TSUKUBA J. MATH.
Vol. 31 No. 1 (2007), 175–196

CURVATURE AND RIGIDITY OF WILLMORE SUBMANIFOLDS*

By

Shichang SHU

Abstract. Let M be an n-dimensional compact Willmore submanifold in an $(n+p)$-dimensional unit sphere S^{n+p}. Denote by S and H the square of the length of the second fundamental form and the mean curvature of M. Let ρ be the non-negative function on M defined by $\rho^2 = S - nH^2$ and K be the function which assigns to each point of M the infimum of the sectional curvature at the point. In this paper, first of all, we prove that, if K, H and ρ satisfy $K \geq \frac{p-1}{2p-1} + (n-2)\frac{H\rho}{\sqrt{n(n-1)}} + H^2$, then either M is totally umbilic; or a Willmore torus $W_{1,n-1}$; or the Veronese surface in S^4; if the Ricci curvature R_{ii}, H and ρ satisfy $R_{ii} \geq (n-2) + (n-2)H\rho + H^2$, for $n \geq 5$, then either M is totally umbilic or a Willmore torus $W_{m,m}$. Secondly, we consider the Willmore submanifold with flat normal connection, we obtain that, if $0 \leq \rho^2 \leq n$ then eigher M is totally umbilic or a Willmore torus $W_{m,n-m}$; if $K \geq (n-2)\frac{H\rho}{\sqrt{n(n-1)}} + H^2$, then M is totally umbilic or $n \leq \rho^2 \leq np$.

1. Introduction

Let M be an n-dimensional compact submanifold of an $(n+p)$-dimensional unit sphere S^{n+p}. Let h_{ij}^{α}, S, $\vec{H}$ and H be the second fundamental form, the square of the length of the second fundamental form, the mean curvature vector and the mean curvature of M. We denote by $W(x)$ the Willmore functional on M (see [1], [12], [14]), that is, $W(x) = \int_M (S - nH^2)^{n/2}\, dv$. From [1], [12] and [14], we know that $W(x)$ is an invariant under Moebius (or conformal) transformations of S^{n+p}. The Willmore submanifold was defined by Li [7], that is, a submanifold is called a Willmore submanifold if it is a extremal submanifold to the Willmore

---

*This work is supported in part by the Natural Science Foundation of China and NSF of Shaanxi.

Received November 11, 2005.
Revised March 6, 2006.

functional. When $n = 2$, the functional essentially coincides with the well-known Willmore functional and its critical points are the Willmore surfaces. In [7] (also see [12], [6]), Li obtained a Willmore equation in terms of Euclidean geometry, which is very important to the study of rigidity and geometry of Willmore submanifold. Li [7] obtained the following.

THEOREM 1.1 ([7]). *Let M be an n-dimensional submanifold in an $(n+p)$-dimensional unit sphere S^{n+p}. Then M is Willmore submanifold if and only if for $n+1 \le \alpha \le n+p$*

$$(1.1)\qquad -\rho^{n-2}\left[SH^\alpha + \sum_{i,j,\beta} H^\beta h_{ij}^\beta h_{ij}^\alpha - \sum_{i,j,k,\beta} h_{ij}^\alpha h_{ik}^\beta h_{kj}^\beta - nH^2 H^\alpha\right]$$
$$+ (n-1)\rho^{n-2}\Delta^\perp H^\alpha + 2(n-1)\sum_i (\rho^{n-2})_i H_{,i}^\alpha + (n-1)H^\alpha \Delta(\rho^{n-2})$$
$$- \square^\alpha(\rho^{n-2}) = 0,$$

where $\Delta(\rho^{n-2}) = \sum_i (\rho^{n-2})_{i,i}$, $\Delta^\perp H^\alpha = \sum_i H_{,ii}^\alpha$, $\square^\alpha(\rho^{n-2}) = \sum_{i,j} (\rho^{n-2})_{i,j}(nH^\alpha \delta_{ij} - h_{ij}^\alpha)$, and $(\rho^{n-2})_{i,j}$ is the Hessian of ρ^{n-2} with respect to the induced metric $dx \cdot dx$, $H_{,i}^\alpha$ and $H_{,ij}^\alpha$ are the components of the first and second covariant derivative of the mean curvature vector field $\vec{H}$.

REMARK 1.1. Fix the index α with $n+1 \le \alpha \le n+p$, define $\square^\alpha : M \to R$ by

$$\square^\alpha f = \sum_{i,j} (nH^\alpha \delta_{ij} - h_{ij}^\alpha) f_{i,j},$$

where f is any smooth function on M. We know that $\square^\alpha$ is a self-adjoint operator (cf. Cheng-Yau [3] and Li [8]). We can see that this operator naturally appears in the Willmore equation (1.1). This operator will play an important role in the proofs of our theorems.

It is well know that in the theory of minimal submanifolds in S^{n+p} (i.e. $H \equiv 0$), Simons J. [13], Chern-Do Carmo-Kobayashi [4], Yau [15] and Ejiri N. [5] had obtained some important rigidity Theorems in terms of the squared norm of the second fundamental form, the sectional curvatures and the Ricci curvatures. It is natural idea to establish the rigidity Theorems of Willmore submanifolds in a unit sphere S^{n+p}. The rigidity generally involve the scalar curvatures, the Ricci curvatures, the sectional curvatures and the mean curvatures of the submanifolds.

In [7], [9] and [10], Li obtained some rigidity Theorems in terms of ρ, which vanishes exactly at the umbilical points of M.

Li [7] obtained the following

THEOREM 1.2 ([7]). *Let M be an n-dimensional ($n \geq 2$) compact Willmore submanifold in S^{n+p}. Then*

$$\int_M \rho^n \left(\frac{n}{2 - \frac{1}{p}} - \rho^2 \right) dv \leq 0. \tag{1.2}$$

In particular, if $0 \leq \rho^2 \leq \frac{n}{2-\frac{1}{p}}$, then either $\rho^2 \equiv 0$ and M is totally umbilic, or $\rho^2 = \frac{n}{2-\frac{1}{p}}$. In that latter case, either $p = 1$ and M is a Willmore torus $W_{m,n-m}$; or $n = 2$, $p = 2$ and M is the Veronese surface.

In this paper, we continue the study of the rigidity of n-dimensional compact Willmore submanifolds in a unit sphere S^{n+p}. First of all, we obtain some important integral equalities of Willmore submanifolds by use of the self-adjoint operator $\square^\alpha$ (see Proposition 3.2 and 3.3). Secondly, we obtain a rigidity Theorem and give a characterization of Willmore torus and Veronese surface in terms of sectional curvatures (see Theorem 4.1). We also study the rigidity of Willmore Submanifols in terms of the Ricci curvatures and obtain Theorem 4.2. Finally, from the Theorem 1.2 of Li [7] and the Theorem 4.1 in section 4 we consider the following

PROBLEM 1.1. Let M be an n-dimensional compact Willmore Submanifold in unit sphere S^{n+p}. If $0 \leq \rho^2 \leq n$ (resp. $K \geq (n-2)\frac{H\rho}{\sqrt{n(n-1)}} + H^2$), is Theorem 1.2 (resp. Theorem 4.1) also true?

This problem seems to be very difficulty. If it is true, all the existing results may be improved. However, it is not known whether it holds even in dimension 2. We obtain some results in section 5 (see Theorem 5.1 and 5.2), which can be considered as partial affirmative answers to above problem. It should note that the assumption of the flat normal connection plays an important role in the proof of these results.

