



# FROM THE BIOT-SAVART LAW TO AMPÈRE'S MAGNETIC CIRCUITAL LAW VIA SYNTHETIC DIFFERENTIAL GEOMETRY

**Hirokazu Nishimura**

Institute of Mathematics

University of Tsukuba

Tsukuba, Ibaraki, 305-8571, Japan

## Abstract

It is well known in classical electrodynamics that the magnetic field given by a current loop and the electric field caused by the corresponding electric dipoles in sheets are very similar, as far as we are far away from the loop, which enables us to deduce Ampère's magnetic circuital law from the Biot-Savart law easily. The principal objective in this paper is to show that synthetic differential geometry, in which nilpotent infinitesimals are available in abundance, furnishes out a natural framework for the exquisite formulation of this similitude and its demonstration. This similitude in heaven enables us to transit from the Biot-Savart law to Ampère's magnetic circuital law like a shot on earth.

## 1. Introduction

It is well known among physicists (see, e.g., [9]) that the magnetic field given by a *current loop* and the electric field caused by the corresponding *electric dipoles* in sheets are very similar, as far as we are far away from the

---

Received: January 25, 2014; Accepted: February 17, 2014

2010 Mathematics Subject Classification: 58A03, 78-02.

Keywords and phrases: Biot-Savart law, Ampère's magnetic circuital law, classical electromagnetism, vector calculus, synthetic differential geometry, Coulomb's law.

loop, which enables us to deduce *Ampère's magnetic circuital law* from the *Biot-Savart law* easily. However, a mathematically satisfactory formulation of this similitude is by no means easy, let alone its proof based upon the Coulomb and Biot-Savart laws.

In good old days of the 17th and 18th centuries, mathematicians and physicists could communicate easily with ones of the other species, and many excellent mathematicians were physicists at the same time and vice versa. The honeymoon was over when mathematicians rushed into eradication of their shabby *nilpotent infinitesimals* by replacing them with their authentic  $\varepsilon$ - $\delta$  arguments.

In the middle of the 20th century, moribund nilpotent infinitesimals were resurrected in not earthly but heavenly manners by *synthetic differential geometers*. They have constructed another world of mathematics, called a *well-adapted model* (a kind of Grothendieck toposes), in which they could indulge themselves in their favorite nilpotent infinitesimals. We have a route from the earth to heaven (*internalization*) and another route in the opposite direction (*externalization*), so that our synthetic formulation and demonstration of the similitude is of earthly significance. For synthetic differential geometry, the reader is referred to [2] and [3].

The very similitude is formulated and established synthetically in Section 4, which is preceded by a synthetical approach to electric dipoles in sheets in Section 3. Once the similitude is firmly established within a well-adapted model, some of its consequences are externalized, which enables us to derive the Ampère's magnetic circuital law from the Biot-Savart law, as is seen in Section 5. In a subsequent paper, we will discuss Vassiliev invariants in knot theory (cf. [7] and [8]) from this standpoint.

## 2. Preliminaries

In this section, we fix our notation for static electric fields and static magnetic fields. Since we would like to concentrate upon mathematical aspects, we omit unnecessary physical constants or the like from this standpoint.

## 2.1. Static electric fields

Given a figure  $\Omega$  in  $\mathbf{R}^3$  and a mapping  $q : \Omega \rightarrow \mathbf{R}$  (as the density of electric charge), the static electric field  $\mathbf{E}_{(\Omega, q)} : \mathbf{R}^3 \rightarrow \mathbf{R}^3$  associated with  $(\Omega, q)$  is given by an integral. Namely, the Coulomb law tells us that

$$\mathbf{E}_{(\Omega, q)}(\mathbf{x}) = \int_{\Omega} \frac{q(\mathbf{p})(\mathbf{x} - \mathbf{p})}{\|\mathbf{x} - \mathbf{p}\|^3} dp$$

for any  $\mathbf{x} \in \mathbf{R}^3$ , where the integral is the volume integral, the surface integral or the line integral according to whether the figure  $\Omega$  is three-dimensional, two-dimensional or one-dimensional. As is well known, the following Maxwell equations obtain:

$$\operatorname{div} \mathbf{E}_{(\Omega, q)} = 4\pi q, \quad (1)$$

$$\operatorname{rot} \mathbf{E}_{(\Omega, q)} = \mathbf{0}. \quad (2)$$

Now we consider electric dipoles. Let  $S$  be an oriented surface in  $\mathbf{R}^3$  and  $\sigma, h \in \mathbf{R}$ . Let  $\mathbf{n}_S : S \rightarrow \mathbf{R}^3$  be the unit normal in the positive direction. We slide the surface  $S$  by  $\frac{h}{2} \mathbf{n}_S$  to get the surface  $S_{\frac{h}{2}}$ . The surface  $S_{\frac{h}{2}}$  endowed with the constant density  $\sigma$  of electric charge gives rise to the static electric field  $\mathbf{E}_{(S, \sigma, h)}^+$  by the Coulomb law. Similarly, we slide the surface  $S$  by  $-\frac{h}{2} \mathbf{n}_S$  to get the surface  $S_{-\frac{h}{2}}$ . The surface  $S_{-\frac{h}{2}}$  endowed with the constant density  $-\sigma$  of electric charge gives rise to the static electric field  $\mathbf{E}_{(S, \sigma, h)}^-$  by the Coulomb law. They together yield the static electric field

$$\mathbf{E}_{(S, \sigma, h)}^{dp} = \mathbf{E}_{(S, \sigma, h)}^+ + \mathbf{E}_{(S, \sigma, h)}^-$$

by the Coulomb law.

