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HOCHSCHILD COHOMOLOGY RING OF THE INTEGRAL GROUP RING OF DIHEDRAL GROUPS†

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Abstract. We will determine the ring structure of the Hochschild cohomology $HH^*(\boldsymbol{Z}D_{2n})$ of the integral group ring of the dihedral group D_{2n} of order $2n$.

Introduction

Let RG be a group ring of a finite group G over a commutative ring R. If G is an abelian group, the multiplicative structure of the Hochschild cohomology $HH^*(RG)$ is explained by Holm [12] and Cibils and Solotar [5]. In the case where G is a non-abelian group, $HH^*(RG)$ can be very complicated ring in general, and it is more difficult to determine the multiplicative structure of $HH^*(RG)$.

The Hochschild cohomology ring $HH^*(RG)$ is isomorphic to the ordinary cohomology ring $H^*(G, {}_{\psi}RG)$, where ${}_{\psi}RG$ is regarded as a left RG-module by conjugation. So it is theoretically possible to calculate the products on the cohomology if an efficient resolution of G is given. Thus we have determined the ring structure of $HH^*(\boldsymbol{Z}Q_t)$ for arbitrary generalized quaternion groups Q_t by calculating the ordinary cup product in $H^*(Q_t, {}_{\psi}\boldsymbol{Z}Q_t)$ using a diagonal approximation on a periodic resolution of period 4 (see [8]).

On the other hand, it is well known that the Hochschild cohomology $HH^n(RG)$ is isomorphic to the direct sum of the ordinary group cohomology of the centralizers of representatives of the conjugacy classes of G (see [1, Theorem 2.11.2], [16, Section 4]):

$$HH^*(RG) \simeq \bigoplus_j H^*(G_j, R).$$

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This isomorphism is not in general multiplicative, however Siegel and Witherspoon [16, Theorem 5.1] define a new product on $\bigoplus_j H^*(G_j, R)$ so that the above additive isomorphism is multiplicative (see [2] for a new proof and a generalization of this result). They show that the multiplicative structure of the Hochschild cohomology of a group ring is described in terms of cup products, corestrictions and restrictions on the ordinary cohomology. This result was conjugated by Cibils [4] and Cibils and Solotar [5]. This new product gives us much helpful information about the Hochschild cohomology ring of group algebras. In this paper, we will calculate the Hochschild cohomology ring of the integral group ring of the dihedral group D_{2n} of order $2n$ for $n \geq 3$ by using this new product.

In Section 1, as preliminaries, we describe some definitions and properties about the Hochschild cohomology, the group cohomology, and the Product Formula given by Siegel and Witherspoon [16, Theorem 5.1].

In Section 2, we state efficient resolutions for dihedral groups and cyclic groups, and we describe the presentations of the integral cohomology rings of these groups. In fact, efficient free resolutions of dihedral groups are given by Wall [18], Hamada [6] and Handel [7]. We state a slightly different version of the Handel's resolution.

In Section 3, we calculate conjugations, restrictions and corestrictions between the integral cohomology rings of subgroups of D_{2n} (Propositions 3.4, 3.7 and 3.9). In order to calculate the cup products using the Product Formula we need their computations. These are given by calculating the images of the generators of the cohomologies on the cochain level by using chain transformations.

In Section 4, we calculate the cup products on $H^*(D_{2n}, {}_{\psi}ZD_{2n})$ ($\simeq HH^*(ZD_{2n})$) using the Product Formula (Propositions 4.1 through 4.7), and as the main result of this paper we determine the ring structure of $H^*(D_{2n}, {}_{\psi}ZD_{2n})$ ($\simeq HH^*(ZD_{2n})$) (Theorem 4.8).

1 Preliminaries

1.1 Hochschild Cohomology and Group Cohomology

Let R be a commutative ring and Λ an R-algebra which is a finitely generated projective R-module. If M is a $\Lambda^e (= \Lambda \otimes_R \Lambda^{op})$-module, then the n-th Hochschild cohomology of Λ with coefficients in M is defined by

$$H^n(\Lambda, M) := \operatorname{Ext}^n_{\Lambda^e}(\Lambda, M).$$

The cup product gives $HH^*(\Lambda) := \bigoplus_{n\geq 0} HH^n(\Lambda)$ a graded ring structure with identity $1 \in Z(\Lambda) \simeq HH^0(\Lambda)$, where $HH^n(\Lambda)$ denotes $H^n(\Lambda,\Lambda)$ and $Z(\Lambda)$ denotes the center of Λ, and $HH^*(\Lambda)$ is called the Hochschild cohomology ring of Λ. The Hochschild cohomology ring $HH^*(\Lambda)$ is graded-commutative, that is, for $\alpha \in HH^p(\Lambda)$ and $\beta \in HH^q(\Lambda)$ we have $\alpha\beta = (-1)^{pq}\beta\alpha$ (see [14, Proposition 1.2] for example).

Suppose that G is a finite group and that A is a G-module. Then we have the definition of the n-th cohomology group of G with coefficients in A:

$$H^n(G,A) := \mathrm{Ext}^n_{RG}(R,A).$$

Let H be a subgroup of G. We denote restriction and corestriction by res^G_H and cor^G_H, respectively (see [3], [17] or [19]):

$$\mathrm{res}^G_H : H^n(G,A) \to H^n(H,A),$$

$$\mathrm{cor}^G_H : H^n(H,A) \to H^n(G,A).$$

Note that

$$\mathrm{cor}^G_H \cdot \mathrm{res}^G_H\, \alpha = |G:H|\alpha \quad \text{for } \alpha \in H^n(G,A). \tag{1.1}$$

Let ${}^gH = gHg^{-1}$ be the conjugacy subgroup of H for $g \in G$. Then there is a homomorphism called conjugation by g:

$$g^* : H^n(H,A) \to H^n({}^gH,A).$$

Note that g^* is the identity for $g \in H$, and note that $(g_1g_2)^* = g_1^*g_2^*$ holds for $g_1, g_2 \in G$. These mappings of the cohomology groups are independent of the choice of resolutions.

About the group ring RG there are close relations between the Hochschild cohomology and the group cohomology. The Hochschild cohomology ring $HH^*(RG)$ is isomorphic to the ordinary cohomology ring $H^*(G, {}_\psi RG)$, where ${}_\psi RG$ is regarded as a left RG-module by conjugation (see [16, Proposition 3.2] or [13] for example).

1.2 Product Formula

Suppose that G is a finite group and R is a commutative ring. Let $g_1 = 1, g_2, \ldots, g_r$ be representatives of the conjugacy classes of G. Fix g_i, and let G_i be the centralizer of g_i. RG_i-homomorphisms $\theta_{g_i} : R \to RG$; $\lambda \mapsto \lambda g_i$ and $\pi_{g_i} : RG \to R$; $\sum_{a\in G} \lambda_a a \mapsto \lambda_{g_i}$ induce $\theta^*_{g_i} : H^n(G_i, R) \to H^n(G_i, {}_\psi RG)$ and

$\pi_{g_i}^*: H^n(G_i, {}_\psi RG) \to H^n(G_i, R)$, respectively. We define $\gamma_i : H^n(G_i, R) \to H^n(G, {}_\psi RG)$ by

$$\gamma_i(\alpha) = \mathrm{cor}_{G_i}^G\, \theta_{g_i}^*(\alpha), \quad \text{for } \alpha \in H^n(G_i, R).$$

Then we have the following isomorphism of graded R-modules

$$\Phi : H^n(G, {}_\psi RG) \xrightarrow{\sim} \bigoplus_i H^n(G_i, R); \quad \zeta \mapsto (\pi_{g_i}^*\, \mathrm{res}_{G_i}^G(\zeta))_i, \tag{1.2}$$

and its inverse is given by $\Phi^{-1}(\alpha) = \gamma_i(\alpha)$ for $\alpha \in H^n(G_i, R)$ (see [16, Section 4]).

Let D be a set of double coset representatives for $G_i\backslash G/G_j$. For each $a \in D$, there is a unique $k = k(a)$ such that

$$g_k = {}^b g_i\, {}^{ba} g_j \tag{1.3}$$

for some $b \in G$. In the above, ${}^x g$ denotes xgx^{-1} for $x, g \in G$. Siegel and Witherspoon [16] define the following new product on $\bigoplus_j H^*(G_j, R)$ so that the above additive isomorphism is multiplicative:

THEOREM 1.1 (Product Formula). *Let $\alpha \in H^*(G_i, R)$, $\beta \in H^*(G_j, R)$. Then the following equation holds in $H^*(G, {}_\psi RG)$:*

$$\gamma_i(\alpha) \smile \gamma_j(\beta) = \sum_{a \in D} \gamma_k(\mathrm{cor}_W^{G_k}(\mathrm{res}_W^{{}^b G_i}\, b^*\alpha \smile \mathrm{res}_W^{{}^{ba} G_j}(ba)^*\beta)), \tag{1.4}$$

where D is a set of double coset representatives for $G_i\backslash G/G_j$, $k = k(a)$ and $b = b(a)$ are chosen to satisfy (1.3), *and $W = {}^{ba}G_j \cap {}^b G_i$.*

Note that the sum in (1.4) is independent of the choices of a and b, and note that γ_1 is a monomorphism between the cohomology rings (see [16, Section 5]).

2 Integral Cohomology Rings of Dihedral Groups and Cyclic Groups

In this section, we describe presentations of the integral cohomology rings of dihedral groups and cyclic groups.

Let D_{2n} denote the dihedral group of order $2n$ for any positive integer $n \geq 2$:

$$D_{2n} = \langle x, y \mid x^n = y^2 = 1, yxy^{-1} = x^{-1} \rangle.$$

Efficient free resolutions of $\boldsymbol{Z}$ over $\boldsymbol{Z}D_{2n}$ are given by Wall [18] and Hamada [6]. Handel [7] reformulates this resolution and determines a diagonal approximation on the new resolution. The boundary operators of the Handel's resolution are right $\boldsymbol{Z}D_{2n}$-homomorphisms. For our convenience, we state a resolution whose boundary operators are left $\boldsymbol{Z}D_{2n}$-homomorphisms. This is a slightly different version of the Handel's resolution.

