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THE SERRE DUALITY THEOREM FOR HOLOMORPHIC VECTOR BUNDLES OVER A STRONGLY PSEUDO-CONVEX MANIFOLD

By

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Abstract. The Serre duality for a holomorphic vector bundle over a compact, complex manifold still holds over a compact, strongly pseudo-convex manifold M. This duality theorem is a vector bundle version of the Serre duality obtained by N. Tanaka in [3] for ordinary (p,q)-forms on M.

§1. Introduction

Let E be a holomorphic vector bundle over a compact, complex manifold M^n and $\Omega^p(E)$ be the sheaf of germs of holomorphic p-forms with values in E. Then we have

$$H^q(M;\Omega^p(E)) \cong H^{n-q}(M;\Omega^{n-p}(E^*)) \tag{1}$$

for any non-negative integers (p,q). Here E^* denotes the dual vector bundle of E. We call this isomorphism the Serre duality, which plays an important role in complex geometry. See, for examples, [1], [2]. When we restrict the bundle E as the trivial complex line bundle, the above then reduces to the ordinary duality

$$H^q(M;\Omega^p) \cong H^{n-q}(M;\Omega^{n-p}) \tag{2}$$

On a compact, complex manifold there is an isomorphism between such cohomology groups and the spaces $H^{p,q}(M;E)$ of harmonic forms taking values in E and then the above dualities are verified in terms of E-valued (p,q)-harmonic forms together with the Hodge star operator(refer to [2]).

N. Tanaka developed the harmonic theory over a compact, strongly pseudo-convex manifold M and derived a similar theorem for the space $H^{p,q}(M)$ of harmonic (p,q)-forms on M(refer to Theorem 7.3 in [3]);

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$$H^{p,q}(M) \cong H^{n-p,n-q-1}(M). \tag{3}$$

for any (p,q). Here $\dim M = 2n-1$.

In this article we consider a holomorphic vector bundle E over a compact, strongly pseudo-convex manifold M. The sub-ellipticity of the Laplacian holds also for the space of smooth E-valued (p,q)-forms on M so that the spaces $H^{p,q}(M,E)$ of E-valued harmonic (p,q)-forms for any (p,q) are finite dimensional whenever $q \neq 0, n-1$.

The aim of this article is to show the duality theorem for a holomorphic vector bundle over a strongly pseudo-convex manifold. We have indeed via the Hodge star operator $\#$.

Theorem 1. *Let M be a compact strongly pseudo-convex manifold and let E be a holomorphic vector bundle over M. Then*

$$H^{p,q}(M;E) \cong H^{n-p,n-q-1}(M;E^*)$$

for any (p,q), where E^* is the dual bundle of E.

A strongly pseudo-convex manifold M is a smooth manifold of dimension $2n-1$ which carries a strongly pseudo-convex structure (S,θ,P,I,g), that is, a complex subbundle S of $T^{\mathbf{C}}M$ satisfying $S \cap \bar{S} = 0$ and $[\Gamma(S),\Gamma(S)] \subset \Gamma(S)$ together with a contact form θ so that M admits the real expression (P,I) of S such that the Levi-form g given by $g(X,Y) = -d\theta(IX,Y)$, $X, Y \in P$ is positive definite.

We notice that our M admits a canonical Riemannian metric $h = g + \theta \otimes \theta$ and the volume form $dv = (n-1)!\theta \wedge (d\theta)^{n-1}$ gives the orientation.

A complex vector bundle E over a strongly pseudo-convex manifold M is said to be *holomorphic*, if there exists a smooth linear differential operator $\bar{\partial}_E = \bar{\partial} : \Gamma(E) \to \Gamma(E \otimes \bar{S}^*)$ satisfying

$$\text{i) } \bar{\partial}(fu) = f\bar{\partial}u + u \otimes d''f, \quad d''f = df|_{\bar{S}}$$

namely, if we set $\bar{\partial}_{\bar{X}}u = \bar{\partial}u(\bar{X})$, then

$$\text{i}') \ \bar{\partial}_{\bar{X}}(fu) = f\bar{\partial}_{\bar{X}}u + (\bar{X}f)u \quad \text{for } u \in \Gamma(E),\ f \in C^{\infty}_{\mathbf{C}}(M),\ X \in \Gamma(S),$$

$$\text{ii) } \bar{\partial}_{\bar{X}}(\bar{\partial}_{\bar{Y}}u) - \bar{\partial}_{\bar{Y}}(\bar{\partial}_{\bar{X}}u) - \bar{\partial}_{[X,Y]}u = 0 \quad \text{for } u \in \Gamma(E),\ X, Y \in \Gamma(S).$$

We call the operator $\bar{\partial}$ on E a holomorphic structure.