## 2.2. The Biot-Savart law and Ampère's magnetic circuital law

The static magnetic field caused by a current loop is given by the so-called Biot-Savart law, so that, given a loop  $C : t \in [0, t_0] \mapsto \mathbf{m}(t) \in \mathbf{R}^3$ , it gives rise to its static magnetic field  $\mathbf{B}_C$  by

$$\begin{aligned} \mathbf{B}_C(\mathbf{x}) &= \frac{1}{4\pi} \int_0^{t_0} \frac{(\mathbf{x} - \mathbf{m}(t)) \times \frac{d\mathbf{m}}{dt}(t)}{\|\mathbf{x} - \mathbf{m}(t)\|^3} dt \\ &= \frac{1}{4\pi} \int_C \frac{(\mathbf{x} - \mathbf{r}) \times d\mathbf{r}}{\|\mathbf{x} - \mathbf{r}\|^3} \end{aligned} \quad (3)$$

for any  $\mathbf{x} \in \mathbf{R}^3$ , where  $\mathbf{r}$  moves along the curve  $C$ . Given another loop  $L : s \in [0, s_0] \mapsto \mathbf{l}(s) \in \mathbf{R}^3$ , Ampère's magnetic circuital law claims that

$$\begin{aligned} &\frac{1}{4\pi} \int_0^{s_0} ds \int_0^{t_0} dt \frac{\left( (\mathbf{l}(s) - \mathbf{m}(t)) \times \frac{d\mathbf{m}}{dt}(t) \right) \cdot \frac{d\mathbf{l}}{ds}(s)}{\|\mathbf{l}(s) - \mathbf{m}(t)\|^3} \\ &= \frac{1}{4\pi} \int_L \int_C \frac{((\mathbf{s} - \mathbf{r}) \times d\mathbf{r}) \cdot d\mathbf{s}}{\|\mathbf{s} - \mathbf{r}\|^3} \\ &= \mathbf{Lk}(C, L), \end{aligned} \quad (4)$$

where  $\mathbf{s}$  moves along the curve  $L$ , and  $\mathbf{Lk}(C, L)$  is defined as follows:

**Definition 1.** Let  $S$  be an oriented surface with its induced oriented boundary  $L$ , which is supposed to be transversal to  $C$  at their intersecting points. They are enumerated as

$$S \cap C = \{\mathbf{p}_1, \dots, \mathbf{p}_k\}.$$

We define  $\varepsilon_i$  ( $i = 1, \dots, k$ ) to be 1 if the tangent of  $C$  at  $\mathbf{p}_i$  transits  $S$  into the part that the orientation of  $S$  selects, and  $-1$  otherwise. Now we define

$$\mathbf{Lk}(C, L) = \sum_{i=1}^n \varepsilon_i.$$

The reader should note that the definition is independent of our choice of  $S$ .

Topology tells us that

**Proposition 2.** *The number  $\mathbf{Lk}(C, L)$  has the following properties:*

(1) *It is symmetric in the sense that*

$$\mathbf{Lk}(C, L) = \mathbf{Lk}(L, C).$$

(2) *For any oriented surface  $S$  with  $\partial S = L \cup -L'$ , if it does not intersect  $C$ , then we have*

$$\mathbf{Lk}(C, L) = \mathbf{Lk}(C, L'),$$

where  $-L'$  denotes the same curve  $L'$  with the orientation reversed.

**Notation 3.** The first and the second formulas of (4) are denoted by

$$\mathbf{A}(C, L).$$

### 3. Synthetic Differential Geometry of Electric Dipoles in Infinitesimal Sheets

In this and the subsequent sections, we are working within a well-adapted model.

**Notation 4.** We denote by  $\mathbb{R}$  the set of real numbers containing nilpotent infinitesimals in abundance (called a line object in synthetic differential geometry). We denote by  $\mathbb{R}_+$  the set

$$\{x \in \mathbb{R} \mid x > 0\}.$$

We denote by  $D$  the set

$$\{d \in \mathbb{R} \mid d^2 = 0\}.$$

Intuitively,  $D$  stands for the set of first-order infinitesimals.

Let  $m$  be an integer and  $n$  be a natural number. For the mapping

$$x \in \mathbb{R}_+ \mapsto x^m \in \mathbb{R},$$

we have

$$(x + d)^m = x^m + mx^{m-1}d \quad (5)$$

for any  $d \in D$ , as is well known. For the mapping

$$x \in \mathbb{R}_+ \mapsto x^{\frac{m}{n}} \in \mathbb{R},$$

we have

**Lemma 5.**

$$(x + d)^{\frac{m}{n}} = x^{\frac{m}{n}} + \frac{m}{n} x^{\frac{m}{n}-1} d.$$

**Proof.** By the Kock-Lawvere axiom, there exists a unique  $a \in \mathbb{R}$  such that

$$(x + d)^{\frac{m}{n}} = x^{\frac{m}{n}} + ad$$

for any  $d \in D$ . On one hand, we have

$$((x + d)^{\frac{m}{n}})^n = (x + d)^m = x^m + mx^{m-1}d$$

by (5). On the other hand, we have

$$(x^{\frac{m}{n}} + ad)^n = (x^{\frac{m}{n}})^n + n(x^{\frac{m}{n}})^{n-1}ad$$

by the binomial theorem. Therefore, we have

$$mx^{m-1} = n(x^{\frac{m}{n}})^{n-1}a$$

so that

$$a = \frac{m}{n} x^{m-1} (x^{\frac{m}{n}})^{1-n} = \frac{m}{n} x^{\frac{n(m-1)+m(1-n)}{n}} = \frac{m}{n} x^{\frac{m}{n}-1}. \quad \square$$