Let M^q denote a direct sum of q copies of a module M. We set $Y_q = (ZD_{2n})^{q+1}$ for $q \geq 0$. As elements of Y_q (or Z^{q+1}), we set

$$c_q^r = \begin{cases} (\underbrace{0,\dots,0,\overset{r}{\check{1}},0,\dots,0}_{q+1}) & (\text{if } 1 \leq r \leq q+1), \\ 0 & (\text{otherwise}). \end{cases}$$

Define left ZD_{2n}-homomorphisms $\varepsilon : Y_0 \to Z;\ c_0^1 \mapsto 1$ and $\delta_q : Y_q \to Y_{q-1}$ $(q > 0)$ given by

$$\delta_q(c_q^r) = \begin{cases} (xy + (-1)^{(q-r)/2})c_{q-1}^{r-1} + (x-1)c_{q-1}^r & \text{for } q \text{ even, } r \text{ even,} \\ (y - (-1)^{(q+r+1)/2})c_{q-1}^{r-1} + Nc_{q-1}^r & \text{for } q \text{ even, } r \text{ odd,} \\ (y - (-1)^{(q+r+1)/2})c_{q-1}^{r-1} - Nc_{q-1}^r & \text{for } q \text{ odd, } r \text{ even,} \\ (xy - (-1)^{(q-r)/2})c_{q-1}^{r-1} + (x-1)c_{q-1}^r & \text{for } q \text{ odd, } r \text{ odd.} \end{cases} \tag{2.1}$$

In the above, N denotes $\sum_{i=0}^{n-1} x^i$. It is easy to check that $\varepsilon \cdot \delta_1 = 0$ and $\delta_q \cdot \delta_{q+1} = 0$ $(q \geq 1)$ hold. To see that the complex (Y, δ) is acyclic, we state a contracting homotopy $T_q : Y_q \to Y_{q+1}$ $(q \geq -1)$, where we set $Y_{-1} = Z$:

$$T_{-1}(1) = c_0^1.$$

If $q(\geq 0)$ is even, then

$$T_q(x^i y^j c_q^r) = \begin{cases} N_i c_{q+1}^1 & (r = 1, 0 \leq i < n, j = 0), \\ (-1)^{q(q+1)/2} N_i c_{q+1}^1 + x^i c_{q+1}^2 & (r = 1, 0 \leq i < n, j = 1), \\ 0 & (r \geq 2, 0 \leq i < n, j = 0), \\ x^{i-1} c_{q+1}^{r+1} & (r(\geq 2) \text{ even, } 0 \leq i < n, j = 1), \\ x^i c_{q+1}^{r+1} & (r(\geq 3) \text{ odd, } 0 \leq i < n, j = 1), \end{cases}$$

where we set

$$N_i = \begin{cases} x^{i-1} + x^{i-2} + \cdots + 1 & (i \geq 1), \\ 0 & (i = 0). \end{cases}$$

If $q(\geq 1)$ is odd, then

$$T_q(x^i y^j c_q^r) = \begin{cases} 0 & (r = 1, 0 \leq i \leq n-2, j = 0), \\ c_{q+1}^1 & (r = 1, i = n-1, j = 0), \\ (-1)^{q(q+1)/2} c_{q+1}^1 + x^{-1} c_{q+1}^2 & (r = 1, i = 0, j = 1), \\ x^{i-1} c_{q+1}^2 & (r = 1, 1 \leq i < n, j = 1), \\ 0 & (r \geq 2, 0 \leq i < n, j = 0), \\ x^i c_{q+1}^{r+1} & (r(\geq 2) \text{ even, } 0 \leq i < n, j = 1), \\ x^{i-1} c_{q+1}^{r+1} & (r(\geq 3) \text{ odd, } 0 \leq i < n, j = 1). \end{cases}$$

For each $q \geq 0$, it is not hard to see that the equation

$$(\delta_{q+1}T_q + T_{q-1}\delta_q)(x^i y^j c_q^r) = x^i y^j c_q^r$$

holds. Therefore (Y, δ) is a free resolution of $\boldsymbol{Z}$ over $\boldsymbol{Z}D_{2n}$ (*cf.* [7, Theorems 2.1 and 3.3]).

Applying the functor $\mathrm{Hom}_{\boldsymbol{Z}D_{2n}}(-, \boldsymbol{Z})$ to the resolution (Y, δ), we have the following complex, where we identify $\mathrm{Hom}_{\boldsymbol{Z}D_{2n}}(Y_q, \boldsymbol{Z})$ with $\boldsymbol{Z}^{q+1}$ using an isomorphism $\mathrm{Hom}_{\boldsymbol{Z}D_{2n}}(Y_q, \boldsymbol{Z}) \to \boldsymbol{Z}^{q+1}$; $f \mapsto (f(c_q^1), f(c_q^2), \dots, f(c_q^{q+1}))$:

$$(\mathrm{Hom}_{\boldsymbol{Z}D_{2n}}(Y, \boldsymbol{Z}), \delta^{\#}) : 0 \to \boldsymbol{Z} \xrightarrow{\delta_1^{\#}} \boldsymbol{Z}^2 \xrightarrow{\delta_2^{\#}} \boldsymbol{Z}^3 \xrightarrow{\delta_3^{\#}} \boldsymbol{Z}^4 \xrightarrow{\delta_4^{\#}} \boldsymbol{Z}^5 \to \cdots,$$

$$\delta_{q+1}^{\#}(a_q^r) = \begin{cases} (x-1)a_{q+1}^r + (y - (-1)^{(q+r-1)/2})a_{q+1}^{r+1} & \text{for } q \text{ even, } r \text{ odd,} \\ -Na_{q+1}^r + (xy - (-1)^{(q-r)/2})a_{q+1}^{r+1} & \text{for } q \text{ even, } r \text{ even,} \\ Na_{q+1}^r + (xy + (-1)^{(q-r)/2})a_{q+1}^{r+1} & \text{for } q \text{ odd, } r \text{ odd,} \\ (x-1)a_{q+1}^r + (y - (-1)^{(q+r-1)/2})a_{q+1}^{r+1} & \text{for } q \text{ odd, } r \text{ even,} \end{cases}$$

$$= \begin{cases} (1 - (-1)^{(q+r-1)/2})a_{q+1}^{r+1} & \text{for } q \text{ even, } r \text{ odd,} \\ -na_{q+1}^r + (1 - (-1)^{(q-r)/2})a_{q+1}^{r+1} & \text{for } q \text{ even, } r \text{ even,} \\ na_{q+1}^r + (1 + (-1)^{(q-r)/2})a_{q+1}^{r+1} & \text{for } q \text{ odd, } r \text{ odd,} \\ (1 - (-1)^{(q+r-1)/2})a_{q+1}^{r+1} & \text{for } q \text{ odd, } r \text{ even.} \end{cases}$$

In the above, a_p^s denotes ac_p^s for $a \in \boldsymbol{Z}$.

If $n \geq 2$ is even, then the module structure of $H^k(D_{2n}, \boldsymbol{Z})$ is represented by the form of the subquotient of the complex as follows:

$$H^k(D_{2n}, \boldsymbol{Z})$$

$$= \begin{cases} \boldsymbol{Z} & \text{for } k = 0, \\ \boldsymbol{Z}c_{4q}^1/n \oplus \bigoplus_{i=1}^{q} \boldsymbol{Z}\left(\frac{n}{2}c_{4q}^{4i-1} + c_{4q}^{4i}\right)\Big/2 \oplus \bigoplus_{i=1}^{q} \boldsymbol{Z}c_{4q}^{4i+1}/2 & \text{for } k = 4q \ (q \neq 0), \\ \bigoplus_{i=1}^{q} \boldsymbol{Z}\left(\frac{n}{2}c_{4q+1}^{4i-2} - c_{4q+1}^{4i-1}\right)\Big/2 \oplus \bigoplus_{i=1}^{q} \boldsymbol{Z}c_{4q+1}^{4i}/2 & \text{for } k = 4q+1, \\ \bigoplus_{i=0}^{q} \boldsymbol{Z}\left(\frac{n}{2}c_{4q+2}^{4i+1} + c_{4q+2}^{4i+2}\right)\Big/2 \oplus \bigoplus_{i=0}^{q} \boldsymbol{Z}c_{4q+2}^{4i+3}/2 & \text{for } k = 4q+2, \\ \bigoplus_{i=1}^{q} \boldsymbol{Z}\left(\frac{n}{2}c_{4q+3}^{4i} - c_{4q+3}^{4i+1}\right)\Big/2 \oplus \bigoplus_{i=0}^{q} \boldsymbol{Z}c_{4q+3}^{4i+2}/2 & \text{for } k = 4q+3. \end{cases} \tag{2.2}$$

In the above, M/s denotes the quotient module M/sM for a $\boldsymbol{Z}$-module M and an element $s \in \boldsymbol{Z}$, and we interpret $\bigoplus_{i=1}^{q}$ term as 0 if $q = 0$. Note that we have the same module structure and ring structure if we use the Handel's resolution. We put $\lambda := c_2^3 \in H^2(D_{2n}, \boldsymbol{Z})$, $\mu := (n/2)c_2^1 + c_2^2 \in H^2(D_{2n}, \boldsymbol{Z})$, $\nu := c_3^2 \in H^3(D_{2n}, \boldsymbol{Z})$ and $\xi := c_4^1 \in H^4(D_{2n}, \boldsymbol{Z})$. Then λ, μ, ν and ξ multiplicatively generate $H^*(D_{2n}, \boldsymbol{Z})$, and the ring structure is given as follows (see [7, Theorem 5.2]):

$$H^*(D_{2n}, \boldsymbol{Z}) = \boldsymbol{Z}[\lambda, \mu, \nu, \xi]/(2\lambda, 2\mu, 2\nu, n\xi, \mu^2 + \lambda\mu + (n^2/4)\xi, \nu^2 + \lambda\xi),$$
$$(\deg \lambda = \deg \mu = 2, \deg \nu = 3, \deg \xi = 4). \tag{2.3}$$

In particular, we have

$$H^*(D_4, \boldsymbol{Z}) = \boldsymbol{Z}[\lambda, \mu, \nu]/(2\lambda, 2\mu, 2\nu, \nu^2 + \lambda\mu^2 + \lambda^2\mu),$$
$$(\deg \lambda = \deg \mu = 2, \deg \nu = 3). \tag{2.4}$$

If n is odd, the cohomology groups of D_{2n} are periodic. The integral cohomology of D_{2n} is as follows:

$$H^k(D_{2n}, \boldsymbol{Z}) = \begin{cases} \boldsymbol{Z} & \text{for } k = 0, \\ \boldsymbol{Z}c_{4q}^1/n \oplus \boldsymbol{Z}c_{4q}^{4q+1}/2 & \text{for } k = 4q \ (q \neq 0), \\ 0 & \text{for } k = 4q+1, \\ \boldsymbol{Z}c_{4q+2}^{4q+3}/2 & \text{for } k = 4q+2, \\ 0 & \text{for } k = 4q+3. \end{cases} \tag{2.5}$$

If we put $\alpha := c_2^3 \in H^2(D_{2n}, \boldsymbol{Z})$ and $\beta := c_4^1 + c_4^5 \in H^4(D_{2n}, \boldsymbol{Z})$, we have the following (see [7, Theorem 5.3]):

$$H^*(D_{2n}, \boldsymbol{Z}) = \boldsymbol{Z}[\alpha, \beta]/(2\alpha, 2n\beta, \alpha^2 - n\beta),$$
$$(\deg \alpha = 2, \deg \beta = 4). \tag{2.6}$$

Next, we describe the integral cohomology ring of the cyclic group. Let $H = \langle a \rangle$ denote the cyclic group of order $l (\geq 2)$. Then the following periodic $\boldsymbol{Z}H$-free resolution for $\boldsymbol{Z}$ of period 2 is well known (see [3, Chapter XII, Section 7] for example):

$$(Z_H, \partial_H) : \cdots \longrightarrow \boldsymbol{Z}H \xrightarrow{(\partial_H)_1} \boldsymbol{Z}H \xrightarrow{(\partial_H)_2} \boldsymbol{Z}H \xrightarrow{(\partial_H)_1} \boldsymbol{Z}H \xrightarrow{\varepsilon} \boldsymbol{Z} \longrightarrow 0,$$
$$(\partial_H)_1(c) = c(a-1),$$
$$(\partial_H)_2(c) = c\sum_{i=0}^{l-1} a^i.$$

Applying the functor $\mathrm{Hom}_{ZH}(-,Z)$ to the above periodic resolution, we have the complex

$$(\mathrm{Hom}_{ZH}(Z_H,Z),(\partial_H)^{\#}):0\longrightarrow Z\xrightarrow{(\partial_H)_1^{\#}}Z\xrightarrow{(\partial_H)_2^{\#}}Z\xrightarrow{(\partial_H)_1^{\#}}Z\longrightarrow\cdots,$$

$$(\partial_H)_1^{\#}(c)=(a-1)c=0,$$

$$(\partial_H)_2^{\#}(c)=\sum_{i=0}^{l-1}a^ic=lc,$$

and we have

$$H^k(H,Z)=\begin{cases}Z & \text{for } k=0,\\ Z/l & \text{for } k\equiv 0 \bmod 2,\ k\neq 0,\\ 0 & \text{for } k\equiv 1 \bmod 2.\end{cases} \tag{2.7}$$

If we put $\chi:=1\in H^2(H,Z)$, then we have the following (see [3, Chapter XII, Section 7]):

$$H^*(H,Z)=Z[\chi]/(l\chi),\quad (\deg\chi=2). \tag{2.8}$$

3 Conjugation, Restriction and Corestriction

In this section, we calculate conjugations, restrictions and corestrictions between the integral cohomology rings of the centralizers of representatives of the conjugacy classes of D_{2n}. These are given by a method similar to [9, Section 2.1].