Every strongly pseudo-convex manifold M admits canonically a holomorphic vector bundle called the holomorphic tangent bundle $\hat{T}M$ of M, the quotient bundle $\hat{T}M = T^{\mathbf{C}}M/\bar{S}$ with the operator $\bar{\partial} = \bar{\partial}_{\hat{T}}$ given by $\bar{\partial}_{\bar{X}}u = \varpi([\bar{X}, Z])$, for $u \in \Gamma(\hat{T}M)$ with $Z \in \Gamma(T^{\mathbf{C}}M)$ such that $\varpi(Z) = u$ and $X \in \Gamma(S)$. Here $\varpi : T^{\mathbf{C}}M \to \hat{T}M$ is the canonical projection.

Notice that like holomorphic vector bundles over a complex manifold the tensor product $E \otimes F$ of holomorphic bundles E, F, the dual bundle E^* and the exterior product bundle $\Lambda^k E$ of a holomorphic bundle E are also holomorphic.

Let $(E, \bar{\partial}_E)$ be a holomorphic vector bundle over a strongly pseudo-convex manifold M. We assume that E admits a smooth Hermitian fiber metric $\langle \cdot, \cdot \rangle_E$.

The tensor product $E \otimes \Lambda^p(\hat{T}M)^*$, $0 \leq p \leq n-1$ carries the holomorphic structure

$$\bar{\partial} = \bar{\partial}_E \otimes id_{\Lambda^p} + id_E \otimes \bar{\partial}_{\Lambda^p}$$

In complex geometry, it is a standard fact that a complex vector bundle E is holomorphic if and only if E admits a locally defined holomorphic frame field around any point. However on a strongly pseudo-convex manifold, it is not obvious whether a holomorphic vector bundle admits a local holomorphic frame fields. With respect to this we have the following theorem ([4]).

THEOREM 2 ([4]). *A holomorphic vector bundle $(E, \bar{\partial})$ over a strongly pseudo-convex manifold M with* $\dim M \geq 7$. *Then, for any point p of M there exists an open neighborhood U of p and a smooth local frame $u_1, \dots, u_r \in \Gamma(U, E)$ such that each u_i satisfies $\bar{\partial} u_i = 0$. Here* $r = \operatorname{rank} E$.

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§ 2. The Proof of Theorem 1

Although the proof of Theorem 1 for a strongly pseudo-convex manifold is quite similar to the proof for a complex manifold, we will give the detailed proof for the sake of readers.

Let E be a holomorphic vector bundle over a compact, strongly pseudo-convex manifold M. We denote by $C^{p,q}(E) = \Gamma(M; E \otimes \Lambda^p \hat{T}M^* \otimes \Lambda^q \bar{S}^*)$ the space of smooth E-valued (p, q)-forms on M. Then the holomorphic structure $\bar{\partial} = \bar{\partial}_E$ of E induces an operator for each p, q in the ordinary way

$$\bar{\partial}^q : C^{p,q}(E) \to C^{p,q+1}(E)$$

for which we use, in abbreviation, the same symbol $\bar{\partial} = \bar{\partial}_E$. For this definition see [3], p. 16.

Let $*$ be the Hodge star operator. Then the operator $*$ is given by the formula

$$h(*\phi, \psi)\, dv = (n-1)!\phi \wedge \psi, \quad \phi \in \Lambda^k T^*M,\ \psi \in \Lambda^{2n-1-k}T^*M.$$

It holds that $*$ is isometric and involutive, that is, $h(*\phi, *\varphi) = h(\phi, \varphi)$ and $*^2 = id$. Moreover, over a strongly pseudo-convex manifold M its complexification exchanges holomorphic forms and anti-holomorphic forms. Thus, $* : \Lambda^p \hat{T}M^* \otimes \Lambda^q \bar{S}^* \to \Lambda^{n-p}\hat{T}M^* \otimes \Lambda^{n-q-1}\bar{S}^*$. If we write $\hat{T}M = \mathbf{C}\xi \oplus S$, then the operator $*$ fulfills $* : \mathbf{C}\theta \otimes \Lambda^{p'}S^* \otimes \Lambda^q \bar{S}^* \to \mathbf{C}\theta \otimes \Lambda^{n-q-1}S^* \otimes \Lambda^{n-p'-1}\bar{S}^*$.