**Corollary 6.** Let  $\mathbf{x}, \mathbf{a} \in \mathbb{R}^3$  with  $\mathbf{x} \neq \mathbf{0}$ . Then we have

$$\|\mathbf{x} + \mathbf{a}d\|^{-3} = \|\mathbf{x}\|^{-3} - 3\|\mathbf{x}\|^{-5}(\mathbf{x} \cdot \mathbf{a})d$$

for any  $d \in D$ , where  $\|\mathbf{x}\|$  is the standard norm of  $\mathbf{x}$  (i.e.,  $\|\mathbf{x}\| = \sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2}$  with  $\mathbf{x} = (x_1, x_2, x_3)$ ) and  $\cdot$  stands for the inner product.

**Proof.** We have

$$\begin{aligned} \|\mathbf{x} + \mathbf{a}d\|^{-3} &= (\|\mathbf{x} + \mathbf{a}d\|^2)^{-\frac{3}{2}} \\ &= ((\mathbf{x} + \mathbf{a}d) \cdot (\mathbf{x} + \mathbf{a}d))^{-\frac{3}{2}} \\ &= ((\mathbf{x} \cdot \mathbf{x}) + 2(\mathbf{x} \cdot \mathbf{a})d)^{-\frac{3}{2}} \\ &= \|\mathbf{x}\|^{-3} - 3\|\mathbf{x}\|^{-5}(\mathbf{x} \cdot \mathbf{a})d \quad [\text{by Lemma 5}]. \quad \square \end{aligned}$$

**Proposition 7.** Let  $d, e, h \in D$ ,  $\sigma \in \mathbb{R}$  and  $\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{r} \in \mathbb{R}^3$  with  $\mathbf{x} \neq \mathbf{r}$ ,  $\mathbf{a} \times \mathbf{b} \neq \mathbf{0}$ . Let  $S$  be the infinitesimal parallelogram spanned by  $\mathbf{x}$ ,  $\mathbf{x} + \mathbf{d}\mathbf{a}$  and  $\mathbf{x} + \mathbf{e}\mathbf{b}$ :

$$\begin{array}{ccccc} & & \mathbf{e}\mathbf{b} & & \\ & & \rightarrow & & \mathbf{x} + \mathbf{e}\mathbf{b} \\ \mathbf{x} & & \downarrow & & \downarrow \\ \mathbf{d}\mathbf{a} & \downarrow & S & \downarrow & \mathbf{d}\mathbf{a} \\ & & \rightarrow & & \\ \mathbf{x} + \mathbf{d}\mathbf{a} & & \mathbf{e}\mathbf{b} & & \mathbf{x} + \mathbf{d}\mathbf{a} + \mathbf{e}\mathbf{b} \end{array}$$

Then we have

$$\mathbf{E}_{(S, \sigma, h)}^{dp}(\mathbf{r}) = \frac{h\sigma de}{\|\mathbf{r} - \mathbf{x}\|^3} \left( 3 \left( \frac{\mathbf{r} - \mathbf{x}}{\|\mathbf{r} - \mathbf{x}\|} \cdot (\mathbf{a} \times \mathbf{b}) \right) \frac{\mathbf{r} - \mathbf{x}}{\|\mathbf{r} - \mathbf{x}\|} - (\mathbf{a} \times \mathbf{b}) \right).$$

**Proof.**

$$\begin{aligned} \mathbf{E}_{(S, \sigma, h)}^+(\mathbf{r}) &= \sigma de \|\mathbf{a} \times \mathbf{b}\| \left\| \mathbf{r} - \left( \mathbf{x} + \frac{h}{2} \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|} \right) \right\|^{-3} \left( \mathbf{r} - \left( \mathbf{x} + \frac{h}{2} \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|} \right) \right) \\ &= \sigma de \|\mathbf{a} \times \mathbf{b}\| \left\| (\mathbf{r} - \mathbf{x}) - \frac{h}{2} \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|} \right\|^{-3} \left( (\mathbf{r} - \mathbf{x}) - \frac{h}{2} \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|} \right) \end{aligned}$$

