1. *Let $x, y \in (W^J)_{\mathrm{af}}$, and write them as $x = w_1 z_{\xi_1} t_{\xi_1}$, $y = w_2 z_{\xi_2} t_{\xi_2}$ with some $w_1, w_2 \in W^J$ and $\xi_1, \xi_2 \in Q^\vee_{J\text{-adj}}$. Assume that there exists a directed path*

$$x = w_1 z_{\xi_1} t_{\xi_1} \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_l} w_2 z_{\xi_2} t_{\xi_2} = y \tag{6.4.1}$$

from x to y in SB^J. *Let*

$$\mathbf{d} : w_1 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_l} w_2$$

be the directed path in QB^J *obtained by applying Proposition 4.3.7 (1) to the directed path* (6.4.1) *repeatedly. Then,* $p_{I\setminus J}(\xi_2 - \xi_1) = p_{I\setminus J}(\mathrm{wt}(\mathbf{d}))$.

Proof. It suffices to show the assertion when $l = 1$. Assume that $x = w_1 z_{\xi_1} t_{\xi_1} \xrightarrow{\beta} r_\beta x = w_2 z_{\xi_2} t_{\xi_2}$ in SB^J for $\beta \in \Delta^+_{\mathrm{af}}$; by Lemma 4.3.5, $\beta = w_1\alpha + \chi\delta \in \Delta^+_{\mathrm{af}}$ for some $\alpha \in \Delta^+ \setminus \Delta^+_J$ and $\chi \in \{0, 1\}$. Note that $r_\beta x = w_1 r_\alpha z_{\xi_1} t_{\xi_1 + \chi z_{\xi_1}^{-1}\alpha^\vee}$, and hence $\xi_2 = \xi_1 + \chi z_{\xi}^{-1}\alpha^\vee$.

Now, the directed path in QB^J obtained from $x \xrightarrow{\beta} r_\beta x$ is

$$\mathbf{d} : w_1 \xrightarrow{\alpha} \lfloor w_1 r_\alpha \rfloor = w_2;$$

note that this edge is Bruhat (resp., quantum) if $\chi = 0$ (resp., $\chi = 1$). If $\chi = 0$, then $\xi_1 - \xi_2 = 0 = \mathrm{wt}(\mathbf{d})$. If $\chi = 1$, then $\xi_2 - \xi_1 = z_{\xi_1}^{-1}\alpha^\vee$ and $\mathrm{wt}(\mathbf{d}) = \alpha^\vee$. Hence, $p_{I\setminus J}(\xi_2 - \xi_1) = p_{I\setminus J}(z_{\xi_1}^{-1}\alpha^\vee) = p_{I\setminus J}(\alpha^\vee) = p_{I\setminus J}(\mathrm{wt}(\mathbf{d}))$. This proves the lemma. □

Proposition 6.4.2. *Let* $\xi, \zeta \in Q^\vee$, *and let* $0 < a \le 1$ *be a rational number. There exists a directed path from* $\Pi^J(t_\zeta)$ *to* $\Pi^J(t_\xi)$ *in* $\mathrm{SB}(\lambda\,; a)$ *if and only if* $p_{I\setminus J}(\xi - \zeta) \in \sum_{i \in I(\lambda\,;a)\setminus J} \mathbb{Z}_{\ge 0}\alpha_i^\vee$.

Proof. Assume that there exists a directed path

$$\Pi^J(t_\zeta) \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_l} \Pi^J(t_\xi) \text{ in } \mathrm{SB}(\lambda\,; a), \tag{6.4.2}$$

with $\beta_1, \dots, \beta_l \in \Delta^+_{\mathrm{af}}$. Recall from Lemma 2.2.6 that $\Pi^J(t_\zeta) = z_\zeta t_{\zeta + \phi_J(\zeta)}$ and $\Pi^J(t_\xi) = z_\xi t_{\xi+\phi_J(\xi)}$. Hence, by applying Proposition 4.3.7 (1) to the directed path (6.4.2) repeatedly, we obtain a directed path $\mathbf{d}$ from e to e in $\mathrm{QB}(\lambda\,; a)$ of the form

$$\mathbf{d} : e \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_l} e \text{ in } \mathrm{QB}(\lambda\,; a) \tag{6.4.3}$$

for some $\gamma_1, \dots, \gamma_l \in \Delta^+ \setminus \Delta^+_J$. Then we have

$$\begin{aligned} p_{I\setminus J}(\mathrm{wt}(\mathbf{d})) &= p_{I\setminus J}(\xi + \phi_J(\xi) - \zeta - \phi_J(\zeta)) && \text{(by Lemma 6.4.1)} \\ &= p_{I\setminus J}(\xi - \zeta) && \text{(since } \phi_J(\xi), \phi_J(\zeta) \in Q_J^\vee). \end{aligned}$$

Therefore, we get $p_{I\setminus J}(\xi - \zeta) = p_{I\setminus J}(\mathrm{wt}(\mathbf{d})) \in \sum_{i \in I(\lambda\,;a)\setminus J} \mathbb{Z}_{\ge 0}\alpha_i^\vee$ by Corollary 6.3.2.

Conversely, let us show that there exists a directed path from $\Pi^J(t_\zeta)$ to $\Pi^J(t_\xi)$ in $\mathrm{SB}(\lambda\,; a)$ for every $\xi, \zeta \in Q^\vee$ such that $p_{I\setminus J}(\xi - \zeta) \in \sum_{i \in I(\lambda\,;a)\setminus J} \mathbb{Z}_{\ge 0}\alpha_i^\vee$; it suffices to prove the assertion in the case that $p_{I\setminus J}(\xi - \zeta) = \alpha_i^\vee$ for some $i \in I(\lambda\,; a) \setminus J$. Let $\mathbf{d}(i)$ be a directed path

$$e = w_0 \xrightarrow[\mathsf{B}]{\gamma_1} w_1 \xrightarrow[\mathsf{B}]{\gamma_2} \cdots \xrightarrow[\mathsf{B}]{\gamma_l} w_l = z_{\tilde{\alpha}_i^\vee} r_{\tilde{\alpha}_i} \xrightarrow[\mathsf{Q}]{\tilde{\alpha}_i} e$$