For the proof of Theorem 1 we need to introduce an essential machinery, namely, the Hodge star operator $\#$. The complex conjugate Hodge star operator $^{-} \circ * : \Lambda^p \hat{T}M^* \otimes \Lambda^q \bar{S}^* \to \Lambda^{n-p}\hat{T}M^* \otimes \Lambda^{n-q-1}\bar{S}^*$ can be naturally extended over the bundle E as

$$\# : E \otimes \Lambda^p \hat{T}M^* \otimes \Lambda^q \bar{S}^* \to E^* \otimes \Lambda^{n-p}\hat{T}M^* \otimes \Lambda^{n-q-1}\bar{S}^*$$

To be precise, let $\{s_i \mid i = 1, \dots, r\}$ be a local frame of E defined over $U \subset M$. Here $r = \operatorname{rank} E$. Set the smooth functions $a_{ij} = \langle s_i, s_j \rangle_E \in C^\infty(U; \mathbf{C})$.

By using a local coframe $\{s^j \mid j = 1, \dots, r\}$, the dual to $\{s_i\}$, we define $\#$ for $\psi = \sum_i \psi^i s_i \in E \otimes \Lambda^p \hat{T}M^* \otimes \Lambda^q \bar{S}^*$,

$$\#\psi = \sum_{j=1}^{r} (\#\psi)_j s^j,$$

where $(\#\psi)_j = \sum_i a_{ji} * \overline{\psi^i}$. Here remark that ψ^i are elements of $\Lambda^p \hat{T}M^* \otimes \Lambda^q \bar{S}^*$, $i = 1, \dots, r$ and the definition is independent of a choice of local frame. So, $\#\psi \in C^{n-p, n-q-1}(E^*)$ for $\psi \in C^{p,q}(E)$.

Similarly, define $\#^* : E^* \otimes \Lambda^{n-p}\hat{T}M^* \otimes \Lambda^{n-q-1}\bar{S}^* \to E \otimes \Lambda^p \hat{T}M^* \otimes \Lambda^q \bar{S}^*$,

$$\#^*\left(\sum_j \alpha_j s^j\right) = \sum_{j,k} \bar{a}^{kj} \overline{* \alpha_j} s_k.$$

Then, it holds $\#^*\#\psi = \psi$ for any $\psi \in E \otimes \Lambda^p \hat{T}M^* \otimes \Lambda^q \bar{S}^*$ at every point of M. In fact,

$$\begin{aligned} \#^*(\#\psi) &= \#^*\left\{\left(\sum_{i,j} \overline{a_{ji}} * \overline{\psi^i}\right) s^j\right\} \\ &= \sum_{i,j,k} \overline{a^{kj}} \overline{*(a_{ji} * \overline{\psi^i})} s_k \end{aligned}$$

$$= \sum_{i,j,k} a^{jk} a_{ij} \overline{*(\overline{*\psi^i})} s_k$$

$$= \sum \delta_i^k \overline{*(\overline{*\psi^i})} s_k = \sum_i \overline{*(\overline{*\psi^i})} s_i$$

$$= \sum_i \psi^i s_i = \psi.$$

Remark that we have also $\#\#^* = 1$ and then $\# : E \otimes \Lambda^p \hat{T}M^* \otimes \Lambda^q \bar{S}^* \to E^* \otimes \Lambda^{n-p} \hat{T}M^* \otimes \Lambda^{n-q-1} \bar{S}^*$ gives a bundle isomorphism.

In order to define the formal adjoint of the operator $\bar{\partial}^q$ we define an L^2-inner product $(\cdot,\cdot)$ on $C^{p,q}(E)$. For $\phi = \sum_i \phi^i s_i$, $\psi = \sum_j \psi^j s_j$ we define a pointwise Hermitian inner product as

$$\langle \phi, \psi \rangle = \sum_{i,j=1}^{r} h(\phi^i, \psi^j) a_{ij},$$

where $h(\phi^i, \psi^j)$ is the inner product of (p,q)-forms ϕ^i, ψ^j defined by

$$h(\phi^i, \psi^j) = \frac{1}{k!} \sum_{i_1,\dots,i_k} \phi^i(X_{i_1},\dots,X_{i_k}) \overline{\psi^j(X_{i_1},\dots,X_{i_k})},$$

where $k = p + q$ and $\{X_i\}$ is a unitary basis of $TM^{\mathbf{C}}$, i.e., $\theta(X_1) = 1$ and $g(X_i, \bar{X}_j) = \delta_{ij}$, $2 \le i, j \le n$. Then we have an L^2-inner product on $C^{p,q}(E)$ by integrating over M; $(\phi, \psi) = \int_M \langle \phi, \psi \rangle \, dv$.