3.1 The Case n Even

In this subsection, we consider the case $n=2m$ $(m\geq 2)$. We take representatives of the conjugacy classes of D_{2n} as follows:

$$g_1=1,\quad g_2=x^m,\quad g_{i+2}=x^i\ (1\leq i<m),\quad g_{m+2}=y,\quad g_{m+3}=xy.$$

Then their centralizers are

$$G_1=G_2=D_{2n},\quad G_{i+2}=\langle x\rangle\ (1\leq i<m),\quad G_{m+2}=\langle x^m,y\rangle,\quad G_{m+3}=\langle x^m,xy\rangle.$$

Note that G_{m+2} and G_{m+3} are isomorphic to D_4.

In the following, we set

$$H^*(G_r,Z)=Z[\lambda,\mu,\nu,\xi]/(2\lambda,2\mu,2\nu,n\xi,\mu^2+\lambda\mu+(n^2/4)\xi,\nu^2+\lambda\xi),$$

$$(\deg\lambda=\deg\mu=2,\deg\nu=3,\deg\xi=4),$$

$$H^*(G_{k+2}, \boldsymbol{Z}) = \boldsymbol{Z}[\sigma]/(n\sigma) \quad (\deg \sigma = 2),$$

$$H^*(G_{m+r+1}, \boldsymbol{Z}) = \boldsymbol{Z}[\lambda_r, \mu_r, \nu_r]/(2\lambda_r, 2\mu_r, 2\nu_r, \nu_r^2 + \lambda_r\mu_r^2 + \lambda_r^2\mu_r)$$
$$(\deg \lambda_r = \deg \mu_r = 2, \deg \nu_r = 3),$$

where $r = 1, 2$ and $1 \leq k \leq m-1$. These presentations follow from (2.3), (2.4) and (2.8). By (1.2), (2.2) and (2.7), we have

$$H^k(D_{2n}, {}_{\psi}\boldsymbol{Z}D_{2n}) = \begin{cases} \boldsymbol{Z}^{m+3} & \text{for } k = 0, \\ 0 & \text{for } k = 1, \\ (\boldsymbol{Z}/2)^{4q} & \text{for } k = 2q+1 \ (q \geq 1), \\ (\boldsymbol{Z}/2)^{8(q+1)} \oplus (\boldsymbol{Z}/n)^{m-1} & \text{for } k = 4q+2 \ (q \geq 0), \\ (\boldsymbol{Z}/2)^{8q+2} \oplus (\boldsymbol{Z}/n)^{m+1} & \text{for } k = 4q \ (q \geq 1). \end{cases}$$

In the above, M^r denotes a direct sum of r copies of a module M.

Moreover we set

$$H^*(\langle x^m \rangle, \boldsymbol{Z}) = \boldsymbol{Z}[\tau]/(2\tau) \quad (\deg \tau = 2).$$

First, we calculate conjugation maps. We need chain transformations in both directions between the standard resolution and the resolution given by (2.1). The following equations are useful for the proof of Lemma 3.1:

$$N_i + x^i N_j = N_{i+j}, \quad N_i(x-1) = x^i - 1, \quad yN_j = x^{-j}N_j xy \ (i, j \geq 0).$$

LEMMA 3.1. *Let (X, d) be the standard resolution of D_{2n} and (Y, δ) the resolution of D_{2n} given by* (2.1).

(i) *An initial part of a chain transformation $u_k : X_k \to Y_k$ lifting the identity map on $\boldsymbol{Z}$ is given as follows:*

$$u_0([\cdot]) = c_0^1;$$

$$u_1([x^i y^p]) = N_i c_1^1 + p x^i c_1^2;$$

$$u_2([x^i y^p \mid x^j y^q]) = \begin{cases} 0 & (\text{if } i+j < n \text{ and } p = 0), \\ c_2^1 & (\text{if } i+j \geq n \text{ and } p = 0), \\ x^{i-j} N_j c_2^2 + q x^{i-j} c_2^3 & (\text{if } i-j \geq 0 \text{ and } p = 1), \\ -c_2^1 + x^{i-j} N_j c_2^2 + q x^{i-j} c_2^3 & (\text{if } i-j < 0 \text{ and } p = 1), \end{cases}$$

where $0 \leq i, j < n$ and $p, q = 0, 1$.

(ii) *An initial part of a chain transformation $v_k : Y_k \to X_k$ lifting the identity map on $\boldsymbol{Z}$ is given as follows:*

$$v_0(c_0^1) = [\cdot];$$

$$v_1(c_1^1) = [x], \quad v_1(c_1^2) = [y];$$

$$v_2(c_2^1) = [N|x], \quad v_2(c_2^2) = [xy \mid x] + [x|y], \quad v_2(c_2^3) = [y+1 \mid y].$$

PROOF. We prove (i) only. It suffices to check that the equation $u_{k-1}d_k = \delta_k u_k$ holds for $k = 1, 2$. In the case $k = 1$, we have

$$u_0 d_1([x^i y^p]) = u_0((x^i y^p - 1)[\cdot]) = (x^i y^p - 1)c_0^1$$
$$= N_i(x-1)c_0^1 + px^i(y-1)c_0^1 = \delta_1 u_1([x^i y^p]).$$

In the case $k = 2$, the proof is divided into four cases.

Case $i + j < n$, $p = 0$:

$$u_1 d_2([x^i \mid x^j y^q]) = u_1(x^i[x^j y^q] - [x^{i+j} y^q] + [x^i])$$
$$= (x^i N_j - N_{i+j} + N_i)c_1^1 = 0 = \delta_2 u_2([x^i \mid x^j y^q]).$$

Case $i + j \geq n$, $p = 0$:

$$u_1 d_2([x^i \mid x^j y^q]) = (N_{i+j} - N_{i+j-n})c_1^1 = (N_{i+j} - (N_{i+j} - x^{i+j-n}N_n))c_1^1$$
$$= Nc_1^1 = \delta_2 u_2([x^i \mid x^j y^q]).$$

Case $i - j \geq 0$, $p = 1$:

$$u_1 d_2([x^i y \mid x^j y^q]) = u_1(x^i y[x^j y^q] - [x^{i-j} y^{1-q}] + [x^i y])$$
$$= (x^i y N_j - N_{i-j} + N_i)c_1^1 + (qx^{i-j}y + (q-1)x^{i-j} + x^i)c_1^2$$
$$= x^{i-j}N_j(xy+1)c_1^1 + x^{i-j}(x^j - 1 + q(y+1))c_1^2$$
$$= x^{i-j}N_j((xy+1)c_1^1 + (x-1)c_1^2) + qx^{i-j}(y+1)c_1^2$$
$$= \delta_2 u_2([x^i y \mid x^j y^q]).$$

Case $i - j < 0$, $p = 1$:

$$u_1 d_2([x^i y \mid x^j y^q]) = (x^i y N_j - N_{i-j+n} + N_i)c_1^1 + (qx^{i-j}y + (q-1)x^{i-j} + x^i)c_1^2$$
$$= -Nc_1^1 + x^{i-j}N_j((xy+1)c_1^1 + (x-1)c_1^2) + qx^{i-j}(y+1)c_1^2$$
$$(\text{since } N_i - N_{i-j+n} = -x^i N_{n-j} = -x^{i-j}(N_n - N_j))$$
$$= \delta_2 u_2([x^i y \mid x^j y^q]).$$

To prove (ii), it suffices to check that the equation $d_k v_k = v_{k-1}\delta_k$ holds for $k = 1, 2$. □

LEMMA 3.2. *Let $H = \langle a \rangle$ denote the cyclic group of order $l (\geq 2)$. (Z_H, ∂_H) denotes the periodic resolution of H and (X_H, d_H) denotes the standard resolution of H.*

(i) *An initial part of a chain transformation $(v_H)_k : (Z_H)_k \to (X_H)_k$ lifting the identity map on $\boldsymbol{Z}$ is given as follows:*

$$(v_H)_0(1) = [\cdot];$$

$$(v_H)_1(1) = [a];$$

$$(v_H)_2(1) = \sum_{i=0}^{l-1} [a^i|a].$$

(ii) *An initial part of a chain transformation $(u_H)_k : (X_H)_k \to (Z_H)_k$ lifting the identity map on $\boldsymbol{Z}$ is given as follows:*

$$(u_H)_0([\cdot]) = 1;$$

$$(u_H)_1([a^i]) = \begin{cases} a^{i-1} + a^{i-2} + \cdots + 1 & (i \geq 1), \\ 0 & (i = 0); \end{cases}$$

$$(u_H)_2([a^i|a^j]) = \begin{cases} 1 & (i + j \geq l), \\ 0 & (i + j < l), \end{cases}$$

for $0 \leq i, j < l$.

PROOF. See [11, Proposition 1] for (i) and [9, Lemma 2.1] for (ii). □

LEMMA 3.3. *Suppose H is a subgroup of a finite group G and A is a G-module. (X_H, d_H) and $(X_{(^gH)}, d_{(^gH)})$ denote the standard resolutions of H and $^gH = gHg^{-1}$ for $g \in G$, respectively. Then the conjugation map $g^* : H^k(H, A) \to H^k(^gH, A)$ is given by the following on the cochain level:*

$$\tilde{g} : \mathrm{Hom}_{ZH}((X_H)_k, A) \to \mathrm{Hom}_{Z(^gH)}((X_{(^gH)})_k, A)$$

$$(\tilde{g}(f))([\cdot]) = gf([\cdot]) \quad (k = 0),$$

$$(\tilde{g}(f))([\rho_1|\rho_2|\ldots|\rho_k]) = gf([g^{-1}\rho_1 g|g^{-1}\rho_2 g|\ldots|g^{-1}\rho_k g]) \quad (k \geq 1),$$

where $\rho_1, \rho_2, \ldots, \rho_k \in {}^gH$.

PROOF. See [19, Proposition 2-5-1]. □

In the following, (X,d) and (Y,δ) denote the standard resolution of D_{2n} and the resolution of D_{2n} given by (2.1), respectively.