$$\begin{aligned}
&= \sigma de \|\mathbf{a} \times \mathbf{b}\| \left( \|\mathbf{r} - \mathbf{x}\|^{-3} + \frac{3h((\mathbf{r} - \mathbf{x}) \cdot (\mathbf{a} \times \mathbf{b}))}{2\|\mathbf{a} \times \mathbf{b}\|} \|\mathbf{r} - \mathbf{x}\|^{-5} \right) \\
&\quad \left( (\mathbf{r} - \mathbf{x}) - \frac{h}{2} \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|} \right) \quad [\text{by Corollary 6}].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\mathbf{E}_{(S, \sigma, h)}^-(\mathbf{r}) &= -\sigma de \|\mathbf{a} \times \mathbf{b}\| \left\| \mathbf{r} - \left( \mathbf{x} - \frac{h}{2} \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|} \right) \right\|^{-3} \left( \mathbf{r} - \left( \mathbf{x} - \frac{h}{2} \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|} \right) \right) \\
&= -\sigma de \|\mathbf{a} \times \mathbf{b}\| \left\| (\mathbf{r} - \mathbf{x}) + \frac{h}{2} \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|} \right\|^{-3} \left( (\mathbf{r} - \mathbf{x}) + \frac{h}{2} \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|} \right) \\
&= -\sigma de \|\mathbf{a} \times \mathbf{b}\| \left( \|\mathbf{r} - \mathbf{x}\|^{-3} - \frac{3h((\mathbf{r} - \mathbf{x}) \cdot (\mathbf{a} \times \mathbf{b}))}{2\|\mathbf{a} \times \mathbf{b}\|} \|\mathbf{r} - \mathbf{x}\|^{-5} \right) \\
&\quad \left( (\mathbf{r} - \mathbf{x}) + \frac{h}{2} \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|} \right) \quad [\text{by Corollary 6}].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\mathbf{E}_{(S, \sigma, h)}^{bp}(\mathbf{r}) \\
&= \mathbf{E}_{(S, \sigma, h)}^+(\mathbf{r}) + \mathbf{E}_{(S, \sigma, h)}^-(\mathbf{r}) \\
&= \sigma de \|\mathbf{a} \times \mathbf{b}\| \left( \frac{3h((\mathbf{r} - \mathbf{x}) \cdot (\mathbf{a} \times \mathbf{b})) \|\mathbf{r} - \mathbf{x}\|^{-5}}{\|\mathbf{a} \times \mathbf{b}\|} (\mathbf{r} - \mathbf{x}) - \frac{h \|\mathbf{r} - \mathbf{x}\|^{-3}}{\|\mathbf{a} \times \mathbf{b}\|} (\mathbf{a} \times \mathbf{b}) \right) \\
&= \frac{h\sigma de}{\|\mathbf{r} - \mathbf{x}\|^3} \left( 3 \left( \frac{\mathbf{r} - \mathbf{x}}{\|\mathbf{r} - \mathbf{x}\|} \cdot (\mathbf{a} \times \mathbf{b}) \right) \frac{\mathbf{r} - \mathbf{x}}{\|\mathbf{r} - \mathbf{x}\|} - (\mathbf{a} \times \mathbf{b}) \right). \quad \square
\end{aligned}$$

#### 4. The Similitude between the Electric Fields of Electric Dipoles in Sheets and the Magnetic Fields of Current Loops within Synthetic Differential Geometry

The principal objective in this section is to establish the similitude



between the electric fields of dipoles in sheets and the magnetic fields of current loops synthetically. The discussion is very similar to that in Stokes' Theorem, for which the reader is referred to [4], [5] and [6]. Let us begin with

**Lemma 8.** *For any vectors  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$  and any unit vector  $\hat{\mathbf{r}} \in \mathbb{R}^3$ , we have*

$$((\mathbf{a} \times \mathbf{b}) \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} = \mathbf{a} \times \mathbf{b} + (\hat{\mathbf{r}} \cdot \mathbf{a})\mathbf{b} \times \hat{\mathbf{r}} - (\hat{\mathbf{r}} \cdot \mathbf{b})\mathbf{a} \times \hat{\mathbf{r}}.$$

**Proof.** Fixing arbitrarily  $\hat{\mathbf{r}} = (\hat{r}_1, \hat{r}_2, \hat{r}_3)$  with  $(\hat{r}_1)^2 + (\hat{r}_2)^2 + (\hat{r}_3)^2 = 1$ , both the left-hand and the right-hand of the above formula can be regarded as functions of  $(\mathbf{a}, \mathbf{b}) = ((\hat{a}_1, \hat{a}_2, \hat{a}_3), (\hat{b}_1, \hat{b}_2, \hat{b}_3)) \in \mathbb{R}^3 \times \mathbb{R}^3$ . It is easy to see that both functions are bilinear, so that it suffices to show the above formula in cases of  $\mathbf{a} = \mathbf{i}, \mathbf{j}, \mathbf{k}$  and  $\mathbf{b} = \mathbf{i}, \mathbf{j}, \mathbf{k}$ , where  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  are the standard base of  $\mathbb{R}^3$ , namely,  $\mathbf{i} = (1, 0, 0)$ ,  $\mathbf{j} = (0, 1, 0)$  and  $\mathbf{k} = (0, 0, 1)$ . In case of  $\mathbf{a} = \mathbf{b}$ , it is easy to see that both sides degenerate into  $\mathbf{0}$ . In case of  $\mathbf{a} = \mathbf{i}$  and  $\mathbf{b} = \mathbf{j}$ , we have  $\mathbf{a} \times \mathbf{b} = \mathbf{k}$ , so that the left-hand is  $(\hat{r}_3\hat{r}_1, \hat{r}_3\hat{r}_2, (\hat{r}_3)^2)$ , while the right-hand is

$$\begin{aligned} & (0, 0, 1) + \hat{r}_1(\mathbf{j} \times \hat{\mathbf{r}}) - \hat{r}_2(\mathbf{i} \times \hat{\mathbf{r}}) \\ &= (0, 0, 1) + (\hat{r}_1\hat{r}_3, 0, -(\hat{r}_1)^2) - (0, -\hat{r}_2\hat{r}_3, (\hat{r}_2)^2) \\ &= (\hat{r}_3\hat{r}_1, \hat{r}_3\hat{r}_2, (\hat{r}_3)^2) \\ & \text{[since } (\hat{r}_1)^2 + (\hat{r}_2)^2 + (\hat{r}_3)^2 \text{ is equal to 1].} \end{aligned}$$