from e to e in $\mathrm{QB}(\lambda\,; a)$ of weight $\tilde{\alpha}_i^\vee$ obtained in (6.3.2); note that $w_l = r_{\gamma_1} r_{\gamma_2} \cdots r_{\gamma_l}$ by Remark 4.2.2 (1), and $\lfloor r_{\gamma_1} r_{\gamma_2} \cdots r_{\gamma_l} r_{\tilde{\alpha}_i} \rfloor = \lfloor w_l r_{\tilde{\alpha}_i} \rfloor = e$. By applying Proposition 4.3.7 (2) to $\mathbf{d}(i)$, we obtain the following directed path

$$\Pi^J(t_\zeta) \xrightarrow{\beta_1} r_{\beta_1}\Pi^J(t_\zeta) \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{l+1}} r_{\beta_{l+1}} \cdots r_{\beta_2} r_{\beta_1} \Pi^J(t_\zeta) \text{ in } \mathrm{SB}(\lambda\,; a), \tag{6.4.4}$$

with $r_{\beta_{l+1}} \cdots r_{\beta_2} r_{\beta_1} = r_{\gamma_1} r_{\gamma_2} \cdots r_{\gamma_l} r_{\tilde{\alpha}_i} t_{\tilde{\alpha}_i^\vee}$; note that $r_{\beta_{l+1}} \cdots r_{\beta_2} r_{\beta_1} \in (W^J)_{\mathrm{af}}$ by Lemma 2.2.8. Therefore, we obtain

$$\begin{aligned} r_{\gamma_1} r_{\gamma_2} \cdots r_{\gamma_l} r_{\tilde{\alpha}_i} t_{\tilde{\alpha}_i^\vee} &= \Pi^J(r_{\gamma_1} r_{\gamma_2} \cdots r_{\gamma_l} r_{\tilde{\alpha}_i} t_{\tilde{\alpha}_i^\vee}) && (\text{since } r_{\gamma_1} r_{\gamma_2} \cdots r_{\gamma_l} r_{\tilde{\alpha}_i} t_{\tilde{\alpha}_i^\vee} \in (W^J)_{\mathrm{af}}) \\ &= \lfloor r_{\gamma_1} r_{\gamma_2} \cdots r_{\gamma_l} r_{\tilde{\alpha}_i} \rfloor z_{\tilde{\alpha}_i^\vee} t_{\tilde{\alpha}_i^\vee} && (\text{by Lemma 2.2.6 (3) along with } \tilde{\alpha}_i^\vee \in Q^\vee_{J\text{-adj}}) \\ &= z_{\tilde{\alpha}_i^\vee} t_{\tilde{\alpha}_i^\vee} && (\text{since } \lfloor r_{\gamma_1} r_{\gamma_2} \cdots r_{\gamma_l} r_{\tilde{\alpha}_i} \rfloor = e), \end{aligned}$$

which implies that $r_{\beta_{l+1}} \cdots r_{\beta_2} r_{\beta_1} = r_{\gamma_1} r_{\gamma_2} \cdots r_{\gamma_l} r_{\tilde{\alpha}_i} t_{\tilde{\alpha}_i^\vee} = z_{\tilde{\alpha}_i^\vee} t_{\tilde{\alpha}_i^\vee}$. Also, we have

$$\begin{aligned} r_{\beta_{l+1}} \cdots r_{\beta_2} r_{\beta_1} \Pi^J(t_\zeta) &= z_{\tilde{\alpha}_i} t_{\tilde{\alpha}_i^\vee} \Pi^J(t_\zeta) && (\text{since } r_{\beta_{l+1}} \cdots r_{\beta_2} r_{\beta_1} = z_{\tilde{\alpha}_i^\vee} t_{\tilde{\alpha}_i^\vee}) \\ &= \Pi^J(t_{\tilde{\alpha}_i^\vee} t_\zeta) && (\text{by Lemma 2.2.4 (2)}) \\ &= \Pi^J(t_\xi) && (\text{since } \zeta + \tilde{\alpha}_i^\vee - \xi \in Q_J^\vee). \end{aligned}$$

By substituting this equality into (6.4.4), we obtain a directed path from $\Pi^J(t_\zeta)$ to $\Pi^J(t_\xi)$ in $\mathrm{SB}(\lambda\,; a)$, as desired. Thus we have proved Proposition 6.4.2. □

6.5 Connected components of $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$

Write $\lambda \in P^+$ as $\lambda = \sum_{i \in I} m_i \varpi_i$ with $m_i \in \mathbb{Z}_{\geq 0}$, $i \in I$. Set

$$\mathrm{Turn}(\lambda) := \{k/m_i \mid i \in I \setminus J,\ 0 \leq k \leq m_i\}.$$

Lemma 6.5.1. *Let $0 < a < 1$ be a rational number, and let $\xi, \zeta \in Q^\vee_{J\text{-adj}}$ with $\xi \neq \zeta$. If there exists a directed path from $z_\zeta t_\zeta$ to $z_\xi t_\xi$ in $\mathrm{SB}(\lambda\,; a)$, then $a \in \mathrm{Turn}(\lambda)$.*

Proof. Let $z_\zeta t_\zeta \xrightarrow{\beta} r_\beta z_\zeta t_\zeta$ be the initial edge of a directed path from $z_\zeta t_\zeta$ to $z_\xi t_\xi$ in $\mathrm{SB}(\lambda\,; a)$. By applying Proposition 4.3.7 (1) to this edge in $\mathrm{SB}(\lambda\,; a)$, we obtain an edge $e \xrightarrow{\gamma} \lfloor r_\gamma \rfloor$ in $\mathrm{QB}(\lambda\,; a)$ for some $\gamma \in \Delta^+ \setminus \Delta_J^+$. Note that $e \xrightarrow{\gamma} \lfloor r_\gamma \rfloor$ is a Bruhat edge by Lemma 6.1.2 (see also Remark 4.2.2 (3)). Hence, $\lfloor r_\gamma \rfloor = r_\gamma$ by Remark 4.2.2 (1), and $1 = \ell(r_\gamma) - \ell(e) = \ell(r_\gamma)$, which implies that $\gamma = \alpha_i$ for some $i \in I$. Then we have

$$am_i = a\langle \alpha_i^\vee, \lambda \rangle = a\langle \gamma^\vee, \lambda \rangle \in \mathbb{Z}_{>0},$$

which implies that $a \in \mathrm{Turn}(\lambda)$. This proves the assertion. □

The next proposition follows immediately from Proposition 6.4.2 and Lemma 6.5.1; for simplicity of notation, we set $T_\xi := \Pi^J(t_\xi) = z_\xi t_\xi \in (W^J)_{\mathrm{af}}$ for $\xi \in Q^\vee_{J\text{-adj}}$.