Moreover, $(X^{(l+1)},d^{(l+1)})$ denotes the standard resolution of $G_{m+l+2} = \langle x^m, x^l y\rangle$ for $l=0,1$, and $(Y^{(l+1)},\delta^{(l+1)})$ denotes the resolution of $G_{m+l+2}(\simeq D_4)$ for $l=0,1$ given by (2.1). The boundaries of $(Y^{(l+1)},\delta^{(l+1)})$ are left $\boldsymbol{Z}G_{m+l+2}$-homomorphisms given by

$$(\delta^{(l+1)})_q(c_q^r) = \begin{cases} (x^{m+l}y + (-1)^{(q-r)/2})c_{q-1}^{r-1} + (x^m - 1)c_{q-1}^r & \text{for } q \text{ even, } r \text{ even,} \\ (x^l y - (-1)^{(q+r+1)/2})c_{q-1}^{r-1} + (x^m + 1)c_{q-1}^r & \text{for } q \text{ even, } r \text{ odd,} \\ (x^l y - (-1)^{(q+r+1)/2})c_{q-1}^{r-1} - (x^m + 1)c_{q-1}^r & \text{for } q \text{ odd, } r \text{ even,} \\ (x^{m+l}y - (-1)^{(q-r)/2})c_{q-1}^{r-1} + (x^m - 1)c_{q-1}^r & \text{for } q \text{ odd, } r \text{ odd,} \end{cases}$$

for $q > 0$. By Lemma 3.1, an initial part of a chain transformation $(u^{(l+1)})_k : (X^{(l+1)})_k \to (Y^{(l+1)})_k$ is given by

$$(u^{(l+1)})_0([\cdot]) = c_0^1;$$

$$(u^{(l+1)})_1([x^{mi+lq}y^q]) = ic_1^1 + qx^{mi}c_1^2;$$

$$(u^{(l+1)})_2([x^{mi+lp}y^p \mid x^{mj+lq}y^q]) = \begin{cases} 0 & (i+j<2, p=0), \\ c_2^1 & (i+j=2, p=0), \\ jx^{m(i-j)}c_2^2 + qx^{m(i-j)}c_2^3 & (i-j\geq 0, p=1), \\ -c_2^1 + jx^{m(i-j)}c_2^2 + qx^{m(i-j)}c_2^3 & (i-j<0, p=1), \end{cases}$$

where $i,j,q = 0,1$, and an initial part of a chain transformation $(v^{(l+1)})_k : (Y^{(l+1)})_k \to (X^{(l+1)})_k$ is given by

$$(v^{(l+1)})_0(c_0^1) = [\cdot];$$

$$(v^{(l+1)})_1(c_1^1) = [x^m], \quad (v^{(l+1)})_1(c_1^2) = [x^l y];$$

$$(v^{(l+1)})_2(c_2^1) = [x^m + 1 \mid x^m], \quad (v^{(l+1)})_2(c_2^2) = [x^{m+l}y \mid x^m] + [x^m \mid x^l y],$$

$$(v^{(l+1)})_2(c_2^3) = [x^l y + 1 \mid x^l y].$$

PROPOSITION 3.4. *The following hold*:

(i) $y^*(\sigma) = -\sigma$,

(ii) $(x^{m/2})^*(\mu_r) = \lambda_r + \mu_r$ *for* m *even and* $r = 1,2$.

Moreover, g^* *maps* $1(\in H^0(G_r, \mathbf{Z}))$ *to* $1(\in H^0({}^gG_r, \mathbf{Z}))$ *for* $1 \le r \le m+3$ *and* $g \in D_{2n}$.

PROOF. We prove $(x^{m/2})^*(\mu_1) = \lambda_1 + \mu_1$ (m even) only. This is given by the composition of the following maps on the cochain level:

$$\begin{aligned}
\mathbf{Z}^3 &\xrightarrow{\beta_2^{\prime -1}} \mathrm{Hom}_{\mathbf{Z}G_{m+2}}((Y^{(1)})_2, \mathbf{Z}) \\
&\xrightarrow{(u^{(1)})_2^{\#}} \mathrm{Hom}_{\mathbf{Z}G_{m+2}}((X^{(1)})_2, \mathbf{Z}) \\
&\xrightarrow{\widetilde{x^l}} \mathrm{Hom}_{\mathbf{Z}G_{m+2}}((X^{(1)})_2, \mathbf{Z}) \\
&\xrightarrow{(v^{(1)})_2^{\#}} \mathrm{Hom}_{\mathbf{Z}G_{m+2}}((Y^{(1)})_2, \mathbf{Z}) \\
&\xrightarrow{\beta_2'} \mathbf{Z}^3,
\end{aligned}$$

where we set $l = m/2$ and β_2' denotes the isomorphism $\mathrm{Hom}_{\mathbf{Z}G_{m+2}}((Y^{(1)})_2, \mathbf{Z}) \xrightarrow{\sim} \mathbf{Z}^3$ stated in Section 2. Since

$$(\widetilde{x^l}(\beta_2^{\prime -1}(\mu_1) \cdot (u^{(1)})_2))((v^{(1)})_2(c_2^1)) = \beta_2^{\prime -1}(\mu_1) \cdot (u^{(1)})_2([x^m + 1 \mid x^m]) = 1,$$

$$(\widetilde{x^l}(\beta_2^{\prime -1}(\mu_1) \cdot (u^{(1)})_2))((v^{(1)})_2(c_2^2)) = \beta_2^{\prime -1}(\mu_1) \cdot (u^{(1)})_2([y|x^m] + [x^m \mid x^m y]) = 1,$$

$$(\widetilde{x^l}(\beta_2^{\prime -1}(\mu_1) \cdot (u^{(1)})_2))((v^{(1)})_2(c_2^3)) = \beta_2^{\prime -1}(\mu_1) \cdot (u^{(1)})_2([x^m y + 1 \mid x^m y]) = 1,$$

it follows that $(x^{m/2})^*(\mu_1) = \lambda_1 + \mu_1$ holds. Similarly, we have $(x^{m/2})^*(\mu_2) = \lambda_2 + \mu_2$. The equation $y^*(\sigma) = -\sigma$ is obtained by using Lemmas 3.2 and 3.3. □

Next, we calculate restriction maps. In the following, (Z_H, ∂_H) denotes the periodic resolution of a cyclic subgroup H of D_{2n}.

LEMMA 3.5. (i) *A chain transformation* $w_k : (Z_{\langle x \rangle})_k \to Y_k$ *lifting the identity map on* $\mathbf{Z}$ *is given by* $w_k(1) = c_k^1$ $(k \ge 0)$.

(ii) *An initial part of a chain transformation* $(w^{(l)})_k : (Y^{(l)})_k \to Y_k$ $(l = 1, 2)$ *lifting the identity map on* $\mathbf{Z}$ *is given as follows*:

$$(w^{(l)})_0(c_0^1) = c_0^1;$$

$$(w^{(l)})_1(c_1^1) = N_m c_1^1, \quad (w^{(l)})_1(c_1^2) = N_{l-1} c_1^1 + x^{l-1} c_1^2;$$

$$(w^{(l)})_2(c_2^r) = \begin{cases} N_{l-1}c_2^{r-1} + c_2^r & (\textit{if } r = 1,3), \\ x^{l-1}N_m c_2^r & (\textit{if } r = 2); \end{cases}$$

$$(w^{(l)})_3(c_3^r) = \begin{cases} N_m c_3^r & (\textit{if } r \textit{ odd}), \\ (-1)^{r/2}N_{l-1}c_3^{r-1} + x^{l-1}c_3^r & (\textit{if } r \textit{ even}); \end{cases}$$

$$(w^{(l)})_4(c_4^r) = \begin{cases} (-1)^{(r-1)/2}N_{l-1}c_4^{r-1} + c_4^r & (\textit{if } r \textit{ odd}), \\ x^{l-1}N_m c_4^r & (\textit{if } r \textit{ even}). \end{cases}$$

PROOF. (i) is easily obtained. To prove (ii), it suffices to check that $\delta_k \cdot (w^{(l)})_k = (w^{(l)})_{k-1} \cdot (\delta^{(l)})_k$ holds for $k = 1,2,3,4$. The proof is straightforward. □

LEMMA 3.6. (i) *A chain transformation* $s_k : (Z_{\langle x^m \rangle})_k \to (Z_{\langle x \rangle})_k$ *lifting the identity map on* $\boldsymbol{Z}$ *is given by* $s_{2k}(1) = 1$; $s_{2k+1}(1) = N_m$ $(k \geq 0)$.

(ii) *A chain transformation* $(s^{(l)})_k : (Z_{\langle x^m \rangle})_k \to (Y^{(l)})_k$ $(l = 1,2)$ *lifting the identity map on* $\boldsymbol{Z}$ *is given by* $(s^{(l)})_k(1) = c_k^1$ $(k \geq 0)$.

PROPOSITION 3.7. *The following hold:*

(i) $\mathrm{res}^{D_{2n}}_{\langle x \rangle} \lambda = 0$, $\mathrm{res}^{D_{2n}}_{\langle x \rangle} \mu = m\sigma$, $\mathrm{res}^{D_{2n}}_{\langle x \rangle} \xi = \sigma^2$.

(ii) $\mathrm{res}^{D_{2n}}_{G_{m+2}} \lambda = \lambda_1$, $\mathrm{res}^{D_{2n}}_{G_{m+2}} \mu = \begin{cases} 0 & (m \textit{ even}), \\ \mu_1 & (m \textit{ odd}), \end{cases}$ $\mathrm{res}^{D_{2n}}_{G_{m+2}} \nu = \nu_1$,
$\mathrm{res}^{D_{2n}}_{G_{m+2}} \xi = \mu_1^2 + \lambda_1\mu_1$.

(iii) $\mathrm{res}^{D_{2n}}_{G_{m+3}} \lambda = \lambda_2$, $\mathrm{res}^{D_{2n}}_{G_{m+3}} \mu = \begin{cases} \lambda_2 & (m \textit{ even}), \\ \lambda_2 + \mu_2 & (m \textit{ odd}), \end{cases}$ $\mathrm{res}^{D_{2n}}_{G_{m+3}} \nu = \nu_2$,
$\mathrm{res}^{D_{2n}}_{G_{m+3}} \xi = \mu_2^2 + \lambda_2\mu_2$.

(iv) $\mathrm{res}^{G_{m+r+1}}_{\langle x^m \rangle} \lambda_r = 0$, $\mathrm{res}^{\langle x \rangle}_{\langle x^m \rangle} \sigma = \mathrm{res}^{G_{m+r+1}}_{\langle x^m \rangle} \mu_r = \tau$ $(r = 1,2)$.

Moreover, $\mathrm{res}^{D_{2n}}_{G_r} 1 = 1 (\in H^0(G_r, \boldsymbol{Z}))$ *and* $\mathrm{res}^{G_r}_{\langle x^m \rangle} 1 = 1 (\in H^0(\langle x^m \rangle, \boldsymbol{Z}))$ *hold for* $3 \leq r \leq m+3$.