The remaining five cases are safely left to the reader. □

**Theorem 9** (The infinitesimal similitude). *Let  $d, e \in D$  and  $\mathbf{x}, \mathbf{a}, \mathbf{b} \in \mathbb{R}^3$  with  $\mathbf{a} \times \mathbf{b} \neq \mathbf{0}$ . Let  $C$  be the infinitesimal oriented curve moving from  $\mathbf{x}$  to  $\mathbf{x} + d\mathbf{a}$  by  $d\mathbf{a}$ , moving from  $\mathbf{x} + d\mathbf{a}$  to  $\mathbf{x} + d\mathbf{a} + e\mathbf{b}$  by  $e\mathbf{b}$ , moving from  $\mathbf{x} + d\mathbf{a} + e\mathbf{b}$  to  $\mathbf{x} + e\mathbf{b}$  by  $-d\mathbf{a}$  and finally moving from  $\mathbf{x} + e\mathbf{b}$  to the start  $\mathbf{x}$  by  $-e\mathbf{b}$ :*

$$\begin{array}{ccccc}
& & -e\mathbf{b} & & \\
& \mathbf{x} & & & \mathbf{x} + e\mathbf{b} \\
& & \leftarrow & & \\
d\mathbf{a} & \downarrow & S & \uparrow & -d\mathbf{a} \\
& & \rightarrow & & \\
\mathbf{x} + d\mathbf{a} & & e\mathbf{b} & & \mathbf{x} + d\mathbf{a} + e\mathbf{b}
\end{array}$$

Let  $S$  be the infinitesimal oriented parallelogram spanned by  $\mathbf{x}$ ,  $\mathbf{x} + d\mathbf{a}$  and  $\mathbf{x} + e\mathbf{b}$  with its induced oriented boundary  $C$ . Let  $h \in D$  and  $\sigma \in \mathbb{R}$ . Then we have

$$\mathbf{E}_{(S,1,h)}^{dp}(\mathbf{r}) = h\mathbf{B}_C(\mathbf{r})$$

for any  $\mathbf{r} \in \mathbb{R}^3$  with  $\mathbf{r} \neq \mathbf{x}$ .

**Proof.** On one hand, thanks to Proposition 7, we have

$$\mathbf{E}_{(S,1,h)}^{dp}(\mathbf{r}) = \frac{hde}{\|\mathbf{r} - \mathbf{x}\|^3} \left( 3 \left( \frac{\mathbf{r} - \mathbf{x}}{\|\mathbf{r} - \mathbf{x}\|} \cdot (\mathbf{a} \times \mathbf{b}) \right) \frac{\mathbf{r} - \mathbf{x}}{\|\mathbf{r} - \mathbf{x}\|} - (\mathbf{a} \times \mathbf{b}) \right).$$

On the other hand, we have

$$\begin{aligned}
& \mathbf{B}_C(\mathbf{r}) \\
&= \frac{d\mathbf{a} \times (\mathbf{r} - \mathbf{x})}{\|\mathbf{r} - \mathbf{x}\|^3} + \frac{e\mathbf{b} \times (\mathbf{r} - (\mathbf{x} + d\mathbf{a}))}{\|\mathbf{r} - (\mathbf{x} + d\mathbf{a})\|^3} - \frac{d\mathbf{a} \times (\mathbf{r} - (\mathbf{x} + e\mathbf{b}))}{\|\mathbf{r} - (\mathbf{x} + e\mathbf{b})\|^3} - \frac{e\mathbf{b} \times (\mathbf{r} - \mathbf{x})}{\|\mathbf{r} - \mathbf{x}\|^3} \\
&= \frac{d\mathbf{a} \times (\mathbf{r} - \mathbf{x})}{\|\mathbf{r} - \mathbf{x}\|^3} + \frac{e\mathbf{b} \times ((\mathbf{r} - \mathbf{x}) - d\mathbf{a})}{\|(\mathbf{r} - \mathbf{x}) - d\mathbf{a}\|^3} - \frac{d\mathbf{a} \times ((\mathbf{r} - \mathbf{x}) - e\mathbf{b})}{\|(\mathbf{r} - \mathbf{x}) - e\mathbf{b}\|^3} - \frac{e\mathbf{b} \times (\mathbf{r} - \mathbf{x})}{\|\mathbf{r} - \mathbf{x}\|^3} \\
&= \frac{d\mathbf{a} \times (\mathbf{r} - \mathbf{x})}{\|\mathbf{r} - \mathbf{x}\|^3} + (e\mathbf{b} \times ((\mathbf{r} - \mathbf{x}) - d\mathbf{a})) (\|\mathbf{r} - \mathbf{x}\|^{-3} + 3\|\mathbf{r} - \mathbf{x}\|^{-5}((\mathbf{r} - \mathbf{x}) \cdot \mathbf{a})d) \\
&\quad - (d\mathbf{a} \times ((\mathbf{r} - \mathbf{x}) - e\mathbf{b})) (\|\mathbf{r} - \mathbf{x}\|^{-3} + 3\|\mathbf{r} - \mathbf{x}\|^{-5}((\mathbf{r} - \mathbf{x}) \cdot \mathbf{b})e) \\
&\quad - \frac{e\mathbf{b} \times (\mathbf{r} - \mathbf{x})}{\|\mathbf{r} - \mathbf{x}\|^3}
\end{aligned}$$