2. Preliminaries

Let $x : M \to S^{n+p}$ be an n-dimensional submanifold in an $(n+p)$-dimensional unit sphere S^{n+p}. Let $\{e_1, \ldots, e_n\}$ be a local orthonormal basis of

M with respect to the induced metric, $\{\theta_1, \ldots, \theta_n\}$ are their dual form. Let $e_{n+1}, \ldots, e_{n+p}$ be the local unit orthonormal normal vector field. We make the following convention on the range of indices:

$$1 \le i, j, k, \ldots \le n; \quad n+1 \le \alpha, \beta, \gamma, \ldots \le n+p$$

Then the structure equations are

$$dx = \sum_i \theta_i e_i, \tag{2.1}$$

$$de_i = \sum_j \theta_{ij} e_j + \sum_{j,\alpha} h_{ij}^\alpha \theta_j e_\alpha - \theta_i x, \tag{2.2}$$

$$de_\alpha = -\sum_{i,j} h_{ij}^\alpha \theta_j e_i + \sum_\beta \theta_{\alpha\beta} e_\beta. \tag{2.3}$$

The Gauss equations are

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{2.4}$$

$$R_{ik} = (n-1)\delta_{ik} + n\sum_\alpha H^\alpha h_{ik}^\alpha - \sum_{j,\alpha} h_{ij}^\alpha h_{jk}^\alpha, \tag{2.5}$$

$$n(n-1)R = n(n-1) + n^2 H^2 - S, \tag{2.6}$$

where $S = \sum_{i,j,\alpha} (h_{ij}^\alpha)^2$, $\vec{H} = \sum_\alpha H^\alpha e_\alpha$, $H^\alpha = \frac{1}{n}\sum_k h_{kk}^\alpha$, $H = |\vec{H}|$. R is the normalized scalar curvature of M.

The first covariant derivative $\{h_{ijk}^\alpha\}$ and the second covariant derivative $\{h_{ijkl}^\alpha\}$ of h_{ij}^α are defined by

$$\sum_k h_{ijk}^\alpha \theta_k = dh_{ij}^\alpha + \sum_k h_{kj}^\alpha \theta_{ki} + \sum_k h_{ik}^\alpha \theta_{kj} + \sum_\beta h_{ij}^\beta \theta_{\beta\alpha}, \tag{2.7}$$

$$\sum_l h_{ijkl}^\alpha \theta_l = dh_{ijk}^\alpha + \sum_l h_{ljk}^\alpha \theta_{li} + \sum_l h_{ilk}^\alpha \theta_{lj} + \sum_l h_{ijl}^\alpha \theta_{lk} + \sum_\beta h_{ijk}^\beta \theta_{\beta\alpha}. \tag{2.8}$$

Then, we have the Codazzi equations and the Ricci identities

$$h_{ijk}^\alpha = h_{ikj}^\alpha, \tag{2.9}$$

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{mj} R_{mikl} + \sum_m h^\alpha_{im} R_{mjkl} + \sum_\beta h^\beta_{ij} R_{\beta\alpha kl}. \tag{2.10}$$

The Ricci equations are

$$R_{\alpha\beta ij} = \sum_k (h^\alpha_{ik} h^\beta_{kj} - h^\beta_{ik} h^\alpha_{kj}). \tag{2.11}$$

Denote by $\rho^2 = S - nH^2$ the non-negative function on M. We know that $\rho^2 = 0$ exactly at the umbilical points of M. Define the first, second covariant derivatives and Laplacian of the mean curvature vector field $\vec{H} = \sum_\alpha H^\alpha e_\alpha$ in the normal bundle $N(M)$ as follows

$$\sum_i H^\alpha_{,i} \theta_i = dH^\alpha + \sum_\beta H^\beta \theta_{\beta\alpha}, \tag{2.12}$$

$$\sum_j H^\alpha_{,ij} \theta_j = dH^\alpha_{,i} + \sum_j H^\alpha_{,j} \theta_{ji} + \sum_\beta H^\beta_{,i} \theta_{\beta\alpha}, \tag{2.13}$$

$$\Delta^\perp H^\alpha = \sum_i H^\alpha_{,ii}, \quad H^\alpha = \frac{1}{n} \sum_k h^\alpha_{kk}. \tag{2.14}$$

Let f be a smooth function on M. The first, second covariant derivatives f_i, $f_{i,j}$ and Laplacian of f are defined by

$$df = \sum_i f_i \theta_i, \quad \sum_j f_{i,j} \theta_j = df_i + \sum_j f_j \theta_{ji}, \quad \Delta f = \sum_i f_{i,i}. \tag{2.15}$$

For the fix index $\alpha (n+1 \le \alpha \le n+p)$, we introduce an operator $\Box^\alpha$ due to Cheng-Yau [3] by

$$\Box^\alpha f = \sum_{i,j} (nH^\alpha \delta_{ij} - h^\alpha_{ij}) f_{i,j}. \tag{2.16}$$

Since M is compact, the operator $\Box^\alpha$ is self-adjoint (see [3]) if and only if

$$\int_M (\Box^\alpha f) g \, dv = \int_M f (\Box^\alpha g) \, dv, \tag{2.17}$$

where f and g are any smooth functions on M.

In general, for a matrix $A = (a_{ij})$ we denote by $N(A)$ the square of the norm of A, that is,

$$N(A) = \operatorname{trace}(A \cdot A^t) = \sum_{i,j} (a_{ij})^2.$$

Clearly, $N(A) = N(T^t A T)$ for any orthogonal matrix T.

We need the following Lemmas due to Chern-Do Carmo-Kobayashi [4], Li [7] and Cheng [2].

Lemma 2.1 ([4]). *Let A and B be symmetric $(n \times n)$-matrices. Then*

$$N(AB - BA) \le 2N(A)N(B), \tag{2.18}$$

and the equality holds for nonzero matrices A and B if and only if A and B can be transformed simultaneously by on orthogonal matrix into multiples of $\tilde{A}$ and $\tilde{B}$ respectively, where

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \tilde{B} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Moreover, if A_1, A_2 and A_3 are $(n \times n)$-symmetric matrices and if

$$N(A_\alpha A_\beta - A_\beta A_\alpha) = 2N(A_\alpha)N(A_\beta), \quad 1 \le \alpha, \beta \le 3$$

then at least one of the matrices A_α must be zero.

Lemma 2.2 ([7]). *Let M be an n-dimensional $(n \ge 2)$ submanifold in S^{n+p}. Then we have*

$$|\nabla h|^2 \ge \frac{3n^2}{n+2} |\nabla^\perp \vec{H}|^2, \tag{2.19}$$

where $|\nabla h|^2 = \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2$, $|\nabla^\perp \vec{H}|^2 = \sum_{i,\alpha} (H^\alpha_{,i})^2$, $H^\alpha_{,i}$ is defined by (2.12).

Lemma 2.3 ([2]). *Let b_i for $i = 1, \dots, n$ be real numbers satisfying $\sum_{i=1}^n b_i = 0$ and $\sum_{i=1}^n b_i^2 = B$. Then*

$$\sum_{i=1}^n b_i^4 - \frac{B^2}{n} \le \frac{(n-2)^2}{n(n-1)} B^2. \tag{2.20}$$

The following well-known Lemma due to M. Okumura [11] is also needed in this paper.

LEMMA 2.4 ([11]). *Let $\{a_i\}_{i=1}^n$ be a set of real numbers satisfying $\sum_i a_i = 0$, $\sum_i a_i^2 = t^2$, where $t \geq 0$. Then we have*

$$(2.21)\qquad -\frac{n-2}{\sqrt{n(n-1)}}t^3 \leq \sum_i a_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}t^3,$$

and the equalities hold if and only if at least $(n-1)$ of the a_i are equal.

We can prove the following Lemma by making use of the method of Lagrane multipliers (see [2], where Cheng obtained the below bound only).

LEMMA 2.5. *Let a_i and b_i for $i = 1, \ldots, n$ be real numbers satisfying $\sum_{i=1}^n a_i = 0$ and $\sum_{i=1}^n a_i^2 = a$. Then*

$$(2.22)\qquad \left|\sum_{i=1}^n a_i b_i^2\right| \leq \sqrt{\sum_{i=1}^n b_i^4 - \frac{\left(\sum_{i=1}^n b_i^2\right)^2}{n}}\sqrt{a}.$$

PROOF. Putting $g(x) = \sum_{i=1}^n x_i b_i^2$, we calculate the maximum (or minimum) of the function $g(x)$ with constraint conditions

$$\sum_{i=1}^n x_i = 0, \quad \sum_{i=1}^n x_i^2 = a.$$

If the function $g(x)$ attains its maximum (or minimum) g_0 at some points x, then we have, at x,

$$b_i^2 + \lambda + 2\mu x_i = 0, \quad \text{for } i = 1, \ldots, n,$$

where λ and μ are Lagrange multipliers. Therefore, we have

$$g_0 = -2\mu a, \quad \lambda = -\frac{\sum_{i=1}^n b_i^2}{n}, \quad \sum_{i=1}^n b_i^4 - \frac{\left(\sum_{i=1}^n b_i^2\right)^2}{n} + 2\mu g_0 = 0.$$

We infer that Lemma 2.5 is true.