PROOF. These are given by using Lemmas 3.5 and 3.6. We calculate $\mathrm{res}^{D_{2n}}_{G_{m+2}} \lambda$ and $\mathrm{res}^{D_{2n}}_{G_{m+2}} \mu$ as examples. These are given by the composition of the following maps on the cochain level:

$$\boldsymbol{Z}^3 \xrightarrow{\alpha_2^{-1}} \mathrm{Hom}_{\boldsymbol{Z}D_{2n}}(Y_2, \boldsymbol{Z})$$

$$\xrightarrow{(w^{(1)})_2^\#} \mathrm{Hom}_{\boldsymbol{Z}G_{m+2}}((Y^{(1)})_2, \boldsymbol{Z})$$

$$\xrightarrow{\beta_2'} \boldsymbol{Z}^3,$$

where α_2 denotes the isomorphism $\mathrm{Hom}_{ZD_{2n}}(Y_2, Z) \xrightarrow{\sim} Z^3$ stated in Section 2. Since

$$\alpha_2^{-1}(\lambda)\cdot(w^{(1)})_2(c_2^1) = \alpha_2^{-1}(\lambda)(c_2^1) = 0,$$

$$\alpha_2^{-1}(\lambda)\cdot(w^{(1)})_2(c_2^2) = \alpha_2^{-1}(\lambda)(N_m c_2^2) = 0,$$

$$\alpha_2^{-1}(\lambda)\cdot(w^{(1)})_2(c_2^3) = \alpha_2^{-1}(\lambda)(c_2^3) = 1,$$

$$\alpha_2^{-1}(\mu)\cdot(w^{(1)})_2(c_2^1) = \alpha_2^{-1}(\mu)(c_2^1) = m,$$

$$\alpha_2^{-1}(\mu)\cdot(w^{(1)})_2(c_2^2) = \alpha_2^{-1}(\mu)(N_m c_2^2) = m,$$

$$\alpha_2^{-1}(\mu)\cdot(w^{(1)})_2(c_2^3) = \alpha_2^{-1}(\mu)(c_2^3) = 0,$$

it follows that $\mathrm{res}^{D_{2n}}_{G_{m+2}}\,\lambda = \lambda_1$ and $\mathrm{res}^{D_{2n}}_{G_{m+2}}\,\mu = m\mu_1$ hold. □

Finally, we calculate corestriction maps. To compute them we need the following lemma:

LEMMA 3.8. *Suppose H is a subgroup of index l of a finite group G and A is a G-module. Fix a set of right coset representatives $S = \{\omega_1(=1), \omega_2, \dots, \omega_l\}$ of H in G, and let $c(g)(\in S)$ denote the representative of the right coset containing $g \in G$. (X_G, d_G) and (X_H, d_H) denote the standard resolutions of G and H, respectively. Then the corestriction map $\mathrm{cor}_H^G : H^k(H, A) \to H^k(G, A)$ is given by the following on the cochain level:*

$$T_H^G : \mathrm{Hom}_{ZH}((X_H)_k, A) \to \mathrm{Hom}_{ZG}((X_G)_k, A)$$

$$(T_H^G(u))([\cdot]) = \sum_{\omega\in S}\omega^{-1}u([\cdot]) \quad (k = 0),$$

$$(T_H^G(u))([\sigma_1|\sigma_2|\dots|\sigma_k]) = \sum_{\omega\in S}\omega^{-1}u([c(\omega)\sigma_1 c(\omega\sigma_1)^{-1}|c(\omega\sigma_1)\sigma_2 c(\omega\sigma_1\sigma_2)^{-1}|\dots$$
$$|c(\omega\sigma_1\cdots\sigma_{k-1})\sigma_k c(\omega\sigma_1\cdots\sigma_k)^{-1}]) \quad (k \geq 1),$$

where $u \in \mathrm{Hom}_{ZH}((X_H)_k, A)$ and $\sigma_1, \sigma_2, \dots, \sigma_k \in G$.

PROOF. See [19, Proposition 2-5-2]. □

PROPOSITION 3.9. *The following hold:*

$$\mathrm{cor}^{D_{2n}}_{\langle x\rangle}\,\sigma = 0, \quad \mathrm{cor}^{D_{2n}}_{G_{m+2}}\,\mu_1 = \lambda + \mu \ (m \text{ even}), \quad \mathrm{cor}^{D_{2n}}_{G_{m+3}}\,\mu_2 = \mu \ (m \text{ even}).$$

PROOF. First we calculate $\mathrm{cor}_{\langle x\rangle}^{D_{2n}}\sigma$. Using Lemmas 3.1, 3.2 and 3.8, this is given by the composition of the following maps on the cochain level:

$$\begin{aligned} Z &\xrightarrow{\beta_2^{-1}} \mathrm{Hom}_{Z\langle x\rangle}((Z_{\langle x\rangle})_2, Z) \\ &\xrightarrow{(u_{\langle x\rangle})_2^{\#}} \mathrm{Hom}_{Z\langle x\rangle}((X_{\langle x\rangle})_2, Z) \\ &\xrightarrow{T_{\langle x\rangle}^{D_{2n}}} \mathrm{Hom}_{ZD_{2n}}(X_2, Z) \\ &\xrightarrow{v_2^{\#}} \mathrm{Hom}_{ZD_{2n}}(Y_2, Z) \\ &\xrightarrow{\alpha_2} Z^3, \end{aligned}$$

where β_2 denotes an isomorphism $\mathrm{Hom}_{Z\langle x\rangle}((Z_{\langle x\rangle})_2, Z) \xrightarrow{\sim} Z$; $f \mapsto f(1)$. Let $\{1, y\}$ be a set of right coset representatives of $\langle x\rangle$ in D_{2n}. Then $c(x^i) = 1$ and $c(x^i y) = y$ hold. Since

$$\begin{aligned} &(T_{\langle x\rangle}^{D_{2n}}(\beta_2^{-1}(\sigma)\cdot(u_{\langle x\rangle})_2))(v_2(c_2^1)) \\ &= \sum_{i=0}^{n-1}\beta_2^{-1}(\sigma)\cdot(u_{\langle x\rangle})_2([c(1)x^i c(x^i)^{-1} \mid c(x^i)xc(x^{i+1})^{-1}] \\ &\qquad + [c(y)x^i c(yx^i)^{-1} \mid c(yx^i)xc(yx^{i+1})^{-1}]) \\ &= \sum_{i=0}^{n-1}\beta_2^{-1}(\sigma)\cdot(u_{\langle x\rangle})_2([x^i|x] + [x^{-i}|x^{-1}]) \\ &= n, \end{aligned}$$

$$\begin{aligned} &(T_{\langle x\rangle}^{D_{2n}}(\beta_2^{-1}(\sigma)\cdot(u_{\langle x\rangle})_2))(v_2(c_2^2)) \\ &= \beta_2^{-1}(\sigma)\cdot(u_{\langle x\rangle})_2([c(1)xyc(xy)^{-1} \mid c(xy)xc(y)^{-1}] \\ &\qquad + [c(y)xyc(x^{-1})^{-1} \mid c(x^{-1})xc(1)^{-1}] \\ &\qquad + [c(1)xc(x)^{-1} \mid c(x)yc(xy)^{-1}] + [c(y)xc(yx)^{-1} \mid c(yx)yc(x^{-1})^{-1}]) \\ &= \beta_2^{-1}(\sigma)\cdot(u_{\langle x\rangle})_2([x|x^{-1}] + [x^{-1}|x] + [x|1] + [x^{-1}|1]) \\ &= 2, \end{aligned}$$

$$\begin{aligned}
&(T_{\langle x\rangle}^{D_{2n}}(\beta_2^{-1}(\sigma)\cdot(u_{\langle x\rangle})_2))(v_2(c_2^3))\\
&\quad=\beta_2^{-1}(\sigma)\cdot(u_{\langle x\rangle})_2([c(1)yc(y)^{-1}\,|\,c(y)yc(1)^{-1}]+[c(y)yc(1)^{-1}\,|\,c(1)yc(y)^{-1}]\\
&\qquad\qquad+[c(1)1c(1)^{-1}\,|\,c(1)yc(y)^{-1}]+[c(y)1c(y)^{-1}\,|\,c(y)yc(1)^{-1}])\\
&\quad=4\beta_2^{-1}(\sigma)\cdot(u_{\langle x\rangle})_2([1|1])\\
&\quad=0,
\end{aligned}$$

we have $\mathrm{cor}_{\langle x\rangle}^{D_{2n}}\,\sigma=2\mu=0$.

Next, let m be even and $\{1,x,\dots,x^{m-1}\}$ a set of right coset representatives of $G_{m+2}=\langle x^m,y\rangle$ in D_{2n}. Then $c(x^i)=c(x^{m+i})=x^i$ $(0\le i\le m-1)$ and $c(x^iy)=c(x^{m+i}y)=x^{m-i}$ $(1\le i\le m)$ hold. Since

$$\begin{aligned}
&(T_{G_{m+2}}^{D_{2n}}(\beta_2'^{-1}(\mu_1)\cdot(u^{(1)})_2))(v_2(c_2^1))\\
&\quad=\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}(\beta_2'^{-1}(\mu_1)\cdot(u^{(1)})_2)([c(x^i)x^jc(x^{i+j})^{-1}\,|\,c(x^{i+j})xc(x^{i+j+1})^{-1}])\\
&\quad=m\beta_2'^{-1}(\mu_1)\cdot(u^{(1)})_2([x^m|x^m])\\
&\quad=m,\\
&(T_{G_{m+2}}^{D_{2n}}(\beta_2'^{-1}(\mu_1)\cdot(u^{(1)})_2))(v_2(c_2^2))\\
&\quad=\sum_{i=0}^{m-1}(\beta_2'^{-1}(\mu_1)\cdot(u^{(1)})_2)([c(x^i)xyc(x^{i+1}y)^{-1}\,|\,c(x^{i+1}y)xc(x^iy)^{-1}]\\
&\qquad\qquad+[c(x^i)xc(x^{i+1})^{-1}\,|\,c(x^{i+1})yc(x^{i+1}y)^{-1}])\\
&\quad=\beta_2'^{-1}(\mu_1)\cdot(u^{(1)})_2([x^my\,|\,x^m])\\
&\quad=1,\\
&(T_{G_{m+2}}^{D_{2n}}(\beta_2'^{-1}(\mu_1)\cdot(u^{(1)})_2))(v_2(c_2^3))\\
&\quad=\sum_{i=0}^{m-1}(\beta_2'^{-1}(\mu_1)\cdot(u^{(1)})_2)([c(x^i)yc(x^iy)^{-1}\,|\,c(x^iy)yc(x^i)^{-1}]\\
&\qquad\qquad+[1\,|\,c(x^i)yc(x^iy)^{-1}])\\
&\quad=\beta_2'^{-1}(\mu_1)\cdot(u^{(1)})_2([y|y]+(m-1)[x^my\,|\,x^my])\\
&\quad=m-1,
\end{aligned}$$

it follows that $\mathrm{cor}_{G_{m+2}}^{D_{2n}}\,\mu_1=(m-1)\lambda+\mu=\lambda+\mu$ holds.

Finally, let m be even and $\{1, x, \dots, x^{m-1}\}$ a set of right coset representatives of $G_{m+3} = \langle x^m, xy \rangle$ in D_{2n}. Then $c(x^i) = c(x^{m+i}) = x^i$ $(0 \le i \le m-1)$ and $c(x^i y) = c(x^{m+i} y) = x^{m-i+1}$ $(2 \le i \le m+1)$ hold. By a similar calculation as above, we have $\mathrm{cor}_{G_{m+3}}^{D_{2n}} \mu_2 = (m-2)\lambda + \mu = \mu$. □

Note that from (1.1), Propositions 3.7 and 3.9, and [19, Proposition 4-3-7], we have

$$\mathrm{cor}_{G_{m+2}}^{D_{2n}}(\lambda_1 \mu_1) = \mathrm{cor}_{G_{m+2}}^{D_{2n}}(\mathrm{res}_{G_{m+2}}^{D_{2n}} \lambda \cdot \mu_1) = \lambda \cdot \mathrm{cor}_{G_{m+2}}^{D_{2n}} \mu_1 = \begin{cases} \lambda^2 + \lambda\mu & (m \text{ even}), \\ \lambda\mu & (m \text{ odd}), \end{cases}$$

and so on.