$$\begin{aligned}
 &= \left\{ (e\mathbf{b} \times ((\mathbf{r} - \mathbf{x}) - d\mathbf{a})) (\|\mathbf{r} - \mathbf{x}\|^{-3} + 3\|\mathbf{r} - \mathbf{x}\|^{-5} ((\mathbf{r} - \mathbf{x}) \cdot \mathbf{a})d) - \frac{e\mathbf{b} \times (\mathbf{r} - \mathbf{x})}{\|\mathbf{r} - \mathbf{x}\|^3} \right\} \\
 &\quad - \left\{ (d\mathbf{a} \times ((\mathbf{r} - \mathbf{x}) - e\mathbf{b})) (\|\mathbf{r} - \mathbf{x}\|^{-3} + 3\|\mathbf{r} - \mathbf{x}\|^{-5} ((\mathbf{r} - \mathbf{x}) \cdot \mathbf{b})e) - \frac{d\mathbf{a} \times (\mathbf{r} - \mathbf{x})}{\|\mathbf{r} - \mathbf{x}\|^3} \right\} \\
 &= \{-de\|\mathbf{r} - \mathbf{x}\|^{-3}(\mathbf{b} \times \mathbf{a}) + 3de\|\mathbf{r} - \mathbf{x}\|^{-5}((\mathbf{r} - \mathbf{x}) \cdot \mathbf{a})(\mathbf{b} \times (\mathbf{r} - \mathbf{x}))\} \\
 &\quad - \{-de\|\mathbf{r} - \mathbf{x}\|^{-3}(\mathbf{a} \times \mathbf{b}) + 3de\|\mathbf{r} - \mathbf{x}\|^{-5}((\mathbf{r} - \mathbf{x}) \cdot \mathbf{b})(\mathbf{a} \times (\mathbf{r} - \mathbf{x}))\} \\
 &= de\|\mathbf{r} - \mathbf{x}\|^{-3} \left\{ 2(\mathbf{a} \times \mathbf{b}) + 3 \left( \frac{\mathbf{r} - \mathbf{x}}{\|\mathbf{r} - \mathbf{x}\|} \cdot \mathbf{a} \right) \left( \mathbf{b} \times \frac{\mathbf{r} - \mathbf{x}}{\|\mathbf{r} - \mathbf{x}\|} \right) \right. \\
 &\quad \left. - 3 \left( \frac{\mathbf{r} - \mathbf{x}}{\|\mathbf{r} - \mathbf{x}\|} \cdot \mathbf{b} \right) \left( \mathbf{a} \times \frac{\mathbf{r} - \mathbf{x}}{\|\mathbf{r} - \mathbf{x}\|} \right) \right\}.
 \end{aligned}$$

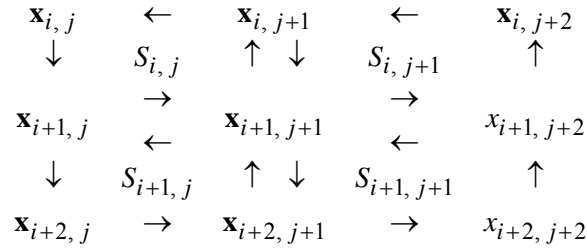
Therefore, the desired result follows by dint of Lemma 8.  $\square$

**Theorem 10** (The general similitude). *Let  $S$  be an oriented surface with its induced oriented boundary  $C$ . Let  $h \in D$ . Then we have*

$$\mathbf{E}_{(S,1,h)}^{dp}(\mathbf{r}) = h\mathbf{B}_C(\mathbf{r})$$

for any  $\mathbf{r} \in \mathbb{R}^3$  with  $\mathbf{r} \notin S$ .

**Proof.** We divide the oriented surface  $S$  into  $MN$  infinitesimal oriented parallelograms, where  $M$  and  $N$  are very great natural numbers. It is depicted partially and schematically in the following diagram:



Then surely we have

$$\mathbf{E}_{(S,1,h)}^{dp}(\mathbf{r}) = \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \mathbf{E}_{(S_i,j,1,h)}^{dp}(\mathbf{r}). \quad (6)$$

Proposition 7 enables us to conclude that

$$\sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \mathbf{E}_{(S_i,j,1,h)}^{dp}(\mathbf{r}) = h \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \mathbf{B}_{C_{i,j}}(\mathbf{r}). \quad (7)$$

The boundary  $C_{i,j}$  of the infinitesimal parallelogram  $S_{i,j}$  consists of the infinitesimal segment from  $\mathbf{x}_{i,j}$  to  $\mathbf{x}_{i+1,j}$ , that from  $\mathbf{x}_{i+1,j}$  to  $\mathbf{x}_{i+1,j+1}$ , that from  $\mathbf{x}_{i+1,j+1}$  to  $\mathbf{x}_{i,j+1}$  and that from  $\mathbf{x}_{i,j+1}$  to  $\mathbf{x}_{i,j}$ . Unless  $i = M - 1$ , the second segment from  $\mathbf{x}_{i+1,j}$  to  $\mathbf{x}_{i+1,j+1}$  is shared by the infinitesimal parallelogram  $S_{i+1,j}$  as its boundary in the opposite direction. Similarly, unless  $j = N - 1$ , the third segment from  $\mathbf{x}_{i+1,j+1}$  to  $\mathbf{x}_{i,j+1}$  is shared by the infinitesimal parallelogram  $S_{i,j+1}$  as its boundary in the opposite direction. Therefore, we have

$$\sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \mathbf{B}_{C_{i,j}}(\mathbf{r}) = \mathbf{B}_C(\mathbf{r}). \quad (8)$$