We denote by $\delta = \delta_E$ the formal adjoint of $\bar{\partial} = \bar{\partial}_E$ with respect to the L^2-inner product;

$$\delta : C^{p,q}(E) \to C^{p,q-1}(E).$$

To prove the following lemma, we need to define some notations. If $\phi \in C^{p,q}(E)$, $\alpha \in C^{s,t}(E^*)$ are locally represented by $\phi = \varphi \otimes u$, $\alpha = \omega \otimes \gamma$, where $\varphi \in C^{p,q}(M)$, $\omega \in C^{s,t}(M)$, $u \in \Gamma(E)$, $\gamma \in \Gamma(E^*)$, we define the product $\phi \wedge \alpha \in C^{p+s,q+t}(M)$ as follows.

$$\phi \wedge \alpha = \varphi \wedge \langle u, \gamma \rangle \omega.$$

Here, $\langle \cdot,\cdot \rangle$ is the pairing of E and E^*. The property of this product is,

Lemma 1. *For* $\phi \in C^{p,q}(E)$, $\alpha \in C^{s,t}(E^*)$,

$$\bar{\partial}_{\Lambda^{p+s}}(\phi \wedge \alpha) = (-1)^s (\bar{\partial}_E \phi) \wedge \alpha + (-1)^q \phi \wedge (\bar{\partial}_{E^*} \alpha).$$

PROOF OF LEMMA 1. Let $\phi = \varphi \otimes u$, $\alpha = \omega \otimes \gamma$, locally. Then by using the formula $d'' = (-1)^p \bar{\partial}_{\Lambda^p}$,

$$\begin{aligned}\bar{\partial}_{\Lambda^{p+s}}(\phi \wedge \alpha) &= \bar{\partial}_{\Lambda^{p+s}}(\varphi \wedge \langle u, \gamma \rangle \omega) \\ &= (-1)^{p+s} d''(\varphi \wedge \langle u, \gamma \rangle \omega) \\ &= (-1)^{p+s} d''\varphi \wedge \langle u, \gamma \rangle \omega + (-1)^{s+q} \varphi \wedge d''\langle u, \gamma \rangle \omega \\ &\quad + (-1)^{s+q} \varphi \wedge \langle u, \gamma \rangle d''\omega \\ &= (-1)^s \bar{\partial}_{\Lambda^p}\varphi \wedge \langle u, \gamma \rangle \omega + (-1)^{s+q} \varphi \wedge \langle \bar{\partial}_E u, \gamma \rangle \omega \\ &\quad + (-1)^{s+q} \varphi \wedge \langle u, \bar{\partial}_{E^*}\gamma \rangle \omega + (-1)^q \varphi \wedge \langle u, \gamma \rangle \bar{\partial}_{\Lambda^s}\omega \\ &= (-1)^s (\bar{\partial}_E \phi) \wedge \alpha + (-1)^q \phi \wedge (\bar{\partial}_{E^*}\alpha).\end{aligned}$$

LEMMA 2.

$$\delta_E \psi = (-1)^{n-k} \#^* \bar{\partial}_{E^*}(\#\psi), \quad \psi \in C^{p,q}(E)$$

PROOF OF LEMMA 2. First, we remark that

$$\int_M \bar{\partial}_{\Lambda^n}(\phi \wedge \#\psi) = 0$$

for $\phi \in C^{p,q-1}(E)$, $\psi \in C^{p,q}(E)$. In fact, the form $\phi \wedge \#\psi$ is a globally defined scalar valued $(n, n-2)$-from on M, so that $\bar{\partial}_{\Lambda^n}(\phi \wedge \#\psi) = (-1)^n d''(\phi \wedge \#\psi) = (-1)^n d(\phi \wedge \#\psi)$.

Thus integrating both sides of the following

$$\bar{\partial}_{\Lambda^n}(\phi \wedge \#\psi) = (-1)^{n-p}(\bar{\partial}_E \phi) \wedge \#\psi + (-1)^{q-1} \phi \wedge (\bar{\partial}_{E^*}\#\psi),$$

which is given by Lemma 1, we have

$$0 = (-1)^{n-p}(\bar{\partial}_E \phi, \psi) + (-1)^{q-1}(\phi, \#^*(\bar{\partial}_{E^*}\#\psi)),$$

that is,

$$(\bar{\partial}_E \phi, \psi) = (-1)^{n-p-q}(\phi, \#^*(\bar{\partial}_{E^*}\#\psi)).$$

for any ϕ. This completes the proof of Lemma 2.