Proposition 6.5.2. *Let $\xi_1, \dots, \xi_{s-1} \in Q^\vee_{J\text{-adj}}$. An element*

$$\eta = (T_{\xi_1}, \dots, T_{\xi_{s-1}}, e\,; a_0, a_1, \dots, a_{s-1}, a_s) \tag{6.5.1}$$

is contained in $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ if and only if $a_u \in \mathrm{Turn}(\lambda)$ and $p_{I \setminus J}(\xi_u - \xi_{u+1}) \in \sum_{i \in I(\lambda\,; a_u) \setminus J} \mathbb{Z}_{\geq 0} \alpha_i^\vee \setminus \{0\}$ for all $1 \leq u \leq s-1$, where we set $\xi_s := 0$.

Proposition 6.5.3. *Each connected component of $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ contains a unique element of the form (6.5.1).*

In order to prove this proposition, we need some notation and lemmas. Let $N \in \mathbb{Z}_{>0}$. For simplicity of notation, we set

$$[y_1, y_2, \dots, y_N] := \eta_{y_1} \otimes \eta_{y_2} \otimes \cdots \otimes \eta_{y_N} \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)^{\otimes N}$$

for $y_1, y_2, \dots, y_N \in (W^J)_{\mathrm{af}}$.

Lemma 6.5.4. *Let $N \in \mathbb{Z}_{>0}$ be as in Lemma 5.2.7. Let $\eta \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$, and write $\sigma_N(\eta)$ as*

$$\sigma_N(\eta) = [y_1, y_2, \dots, y_N] \quad \text{with some } y_1, y_2, \dots, y_N \in (W^J)_{\mathrm{af}}.$$

Let X be a monomial in the root operators e_i and f_i, $i \in I_{\mathrm{af}}$, and assume that $X\eta \neq \mathbf{0}$. Then,

$$\sigma_N(X\eta) = [v_1 y_1, v_2 y_2, \dots, v_N y_N]$$

with some $v_1, v_2, \dots, v_N \in W_{\mathrm{af}}$ such that $v_n y_n \in (W^J)_{\mathrm{af}}$ for all $1 \le n \le N$.

Proof. It is essential to verify the assertion in the case where $X = e_i$ or f_i, $i \in I_{\mathrm{af}}$; the assertion for a general X follows immediately by an inductive argument. By the definition of the root operators (see also the proof of Theorem 3.1.6) and Lemma 5.2.7, we deduce that

$$\sigma_N(X\eta) = [y_1, \dots, y_{k-1}, r_i y_k, \dots, r_i y_m, y_{m+1}, \dots, y_N]$$

for some $1 \le k \le m \le N$, with $r_i y_l \in (W^J)_{\mathrm{af}}$ for all $k \le l \le m$. Thus we have proved the lemma. □

Let $N \in \mathbb{Z}_{>0}$ be as in Lemma 5.2.7. Let $\eta \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ be of the form (6.5.1); by Lemma 5.2.7, $\sigma_N(\eta)$ is of the form:

$$\sigma_N(\eta) = [T_{\zeta_1}, T_{\zeta_2}, \dots, T_{\zeta_N}]$$

for some $\zeta_1, \zeta_2, \dots, \zeta_N \in Q^{\vee}_{J\text{-adj}}$. Let X be a monomial in the root operators e_i and f_i, $i \in I_{\mathrm{af}}$. Assume that $X\eta \neq \mathbf{0}$, and write $\sigma_N(X\eta)$ as (see Lemma 6.5.4):

$$\sigma_N(X\eta) = [v_1 T_{\zeta_1}, v_2 T_{\zeta_2}, \dots, v_N T_{\zeta_N}]$$

with some $v_1, v_2, \dots, v_N \in W_{\mathrm{af}}$ such that $v_n T_{\zeta_n} = v_n z_{\zeta_n} t_{\zeta_n} \in (W^J)_{\mathrm{af}}$, $1 \le n \le N$; note that $v_n \in (W^J)_{\mathrm{af}}$ for all $1 \le n \le N$ by Lemma 2.2.8.

Lemma 6.5.5. *Keep the notation and setting above. Let $\eta' \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ be also of the form (6.5.1), and write $\sigma_N(\eta')$ as*

$$\sigma_N(\eta') = [T_{\zeta'_1}, T_{\zeta'_2}, \dots, T_{\zeta'_N}]$$

for some $\zeta'_1, \zeta'_2, \dots, \zeta'_N \in Q^{\vee}_{J\text{-adj}}$. Then, $X\eta' \neq \mathbf{0}$, and

$$\sigma_N(X\eta') = [v_1 T_{\zeta'_1}, v_2 T_{\zeta'_2}, \dots, v_N T_{\zeta'_N}];$$

note that $v_n T_{\zeta'_n} \in (W^J)_{\mathrm{af}}$ for all $1 \le n \le N$ by Lemma 2.2.8 since $v_n \in (W^J)_{\mathrm{af}}$.