3. Integral Equalities of Willmore Submanifolds

In this section we shall obtain some integral equalities of Willmore submanifolds. We should note that the self-adjoint operator $\square^\alpha$, which appears in Willmore equation (1.1) naturally, will play an important role in the proof of these integral equalities.

Define tensors

$$\tilde{h}_{ij}^\alpha = h_{ij}^\alpha - H^\alpha \delta_{ij}, \tag{3.1}$$

$$\tilde{\sigma}_{\alpha\beta} = \sum_{i,j} \tilde{h}_{ij}^\alpha \tilde{h}_{ij}^\beta, \quad \sigma_{\alpha\beta} = \sum_{i,j} h_{ij}^\alpha h_{ij}^\beta. \tag{3.2}$$

Then the $(p \times p)$-matrix $(\tilde{\sigma}_{\alpha\beta})$ is symmetric and can be assumed to be diagonized for a suitable choice of $e_{n+1}, \dots, e_{n+p}$. We set

$$\tilde{\sigma}_{\alpha\beta} = \tilde{\sigma}_\alpha \delta_{\alpha\beta}. \tag{3.3}$$

By a direct calculation, we have

$$\sum_k \tilde{h}_{kk}^\alpha = 0, \quad \tilde{\sigma}_{\alpha\beta} = \sigma_{\alpha\beta} - nH^\alpha H^\beta, \quad \rho^2 = \sum_\alpha \tilde{\sigma}_\alpha = S - nH^2, \tag{3.4}$$

$$\sum_{i,j,k,\alpha} h_{kj}^\beta h_{ij}^\alpha h_{ik}^\alpha = \sum_{i,j,k,\alpha} \tilde{h}_{kj}^\beta \tilde{h}_{ij}^\alpha \tilde{h}_{ik}^\alpha + 2\sum_{i,j,\alpha} H^\alpha \tilde{h}_{ij}^\alpha \tilde{h}_{ij}^\beta + H^\beta \rho^2 + nH^2 H^\beta. \tag{3.5}$$

From Li [7], the Willmore equation (1.1) can be rewritten as

PROPOSITION 3.1 ([7]). *Let M be an n-dimensional submanifold in an $(n+p)$-dimensional unit sphere S^{n+p}. Then M is a Willmore submanifold if and only if for $n+1 \le \alpha \le n+p$*

$$\begin{aligned} \square^\alpha(\rho^{n-2}) &= (n-1)\rho^{n-2}\Delta^\perp H^\alpha + 2(n-1)\sum_i (\rho^{n-2})_{,i} H_{,i}^\alpha \\ &\quad + (n-1)H^\alpha \Delta(\rho^{n-2}) + \rho^{n-2}\left(\sum_\beta H^\beta \tilde{\sigma}_{\alpha\beta} + \sum_{i,j,k,\beta} \tilde{h}_{ij}^\alpha \tilde{h}_{ik}^\beta \tilde{h}_{kj}^\beta\right). \end{aligned} \tag{3.6}$$

Setting $f = nH^\alpha$ in (2.16), we have

$$\begin{aligned} \square^\alpha(nH^\alpha) &= \sum_{i,j} (nH^\alpha \delta_{ij} - h_{ij}^\alpha)(nH^\alpha)_{,i,j} \\ &= \sum_i (nH^\alpha)(nH^\alpha)_{,i,i} - \sum_{i,j} h_{ij}^\alpha (nH^\alpha)_{,i,j}. \end{aligned} \tag{3.7}$$

We also have

$$(3.8)\qquad \frac{1}{2}\Delta(nH)^2 = \frac{1}{2}\Delta\sum_\alpha (nH^\alpha)^2 = \frac{1}{2}\sum_\alpha \Delta(nH^\alpha)^2$$
$$= \frac{1}{2}\sum_{\alpha,i}[(nH^\alpha)^2]_{i,i} = \sum_{\alpha,i}[(nH^\alpha)_{,i}]^2 + \sum_{\alpha,i}(nH^\alpha)(nH^\alpha)_{i,i}$$
$$= n^2|\nabla^\perp \vec{H}|^2 + \sum_{\alpha,i}(nH^\alpha)(nH^\alpha)_{i,i}.$$

Therefore, from (3.7), (3.8), *we get*

$$(3.9)\qquad \sum_\alpha \Box^\alpha(nH^\alpha) = \frac{1}{2}\Delta(nH)^2 - n^2|\nabla^\perp \vec{H}|^2 - \sum_{i,j,\alpha} h_{ij}^\alpha (nH^\alpha)_{i,j}$$
$$= \frac{1}{2}\Delta[n(n-1)H^2 - \rho^2 + S] - n^2|\nabla^\perp \vec{H}|^2 - \sum_{i,j,\alpha} h_{ij}^\alpha (nH^\alpha)_{i,j}$$
$$= \frac{1}{2}\Delta S + \frac{1}{2}n(n-1)\Delta H^2 - \frac{1}{2}\Delta\rho^2 - n^2|\nabla^\perp \vec{H}|^2$$
$$- \sum_{i,j,\alpha} h_{ij}^\alpha (nH^\alpha)_{i,j}.$$

On the other hand, we have

$$(3.10)\qquad \frac{1}{2}\Delta S = \sum_{i,j,k,\alpha}(h_{ijk}^\alpha)^2 + \sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha$$
$$= |\nabla h|^2 + \sum_{i,j,\alpha} h_{ij}^\alpha (nH^\alpha)_{i,j} + \sum_\alpha \sum_{i,j,k,l} h_{ij}^\alpha (h_{kl}^\alpha R_{lijk} + h_{li}^\alpha R_{lkjk})$$
$$+ \sum_{\alpha,\beta}\sum_{i,j,k} h_{ij}^\alpha h_{ki}^\beta R_{\beta\alpha jk}.$$

Putting (3.10) *into* (3.9), *we have*

$$(3.11)\qquad \sum_\alpha \Box^\alpha(nH^\alpha) = |\nabla h|^2 - n^2|\nabla^\perp \vec{H}|^2 + \frac{1}{2}n(n-1)\Delta H^2 - \frac{1}{2}\Delta\rho^2$$
$$+ \sum_\alpha \sum_{i,j,k,l} h_{ij}^\alpha (h_{kl}^\alpha R_{lijk} + h_{li}^\alpha R_{lkjk}) + \sum_{\alpha,\beta}\sum_{i,j,k} h_{ij}^\alpha h_{ki}^\beta R_{\beta\alpha jk}.$$

Multiplying (3.11) *by* ρ^{n-2} *and taking integration, using* (2.17), *we have*

$$(3.12)\quad \sum_{\alpha}\int_M (nH^\alpha)\square^\alpha(\rho^{n-2})\,dv = \int_M \rho^{n-2}(|\nabla h|^2 - n^2|\nabla^\perp \vec{H}|^2)\,dv$$
$$+\frac{1}{2}n(n-1)\int_M \rho^{n-2}\Delta H^2\,dv - \frac{1}{2}\int_M \rho^{n-2}\Delta\rho^2\,dv$$
$$+\int_M \rho^{n-2}\sum_{\alpha}\sum_{i,j,k,l} h_{ij}^\alpha(h_{kl}^\alpha R_{lijk} + h_{li}^\alpha R_{lkjk})\,dv$$
$$+\int_M \rho^{n-2}\sum_{\alpha,\beta}\sum_{i,j,k} h_{ij}^\alpha h_{ki}^\beta R_{\beta\alpha jk}\,dv.$$