3.2 The Case n Odd

In the case n odd, the calculations are easily obtained. We set $t = (n-1)/2$. We take representatives of the conjugacy classes of D_{2n} are

$$g_1 = 1, \quad g_{i+1} = x^i \ (1 \le i \le t), \quad g_{t+2} = y,$$

and their centralizers are

$$G_1 = D_{2n}, \quad G_{i+1} = \langle x \rangle \ (1 \le i \le t), \quad G_{t+2} = \langle y \rangle,$$

respectively. We set

$$H^*(G_1, \boldsymbol{Z}) = \boldsymbol{Z}[\alpha, \beta]/(2\alpha, 2n\beta, \alpha^2 - n\beta) \quad (\deg \alpha = 2, \deg \beta = 4),$$
$$H^*(G_{k+1}, \boldsymbol{Z}) = \boldsymbol{Z}[\rho]/(n\rho) \quad (\deg \rho = 2),$$
$$H^*(G_{t+2}, \boldsymbol{Z}) = \boldsymbol{Z}[\chi]/(2\chi) \quad (\deg \chi = 2),$$

where $1 \le k \le t$. These presentations follow from (2.6) and (2.8). By (1.2), (2.5) and (2.7), we have

$$H^k(D_{2n}, {}_\psi \boldsymbol{Z} D_{2n}) = \begin{cases} \boldsymbol{Z}^{t+2} & \text{for } k = 0, \\ 0 & \text{for } k \equiv 1 \ (4), \\ (\boldsymbol{Z}/2)^2 \oplus (\boldsymbol{Z}/n)^t & \text{for } k \equiv 2 \ (4), \\ 0 & \text{for } k \equiv 3 \ (4), \\ (\boldsymbol{Z}/2)^2 \oplus (\boldsymbol{Z}/n)^{t+1} & \text{for } k \equiv 0 \ (4),\ k \ne 0. \end{cases}$$

By computations similar to Propositions 3.4, 3.7 and 3.9, we have

$$y^*(\rho) = -\rho, \quad \mathrm{res}_{\langle x \rangle}^{D_{2n}} \alpha = 0, \quad \mathrm{res}_{\langle x \rangle}^{D_{2n}} \beta = \rho^2, \quad \mathrm{cor}_{\langle x \rangle}^{D_{2n}} \rho = 0.$$

Moreover, by (1.1), (2.6) and (2.8), we have $\mathrm{res}_{\langle y \rangle}^{D_{2n}} \alpha = \chi$, $\mathrm{res}_{\langle y \rangle}^{D_{2n}} \beta = \chi^2$.

4 Products on $H^*(D_{2n}, {}_{\psi}ZD_{2n})$

In this section, we will determine the ring structure of the Hochschild cohomology $H^*(D_{2n}, {}_{\psi}ZD_{2n})(\simeq HH^*(ZD_{2n}))$ by using the Product Formula. In the following, we write XY in place of $X \smile Y$ for brevity.

4.1 The Case n Even

In this subsection, we calculate the products on the Hochschild cohomology $H^*(D_{2n}, {}_{\psi}ZD_{2n})(\simeq HH^*(ZD_{2n}))$ for the case $n = 2m$ $(m \geq 2)$. In the following, we set

$$A_2 = \gamma_1(\lambda), \quad B_2 = \gamma_1(\mu), \quad A_3 = \gamma_1(\nu), \quad A_4 = \gamma_1(\xi), \quad C_0 = \gamma_2(1),$$

$$(E_i)_0 = \gamma_{i+2}(1) \ (1 \leq i \leq m-1), \quad (E_i)_2 = \gamma_{i+2}(\sigma) \ (1 \leq i \leq m-1),$$

$$S_0 = \gamma_{m+2}(1), \quad S_2 = \gamma_{m+2}(\mu_1), \quad T_0 = \gamma_{m+3}(1), \quad T_2 = \gamma_{m+3}(\mu_2).$$

Moreover, we set $F_i = (E_1)_i$ $(i = 0, 2)$,

$$U_k = \begin{cases} (E_{-k})_0 & (-m < k < 0), \\ 2 & (k = 0), \\ (E_k)_0 & (0 < k < m), \\ 2C_0 & (k = m), \\ (E_{n-k})_0 & (m < k < n), \end{cases} \qquad V_k = \begin{cases} -(E_{-k})_2 & (-m < k < 0), \\ 0 & (k = 0), \\ (E_k)_2 & (0 < k < m), \\ 0 & (k = m), \\ -(E_{n-k})_2 & (m < k < n). \end{cases}$$

In the following, we interpret $\sum_{l=1}^{(m/2)-1}$ term as 0 if $m = 2$.

First, we calculate products in degree 0. The products in degree 0 correspond to the multiplication in the center of ZD_{2n}. By using this identification, we have the following proposition:

PROPOSITION 4.1. (i) *If m is even, the following equations hold in* $H^0(D_{2n}, {}_{\psi}ZD_{2n})$:

$$C_0^2 = 1, \quad C_0(E_i)_0 = (E_{m-i})_0, \quad C_0S_0 = S_0, \quad C_0T_0 = T_0,$$

$$(E_i)_0S_0 = \begin{cases} 2S_0 & (i \text{ even}), \\ 2T_0 & (i \text{ odd}), \end{cases} \quad (E_i)_0T_0 = \begin{cases} 2T_0 & (i \text{ even}), \\ 2S_0 & (i \text{ odd}), \end{cases}$$

$$S_0^2 = T_0^2 = m(1 + C_0) + m\sum_{l=1}^{(m/2)-1}(E_{2l})_0, \quad S_0T_0 = m\sum_{l=1}^{m/2}(E_{2l-1})_0,$$

$$(E_i)_0(E_j)_0 = U_{i+j} + U_{i-j}.$$

(ii) *If m is odd, the following equations hold in* $H^0(D_{2n}, {}_\psi ZD_{2n})$:

$$C_0^2 = 1, \quad C_0(E_i)_0 = (E_{m-i})_0, \quad C_0S_0 = T_0, \quad (E_i)_0S_0 = \begin{cases} 2S_0 & (i \textit{ even}), \\ 2T_0 & (i \textit{ odd}), \end{cases}$$

$$S_0^2 = m + m\sum_{l=1}^{(m-1)/2}(E_{2l})_0, \quad (E_i)_0(E_j)_0 = U_{i+j} + U_{i-j}.$$

REMARK 1. Since the equations $(E_2)_0 = F_0^2 - 2$ and $(E_k)_0 = F_0(E_{k-1})_0 - (E_{k-2})_0$ $(3 \le k \le m-1)$ hold, it follows that the powers of F_0 generate $(E_k)_0$ $(2 \le k \le m-1)$. Hence $H^0(D_{2n}, {}_\psi ZD_{2n})$ is generated by the products of C_0, F_0, S_0 and T_0 (resp. C_0, F_0 and S_0) for the case m even (resp. m odd).

Next, we compute cup products for generators of $H^0(D_{2n}, {}_\psi ZD_{2n})$ and generators of $H^2(D_{2n}, {}_\psi ZD_{2n})$.

PROPOSITION 4.2. (i) *If m is even, the following equations hold in* $H^2(D_{2n}, {}_\psi ZD_{2n})$:

(1) $C_0A_2 = \gamma_2(\lambda)$, $C_0B_2 = \gamma_2(\mu)$, $C_0(E_i)_2 = -(E_{m-i})_2$, $C_0S_2 = S_0A_2 + S_2$, $C_0T_2 = T_0A_2 + T_2$.

(2) $(E_i)_0A_2 = (E_i)_0S_2 = (E_i)_0T_2 = 0$, $(E_i)_0B_2 = m(E_i)_2$, $(E_i)_0(E_j)_2 = V_{i+j} + V_{j-i}$.

(3) $S_0A_2 = \gamma_{m+2}(\lambda_1)$, $S_0B_2 = S_0(E_i)_2 = 0$, $S_0S_2 = (1 + C_0)(A_2 + B_2) + m\sum_{l=1}^{(m/2)-1}(E_{2l})_2$, $S_0T_2 = m\sum_{l=1}^{m/2}(E_{2l-1})_2$.

(4) $T_0A_2 = T_0B_2 = \gamma_{m+3}(\lambda_2)$, $T_0(E_i)_2 = 0$, $T_0S_2 = m\sum_{l=1}^{m/2}(E_{2l-1})_2$, $T_0T_2 = (1 + C_0)B_2 + m\sum_{l=1}^{(m/2)-1}(E_{2l})_2$.

(ii) *If m is odd, the following equations hold in* $H^2(D_{2n}, {}_\psi ZD_{2n})$:

(1) $C_0A_2 = \gamma_2(\lambda)$, $C_0B_2 = \gamma_2(\mu)$, $C_0(E_i)_2 = -(E_{m-i})_2$.

(2) $(E_i)_0A_2 = 0$, $(E_i)_0B_2 = m(E_i)_2$, $(E_i)_0(E_j)_2 = V_{i+j} + V_{j-i}$.

(3) $S_0A_2 = \gamma_{m+2}(\lambda_1)$, $S_0B_2 = \gamma_{m+2}(\mu_1)$, $S_0(E_i)_2 = 0$, $C_0S_0A_2 = \gamma_{m+3}(\lambda_2)$, $C_0S_0(A_2 + B_2) = \gamma_{m+3}(\mu_2)$.

PROOF. Table 1 is useful for computations.
We prove (i) only. By the Product Formula, we have

$$\gamma_1(\alpha)\gamma_r(\beta) = \gamma_r(\mathrm{cor}_{G_r}^{G_r}(\mathrm{res}_{G_r}^{D_{2n}}\alpha \cdot \mathrm{res}_{G_r}^{G_r}\beta)) = \gamma_r(\mathrm{res}_{G_r}^{D_{2n}}\alpha \cdot \beta) \tag{4.1}$$

Table 1. Data for the Product Formula.

i	j	a	$g_i{}^a g_j$	b	k	${}^bG_i$	${}^{ba}G_j$	$W = {}^bG_i \cap {}^{ba}G_j$
1	r $(1 \le r \le m+3)$	1	g_r	1	r	D_{2n}	G_r	G_r
2	2	1	1	1	1	D_{2n}	D_{2n}	D_{2n}
2	$r+2$ $(1 \le r < m)$	1	x^{m+r}	y	$m-r+2$	D_{2n}	$\langle x \rangle$	$\langle x \rangle$
2	$m+2$ (m even)	1	$x^m y$	$x^{m/2}$	$m+2$	D_{2n}	G_{m+2}	G_{m+2}
2	$m+2$ (m odd)	1	$x^m y$	$x^{(m+1)/2}$	$m+3$	D_{2n}	G_{m+3}	G_{m+3}
2	$m+3$ (m even)	1	$x^{m+1} y$	$x^{m/2}$	$m+3$	D_{2n}	G_{m+3}	G_{m+3}
2	$m+3$ (m odd)	1	$x^{m+1} y$	$x^{(m-1)/2}$	$m+2$	D_{2n}	G_{m+2}	G_{m+2}
$r+2$	$s+2$ $(1 \le r, s < m)$	1 y	x^{r+s} x^{r-s}	1 $(r+s<m)$ 1 $(r+s=m)$ y $(r+s>m)$ 1 $(r>s)$ 1 $(r=s)$ y $(s>r)$	$r+s+2$ 2 $n-r-s+2$ $r-s+2$ 1 $s-r+2$	$\langle x \rangle$ $\langle x \rangle$ $\langle x \rangle$ $\langle x \rangle$ $\langle x \rangle$ $\langle x \rangle$	$\langle x \rangle$ $\langle x \rangle$ $\langle x \rangle$ $\langle x \rangle$ $\langle x \rangle$ $\langle x \rangle$	$\langle x \rangle$ $\langle x \rangle$ $\langle x \rangle$ $\langle x \rangle$ $\langle x \rangle$ $\langle x \rangle$
$r+2$	$m+2$ $(1 \le r < m)$	1	$x^r y$	$x^{-r/2}$ (r even) $x^{(1-r)/2}$ (r odd)	$m+2$ $m+3$	$\langle x \rangle$ $\langle x \rangle$	$\langle x^m, x^{-r} y \rangle$ $\langle x^m, x^{1-r} y \rangle$	$\langle x^m \rangle$ $\langle x^m \rangle$
$r+2$	$m+3$ $(1 \le r < m)$	1	$x^{r+1} y$	$x^{-r/2}$ (r even) $x^{(-r-1)/2}$ (r odd)	$m+3$ $m+2$	$\langle x \rangle$ $\langle x \rangle$	$\langle x^m, x^{1-r} y \rangle$ $\langle x^m, x^{-r} y \rangle$	$\langle x^m \rangle$ $\langle x^m \rangle$
$m+2$	$m+2$ (m even)	1 x^l $(1 \le l < \frac{m}{2})$ $x^{m/2}$	1 x^{-2l} x^m	1 y 1	1 $2l+2$ 2	G_{m+2} G_{m+2} G_{m+2}	G_{m+2} $\langle x^m, x^{-2l} y \rangle$ G_{m+2}	G_{m+2} $\langle x^m \rangle$ G_{m+2}
$m+2$	$m+3$ (m even)	x^l $(0 \le l < \frac{m}{2})$	x^{-2l-1}	y	$2l+3$	G_{m+2}	$\langle x^m, x^{-2l-1} y \rangle$	$\langle x^m \rangle$
$m+3$	$m+3$ (m even)	1 x^l $(1 \le l < \frac{m}{2})$ $x^{m/2}$	1 x^{-2l} x^m	1 y 1	1 $2l+2$ 2	G_{m+3} $\langle x^m, x^{-1} y \rangle$ G_{m+3}	G_{m+3} $\langle x^m, x^{-2l-1} y \rangle$ G_{m+3}	G_{m+3} $\langle x^m \rangle$ G_{m+3}

for $\alpha \in H^*(D_{2n}, \boldsymbol{Z})$, $\beta \in H^*(G_r, \boldsymbol{Z})$ $(1 \le r \le m+3)$. By the above equation and Proposition 3.7 we have $C_0A_2 = \gamma_2(\lambda)$, $C_0B_2 = \gamma_2(\mu)$, and so on.