Proof. Let $X = g_p g_{p-1} \cdots g_2 g_1$, where $g_q \in \{e_i, f_i \mid i \in I_{\text{af}}\}$ for each $1 \le q \le p$. We show the assertion by induction on p. If $p = 0$, then the assertion is obvious since $X = \text{id}$. Assume that $p > 0$. Set $Y := g_{p-1} \cdots g_2 g_1$. Since $X\eta \ne \mathbf{0}$, it follows that $Y\eta \ne \mathbf{0}$. Write $\sigma_N(Y\eta)$ as

$$\sigma_N(Y\eta) = [u_1 T_{\zeta_1}, u_2 T_{\zeta_2}, \ldots, u_N T_{\zeta_N}]$$

with some $u_1, u_2, \ldots, u_N \in W_{\text{af}}$ such that $u_n T_{\zeta_n} = u_n z_{\zeta_n} t_{\zeta_n} \in (W^J)_{\text{af}}$ for all $1 \le n \le N$. By induction hypothesis, we have $Y\eta' \ne \mathbf{0}$, and

$$\sigma_N(Y\eta') = [u_1 T_{\zeta_1'}, u_2 T_{\zeta_2'}, \ldots, u_N T_{\zeta_N'}].$$

Now, assume that g_p is either e_i or f_i for some $i \in I_{\text{af}}$. We see from the definition of the root operator e_i or f_i and Lemma 5.2.7 that

$$\begin{aligned}\sigma_N(X\eta) &= \sigma_N(g_p Y\eta) \\ &= [u_1 T_{\zeta_1}, \ldots, u_{k-1} T_{\zeta_{k-1}}, r_i u_k T_{\zeta_k}, \ldots, r_i u_m T_{\zeta_m}, u_{m+1} T_{\zeta_{m+1}}, \ldots, u_N T_{\zeta_N}]\end{aligned}$$

for some $0 \le k \le m \le N$, with $r_i u_l T_{\zeta_l} \in (W^J)_{\text{af}}$ for every $k \le l \le m$; we should remark that k and m are determined by the function $H_i^{\pi_{Y\eta}}(t) = \langle \alpha_i^\vee, \pi_{Y\eta}(t) \rangle$ (see the definition of $t_0, t_1 \in [0,1]$ in Definition 3.1.5 and Lemma 5.2.7). Since

$$\langle \alpha_i^\vee, u_n T_{\zeta_n'} \lambda \rangle = \langle \alpha_i^\vee, u_n \lambda \rangle = \langle \alpha_i^\vee, u_n T_{\zeta_n} \lambda \rangle \quad \text{for all } 1 \le n \le N,$$

we deduce that $H_i^{\pi_{Y\eta}}(t) = H_i^{\pi_{Y\eta'}}(t)$ for all $t \in [0,1]$, which implies that t_0, t_1 for $Y\eta'$ coincides with those for $Y\eta$, respectively. Therefore, if follows from the definition of the root operator e_i or f_i and Lemma 5.2.7 that $X\eta' = g_p Y\eta' \ne \mathbf{0}$, and

$$\begin{aligned}\sigma_N(X\eta') &= \sigma_N(g_p Y\eta') \\ &= [u_1 T_{\zeta_1'}, \ldots, u_{k-1} T_{\zeta_{k-1}'}, r_i u_k T_{\zeta_k'}, \ldots, r_i u_m T_{\zeta_m'}, u_{m+1} T_{\zeta_{m+1}'}, \ldots, u_N T_{\zeta_N'}].\end{aligned}$$

Thus we have proved the lemma. □

Proof of Proposition 6.5.2. First we show that each connected component of $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ contains an element of the form (6.5.1). Let $\eta \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$; recall that $\pi_\eta \in \mathbb{B}(\lambda)$. By [NS08, Theorem 3.1 (2)], there exists a monomial X in the root operators e_i and f_i, $i \in I_{\text{af}}$, such that $(X\pi_\eta)(t) \equiv t\lambda \bmod \mathbb{R}\delta$ for all $t \in [0,1]$. Because the map $\mathbb{B}^{\frac{\infty}{2}}(\lambda) \to \mathbb{B}(\lambda)$, $\eta \mapsto \pi_\eta$, is a strict morphism of crystals (see Remark 4.4.1 (1)), it follows immediately that $X\eta \ne \mathbf{0}$, and $X\eta \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ is of the form:

$$X\eta = (T_{\xi_1'}, T_{\xi_2'}, \ldots, T_{\xi_{s-1}'}, T_\xi\,; a_0, a_1, \ldots, a_{s-1}, a_s)$$

for some $\xi_1', \xi_2', \ldots, \xi_{s-1}', \xi \in Q^\vee_{J\text{-adj}}$.

Now, observe that $\eta_{T_\xi} = (T_\xi\,; 0, 1) \in \mathbb{B}_0^{\frac{\infty}{2}}(\lambda)$ (see §5.1). Let Y be a monomial in the root operators e_i and f_i, $i \in I_{\text{af}}$, such that $Y\eta_{T_\xi} = \eta_e$. Take $N \in \mathbb{Z}_{>0}$ as in Lemma 5.2.7. Then,

$$\sigma_N(\eta_{T_\xi}) = [T_\xi, T_\xi, \ldots, T_\xi] \quad (N \text{ times}),$$
$$\sigma_N(\eta_e) = [e, e, \ldots, e] \quad (N \text{ times}).$$

Also, we see from Lemma 6.5.5 that

$$\sigma_N(Y\eta_{T_\xi}) = [v_1T_\xi, v_2T_\xi, \dots, v_NT_\xi]$$

for some $v_1, v_2, \dots, v_N \in W_{\mathrm{af}}$ with $v_nT_\xi \in (W^J)_{\mathrm{af}}$ for every $1 \le n \le N$. Because $Y\eta_{T_\zeta} = \eta_e$, and hence $\sigma_N(Y\eta_{T_\zeta}) = \sigma_N(\eta_e)$, it follows that $v_nT_\xi = e$ for all $1 \le n \le N$. Thus, $v_n = T_\xi^{-1}$ for all $1 \le n \le N$.