Taking the Willmore equation (3.6) into (3.12) and making use of the following

$$\int_M \rho^{n-2}\sum_{\alpha} H^\alpha \Delta^\perp H^\alpha\,dv = \frac{1}{2}\int_M \rho^{n-2}\sum_{\alpha}\Delta^\perp(H^\alpha)^2\,dv - \int_M \rho^{n-2}\sum_{i,\alpha}(H_{,i}^\alpha)^2\,dv$$
$$= \frac{1}{2}\int_M \rho^{n-2}\Delta H^2\,dv - \int_M \rho^{n-2}|\nabla\vec{H}|^2\,dv,$$

$$\int_M H^2\Delta(\rho^{n-2})\,dv = \int_M \sum_{\alpha}(H^\alpha)^2\sum_{i}(\rho^{n-2})_{i,i}\,dv$$
$$= \sum_{\alpha,i}\int_M (H^\alpha)^2(\rho^{n-2})_{i,i}\,dv = -\sum_{\alpha,i}\int_M (\rho^{n-2})_i((H^\alpha)^2)_{,i}\,dv$$
$$= -2\int_M \sum_{\alpha} H^\alpha \sum_{i}(\rho^{n-2})_i H_{,i}^\alpha\,dv,$$

$$-\frac{1}{2}\int_M \rho^{n-2}\Delta\rho^2\,dv = -\frac{1}{2}\sum_{i}\int_M \rho^{n-2}(\rho^2)_{i,i}\,dv$$
$$= \frac{1}{2}\sum_{i}\int_M (\rho^2)_i(\rho^{n-2})_i\,dv = (n-2)\int_M \rho^{n-2}|\nabla\rho|^2\,dv,$$

we have, by a direct calculation, the following

PROPOSITION 3.2. *For any n-dimensional compact Willmore submanifold in unit sphere S^{n+p}, there holds the following integral equality*

$$(3.13)\quad \int_M \rho^{n-2}(|\nabla h|^2 - n|\nabla^\perp \vec{H}|^2)\,dv + (n-2)\int_M \rho^{n-2}|\nabla\rho|^2\,dv$$
$$-\int_M \rho^{n-2}\sum_{\alpha,\beta} nH^\alpha\left(H^\beta\tilde{\sigma}_{\alpha\beta} + \sum_{i,j,k}\tilde{h}_{ij}^\alpha\tilde{h}_{ik}^\beta\tilde{h}_{kj}^\beta\right)dv$$

$$+\int_M \rho^{n-2}\sum_\alpha \sum_{i,j,k,l} h^\alpha_{ij}(h^\alpha_{kl}R_{lijk}+h^\alpha_{li}R_{lkjk})\,dv$$

$$+\int_M \rho^{n-2}\sum_{\alpha,\beta}\sum_{i,j,k} h^\alpha_{ij}h^\beta_{ki}R_{\beta\alpha jk}\,dv=0.$$

From (2.11), *we have*

$$\begin{aligned}(3.14)\qquad \sum_{\alpha,\beta}\sum_{i,j,k} h^\alpha_{ij}h^\beta_{ki}R_{\beta\alpha jk} &= \sum_{\alpha,\beta}\sum_{i,j,k,l} h^\alpha_{ij}h^\beta_{ki}(h^\beta_{jl}h^\alpha_{lk}-h^\beta_{kl}h^\alpha_{lj})\\ &= -\frac{1}{2}\sum_{\alpha,\beta,j,k}\left(\sum_l h^\beta_{jl}h^\alpha_{lk}-\sum_l h^\alpha_{jl}h^\beta_{lk}\right)^2\\ &= -\frac{1}{2}\sum_{\alpha,\beta,j,k}\left(\sum_l \tilde{h}^\beta_{jl}\tilde{h}^\alpha_{lk}-\sum_l \tilde{h}^\alpha_{jl}\tilde{h}^\beta_{lk}\right)^2\\ &= -\frac{1}{2}\sum_{\alpha,\beta} N(\tilde{A}_\alpha\tilde{A}_\beta-\tilde{A}_\beta\tilde{A}_\alpha),\end{aligned}$$

where $\tilde{A}_\alpha := (\tilde{h}^\alpha_{ij}) = (h^\alpha_{ij}-H^\alpha\delta_{ij})$.

By use of (2.4), (2.11), (3.2), (3.4), (3.5) *and* (3.14), *we conclude*

$$\begin{aligned}(3.15)\quad &\sum_\alpha\sum_{i,j,k,l} h^\alpha_{ij}(h^\alpha_{kl}R_{lijk}+h^\alpha_{li}R_{lkjk})\\ &= n\rho^2-\sum_{\alpha,\beta}\sum_{i,j,k,l} h^\alpha_{ij}h^\beta_{ij}h^\alpha_{lk}h^\beta_{lk}+n\sum_{\alpha,\beta}\sum_{i,j,k} H^\beta h^\beta_{kj}h^\alpha_{ij}h^\alpha_{ik}+\sum_{\alpha,\beta,i,j,k} h^\alpha_{ji}h^\beta_{ik}R_{\beta\alpha jk}\\ &= n\rho^2-\sum_{\alpha,\beta}\sigma^2_{\alpha\beta}+n\sum_{\alpha,\beta}\sum_{i,j,k} H^\beta\tilde{h}^\beta_{kj}\tilde{h}^\alpha_{ij}\tilde{h}^\alpha_{ik}+2n\sum_{\alpha,\beta}\sum_{i,j} H^\alpha H^\beta\tilde{h}^\alpha_{ij}\tilde{h}^\beta_{ij}\\ &\quad +n\sum_\beta (H^\beta)^2\rho^2+n^2H^2\sum_\beta (H^\beta)^2-\frac{1}{2}\sum_{\alpha,\beta} N(\tilde{A}_\alpha\tilde{A}_\beta-\tilde{A}_\beta\tilde{A}_\alpha)\\ &= n\rho^2-\sum_{\alpha,\beta}\tilde{\sigma}^2_{\alpha\beta}+nH^2\rho^2+n\sum_{\alpha,\beta}\sum_{i,j,k} H^\beta\tilde{h}^\beta_{kj}\tilde{h}^\alpha_{ij}\tilde{h}^\alpha_{ik}\\ &\quad -\frac{1}{2}\sum_{\alpha,\beta} N(\tilde{A}_\alpha\tilde{A}_\beta-\tilde{A}_\beta\tilde{A}_\alpha).\end{aligned}$$

Putting (3.14) and (3.15) into (3.13), we have the following

PROPOSITION 3.3. *For any n dimensional compact Willmore submanifold in unit sphere S^{n+p}, there holds the following integral equality*

$$
\begin{aligned}
(3.16)\qquad & \int_M \rho^{n-2}(|\nabla h|^2 - n|\nabla^\perp \vec{H}|^2)\,dv + (n-2)\int_M \rho^{n-2}|\nabla\rho|^2\,dv \\
&+ n\int_M \rho^{n-2}\left(H^2\rho^2 - \sum_{\alpha,\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta}\right) dv + n\int_M \rho^n\,dv \\
&- \int_M \rho^{n-2}\sum_{\alpha,\beta}(N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) + \tilde{\sigma}_{\alpha\beta}^2)\,dv = 0.
\end{aligned}
$$

If the codimension $p = 1$, denote $h_{ij}^{n+1} = h_{ij}$, $\tilde{h}_{ij}^{n+1} = \tilde{h}_{ij}$, for any i, j and note that $H = H^{n+1}$, $|\nabla^\perp \vec{H}|^2 = |\nabla H|^2$ and $\tilde{\sigma}_{n+1,n+1} = \rho^2$, we have from (3.13) and (3.16).