Let $\langle \cdot, \cdot \rangle_{E^*}$ be the Hermitian fiber metric on the dual bundle E^* induced from the fiber metric $\langle \cdot, \cdot \rangle_E$ and set $a^{ij} = \langle s^i, s^j \rangle_{E^*}$ with respect to the dual frame

$\{s^i \mid i = 1, \dots, r\}$ of E^*. The space $C^{n-p,n-q-1}(E^*)$ also admits the L^2-inner product

$$(\alpha, \beta)_{E^*} = \frac{1}{(n-1)!} \int_M \sum_{i,j} \alpha_i \wedge \overline{*\beta_j a^{ij}}$$

for $\alpha = \sum_i \alpha_i s^i$, $\beta = \sum_j \beta_j s^j$.

Then the Hodge star operator $\#$ enjoys being an isometry with respect to the L^2-inner products, that is,

$$(\#\phi, \#\psi)_{E^*} = (\psi, \phi)$$

This is shown in a straightforward manner as

$$\begin{aligned}(n-1)!(\#\phi, \#\psi)_{E^*} &= \int_M \sum (\#\phi)_i \wedge \overline{*(\#\psi)_j a^{ij}} \\ &= \int_M \sum (\#\phi)_i \wedge \overline{(*\overline{a^{ij}}(\#\psi)_j)}\end{aligned}$$

which is

$$\int_M \sum (\#\phi)_i \wedge \#^*(\#\psi)^i = \int_M \sum (\#\phi)_i \wedge \psi^i = \int_M \psi^i \wedge (\#\psi)_i,$$

which is written as

$$(n-1)!(\psi, \phi).$$

Moreover, for $\phi \in C^{p,q}(E)$, $\psi \in C^{p,q-1}(E)$

$$\begin{aligned}(\bar{\partial}_{E^*}\#\phi)_i \wedge \overline{*(\#\psi)_j a^{ij}} &= \sum (\bar{\partial}_{E^*}\#\phi)_i \wedge \psi^j \\ &= (\bar{\partial}_{E^*}\#\phi) \wedge \psi \\ &= \psi \wedge (\bar{\partial}_{E^*}\#\phi) \\ &= (-1)^{q-1} \bar{\partial}_{\Lambda^p}(\psi \wedge \#\phi) + (-1)^{n-k} (\bar{\partial}_E \psi) \wedge \#\phi.\end{aligned}$$

Therefore, it turn out that

$$\begin{aligned}(\bar{\partial}_{E^*}\#\phi, \#\psi)_{E^*} &= (-1)^{n-k} (\bar{\partial}_E \psi, \phi)_E \\ &= (-1)^{n-k} (\#\delta\phi, \#\psi)_{E^*} \\ &= (-1)^{n-k} (\#\phi, \#\bar{\partial}_E \psi)_{E^*}.\end{aligned}$$

This implies $\bar{\partial}_{E^*} = (-1)^{n-k}\#\delta_E\#^*$. Hence, the formal adjoint of $\bar{\partial}_{E^*}$ becomes $(-1)^{n-k}\#\bar{\partial}_E\#^*$.

We are now in a position to show Theorem 1.

Take $\psi \in \boldsymbol{H}^{p,q}(M;E)$. Then it holds from definition $\bar{\partial}_E\psi = 0$ and $\delta_E\psi = 0$.

From Lemma 2 we have, since $\#\#^* = id$

$$\bar{\partial}_{E^*}(\#\psi) = (-1)^{n-k}\#\delta_E\psi = 0$$

On the other hand from the above consideration the formal adjoint δ_{E^*} of $\bar{\partial}_{E^*}$ is $(-1)^{n-k}\#\bar{\partial}_E\#^*$ so that we have

$$\delta_{E^*}\#\psi = (-1)^{n-k}\#\bar{\partial}_E\#^*(\#\psi) = (-1)^{n-k}\#\bar{\partial}_E\psi = 0$$

Therefore we have $\#\psi \in \boldsymbol{H}^{n-p,n-q-1}(E^*)$

The inverse implication is similarly shown.

So we see

$$\psi \in \boldsymbol{H}^{p,q}(M;E) \Leftrightarrow \#\psi \in \boldsymbol{H}^{n-p,n-q-1}(M;E^*).$$

In particular, $\# : \boldsymbol{H}^{p,q}(M;E) \to \boldsymbol{H}^{n-p,n-q-1}(M;E^*)$ is a complex conjugate linear isomorphism.

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