We see from Lemma 5.2.7 that $\sigma_N(X\eta)$ is of the form:

$$\sigma_N(X\eta) = [T_{\zeta'_1}, T_{\zeta'_2}, \dots, T_{\zeta'_N}]$$

for some $\zeta'_1, \zeta'_2, \dots, \zeta'_N \in Q^\vee_{J\text{-adj}}$; we should remark that $\zeta'_N = \xi$. Then we deduce from Lemma 6.5.5 that

$$\begin{aligned}\sigma_N(YX\eta) &= [v_1T_{\zeta'_1}, v_2T_{\zeta'_2}, \dots, v_NT_{\zeta'_N}] \\ &= [T_\xi^{-1}T_{\zeta'_1}, T_\xi^{-1}T_{\zeta'_2}, \dots, \underbrace{T_\xi^{-1}T_{\zeta'_N}}_{=e}]\end{aligned}$$

with $v_nT_{\zeta'_n} = T_\xi^{-1}T_{\zeta'_n} \in (W^J)_{\mathrm{af}}$ for every $1 \le n \le N$. Since

$$(W^J)_{\mathrm{af}} \ni T_\xi^{-1}T_{\zeta'_n} = t_{-\xi}z_\xi^{-1}z_{\zeta'_n}t_{\zeta'_n} = z_\xi z_{\zeta'_n}t_{\zeta_n}, \quad \text{with } \zeta_n := \zeta'_n - z_{\zeta'_n}^{-1}z_\xi\xi,$$

we see from (2.2.5) that $\zeta_n \in Q^\vee_{J\text{-adj}}$ and $z_\xi z_{\zeta'_n} = z_{\zeta_n}$. Thus we get

$$\sigma_N(YX\eta) = [T_{\zeta_1}, T_{\zeta_2}, \dots, T_{\zeta_{n-1}}, e].$$

Because the final factor above is equal to e, we deduce from Lemma 5.2.7 that $YX\eta$ is of the form (6.5.1). Thus we have shown that each connected component of $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ contains an element of the form (6.5.1).

Next we prove the uniqueness. Let $\eta, \eta' \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$, $\eta \ne \eta'$, be of the form (6.5.1), and suppose that $X\eta = \eta'$ for some monomial X in the root operators e_i and f_i, $i \in I_{\mathrm{af}}$. As above, let $N \in \mathbb{Z}_{>0}$ be as in Lemma 5.2.7. Then, $\sigma_N(\eta)$ and $\sigma_N(\eta')$ are of the form (note that $e = T_0$):

$$\sigma_N(\eta) = [T_{\zeta_1}, T_{\zeta_2}, \dots, T_{\zeta_{N-1}}, T_0], \qquad \sigma_N(\eta') = [T_{\zeta'_1}, T_{\zeta'_2}, \dots, T_{\zeta'_{N-1}}, T_0]$$

for some $\zeta_n, \zeta'_n \in Q^\vee_{J\text{-adj}}$, $1 \le n \le N-1$. Since $\eta \ne \eta'$ and the map $\sigma_N : \mathbb{B}^{\frac{\infty}{2}}(\lambda) \hookrightarrow \mathbb{B}^{\frac{\infty}{2}}(\lambda)^{\otimes N}$ is injective, there exists $1 \le n \le N-1$ such that $\zeta_n \ne \zeta'_n$; set

$$m := \max\{1 \le n \le N-1 \mid \zeta_n \ne \zeta'_n\}.$$

Then we see that $\zeta_m - \zeta'_m \notin Q^\vee_J$. Indeed, suppose that $\zeta_m - \zeta'_m \in Q^\vee_J$. Since $t_{\zeta_m - \zeta'_m} \in (W_J)_{\mathrm{af}}$,

$$T_{\zeta_m} = \Pi^J(t_{\zeta_m}) = \Pi^J(t_{\zeta'_m}t_{\zeta_m - \zeta'_m}) = \Pi^J(t_{\zeta'_m}) = T_{\zeta'_m},$$

which is a contradiction. Thus, $p_{I\setminus J}(\zeta_m - \zeta'_m) \ne 0$; we may assume, without loss of generality, that

$$p_{I\setminus J}(\zeta'_m - \zeta_m) \notin \sum_{i \in I\setminus J} \mathbb{Z}_{\ge 0}\alpha_i^\vee.$$

Now, by Lemma 6.5.4,

$$\sigma_N(X\eta) = [v_1 T_{\zeta_1}, v_2 T_{\zeta_2}, \dots, v_{N-1} T_{\zeta_{N-1}}, v_N T_0]$$

for some $v_1, v_2, \dots, v_N \in W_{\mathrm{af}}$ with $v_n T_{\zeta_n} \in (W^J)_{\mathrm{af}}$ for every $1 \le n \le N-1$ and $v_N T_0 \in (W^J)_{\mathrm{af}}$; note that $v_1, v_2, \dots, v_N \in (W^J)_{\mathrm{af}}$ by Lemma 2.2.8. Since $X\eta = \eta'$, it follows that $v_n T_{\zeta_n} = T_{\zeta'_n}$ for every $1 \le n \le N-1$, and $v_N T_0 = T_0$. Hence, we get $v_n = T_{\zeta'_n} T_{\zeta_n}^{-1}$ for every $1 \le n \le N-1$, and $v_N = e$; in particular, $v_{m+1} = \cdots = v_{N-1} = v_N = e$ by the definition of m. Also, as above, we see that $v_n = T_{\zeta'_n} T_{\zeta_n}^{-1} \in (W^J)_{\mathrm{af}}$, $1 \le n \le m$, is of the form: $v_n = T_{\zeta''_n}$ for some $\zeta''_n \in Q^\vee_{J\text{-adj}}$ such that $p_{I\setminus J}(\zeta''_n) = p_{I\setminus J}(\zeta'_n - \zeta_n)$. We deduce from Lemma 6.5.5 that $X\eta_e \ne \mathbf{0}$, and

$$\sigma_N(X\eta_e) = [v_1, v_2, \dots, v_N] = [T_{\zeta''_1}, T_{\zeta''_2}, \dots, T_{\zeta''_m}, e, \dots, e].$$

Then, we see that $X\eta_e \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ is of the form:

$$X\eta_e = (T_{\xi_1}, \dots, T_{\xi_{s-1}}, e\,; a_0, a_1, \dots, a_{s-1}, a_s)$$

for some $\xi_1, \dots, \xi_{s-1} \in Q^\vee_{J\text{-adj}}$ with $\xi_{s-1} = \zeta''_m$. However, since

$$p_{I\setminus J}(\xi_{s-1}) = p_{I\setminus J}(\zeta''_m) = p_{I\setminus J}(\zeta'_m - \zeta_m) \notin \sum_{i \in I\setminus J} \mathbb{Z}_{\ge 0}\alpha_i^\vee,$$

this contradicts Proposition 6.5.2. This completes the proof of Proposition 6.5.3. □

Proposition 6.5.6. *There exists a one-to-one correspondence between* $\mathrm{Par}(\lambda)$ *and the connected components of* $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$.