Therefore, the desired formula follows readily from (6), (7) and (8).  $\square$

**Corollary 11.** *With the same notation and assumptions in the above theorem, we have*

$$(\text{rot } \mathbf{B}_C)(\mathbf{r}) = \mathbf{0}.$$

**Proof.** We have

$$\begin{aligned} h(\text{rot } \mathbf{B}_C)(\mathbf{r}) &= (\text{rot } \mathbf{E}_{(S,1,h)}^{dp})(\mathbf{r}) \quad [\text{by Theorem 10}] \\ &= \mathbf{0} \quad [\text{by (2)}] \end{aligned}$$

for any  $h \in D$ , so that we have

$$(\text{rot } \mathbf{B}_C)(\mathbf{r}) = \mathbf{0}. \quad \square$$

### 5. From the Biot-Savart Law to Ampère's Circuital Law

This section owes much to [1].

**Proposition 12.** *For any  $\mathbf{x} \notin C$ , we have*

$$(\text{rot } \mathbf{B}_C)(\mathbf{x}) = \mathbf{0}.$$

**Proof.** Since  $\mathbf{x} \notin C$ , it is not difficult to find a surface  $S$  dodging  $\mathbf{x}$  with its boundary being  $C$ . By internalizing these entities in a well-adapted model and externalizing Corollary 11, we get the desired result.  $\square$

**Proposition 13.** *The number  $\mathbf{A}(C, L)$  has the following properties:*

(1) *It is symmetric in the sense that*

$$\mathbf{A}(C, L) = \mathbf{A}(L, C).$$

(2) *For any oriented surface  $S$  with  $\partial S = L \cup -L'$ , if it does not intersect  $C$ , then we have*

$$\mathbf{A}(C, L) = \mathbf{A}(C, L').$$

**Proof.** The first property follows simply from

$$\begin{aligned} & \left( (\mathbf{l}(s) - \mathbf{m}(t)) \times \frac{d\mathbf{m}}{dt}(t) \right) \cdot \frac{d\mathbf{l}}{ds}(s) \\ &= \det \begin{pmatrix} \mathbf{l}(s) - \mathbf{m}(t) \\ \frac{d\mathbf{m}}{dt}(t) \\ \frac{d\mathbf{l}}{ds}(s) \end{pmatrix} \\ &= \det \begin{pmatrix} \mathbf{m}(t) - \mathbf{l}(s) \\ \frac{d\mathbf{l}}{ds}(s) \\ \frac{d\mathbf{m}}{dt}(t) \end{pmatrix} \\ &= \left( (\mathbf{m}(t) - \mathbf{l}(s)) \times \frac{d\mathbf{l}}{ds}(s) \right) \cdot \frac{d\mathbf{m}}{dt}(t). \end{aligned}$$

The second property follows simply from Stokes' Theorem, as is seen in the following computation:

$$\begin{aligned}
 \mathbf{A}(C, L) - \mathbf{A}(C, L') &= \frac{1}{4\pi} \int_{L \cup -L'} \mathbf{B}_L \cdot d\mathbf{r} \\
 &= \frac{1}{4\pi} \int_S (\text{rot } \mathbf{B}_C) \cdot d\mathbf{S} \quad [\text{by Stokes' Theorem}] \\
 &= 0 \quad [\text{by Proposition 12}]. \quad \square
 \end{aligned}$$

**Lemma 14.** *Let  $n$  be a natural number with  $n \geq 2$ . The curve  $L$  is the unit circle on the  $xy$  plane with center  $(0, 0, 0)$  rounding counterclockwise against the positive part of the  $z$  axis. The curve  $C_n$ , to begin with, moves up straight from  $(0, 0, -n)$  to  $(0, 0, n)$ , moves horizontally from  $(0, 0, n)$  to  $(n, 0, n)$ , moves down straight from  $(n, 0, n)$  to  $(n, 0, -n)$ , and finally moves horizontally from  $(n, 0, -n)$  to  $(0, 0, -n)$ :*

$$\begin{array}{ccc}
 (0, 0, n) & \rightarrow & (n, 0, n) \\
 \uparrow & & \downarrow \\
 \curvearrowright & & \downarrow \\
 \uparrow & & \downarrow \\
 (0, 0, -n) & \leftarrow & (n, 0, -n)
 \end{array}$$