The other equations are obtained by using Theorem 1.1, Propositions 3.4, 3.7 and 3.9 and Table 1. We calculate S_0S_2 as an example.

$$\begin{aligned} S_0S_2 &= \gamma_{m+2}(1)\gamma_{m+2}(\mu_1) \\ &= \gamma_1(\mathrm{cor}_{G_{m+2}}^{D_{2n}}\mu_1) + \gamma_2(\mathrm{cor}_{G_{m+2}}^{D_{2n}}(x^{m/2})^*\mu_1) \\ &\quad + \sum_{l=1}^{(m/2)-1}\gamma_{2l+2}(\mathrm{cor}_{\langle x^m\rangle}^{\langle x\rangle}(\mathrm{res}_{\langle x^m\rangle}^{\langle x^m, x^{-2l}y\rangle}(yx^l)^*(\mu_1))) \\ &= \gamma_1(\lambda+\mu) + \gamma_2(\lambda+\mu) + m\sum_{l=1}^{(m/2)-1}\gamma_{2l+2}(\sigma) \\ &= (1+C_0)(A_2+B_2) + m\sum_{l=1}^{(m/2)-1}(E_{2l})_2. \end{aligned}$$

In the above calculation, note that restriction maps commute with conjugation maps and the conjugation maps are the identity on $H^2(\langle x^m\rangle, \boldsymbol{Z}) = \boldsymbol{Z}/2$. The other computations are similar. □

REMARK 2. By Proposition 4.2, we have

$$(E_i)_2 = \begin{cases} F_2 + \sum_{l=1}^{(i-1)/2}(E_{2l})_0F_2 & (i(\ge 3) \text{ odd}), \\ \sum_{l=1}^{i/2}(E_{2l-1})_0F_2 & (i \text{ even}). \end{cases}$$

By Remark 1, $(E_i)_2$ is generated by the products of F_0 and F_2. Therefore, $H^2(D_{2n}, {}_\psi\boldsymbol{Z}D_{2n})$ is generated by the products of C_0, F_0, S_0, T_0, A_2, B_2, F_2, S_2 and T_2 (resp. C_0, F_0, S_0, A_2, B_2 and F_2) for the case m even (resp. m odd).

PROPOSITION 4.3. *The following equations hold in* $H^3(D_{2n}, {}_\psi\boldsymbol{Z}D_{2n})$:

$$C_0A_3 = \gamma_2(\nu), \quad (E_i)_0A_3 = 0, \quad S_0A_3 = \gamma_{m+2}(\nu_1), \quad T_0A_3 = \gamma_{m+3}(\nu_2).$$

PROOF. These are immediate from (4.1) and Proposition 3.7. □

REMARK 3. $H^3(D_{2n}, {}_\psi ZD_{2n})$ is generated by the products of C_0, S_0, T_0 and A_3 (resp. C_0, S_0 and A_3) for the case m even (resp. m odd).

PROPOSITION 4.4. (i) *If m is even, the following equations hold in $H^4(D_{2n}, {}_\psi ZD_{2n})$:*

(1) $A_2^2C_0 = \gamma_2(\lambda^2)$, $A_2B_2C_0 = \gamma_2(\lambda\mu)$, $A_2^2S_0 = \gamma_{m+2}(\lambda_1^2)$, $A_2^2T_0 = \gamma_{m+3}(\lambda_2^2)$.

(2) $A_4C_0 = \gamma_2(\xi)$, $A_4(E_i)_0 = \gamma_{i+2}(\sigma^2)$, $A_4S_0 = \gamma_{m+2}(\lambda_1\mu_1 + \mu_1^2)$, $A_4T_0 = \gamma_{m+3}(\lambda_2\mu_2 + \mu_2^2)$.

(3) $A_2(E_i)_2 = B_2S_2 = (E_i)_2S_2 = (E_i)_2T_2 = 0$, $A_2S_2 = \gamma_{m+2}(\lambda_1\mu_1)$, $A_2T_2 = B_2T_2 = \gamma_{m+3}(\lambda_2\mu_2)$, $B_2(E_i)_2 = mA_4(E_i)_0$, $(E_i)_2(E_j)_2 = A_4(U_{i+j} - U_{i-j})$, $S_2^2 + A_2^2 = T_2^2 = A_2B_2 + A_4S_0^2$, $S_2T_2 = A_4S_0T_0$.

(ii) *If m is odd, the following equations hold in $H^4(D_{2n}, {}_\psi ZD_{2n})$:*

(1) $A_2^2C_0 = \gamma_2(\lambda^2)$, $A_2B_2C_0 = \gamma_2(\lambda\mu)$, $A_2^2S_0 = \gamma_{m+2}(\lambda_1^2)$, $A_2B_2S_0 = \gamma_{m+2}(\lambda_1\mu_1)$, $B_2^2S_0 = \gamma_{m+2}(\mu_1^2)$, $A_2^2C_0S_0 = \gamma_{m+3}(\lambda_2^2)$, $A_2B_2C_0S_0 = \gamma_{m+3}(\lambda_2^2 + \lambda_2\mu_2)$, $B_2^2C_0S_0 = \gamma_{m+3}(\lambda_2^2 + \mu_2^2)$.

(2) $A_4C_0 = \gamma_2(\xi)$, $A_4(E_i)_0 = \gamma_{i+2}(\sigma^2)$.

(3) $A_2(E_i)_2 = 0$, $B_2(E_i)_2 = mA_4(E_i)_0$, $(E_i)_2(E_j)_2 = A_4(U_{i+j} - U_{i-j})$.

PROOF. Note that γ_1 is a monomorphism between the cohomology rings (see [16, Section 5]). Thus the products of $\gamma_1(-)$ and $\gamma_r(-)$ $(1 \le r \le m+3)$ are obtained by using (4.1) and Proposition 3.7. The other equations are obtained by using Theorem 1.1, Propositions 3.4, 3.7 and 3.9 and Table 1. □

REMARK 4. By Proposition 4.4 and Remark 1, note that $A_4(E_i)_0$ is generated by the products of A_4 and F_0. Hence $H^4(D_{2n}, {}_\psi ZD_{2n})$ is generated by the products of C_0, F_0, S_0, T_0, A_2, B_2, S_2, T_2 and A_4 (resp. C_0, F_0, S_0, A_2, B_2 and A_4) for the case m even (resp. m odd).

REMARK 5. By (4.1), $\gamma_2(\lambda^i\mu^j\nu^k\xi^l)$ is generated by the products of A_2, B_2, A_3, A_4 and C_0. Similarly, from (4.1) and Propositions 3.7 and 4.4, we have

$$\gamma_{r+2}(\sigma^{2i+p}) = \gamma_1(\xi^i)\gamma_{r+2}(\sigma^p) \quad (i \ge 0, p = 0, 1),$$

where $1 \le r \le m-1$. By summarizing Remarks 1 and 2, it follows that $\gamma_{r+2}(\sigma^i)$ is generated by the products of A_4, F_0 and F_2.

Moreover, from (4.1) and Propositions 3.7 and 4.4, we have

$$\begin{aligned}\gamma_{m+r+1}(\mu_r^{i+2}) &= \gamma_{m+r+1}((\mathrm{res}^{D_{2n}}_{G_{m+r+1}}\lambda)\mu_r^{i+1} + (\mathrm{res}^{D_{2n}}_{G_{m+r+1}}\xi)\mu_r^i) \\ &= \gamma_1(\lambda)\gamma_{m+r+1}(\mu_r^{i+1}) + \gamma_1(\xi)\gamma_{m+r+1}(\mu_r^i) \quad (i \ge 0, r = 1, 2). \qquad (4.2)\end{aligned}$$

Using Proposition 4.4 and (4.2), it is shown that $\gamma_{m+r+1}(\mu_r^k)$ is generated by the products of $\gamma_1(\lambda)$, $\gamma_1(\nu)$, $\gamma_{m+r+1}(1)$ and $\gamma_{m+r+1}(\mu_r)$ by the induction on k. Since

$$\gamma_{m+r+1}(\lambda_r^i \mu_r^j \nu_r^k) = \gamma_1(\lambda)^i \gamma_1(\nu)^k \gamma_{m+r+1}(\mu_r^j) \quad (i, j, k \geq 0, r = 1, 2),$$

it follows that $\gamma_{m+r+1}(\lambda_r^i \mu_r^j \nu_r^k)$ is multiplicatively generated by $\gamma_1(\lambda)$, $\gamma_1(\mu)$, $\gamma_1(\nu)$, $\gamma_1(\xi)$, $\gamma_{m+r+1}(1)$ and $\gamma_{m+r+1}(\mu_r)$.

Hence, $H^k(D_{2n}, {}_\psi ZD_{2n})$ $(k \geq 5)$ is generated by the products of C_0, F_0, S_0, T_0, A_2, B_2, F_2, S_2, T_2 and A_4 (resp. C_0, F_0, S_0, A_2, B_2, F_2 and A_4) for the case m even (resp. m odd).

4.2 The Case n Odd

In this subsection, let $n(\geq 3)$ be odd. We put $t = (n-1)/2$. The computations of the products on the Hochschild cohomology $H^*(D_{2n}, {}_\psi ZD_{2n}) (\simeq HH^*(ZD_{2n}))$ are similar to Section 4.1. In the following, we set

$$A_2 = \gamma_1(\alpha), \quad A_4 = \gamma_1(\beta),$$
$$(E_i)_0 = \gamma_{i+1}(1) \ (1 \leq i \leq t), \quad (E_i)_2 = \gamma_{i+1}(\rho) \ (1 \leq i \leq t),$$
$$S_0 = \gamma_{t+2}(1).$$

Moreover, we set $F_i = (E_1)_i$ $(i = 0, 2)$ and

$$U_k = \begin{cases} (E_{-k})_0 & (-t < k < 0), \\ 2 & (k = 0), \\ (E_k)_0 & (1 \leq k \leq t), \\ (E_{n-k})_0 & (t < k < n), \end{cases} \qquad V_k = \begin{cases} -(E_{-k})_2 & (-t < k < 0), \\ 0 & (k = 0), \\ (E_k)_2 & (1 \leq k \leq t), \\ -(E_{n-k})_2 & (t < k < n), \end{cases}$$

First, we calculate the products in degree 0. These are obtained by computations similar to Proposition 4.1.