Proof. Let $\mathrm{Conn}(\lambda)$ be the set of connected components of $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$. First we define a map $\Theta : \mathrm{Conn}(\lambda) \to \mathrm{Par}(\lambda)$ as follows: let $C \in \mathrm{Conn}(\lambda)$. By Proposition 6.5.2, C contains a unique element of the form

$$\eta = (T_{\xi_1}, T_{\xi_2}, \dots, T_{\xi_{s-1}}, e\,; a_0, a_1, \dots, a_{s-1}, a_s)$$

for some $\xi_1, \xi_2, \dots, \xi_{s-1} \in Q^\vee_{J\text{-adj}}$; by convention, we set $\xi_s := 0$. Write $\mathrm{Turn}(\lambda) = \{k/m_i \mid i \in I \setminus J,\ 0 \le k \le m_i\}$ as:

$$\mathrm{Turn}(\lambda) = \{0 = \tau_0 < \tau_1 < \cdots < \tau_\kappa = 1\};$$

note that $i \in I(\lambda\,;\tau_l) \setminus J$ if and only if $i \notin J$ and $\tau_l = k/m_i$ for some $0 \le k \le m_i$. Recall from Proposition 6.5.2 that $a_0, a_1, \dots, a_{s-1}, a_s \in \mathrm{Turn}(\lambda)$. So, for each $0 \le u \le s$, let $0 \le \kappa_u \le \kappa$ be such that $a_u = \tau_{\kappa_u}$. Then we define ζ_l, $1 \le l \le \kappa$, by

$$\zeta_l := \xi_u \quad \text{if } \kappa_{u-1} + 1 \le l \le \kappa_u,$$

that is,

$$\underbrace{\zeta_1, \dots, \zeta_{\kappa_1}}_{:=\xi_1}, \underbrace{\zeta_{\kappa_1+1}, \dots, \zeta_{\kappa_2}}_{:=\xi_2}, \dots, \underbrace{\zeta_{\kappa_{s-2}+1}, \dots, \zeta_{\kappa_{s-1}}}_{:=\xi_{s-1}}, \underbrace{\zeta_{\kappa_{s-1}+1}, \dots, \zeta_\kappa}_{:=\xi_s=0}.$$

Then we deduce from Proposition 6.5.2 that

$$p_{I\setminus J}(\zeta_l - \zeta_{l+1}) \in \sum_{i \in I(\lambda\,;\tau_l)\setminus J} \mathbb{Z}_{\ge 0}\alpha_i^\vee \ \text{ for } 1 \le l \le \kappa - 1. \tag{6.5.2}$$

For each $i \in I \setminus J$, let $p_l^{(i)}$ be a coefficient of $\alpha_i^\vee$ in ζ_l for $1 \le l \le \kappa$. We see from (6.5.2) that

$$p_1^{(i)} \ge p_2^{(i)} \ge \cdots \ge p_{\kappa-1}^{(i)} \ge p_\kappa^{(i)} = 0,$$

and $p_l^{(i)} = p_{l+1}^{(i)}$ for $1 \le l \le \kappa - 1$ such that $i \notin I(\lambda\,;\tau_l) \setminus J$, that is, $\tau_l \notin \{k/m_i \mid 0 \le k \le m_i\}$. So, for each $1 \le k \le m_i$, set $\rho_k^{(i)} := p_l^{(i)}$ if $k/m_i = \tau_l$, and define a partition $\rho^{(i)}$ of length less than m_i by

$$\rho^{(i)} := (\rho_1^{(i)} \ge \rho_2^{(i)} \ge \dots \rho_{m_i-1}^{(i)} \ge \rho_{m_i}^{(i)} = 0).$$

Then we define $\Theta(C) := (\rho^{(i)})_{i\in I} \in \mathrm{Par}(\lambda)$, where for every $j \in J$, we set $\rho^{(j)}$ to be the empty partition for every $j \in J$.

Next we define a map $\Xi : \mathrm{Par}(\lambda) \to \mathrm{Conn}(\lambda)$ as follows: let $\boldsymbol{\rho} = (\rho^{(i)}) \in \mathrm{Par}(\lambda)$ with $\rho^{(i)} = (\rho_1^{(i)} \ge \rho_2^{(i)} \ge \cdots \ge \rho_{m_i-1}^{(i)} \ge 0)$ for $i \in I \setminus J$. Define $\zeta_l \in Q^\vee$, $1 \le l \le \kappa$, by

$$\zeta_\kappa = 0, \quad \zeta_l - \zeta_{l+1} = \sum_{i\in I(\lambda\,;\tau_l)\setminus J} \rho_{m_i\tau_l}^{(i)} \alpha_i^\vee \text{ for } 1 \le l \le \kappa - 1;$$

note that for $1 \le l \le \kappa - 1$, if $i \in I(\lambda\,;\tau_l) \setminus J$, then $m_i\tau_l \in \mathbb{Z}$ with $1 \le m_i\tau_l \le m_i - 1$. Set $s := \#\{1 \le l \le \kappa - 1 \mid \zeta_l \ne \zeta_{l+1}\} + 1$, and assume that

$$\{1 \le l \le \kappa - 1 \mid \zeta_l \ne \zeta_{l+1}\} = \{\kappa_1 < \kappa_2 < \cdots < \kappa_{s-1}\}.$$

Namely,

$$\zeta_1 = \cdots = \zeta_{\kappa_1} \ne \zeta_{\kappa_1+1} = \cdots = \zeta_{\kappa_2} \ne \cdots \ne \zeta_{\kappa_{s-1}+1} = \cdots = \zeta_\kappa = 0.$$

Then we define (for the definition of $\phi_J(\zeta_{\kappa_u})$, see Lemma 2.2.6)

$$\xi_s := 0, \quad \xi_u := \zeta_{\kappa_u} + \phi_J(\zeta_{\kappa_u}) \ \text{ for } 1 \le u \le s - 1,$$

and

$$a_0 := 0; \quad a_u := \tau_{\kappa_u} \ \text{ for } 1 \le u \le s - 1; \quad a_s := 1.$$

We deduce from Proposition 6.5.2 that

$$\eta_{\boldsymbol{\rho}} := (T_{\xi_1}, T_{\xi_2}, \dots, T_{\xi_{s-1}}, e\,; a_0, a_1, \dots, a_{s-1}, a_s) \in \mathbb{B}^{\frac{\infty}{2}}(\lambda). \tag{6.5.3}$$

So, let us define $\Xi(\boldsymbol{\rho})$ to be the connected component of $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ containing this $\eta_{\boldsymbol{\rho}}$.