Then we have

$$\mathbf{A}(C_n, L) = 1$$

while trivially we have

$$\mathbf{Lk}(C_n, L) = 1.$$

**Proof.** Thanks to Proposition 13, we are sure that  $\mathbf{A}(C_n, L)$  is independent of  $n$ , for we have

$$\mathbf{A}(C_n \cup -C_{n+1}, L) = 0$$

as is to be seen easily. The curve  $C_n$  is composed of the curve  $C_n^1$  moving up straight from  $(0, 0, -n)$  to  $(0, 0, n)$  and the curve  $C_n^2$  moving horizontally from  $(0, 0, n)$  to  $(n, 0, n)$ , then moving down straight from  $(n, 0, n)$  to  $(n, 0, -n)$  and finally moves horizontally from  $(n, 0, -n)$  to  $(0, 0, -n)$ . Now we have

$$\mathbf{A}(C_n, L) = \frac{1}{4\pi} \int_L \int_{C_n^1} \frac{((\mathbf{s} - \mathbf{r}) \times d\mathbf{r}) \cdot d\mathbf{s}}{\|\mathbf{s} - \mathbf{r}\|^3} + \frac{1}{4\pi} \int_L \int_{C_n^2} \frac{((\mathbf{s} - \mathbf{r}) \times d\mathbf{r}) \cdot d\mathbf{s}}{\|\mathbf{s} - \mathbf{r}\|^3},$$

where  $\mathbf{s}$  moves along the curve  $L$  and  $\mathbf{r}$  moves along the curve  $C_n^1$  or  $C_n^2$ . It is easy to see that we have

$$\frac{1}{4\pi} \int_L \int_{C_n^2} \frac{((\mathbf{s} - \mathbf{r}) \times d\mathbf{r}) \cdot d\mathbf{s}}{\|\mathbf{s} - \mathbf{r}\|^3} \rightarrow 0$$

as  $n \rightarrow \infty$ , while we have

$$\frac{1}{4\pi} \int_L \int_{C_n^1} \frac{((\mathbf{s} - \mathbf{r}) \times d\mathbf{r}) \cdot d\mathbf{s}}{\|\mathbf{s} - \mathbf{r}\|^3} \rightarrow \frac{1}{4\pi} \int_L \int_{C^\infty} \frac{((\mathbf{s} - \mathbf{r}) \times d\mathbf{r}) \cdot d\mathbf{s}}{\|\mathbf{s} - \mathbf{r}\|^3}$$

as  $n \rightarrow \infty$ , where the curve  $C^\infty$  is no other than the  $z$ -axis moving from  $-\infty$  to  $+\infty$ . It is well known that

$$\frac{1}{4\pi} \int_L \int_{C^\infty} \frac{((\mathbf{s} - \mathbf{r}) \times d\mathbf{r}) \cdot d\mathbf{s}}{\|\mathbf{s} - \mathbf{r}\|^3} = 1.$$

Therefore, we are done.  $\square$

**Theorem 15** (The general Ampère's circuital law). *The Ampère's law (4) obtains.*

**Proof.** Let  $\varepsilon$  be a very small positive number. To each  $t \in [0, t_0]$ , we consider the circle  $\mathcal{C}_\varepsilon(t)$  with its center  $\mathbf{m}(t)$  and its radius  $\varepsilon$  in the plane perpendicular to  $\frac{d\mathbf{m}}{dt}(t)$ . Then the totality of  $\mathcal{C}(t)$  with  $t$  ranging over  $[0, T]$

forms a cylinder-like figure, which cuts out  $k$  circle-like curves from  $S$ . They are denoted by  $L_1, \dots, L_k$ , which surround the surfaces  $S_1, \dots, S_k$  containing  $\mathbf{p}_1, \dots, \mathbf{p}_k$ , respectively. They are endowed with the orientations induced from that of the surface  $S$ . Then the surface  $S'$  carved out by the curve  $L \cup (-L_1) \cup \dots \cup (-L_k)$  from  $S$  no longer intersects the curve  $C$ , so that we have

$$\mathbf{A}(C, L \cup (-L_1) \cup \dots \cup (-L_k)) = 0$$

by dint of Stokes' Theorem and Proposition 12. On the other hand, we are sure by the very definition that

$$\mathbf{A}(C, L \cup (-L_1) \cup \dots \cup (-L_k)) = \mathbf{A}(C, L) - \sum_{i=1}^k \mathbf{A}(C, L_i)$$

while we have

$$\mathbf{A}(C, L_i) = \mathbf{Lk}(C, L_i)$$

by dint of Lemma 14 with the aid of Proposition 13. Therefore, we are done. □

### References

- [1] K. Fukaya, *Electromagnetic Fields and Vector Calculus*, Iwanami, Tokyo, 1995 (in Japanese).
- [2] A. Kock, *Synthetic Differential Geometry*, 2nd ed., Cambridge University Press, Cambridge, 2006.
- [3] R. Lavendhomme, *Basic Concepts of Synthetic Differential Geometry*, Kluwer Academic Publishers, Dordrecht, 1996.
- [4] H. Nishimura, Synthetic vector analysis, *Internat. J. Theoret. Phys.* 41 (2002), 1165-1190.
- [5] H. Nishimura, Synthetic vector analysis II, *Internat. J. Theoret. Phys.* 43 (2004), 505-517.
- [6] H. Nishimura, Synthetic vector analysis III, from vector analysis to differential forms, *Far East J. Math. Sci. (FJMS)* 32(3) (2009), 335-346.



- [7] V. A. Vassiliev, Cohomology of knot spaces. Theory of singularities and its applications, Adv. Sov. Math. 1 (1990), 23-69.
- [8] V. A. Vassiliev, Combinatorial formulas for cohomology of knot spaces, Mosc. Math. J. 1 (2001), 91-123.
- [9] S. Wada, Essentials of Electromagnetism, Iwanami, Tokyo, 1994 (in Japanese).