COROLLARY 3.1. *For any n-dimensional compact Willmore hypersurface in unit sphere S^{n+1}, there hold the following integral equalities*

$$
\begin{aligned}
(3.17)\qquad & \int_M \rho^{n-2}(|\nabla h|^2 - n|\nabla H|^2)\,dv + (n-2)\int_M \rho^{n-2}|\nabla\rho|^2\,dv \\
&- \int_M \rho^{n-2}\left(nH^2\rho^2 + nH\sum_{i,j,k}\tilde{h}_{ij}\tilde{h}_{ik}\tilde{h}_{kj}\right) dv \\
&+ \int_M \rho^{n-2}\sum_{i,j,k,l} h_{ij}(h_{kl}R_{lijk} + h_{li}R_{lkjk})\,dv = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
(3.18)\qquad & \int_M \rho^{n-2}(|\nabla h|^2 - n|\nabla H|^2)\,dv + (n-2)\int_M \rho^{n-2}|\nabla\rho|^2\,dv \\
&+ \int_M \rho^n(n-\rho^2)\,dv = 0.
\end{aligned}
$$

REMARK 3.1. From (3.16) and (3.18), we can obtain the results of Li [7] and [9] easily (see Li [7], [9]). From (3.13) and (3.17), we shall obtain the rigidity Theorem in terms of sectional curvatures.

4. Rigidity Theorems in Terms of Sectional and Ricci Curvatures

In this section, we shall obtain some rigidity Theorems of n-dimensional Willmore submanifolds in unit sphere S^{n+p} in terms of sectional and Ricci curvatures. We should note that the integral equalities (3.13), (3.16) and (3.17) will play an important role in the proofs of these Theorems.

In order to prove our Theorems, first of all, we review the following typical example.

Typical example 4.1 (see [7] or [6]). The torus

$$W_{m,n-m} = S^m\left(\sqrt{\frac{n-m}{n}}\right) \times S^{n-m}\left(\sqrt{\frac{m}{n}}\right), \quad 1 \le m \le n-1$$

are Willmore hypersurfaces in S^{n+1}, which are called Willmore torus. The principal curvatures $\lambda_1, \dots, \lambda_n$ of $W_{m,n-m}$ are

$$\lambda_1 = \cdots = \lambda_m = \sqrt{\frac{m}{n-m}}, \quad \lambda_{m+1} = \cdots = \lambda_n = -\sqrt{\frac{n-m}{m}}.$$

Then, we have

$$H = \frac{1}{n}\left(m\sqrt{\frac{m}{n-m}} - (n-m)\sqrt{\frac{n-m}{m}}\right), \quad S = \frac{m^2}{n-m} + \frac{(n-m)^2}{m}, \quad \rho^2 = n.$$

We also have by a direct calculation

$$R_{ijij} = 0, \quad R_{ii} = \frac{n(m-1)}{n-m}, \quad 1 \le i \le m, \quad R_{ii} = \frac{n(n-m-1)}{m}, \quad m+1 \le i \le n,$$

where R_{ijij} $(i \neq j)$ denotes the sectional curvature of the plane section spanned by $\{e_i, e_j\}$ and R_{ii} denotes the Ricci curvature of $W_{m,n-m}$, respectively.

Remark 4.1. From the typical example 4.1, we know that $R_{ijij} = \frac{n-2}{\sqrt{n(n-1)}} H\rho + H^2$, if and only if $m = 1$ and $R_{ii} = (n-2) + (n-2)H\rho + H^2$, $1 \le i \le n$, if and only if $m = \frac{n}{2}$.

From (3.13), (3.14) and (3.15), we know that for any real number a, the following integral equality holds

$$\begin{aligned}(4.1)\quad & \int_M \rho^{n-2}(|\nabla h|^2 - n|\nabla^\perp \vec{H}|^2)\,dv + (n-2)\int_M \rho^{n-2}|\nabla\rho|^2\,dv \\ & + n\int_M \rho^{n-2}\left(H^2\rho^2 - \sum_{\alpha,\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta}\right) dv - (a+1)n\int_M H^2\rho^n\,dv \\ & + (1+a)\int_M \rho^{n-2}\sum_\alpha \sum_{i,j,k,l} h_{ij}^\alpha (h_{kl}^\alpha R_{lijk} + h_{li}^\alpha R_{lkjk})\,dv\end{aligned}$$

$$- (1+a)n\int_M \rho^{n-2}\sum_{\alpha,\beta}\sum_{i,j,k} H^\alpha \tilde h^\alpha_{ij}\tilde h^\beta_{ik}\tilde h^\beta_{kj}\,dv - an\int_M \rho^n\,dv$$

$$+ a\int_M \rho^{n-2}\sum_{\alpha,\beta}\tilde\sigma^2_{\alpha\beta}\,dv - \frac{1-a}{2}\int_M \rho^{n-2}\sum_{\alpha,\beta} N(\tilde A_\alpha\tilde A_\beta - \tilde A_\beta\tilde A_\alpha)\,dv = 0.$$

Now, denote by K the function which assigns to each point of M the infimum of the sectional curvature at that point, we have the following

THEOREM 4.1. *Let M be an n-dimensional $(n\ge 2)$ compact Willmore submanifold in $(n+p)$-dimensional unit sphere S^{n+p}. If K, H and ρ satisfy*

$$K \ge \frac{p-1}{2p-1} + \frac{n-2}{\sqrt{n(n-1)}}H\rho + H^2,$$

then either M is totally umbilic; or M is a Willmore torus $W_{1,n-1} = S^1\left(\sqrt{\frac{n-1}{n}}\right)\times S^{n-1}\left(\sqrt{\frac{1}{n}}\right)$; or M is the Veronese surface in S^4.

PROOF. For a fixed α, $n+1\le\alpha\le n+p$, we can take a local orthonormal frame field $\{e_1,\dots,e_n\}$ such that $h^\alpha_{ij} = \lambda^\alpha_i\delta_{ij}$. Then, $\tilde h^\alpha_{ij} = \mu^\alpha_i\delta_{ij}$ with $\mu^\alpha_i = \lambda^\alpha_i - H^\alpha$, $\sum_i \mu^\alpha_i = 0$. Thus

$$\sum_{\alpha,i,j,k,l} h^\alpha_{ij}(h^\alpha_{kl}R_{lijk} + h^\alpha_{li}R_{lkjk}) = \frac12\sum_{\alpha,i,j}(\lambda^\alpha_i - \lambda^\alpha_j)^2 R_{ijij} \tag{4.2}$$
$$= \frac12\sum_{\alpha,i,j}(\mu^\alpha_i - \mu^\alpha_j)^2 R_{ijij}$$
$$\ge nK\rho^2,$$

and the equality in (4.2) holds if and only if $R_{ijij} = K$ for any $i\ne j$.

Let $\sum_i(\tilde h^\beta_{ii})^2 = \tau_\beta$. Then $\tau_\beta \le \sum_{i,j}(\tilde h^\beta_{ij})^2 = \tilde\sigma_\beta$. Since $\sum_i \tilde h^\beta_{ii} = 0$, $\sum_i \mu^\alpha_i = 0$ and $\sum_i(\mu^\alpha_i)^2 = \tilde\sigma_\alpha$. We have from Lemma 2.3 and Lemma 2.5

$$\sum_{\alpha,\beta}\sum_{i,j,k} H^\alpha\tilde h^\alpha_{ij}\tilde h^\beta_{kj}\tilde h^\beta_{ik} = \sum_{\beta,\alpha}\sum_{i,j,k} H^\beta\tilde h^\beta_{ij}\tilde h^\alpha_{kj}\tilde h^\alpha_{ik} \tag{4.3}$$
$$= \sum_{\alpha,\beta} H^\beta\sum_i \tilde h^\beta_{ii}(\mu^\alpha_i)^2 \le \frac{n-2}{\sqrt{n(n-1)}}\sum_{\alpha,\beta}|H^\beta|\tilde\sigma_\alpha\sqrt{\tau_\beta}$$

$$\leq \frac{n-2}{\sqrt{n(n-1)}}\sum_\alpha \tilde{\sigma}_\alpha \sum_\beta |H^\beta|\sqrt{\tilde{\sigma}_\beta}$$

$$\leq \frac{n-2}{\sqrt{n(n-1)}}\rho^2\sqrt{\sum_\beta (H^\beta)^2 \sum_\beta \tilde{\sigma}_\beta} = \frac{n-2}{\sqrt{n(n-1)}}H\rho^3.$$