We deduce from the definitions that the maps Θ and Ξ are mutually inverse. Thus we have proved the proposition. □

6.6 Proof of Proposition 3.2.4

Let $\boldsymbol{\rho} \in \mathrm{Par}(\lambda)$, and let $\eta_{\boldsymbol{\rho}} \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ be defined as (6.5.3), which is a unique element of the form (6.5.1) contained in the connected component $\mathbb{B}_{\boldsymbol{\rho}}^{\frac{\infty}{2}}(\lambda) := \Xi(\boldsymbol{\rho})$. We will show that there exists a unique isomorphism $\mathbb{B}_{\boldsymbol{\rho}}^{\frac{\infty}{2}}(\lambda) \xrightarrow{\cong} \{\boldsymbol{\rho}\} \otimes \mathbb{B}_0^{\frac{\infty}{2}}(\lambda)$ of crystals that maps $\eta_{\boldsymbol{\rho}}$ to $\boldsymbol{\rho} \otimes \eta_e$. For this, it suffices to show that the following hold for monomials X, Y in the Kashiwara operators:

(1) $X\eta_{\boldsymbol{\rho}} \ne \mathbf{0}$ in $\mathbb{B}_{\boldsymbol{\rho}}^{\frac{\infty}{2}}(\lambda)$ if and only if $X(\boldsymbol{\rho} \otimes \eta_e) \ne \mathbf{0}$ in $\{\boldsymbol{\rho}\} \otimes \mathbb{B}_0^{\frac{\infty}{2}}(\lambda)$,

(2) $X\eta_{\boldsymbol{\rho}} = Y\eta_{\boldsymbol{\rho}}$ in $\mathbb{B}^{\frac{\infty}{2}}_{\boldsymbol{\rho}}(\lambda)$ if and only if $X(\boldsymbol{\rho} \otimes \eta_e) = Y(\boldsymbol{\rho} \otimes \eta_e)$ in $\{\boldsymbol{\rho}\} \otimes \mathbb{B}^{\frac{\infty}{2}}_0(\lambda)$.

Assertion (1) follows immediately from Lemma 6.5.5 and the equality $X(\boldsymbol{\rho} \otimes \eta_e) = \boldsymbol{\rho} \otimes X\eta_e$. Let us show assertion (2). We give a proof only for the "only if" part; the proof for the "if" part is similar. Assume that $X\eta_{\boldsymbol{\rho}} = Y\eta_{\boldsymbol{\rho}}$. Let $N \in \mathbb{Z}_{>0}$ be a multiple of N_λ (see Lemma 5.2.7), and write $\sigma_N(\eta_{\boldsymbol{\rho}})$ as $\sigma_N(\eta_{\boldsymbol{\rho}}) = [T_{\xi_1}, T_{\xi_2}, \ldots, T_{\xi_N}]$ with some $\xi_1, \xi_2, \ldots, \xi_N \in Q^\vee_{J\text{-adj}}$. Also, by Lemmas 2.2.8 and 5.2.6, $\sigma_N(X\eta_{\boldsymbol{\rho}})$ and $\sigma_N(Y\eta_{\boldsymbol{\rho}})$ are of the form

$$\sigma_N(X\eta_{\boldsymbol{\rho}}) = [u_1T_{\xi_1}, u_2T_{\xi_2}, \ldots, u_NT_{\xi_N}], \qquad \sigma_N(Y\eta_{\boldsymbol{\rho}}) = [v_1T_{\xi_1}, v_2T_{\xi_2}, \ldots, v_NT_{\xi_N}]$$

with some $u_1, u_2, \ldots, u_N \in (W^J)_{\text{af}}$ and $v_1, v_2, \ldots, v_N \in (W^J)_{\text{af}}$, respectively. Then, by Lemma 6.5.5,

$$\sigma_N(X\eta_e) = [u_1, u_2, \ldots, u_N], \qquad \sigma_N(Y\eta_e) = [v_1, v_2, \ldots, v_N].$$

Since $X\eta_{\boldsymbol{\rho}} = Y\eta_{\boldsymbol{\rho}}$, we have $u_p = v_p$ for every $1 \le p \le N$. Therefore, we obtain $\sigma_N(X\eta_e) = \sigma_N(Y\eta_e)$, and hence $X\eta_e = Y\eta_e$ by the injectivity of σ_N. Thus we have proved the "only if" part of assertion (2). Thus we obtain an isomorphism $\mathbb{B}^{\frac{\infty}{2}}_{\boldsymbol{\rho}}(\lambda) \cong \{\boldsymbol{\rho}\} \otimes \mathbb{B}^{\frac{\infty}{2}}_0(\lambda)$ of crystals for each $\boldsymbol{\rho} \in \text{Par}(\lambda)$. Hence,

$$\mathbb{B}^{\frac{\infty}{2}}(\lambda) = \bigsqcup_{\boldsymbol{\rho} \in \text{Par}(\lambda)} \mathbb{B}^{\frac{\infty}{2}}_{\boldsymbol{\rho}}(\lambda) \cong \bigsqcup_{\boldsymbol{\rho} \in \text{Par}(\lambda)} \{\boldsymbol{\rho}\} \otimes \mathbb{B}^{\frac{\infty}{2}}_0(\lambda) = \text{Par}(\lambda) \otimes \mathbb{B}^{\frac{\infty}{2}}_0(\lambda).$$

Thus we have proved Proposition 3.2.4. □

A Appendix

A.1 Another definition of semi-infinite Bruhat order

Let J be a subset of I. For $vt_\zeta \in W_{\text{af}}$ with $v \in W$ and $\zeta \in Q^\vee$, we define

$$\ell^{\frac{\infty}{2}}_J(vt_\zeta) := \ell(\lfloor v \rfloor) + 2\langle \zeta, \rho - \rho_J \rangle. \tag{A.1.1}$$

Lemma A.1.1. *The equalities* $\ell^{\frac{\infty}{2}}_J(x) = \ell^{\frac{\infty}{2}}_J(\Pi^J(x)) = \ell^{\frac{\infty}{2}}(\Pi^J(x))$ *hold for all* $x \in W_{\text{af}}$.