PROPOSITION 4.5. *The following equations hold in* $H^0(D_{2n}, {}_\psi ZD_{2n})$:

$$(E_i)_0 S_0 = 2S_0, \quad S_0^2 = n + n\sum_{i=1}^{t}(E_i)_0, \quad (E_i)_0(E_j)_0 = U_{i+j} + U_{i-j}.$$

REMARK 6. $H^0(D_{2n}, {}_\psi ZD_{2n})$ is generated by the products of F_0 and S_0 (*cf.* Remark 1).

Next, we calculate the products in degree 2.

Proposition 4.6. *The following equations hold in* $H^2(D_{2n}, {}_\psi ZD_{2n})$:

$$(E_i)_0 A_2 = S_0(E_i)_2 = 0, \quad S_0 A_2 = \gamma_{t+2}(\chi), \quad (E_i)_0(E_j)_2 = V_{i+j} + V_{j-i}.$$

Remark 7. $H^2(D_{2n}, {}_\psi ZD_{2n})$ is generatedby the products of F_0, S_0, A_2 and F_2 (*cf.* Remark 2).

Remark 8. Since the equations

$$\mathrm{res}^{D_{2n}}_{\langle x\rangle} \beta^k = \rho^{2k}, \quad \mathrm{res}^{D_{2n}}_{\langle y\rangle} \beta^k = \chi^{2k} \ (k \geq 0)$$

hold, by (4.1) the cup product with $A_4 = \gamma_1(\beta)$ gives a periodicity isomorphism

$$A_4 \smile - : H^k(D_{2n}, {}_\psi ZD_{2n}) \xrightarrow{\sim} H^{k+4}(D_{2n}, {}_\psi ZD_{2n})$$

for all $k \geq 1$.

Finally, we have the following proposition.

Proposition 4.7. *The following equations hold in* $H^4(D_{2n}, {}_\psi ZD_{2n})$:

$$A_4(E_i)_0 = \gamma_{i+1}(\rho^2), \quad A_4 S_0 = \gamma_{t+2}(\chi^2), \quad A_2(E_i)_2 = 0, \quad (E_i)_2(E_j)_2 = A_4(U_{i+j} - U_{i-j}).$$

4.3 Ring Structure

We will state the ring structure of $H^*(D_{2n}, {}_\psi ZD_{2n})$ by summarizing Sections 4.1 and 4.2.

Theorem 4.8. *Let D_{2n} denote the dihedral group of order $2n$ for $n \geq 3$.*

(i) *Let n be even. We set $m = n/2$.*

(1) *If m is even, the Hochschild cohomology ring $H^*(D_{2n}, {}_\psi ZD_{2n})$ ($\simeq HH^*(ZD_{2n})$) is commutative, generated by elements*

$$C_0, F_0, S_0, T_0 \in H^0(D_{2n}, {}_\psi ZD_{2n}), \quad A_2, B_2, F_2, S_2, T_2 \in H^2(D_{2n}, {}_\psi ZD_{2n}),$$

$$A_3 \in H^3(D_{2n}, {}_\psi ZD_{2n}), \quad A_4 \in H^4(D_{2n}, {}_\psi ZD_{2n}).$$

The relations follow from Table 2.

(2) *If m is odd, the Hochschild cohomology ring $H^*(D_{2n}, {}_\psi ZD_{2n})$ ($\simeq HH^*(ZD_{2n})$) is commutative, generated by elements*

$$C_0, F_0, S_0 \in H^0(D_{2n}, {}_\psi ZD_{2n}), \quad A_2, B_2, F_2 \in H^2(D_{2n}, {}_\psi ZD_{2n}),$$

$$A_3 \in H^3(D_{2n}, {}_\psi ZD_{2n}), \quad A_4 \in H^4(D_{2n}, {}_\psi ZD_{2n}).$$

The relations follow from Table 3.

Table 2. Cohomology ring $H^*(D_{2n}, {}_{\psi}\mathbf{Z}D_{2n}) (\simeq HH^*(\mathbf{Z}D_{2n}))$ for $n = 2m$ $(m \geq 2$ even).

	C_0	F_0	S_0	T_0	A_2	B_2	F_2	S_2	T_2	A_3	A_4
C_0	1	$(E_{m-1})_0$	S_0	T_0	C_0A_2	C_0B_2	$-(E_{m-1})_2$	$S_0A_2 + S_2$	$T_0A_2 + T_2$	C_0A_3	${}_nC_0A_4$
F_0		$\begin{cases} 2(C_0+1) & (m=2) \\ (E_2)_0 + 2 & (m \geq 4) \end{cases}$	$2T_0$	$2S_0$	0	mF_2	$\begin{cases} 0 & (m=2) \\ (E_2)_2 & (m \geq 4) \end{cases}$	0	0	0	${}_nF_0A_4$
S_0			I	J	S_0A_2	0	0	K	L	S_0A_3	${}_2S_0A_4$
T_0				I	T_0A_2	T_0A_2	0	L	$K + (1 + C_0)A_2$	T_0A_3	${}_2T_0A_4$
${}_2A_2$					A_2^2	A_2B_2	0	A_2S_2	A_2T_2	A_2A_3	A_2A_4
${}_2B_2$						A_2B_2	mF_0A_4	0	A_2T_2	B_2A_3	B_2A_4
${}_nF_2$							$\begin{cases} 2A_4(C_0-1) & (m=2) \\ A_4((E_2)_0-2) & (m \geq 4) \end{cases}$	0	0	0	F_2A_4
${}_2S_2$								$A_4I + A_2^2 + A_2B_2$	A_4J	S_2A_3	S_2A_4
${}_2T_2$									$A_4I + A_2B_2$	T_2A_3	T_2A_4
${}_2A_3$										A_2A_4	A_3A_4
${}_nA_4$											A_4^2

${}_lW_k$ means that l is the order of $W_k \in H^k(D_{2n}, {}_{\psi}\mathbf{Z}D_{2n})$ as a $\mathbf{Z}$-module.

$(E_i)_0$ $(1 \leq i \leq m-1)$ are defined inductively by $(E_1)_0 = F_0$, $(E_2)_0 = F_0^2 - 2$ and $(E_k)_0 = F_0(E_{k-1})_0 - (E_{k-2})_0$ $(3 \leq i \leq m-1)$.

$(E_i)_2$ $(1 \leq i \leq m-1)$ are defined by $(E_1)_2 = F_2$, $(E_k)_2 = \begin{cases} F_2 + \sum_{l=1}^{(k-1)/2} (E_{2l})_0 F_2 & (k \geq 3 \text{ odd}), \\ \sum_{l=1}^{k/2} (E_{2l-1})_0 F_2 & (k \text{ even}). \end{cases}$

$I = m(C_0+1) + m \sum_{l=1}^{(m/2)-1} (E_{2l})_0$, $J = m \sum_{l=1}^{m/2} (E_{2l-1})_0$, $K = (C_0+1)(A_2+B_2) + m \sum_{l=1}^{(m/2)-1} (E_{2l})_2$, $L = m \sum_{l=1}^{m/2} (E_{2l-1})_2$.

In the above, we interpret $\sum_{l=1}^{(m/2)-1}$ term as 0 if $m = 2$.

Table 3. Cohomology ring $H^*(D_{2n}, {}_{\psi}ZD_{2n})(\simeq HH^*(ZD_{2n}))$ for $n = 2m$ ($m \geq 3$ odd).

	C_0	F_0	S_0	A_2	B_2	F_2	A_3	A_4
C_0	1	$(E_{m-1})_0$	C_0S_0	C_0A_2	C_0B_2	$-(E_{m-1})_2$	C_0A_3	${}_nC_0A_4$
F_0		$(E_2)_0+2$	$2C_0S_0$	0	mF_2	$(E_2)_2$	0	${}_nF_0A_4$
S_0			$m+m\sum_{l=1}^{(m-1)/2}(E_{2l})_0$	S_0A_2	S_0B_2	0	S_0A_3	${}_2S_0A_4$
${}_2A_2$				A_2^2	A_2B_2	0	A_2A_3	A_2A_4
${}_2B_2$					$A_2B_2-mA_4$	mF_0A_4	B_2A_3	B_2A_4
${}_nF_2$						$A_4((E_2)_0-2)$	0	F_2A_4
${}_2A_3$							A_2A_4	A_3A_4
${}_nA_4$								A_4^2

${}_lW_k$ means that l is the order of $W_k \in H^k(D_{2n}, {}_{\psi}ZD_{2n})$ as a $\boldsymbol{Z}$-module.
$(E_i)_0$ $(1 \leq i \leq m-1)$ are defined inductively by $(E_1)_0 = F_0$, $(E_2)_0 = F_0^2 - 2$ and $(E_k)_0 = F_0(E_{k-1})_0 - (E_{k-2})_0$ $(3 \leq i \leq m-1)$.

$(E_i)_2$ $(1 \leq i \leq m-1)$ are defined by $(E_1)_2 = F_2$, $(E_k)_2 = \begin{cases} F_2 + \sum_{l=1}^{(k-1)/2}(E_{2l})_0F_2 & (k \geq 3 \text{ odd}), \\ \sum_{l=1}^{k/2}(E_{2l-1})_0F_2 & (k \text{ even}). \end{cases}$

(ii) *If n is odd, the Hochschild cohomology ring $H^*(D_{2n}, {}_{\psi}ZD_{2n})$ $(\simeq HH^*(ZD_{2n}))$ is commutative, generated by elements*

$$F_0, S_0 \in H^0(D_{2n}, {}_{\psi}ZD_{2n}), \quad A_2 \in H^2(D_{2n}, {}_{\psi}ZD_{2n}), \quad A_4 \in H^4(D_{2n}, {}_{\psi}ZD_{2n}).$$

The relations follow from Table 4.

Table 4. Cohomology ring $H^*(D_{2n}, {}_\psi ZD_{2n}) (\simeq HH^*(ZD_{2n}))$ for $n = 2t+1$ $(t \geq 1)$.

	F_0	S_0	A_2	F_2	A_4
F_0	$\begin{cases} F_0+2 & (t=1) \\ (E_2)_0+2 & (t \geq 2) \end{cases}$	$2S_0$	0	$\begin{cases} -F_2 & (t=1) \\ (E_2)_2 & (t \geq 2) \end{cases}$	${}_nF_0A_4$
S_0		$n + n\sum_{i=1}^{t}(E_i)_0$	S_0A_2	0	${}_nS_0A_4$
${}_2A_2$			nA_4	0	A_2A_4
${}_nF_2$				$\begin{cases} A_4(F_0-2) & (t=1) \\ A_4((E_2)_0-2) & (t \geq 2) \end{cases}$	F_2A_4
${}_{2n}A_4$					A_4^2

${}_lW_m$ means that l is the order of $W_m \in H^m(D_{2n}, {}_\psi ZD_{2n})$ as a Z-module.
$(E_i)_0$ $(1 \leq i \leq t)$ are defined inductively by $(E_1)_0 = F_0$, $(E_2)_0 = F_0^2 - 2$ and $(E_k)_0 = F_0(E_{k-1})_0 - (E_{k-2})_0$ $(3 \leq i \leq t)$.

$(E_i)_2$ $(1 \leq i \leq t)$ are defined by $(E_1)_2 = F_2$, $(E_k)_2 = \begin{cases} F_2 + \sum_{l=1}^{(k-1)/2} (E_{2l})_0F_2 & (k \geq 3 \text{ odd}), \\ \sum_{l=1}^{k/2}(E_{2l-1})_0F_2 & (k \text{ even}). \end{cases}$

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