From (3.3), we get

$$\sum_{\alpha,\beta}\tilde{\sigma}_{\alpha\beta}^2 = \sum_\alpha \tilde{\sigma}_\alpha^2 \geq \frac{1}{p}\left(\sum_\alpha \tilde{\sigma}_\alpha\right)^2 = \frac{1}{p}\rho^4. \tag{4.4}$$

From Lemma 2.1, (3.2) and (3.3), we have

$$\begin{aligned} \sum_{\alpha,\beta} N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) &\leq 2\sum_{\alpha\neq\beta}\tilde{\sigma}_\alpha\tilde{\sigma}_\beta = 2\left(\sum_\alpha \tilde{\sigma}_\alpha\right)^2 - 2\sum_\alpha \tilde{\sigma}_\alpha^2 \\ &\leq 2\rho^4 - 2\frac{1}{p}\left(\sum_\alpha \tilde{\sigma}_\alpha\right)^2 = 2\frac{p-1}{p}\rho^4. \end{aligned} \tag{4.5}$$

Therefore, from (4.1), Lemma 2.2, (4.2)–(4.5), we obtain for $0 \leq a \leq 1$

$$\begin{aligned} 0 \geq &\int_M \rho^{n-2}(|\nabla h|^2 - n|\nabla^\perp \vec{H}|^2)\,dv + (n-2)\int_M \rho^{n-2}|\nabla\rho|^2\,dv \\ &+ n\int_M \rho^{n-2}\left(H^2\rho^2 - \sum_{\alpha,\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta}\right) dv - (1+a)n\int_M H^2\rho^n\,dv \\ &+ (1+a)\int_M \rho^{n-2} nK\rho^2\,dv - (1+a)n\int_M \rho^{n-2}\frac{n-2}{\sqrt{n(n-1)}}H\rho^3\,dv \\ &- an\int_M \rho^n\,dv + a\int_M \rho^{n-2}\frac{1}{p}\rho^4\,dv - (1-a)\int_M \rho^{n-2}\frac{p-1}{p}\rho^4\,dv \\ \geq &\,(1+a)n\int_M \rho^n\left(K - \frac{n-2}{\sqrt{n(n-1)}}H\rho - H^2\right) dv - an\int_M \rho^n\,dv \\ &+ \left[\frac{a}{p} - (1-a)\frac{p-1}{p}\right]\int_M \rho^{n+2}\,dv, \end{aligned} \tag{4.6}$$

where we used

$$\sum_{\alpha,\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta} = \sum_\alpha (H^\alpha)^2\tilde{\sigma}_\alpha \leq \sum_\alpha (H^\alpha)^2 \sum_\beta \tilde{\sigma}_\beta = H^2\rho^2. \tag{4.7}$$

Putting $a=\frac{p-1}{p}$, we have

$$(4.8)\qquad 0\geq\int_M\rho^n\left(K-\frac{p-1}{2p-1}-\frac{n-2}{\sqrt{n(n-1)}}H\rho-H^2\right)dv.$$

If $K\geq\frac{p-1}{2p-1}+\frac{n-2}{\sqrt{n(n-1)}}H\rho+H^2$, from (4.8) we have $\rho^2=0$, that is M is totally umbilic, or $K=\frac{p-1}{2p-1}+\frac{n-2}{\sqrt{n(n-1)}}H\rho+H^2$ on M. In the second case, the equality in (4.8) holds. Thus, the equality in (4.2) holds, we have $R_{ijij}=K$ for any $i\neq j$. We can prove that M is not totally umbilic. In fact, if $\rho=0$, we know that the principal curvatures λ_i^α are equal for all i and α, that is, for all α, $\lambda_1^\alpha=\cdots=\lambda_n^\alpha$. Therefore, we have $1+\sum_\alpha(\lambda_i^\alpha)^2=R_{ijij}=K=\frac{p-1}{2p-1}+H^2=\frac{p-1}{2p-1}+\sum_\alpha(\lambda_i^\alpha)^2$, then $p=0$, this is a contradiction. Now, we may consider the case $p=1$ and $p\geq 2$ separately.

Case (i). If $p=1$, then $K=\frac{n-2}{\sqrt{n(n-1)}}H\rho+H^2$ on M. Take a local orthonormal frame field $\{e_1,\dots,e_n\}$ such that $h_{ij}=\lambda_i\delta_{ij}$. Then $\tilde{h}_{ij}=\mu_i\delta_{ij}$ with $\mu_i=\lambda_i-H$. From (3.17), Lemma 2.2, Lemma 2.4 and (4.2), we have

$$\begin{aligned}(4.9)\qquad 0\geq&\int_M\rho^{n-2}\left(|\nabla h|^2-\frac{3n^2}{n+2}|\nabla H|^2\right)dv+\int_M\rho^{n-2}\left(\frac{3n^2}{n+2}-n\right)|\nabla H|^2\,dv\\&+(n-2)\int_M\rho^{n-2}|\nabla\rho|^2\,dv-\int_M\rho^{n-2}\left(nH^2\rho^2+nH\sum_i\mu_i^3\right)dv\\&+\int_M\rho^{n-2}nK\rho^2\,dv\\\geq&\, n\int_M\rho^n\left(K-\frac{n-2}{\sqrt{n(n-1)}}H\rho-H^2\right)dv.\end{aligned}$$

Since $K=\frac{n-2}{\sqrt{n(n-1)}}H\rho+H^2$, the equalities in (4.9) hold. Therefore, we know that the equalities in (4.2), Lemma 2.2 and Lemma 2.4 hold. Since in the second case, we know that M is not totally umbilic. From (4.9) and Lemma 2.2, we conclude that $\nabla H=0$, i.e. $H=const.$ and $\nabla h=0$, i.e., the second fundamental form of M is parallel. From Lemma 2.4, we conclude that M has only two distinct principal curvatures. We easily follow that M is an isoparametric Willmore hypersurface with two distinct principal curvatures. From the result of Li [9] (cf. Theorem 5.1 in Li [9]), we know that M is one of the Willmore torus $W_{m,n-m}$, $1\leq m\leq n-1$. Since the equality in (4.2) holds, we infer that $R_{ijij}=K=\frac{n-2}{\sqrt{n(n-1)}}H\rho+H^2$. From Remark 4.1, we know that $m=1$. Therefore, we have M is the Willmore torus $W_{1,n-1}=S^1\left(\sqrt{\frac{n-1}{n}}\right)\times S^{n-1}\left(\sqrt{\frac{1}{n}}\right)$.

Case (ii). If $p \geq 2$, from $K = \frac{p-1}{2p-1} + \frac{n-2}{\sqrt{n(n-1)}} H\rho + H^2$ on M, we know that the equalities in (4.8) or (4.6) hold. Therefore the equalities in Lemma 2.1, Lemma 2.2, (4.4) and (4.7) hold. Since we know that M is not totally umbilic, we have

$$(4.10)\qquad \begin{aligned} &\nabla^{\perp}\vec{H} = 0, \quad \nabla h = 0, \\ &N(\tilde{A}_\alpha \tilde{A}_\beta - \tilde{A}_\beta \tilde{A}_\alpha) = 2N(\tilde{A}_\alpha)N(\tilde{A}_\beta), \quad \alpha \neq \beta, \\ &p \sum_\alpha \tilde{\sigma}_\alpha^2 = \left(\sum_\alpha \tilde{\sigma}_\alpha\right)^2, \end{aligned}$$

that is

$$(4.11)\qquad \tilde{\sigma}_{n+1} = \cdots = \tilde{\sigma}_{n+p},$$

$$(4.12)\qquad \sum_{\alpha,\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta} = H^2\rho^2.$$

From Lemma 2.1, we know that at most two of $\tilde{A}_\alpha = (\tilde{h}_{ij}^\alpha)$, $\alpha = n+1, \ldots, n+p$, are different from zero. If all of $\tilde{A}_\alpha = (\tilde{h}_{ij}^\alpha)$ are zero, which is contradiction with M is not totally umbilic. If only one of them, say $\tilde{A}_\alpha$, is different from zero, which is contradiction with (4.11). Therefore, we may assume that

$$\tilde{A}_{n+1} = \lambda \tilde{A}, \quad \tilde{A}_{n+2} = \mu \tilde{B}, \quad \lambda, \mu \neq 0,$$
$$\tilde{A}_\alpha = 0, \quad \alpha \geq n+3,$$

where $\tilde{A}$ and $\tilde{B}$ are defined in Lemma 2.1.