Proof. Write $\Pi^J(x) = wz_\xi t_\xi$ with $w \in W^J$ and $\xi \in Q^\vee_{J\text{-adj}}$. The second equality follows from Lemma 4.3.6 and $\ell(wz_\xi) = \ell(w) + \ell(z_\xi)$.

In order to prove the first equality, we write $x = x_1x_2$ with $x_1 \in (W^J)_{\text{af}}$ and $x_2 \in (W_J)_{\text{af}}$; note that $\Pi^J(x) = x_1$. By (2.2.5), $x_1 = w_1z_{\xi_1}t_{\xi_1}$ for some $w_1 \in W^J$ and $\xi_1 \in Q^\vee_{J\text{-adj}}$, and by (2.2.3), $x_2 = w_2t_{\xi_2}$ for some $w_2 \in W_J$ and $\xi_2 \in Q^\vee_J$. Since $x = x_1x_2 = w_1z_{\xi_1}w_2t_{w_2^{-1}\xi_1+\xi_2}$, we have

$$\begin{aligned}
\ell^{\frac{\infty}{2}}_J(x) &= \ell(\lfloor w_1z_{\xi_1}w_2 \rfloor) + 2\langle w_2^{-1}\xi_1 + \xi_2, \rho - \rho_J \rangle \\
&= \ell(w_1) + 2\langle w_2^{-1}\xi_1 + \xi_2, \rho - \rho_J \rangle && \text{(since } w_1 \in W^J \text{ and } z_{\xi_1}w_2 \in W_J) \\
&= \ell(w_1) + 2\langle \xi_1, \rho - \rho_J \rangle && \text{(since } \xi_2 \in Q^\vee_J \text{ and } w_2 \in W_J \text{ (see (2.1.7)))} \\
&= \ell^{\frac{\infty}{2}}_J(w_1z_{\xi_1}t_{\xi_1}) = \ell^{\frac{\infty}{2}}_J(\Pi^J(x)).
\end{aligned}$$

Thus we have proved the lemma. □

Proposition A.1.2. *Let* $x, y \in (W^J)_{\text{af}}$ *and* $\beta \in \Delta^+_{\text{af}}$. *We have* $x \xrightarrow{\beta} y$ *in* SB^J *if and only if the following three conditions hold:*

(a) $y = \Pi^J(r_\beta x)$;

(b) $\ell_J^{\frac{\infty}{2}}(r_\beta x) = \ell_J^{\frac{\infty}{2}}(x) + 1$;

(c) *Write x as $x = wz_\xi t_\xi$ with $w \in W^J$ and $\xi \in Q^\vee_{J\text{-adj}}$. Then, $\beta = w\alpha + \chi\delta$ for some $\alpha \in \Delta^+ \setminus \Delta_J^+$ and $\chi \in \{0, 1\}$.*

Proof. The "only if" part follows immediately from Lemmas 4.3.5 and A.1.1. We show the "if" part. By (c), we have $r_\beta x = wr_\alpha z_\xi t_{\xi + \chi z_\xi^{-1}\alpha^\vee}$. We deduce that

$$\begin{aligned} 1 &= \ell_J^{\frac{\infty}{2}}(r_\beta x) - \ell_J^{\frac{\infty}{2}}(x) && \text{(by (b))} \\ &= \ell(\lfloor wr_\alpha \rfloor) + 2\langle \xi + \chi z_\xi^{-1}\alpha^\vee, \rho - \rho_J \rangle - \ell(w) - 2\langle \xi, \rho - \rho_J \rangle \\ &= \ell(\lfloor wr_\alpha \rfloor) - \ell(w) + 2\chi\langle \alpha^\vee, \rho - \rho_J \rangle, && \text{(By (2.1.7))} \end{aligned}$$

which implies that $w \xrightarrow[\mathsf{B}]{\alpha} \lfloor wr_\alpha \rfloor$ (resp., $w \xrightarrow[\mathsf{Q}]{\alpha} \lfloor wr_\alpha \rfloor$) in QB^J if $\chi = 0$ (resp., $\chi = 1$). Therefore, we obtain $y = r_\beta x \in (W^J)_{\mathrm{af}}$ and $x \xrightarrow{\beta} y$ in SB^J by Proposition 4.3.7 (2). □

A.2 Relation between the semi-infinite Bruhat order and the generic Bruhat order

In this subsection, we assume that $J = \emptyset$; note that $(W^J)_{\mathrm{af}} = W_{\mathrm{af}}$. Fix an arbitrary element $\xi \in Q^\vee$ such that $\langle \xi, \alpha_i \rangle > 0$ for all $i \in I$. We know from [Pet97] (see also [LNSSS13a, Theorem 5.2] together with Proposition 4.3.7) that for $x, y \in (W^J)_{\mathrm{af}} = W_{\mathrm{af}}$, $x \le_{\frac{\infty}{2}} y$ if and only if there exists $N \in \mathbb{Z}_{\ge 0}$, depending on x, y, and ξ, such that $yt_{-n\xi} \le xt_{-n\xi}$ (or equivalently, $t_{n\xi}y^{-1} \le t_{n\xi}x^{-1}$) for all $n \in \mathbb{Z}_{\ge N}$; here, recall that $\le$ is an (ordinary) Bruhat order on W_{af}. On the other hand, in [Lus80, §1.5], Lusztig introduced a partial order $\le_L$ on W_{af}, which we call Lusztig's generic Bruhat order; we know from [Soe97, Claim 4.14 in the proof of Lemma 4.13] that $x \le_L y$ if and only if there exists $N \in \mathbb{Z}_{\ge 0}$, depending on x, y, and ξ, such that $t_{n\xi}x \le t_{n\xi}y$ for all $n \in \mathbb{Z}_{\ge N}$. Combining these facts, we obtain

Lemma A.2.1. *Let $x, y \in W_{\mathrm{af}}$. We have $x \le_{\frac{\infty}{2}} y$ if and only if $y^{-1} \le_L x^{-1}$.*

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