From (4.12), we have

$$\lambda^2 (H^{n+1})^2 + \mu^2 (H^{n+2})^2 = (\lambda^2 + \mu^2) \sum_\alpha (H^\alpha)^2.$$

Since $\lambda, \mu \neq 0$, we infer that $H^\alpha = 0$, $n+1 \leq \alpha \leq n+p$, that is, $\vec{H} = 0$, i.e., M is a minimal submanifold in S^{n+p} and $K = \frac{p-1}{2p-1}$ on M. From the Theorem of S. T. Yau [15] (cf. Theorem 15 of Yau [15]), we know that M is Veronese surface in S^4. This completes the proof of Theorem 4.1.

Now, we consider the rigidity of Willmore submanifolds in terms of Ricci curvatures. We need the following

LEMMA 4.1. *For any n-dimensional submanifold in an $(n+p)$-dimensional unit sphere S^{n+p}, if the Ricci curvature $R_{ii} \geq (n-2)+(n-2)H\rho+H^2$, then there hold the following*

$$\rho^2 \leq n, \tag{4.13}$$

$$\sum_{\alpha,\beta} N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) \leq 4\rho^2 - \frac{4}{n}\sum_\alpha \tilde{\sigma}_\alpha^2. \tag{4.14}$$

PROOF. From Gauss equation (2.5) and (3.1), we have

$$R_{ik} = (n-1)\delta_{ik} + (n-2)\sum_\alpha H^\alpha \tilde{h}_{ik}^\alpha + (n-1)H^2\delta_{ik} - \sum_{\alpha,j} \tilde{h}_{ij}^\alpha \tilde{h}_{jk}^\alpha.$$

Thus, we get

$$R_{ii} = (n-1) + (n-2)\sum_\alpha H^\alpha h_{ii}^\alpha + H^2 - \sum_{\alpha,j} (\tilde{h}_{ij}^\alpha)^2. \tag{4.15}$$

By Cauchy-Schwarz inequality, we have

$$\sum_\alpha H^\alpha h_{ii}^\alpha \leq \sqrt{\sum_\alpha (H^\alpha)^2}\sqrt{\sum_\alpha (h_{ii}^\alpha)^2} \leq H\rho. \tag{4.16}$$

(4.16) and the assumption of Lemma 4.1 infer that

$$R_{ii} \geq (n-2) + (n-2)\sum_\alpha H^\alpha h_{ii}^\alpha + H^2. \tag{4.17}$$

(4.15) and (4.17) imply that

$$1 - \sum_{\alpha,j} (\tilde{h}_{ij}^\alpha)^2 \geq 0, \tag{4.18}$$

that is, we have $\rho^2 \leq n$.

From (4.18) and $\tilde{h}_{ij}^\alpha = \mu_i^\alpha \delta_{ij}$, it is easy to see

$$\begin{aligned}
\sum_\beta N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) &= \sum_{\beta\neq\alpha,i,l} (\tilde{h}_{il}^\beta)^2(\mu_i^\alpha - \mu_l^\alpha)^2 \\
&\leq 4\sum_{\beta\neq\alpha,i,l} (\tilde{h}_{il}^\beta)^2(\mu_l^\alpha)^2 \leq 4\sum_l (1-(\mu_l^\alpha)^2)(\mu_l^\alpha)^2 \\
&= 4\sum_l (\mu_l^\alpha)^2 - 4\sum_l (\mu_l^\alpha)^4 \leq 4\sum_l (\mu_l^\alpha)^2 - \frac{4}{n}\left(\sum_l (\mu_l^\alpha)^2\right)^2.
\end{aligned}$$

Therefore, we know (4.14) holds. This completes the proof of Lemma 4.1.

REMARK 4.2. From the proof of Lemma 4.1, we infer that if $\rho^2 = n$, then $R_{ii} = (n-2) + (n-2)H\rho + H^2$.

THEOREM 4.2. *Let M be an n-dimensional ($n \geq 5$) compact Willmore submanifold in $(n+p)$-dimensional unit sphere S^{n+p}. If Ricci curvature R_{ii} of M, H and ρ satisfy*

$$R_{ii} \geq (n-2) + (n-2)H\rho + H^2,$$

then either M is totally umbilic, or M is the Willmore torus $W_{m,m} = S^m\left(\sqrt{\frac{1}{2}}\right) \times S^m\left(\sqrt{\frac{1}{2}}\right)$.

PROOF. From (3.16), Lemma 2.2, (3.3), (4.7) and Lemma 4.1, we have

$$\begin{aligned} 0 &\geq n\int_M \rho^n \, dv - \int_M \rho^{n-2}\left(4\rho^2 + \frac{n-4}{n}\sum_\alpha \tilde{\sigma}_\alpha^2\right) dv \\ &\geq (n-4)\int_M \rho^n \, dv - \frac{n-4}{n}\int_M \rho^{n-2}\left(\sum_\alpha \tilde{\sigma}_\alpha\right)^2 dv \\ &= \frac{n-4}{n}\int_M \rho^n(n-\rho^2) \, dv \geq 0, \end{aligned} \tag{4.19}$$

where we used

$$\sum_\alpha \tilde{\sigma}_\alpha^2 \leq \left(\sum_\alpha \tilde{\sigma}_\alpha\right)^2 = \rho^4. \tag{4.20}$$

From (4.19), we conclude $\rho^n = 0$, that is M is totally umbilic, or $\rho^2 = n$. In the latter case, we have the equalities in (4.19) and (4.20) hold. From $\sum_\alpha \tilde{\sigma}_\alpha^2 = \left(\sum_\alpha \tilde{\sigma}_\alpha\right)^2$, we have $\sum_{\alpha \neq \beta} \tilde{\sigma}_\alpha \tilde{\sigma}_\beta = 0$. Therefore, we infer that $(p-1)$ of the $\tilde{\sigma}_\alpha$ must be zero, this implies that $(p-1)$ of the $\tilde{A}_\alpha = (\tilde{h}_{ij}^\alpha)$ must be zero so that $p = 1$. Since $p = 1$ and $\rho^2 = n$, from the Theorem 1.2 due to Li [7] or [9], we know that M is a Willmore torus $W_{m,n-m}$, $1 \leq m \leq n-1$. From Remark 4.2 and Remark 4.1, we infer that $m = \frac{n}{2}$. Therefore, we know that M is the Willmore torus $W_{m,m} = S^m\left(\sqrt{\frac{1}{2}}\right) \times S^m\left(\sqrt{\frac{1}{2}}\right)$. This completes the proof of Theorem 4.2.

5. Willmore Submanifolds with Flat Normal Connection

In this section, we try to solve Problem 1.1. If Problem 1.1 is true, we know that all the existing results may be improved. However, this problem seems to be a very hard problem, we try to give some partial affirmative answers to this problem.

THEOREM 5.1. *Let M be an n-dimensional $(n \geq 2)$ compact Willmore submanifold in $(n+p)$-dimensional unit sphere S^{n+p} with flat normal connection. If $0 \leq \rho^2 \leq n$, then either M is totally umbilic or M is a Willmore torus $W_{m,n-m}$.*

PROOF. Since the normal connection of M is flat, we know that $R_{\beta\alpha jk} = 0$. From (3.14), we have

$$\sum_{\alpha,\beta} N(\tilde{A}_\alpha \tilde{A}_\beta - \tilde{A}_\beta \tilde{A}_\alpha) = 0.$$

Therefore, from (3.16), Lemma 2.2, (3.3) and (4.7), we infer that

$$\begin{aligned} 0 &\geq n\int_M \rho^n \, dv - \int_M \rho^{n-2} \sum_\alpha \tilde{\sigma}_\alpha^2 \, dv \\ &\geq n\int_M \rho^n \, dv - \int_M \rho^{n-2} \left(\sum_\alpha \tilde{\sigma}_\alpha\right)^2 dv \\ &= \int_M \rho^n (n - \rho^2) \, dv. \end{aligned} \tag{5.1}$$

If $0 \leq \rho^2 \leq n$, from (5.1), we have $\rho^n = 0$, i.e., M is totally umbilic, or $\rho^2 = n$. In the latter case, we know that the equalities in (5.1) hold. Hence, we conclude that $\sum_\alpha \tilde{\sigma}_\alpha^2 = (\sum_\alpha \tilde{\sigma}_\alpha)^2$, this implies that $(p-1)$ of the $\tilde{A}_\alpha = (\tilde{h}_{ij}^\alpha)$ must be zero so that $p = 1$. From the Theorem 1.2 due to Li [7] or [9], we know that M is a Willmore torus $W_{m,n-m}$. This completes the proof of the Theorem 5.1.

REMARK 5.1. If M is an n-dimensional compact Willmore hypersurface in unit sphere S^{n+1}, then the normal connection of M is flat. Therefore, we know that Theorem 5.1 reduces to the first case $(p = 1)$ of Theorem 1.2 of Li [7] or [9].

THEOREM 5.2. *Let M be an n-dimensional $(n \geq 2)$ compact Willmore submanifold in $(n+p)$-dimensional unit sphere S^{n+p} with flat normal connection. If $K \geq \frac{n-2}{\sqrt{n(n-1)}} H\rho + H^2$, then M is totally umbilic or $n \leq \rho^2 \leq np$.*

PROOF. Since the normal connection of M is flat. From (3.15), (4.2), (4.3) and the assumption of Theorem 5.2, we have

$$(5.2)\qquad n\rho^2 - \sum_{\alpha,\beta}\tilde{\sigma}_{\alpha\beta}^2 = \sum_{\alpha}\sum_{i,j,k,l} h_{ij}^{\alpha}(h_{kl}^{\alpha}R_{lijk} + h_{li}^{\alpha}R_{lkjk})$$
$$- nH^2\rho^2 - n\sum_{\alpha,\beta}\sum_{i,j,k} H^{\beta}\tilde{h}_{kj}^{\beta}\tilde{h}_{ij}^{\alpha}\tilde{h}_{ik}^{\alpha}$$
$$\geq n\rho^2\left(K - \frac{n-2}{\sqrt{n(n-1)}}H\rho - H^2\right) \geq 0.$$

Therefore, from (3.16), Lemma 2.2, (4.7) and (5.2), we obtain

$$(5.3)\qquad 0 \geq \int_M \rho^{n-2}\left(n\rho^2 - \sum_{\alpha,\beta}\tilde{\sigma}_{\alpha\beta}^2\right) dv$$
$$\geq \int_M n\rho^n\left(K - \frac{n-2}{\sqrt{n(n-1)}}H\rho - H^2\right) dv \geq 0,$$

this implies that $\rho^n = 0$, i.e., M is totally umbilic, or $K = \frac{n-2}{\sqrt{n(n-1)}}H\rho + H^2$. In the latter case, we know that the equalities in (5.3) hold. Therefore, we have

$$(5.4)\qquad \sum_{\alpha,\beta}\tilde{\sigma}_{\alpha\beta}^2 = n\rho^2.$$

From (5.4) and (3.3), we have

$$n\rho^2 = \sum_{\alpha}\tilde{\sigma}_{\alpha}^2 \leq \left(\sum_{\alpha}\tilde{\sigma}_{\alpha}\right)^2 = \rho^4.$$

Thus, we know that $\rho^2 \geq n$.

On the other hand, from (5.4) and (3.3), we have $\sum_{\alpha}\tilde{\sigma}_{\alpha}^2 = n\sum_{\alpha}\tilde{\sigma}_{\alpha}$, that is, $\sum_{\alpha}(\tilde{\sigma}_{\alpha} - \frac{n}{2})^2 = \frac{n^2 p}{4}$.

Hence, we have

$$\frac{n^2 p}{4} = \sum_{\alpha}\left(\tilde{\sigma}_{\alpha} - \frac{n}{2}\right)^2 \geq \frac{1}{p}\left(\sum_{\alpha}\left(\tilde{\sigma}_{\alpha} - \frac{n}{2}\right)\right)^2 = \frac{1}{p}\left(\rho^2 - \frac{np}{2}\right)^2,$$

this infers that $\rho^2 \leq np$. We complete the proof of Theorem 5.2.

Remark 5.2. If M is an n-dimensional compact Willmore hypersurface in unit sphere S^{n+1}, we know that the normal connection of M is flat. From Theorem 5.2, we have M is totally umbilic or $\rho^2 = n$. In the latter case, M is a Willmore torus $W_{m,n-m}$. Furthermore, from the proof of Theorem 5.2, we know that $R_{ijij} = K = \frac{n-2}{\sqrt{n(n-1)}} H\rho + H^2$. From Remark 4.1, we have $m = 1$. Therefore, we infer that Theorem 5.2 reduces to the first case $(p = 1)$ of Theorem 4.1.

Acknowledgements

The author has done this research work during his stay in Department of Mathematics, Faculty of Science and Engineering, Saga University, Japan as a visiting professor. He would like to express his thanks to Professor Qing-Ming Cheng for his help and useful discussions.

References

[1] Chen B. Y., Some conformal invariants of submanifolds and their applications, Boll. Un. Mat. Ital. **10** (1974), 380–385.
[2] Cheng Q.-M., Submanifolds with constant scalar curvature, Proc. of the Royal Soc. of Edinburgh, **132A** (2002), 1163–1183.
[3] Cheng S. Y. and Yau S.-T., Hypersurfaces with constant scalar curvature, Math. Ann. **225** (1977), 195–204.
[4] Chern S. S., Do Carmo M. and Kobayashi S., Minimal submanifolds of a sphere with second fundamental form of constant length, in Functional Analysis and Related Fields (F. Brower, ed.), Springer-Verlag, Berlin, (1970), 59–75.
[5] Ejiri N., Compact minimal submanifolds of a sphere with positive Ricci curvature, J. Math. Soc. Japan, **31** (1979), 251–256.
[6] Guo Z., Li H. and Wang C. P., The second variation formula for Willmore submanifolds in S^n, Results in Math. **40** (2001), 205–225.
[7] Li H., Willmore submanifolds in a sphere. Math. Res. Letters **9** (2002), 771–790.
[8] Li H., Global rigidity theorems of hypersurfaces, Ark. Mat. **35** (1997), 327–351.
[9] Li H., Willmore hypersurfaces in a sphere, Asian J. of Math. **5** (2001), 365–378.
[10] Li H., Willmore surfaces in S^n, Ann. Global Anal. Geom. **21** (2002), 203–213.
[11] Okumura M., Hypersurfaces and a pinching problem on the second fundamental tensor. Amer. J. Math. **96** (1974), 207–213.
[12] Pedit F. J. and Willmore T. J., Conformal geometry, Atti Sem. Mat. Fis, Univ Modena, **XXXVI** (1988), 237–245.
[13] Simons J., Minimal varieties in Riemannian manifolds, Ann. of Math., (2) **88** (1968), 62–105.
[14] Wang C. P., Moebius geometry of submanifolds in S^n, Manuscripta Math. **96** (1998), 517–534.
[15] Yau S. T., Submanifolds with constant mean curvature (II). Amer. J. Math. **97** (1975), 76–100.

Department of Mathematics
Xianyang Teachers' University
Xianyang, 712000, P.R. China
e-mail: shushichang@126.com