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THE INTEGRATED DENSITY OF STATES OF ONE-DIMENSIONAL RANDOM SCHRÖDINGER OPERATOR WITH WHITE NOISE POTENTIAL AND BACKGROUND

By

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1. Introduction

We consider the Integrated Density of States (IDS), $N(\lambda)$, $\lambda \in \mathbf{R}$, of the fomally defined operator H,

$$H = -\frac{1}{r(t)}\frac{d}{dt}\left(\frac{1}{p(t)}\frac{d}{dt}\right) + \frac{q(t)}{r(t)} + \frac{cB'(t)}{r(t)}, \quad 0 \leq t < \infty, \tag{1.1}$$

i.e., the limit of the normalized distribution function of the eigenvalues of H_l which is the restriction of H to $L^2((0,l) : r(t)\,dt)$ under the boundary conditions,

$$(b.c)_{\alpha,\beta} \quad \begin{cases} \varphi(0)\cos\alpha - \frac{1}{p(0)}\varphi'(0)\sin\alpha = 0, \\ \varphi(l)\cos\beta - \frac{1}{p(l)}\varphi'(l)\sin\beta = 0, \end{cases}$$

where $(B(t))_{t\geq 0}$ is the standard Brownian motion and $B'(t)$ is the derivative of its sample function, namely the white noise. $(p(t))_{t\geq 0}$, $(q(t))_{t\geq 0}$ and $(r(t))_{t\geq 0}$ are bounded semi-martingales which we shall call the background, and c is a *coupling constant*.

$N(\lambda)$ is defined by

$$N(\lambda) := \lim_{l\to\infty} \frac{1}{l} N(l, \lambda, \omega),$$

where we denote by $N(l, \lambda, \omega)$ the number of eigenvalues of H_l which are less than or equal to λ.

The main purpose of this paper is to improve Theorem of [5] and Theorem (*b*) of [12] cited below, simplifying their proofs at the same time.

PROPOSITION 1.1 ([5]). *Suppose that* $p(t) \equiv 1$, $q(t) \equiv 0$, $r(t) \equiv 1$ *and* $c = 1$. *Then*

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$$N(\lambda) = \left(\sqrt{2\pi}\int_0^\infty \frac{1}{\sqrt{x}} \exp\left\{-\frac{1}{6}x^3 - 2\lambda x\right\} dx\right)^{-1}.$$

PROPOSITION 1.2 ([12]). *Suppose that* $q(t) \equiv 0$, $c = 1$ *and*

(i) $(p(t))$, $(r(t))$ *are nonanticipating with respect to* $\sigma(B(s)\colon 0 \le s \le t)$,

(ii) $p_1 \le p(t) \le p_2$, $r_0 \le r(t)$ *for some* p_1, p_2 *and* $r_0 \in (0, \infty)$,

(iii) *There exists an ergodic homogeneous stochastic processes* $M(T, \omega)$, *and a positive function* $\eta(T)$ *such that* $\sup_{T \le t \le T+2\pi}(|p'(t)| + |r'(t)|) \le \eta(T)M(T)$ *and* $\eta(T) \to 0$ *as* $T \to \infty$,

(iv) $p(t) \to p(\infty)$ *and* $r(t) \to r(\infty)$ *as* $t \to \infty$.

Then

$$N(\lambda) = \left(\int_0^\pi u(x)\, dx\right)^{-1},$$

where $u(x)$ *is the bounded solution of the equation*

$$\frac{1}{2}\sin^4 x u'(x) + b(x)u(x) = 1, \quad 0 < x < \pi,$$

where $b(x) = p(\infty)\cos^2 x + \lambda r(\infty)\sin^2 x + \sin^3 x \cos x$.

We shall derive the IDS concretely when the background is continuous semi-martingales that have limit at ∞. To state the main result, we assume the following conditions: let $(p_\omega(t))_{t\ge 0}$, $(q_\omega(t))_{t\ge 0}$, $(r_\omega(t))_{t\ge 0}$ be continuous semi-martingales on a probability space $(\Omega, \mathscr{F}, P)$ with a filtration $(\mathscr{F}_t)_{t\ge 0}$, namely $p(t)$ is expressed as $p_\omega(t) = p(t) = p(0) + M^p(t) + A^p(t)$ where M^p ($M(0) = 0$ *a.s*) is a continuous local $(\mathscr{F}_t)$-martingale and $A^p(t)$ ($A(0) = 0$ *a.s*) is a continuous $(\mathscr{F}_t)$-adapted process whose sample functions $(t \mapsto A^p)$ are of bounded variation on any finite interval a.s., and $p(0)$ is an $\mathscr{F}_0$-measurable random variable. $(B_\omega(t))_{t\ge 0}$ is an $(\mathscr{F}_t)$-Brownian motion. Moreover $p(t)$, $M^p(t)$ and $A^p(t)$ satisfy following conditions.

(A.1): there exist that $M^p(\infty) := \lim_{t\to\infty} M^p(t)$, $A^p(\infty) := \lim_{t\to\infty} A^p(t)$ a.s

(A.2): For some $0 < c_1 < c_2$, $c_3 \in \mathbf{R}$, which are independent of ω, $c_1 \le p(t)$, $r(t) \le c_2$, $|q(t)| \le c_3$.

(A.3): $\int_0^l t|dA^p| = o(l)$ as $l \to \infty$, $\int_0^l t^2 d\langle M^p\rangle = O(l^\delta)$ for some $0 < \delta < 2$ as $l \to \infty$.

When $q(t)$ and $r(t)$ are expressed similarly, we suppose that each martingale part and each part of bounded variation part also satisfy the above conditions.

Then the main result is the following.

THEOREM 1.1. *Under the assumptions* (A.1), (A.2) *and* (A.3), *we have*

$$N(\lambda) = \left(\int_0^{\pi} u(x; p(\infty), q(\infty), r(\infty))\, dx\right)^{-1},$$

where, for each $(p, q, r) \in \mathbf{R}^3$, *the function* $u(x) = u(x; p, q, r)$, $0 < x < \infty$, *is the bounded solution of the equation*

$$\frac{1}{2}\sigma^2(x)u'(x) + b(x; p, q, r)u(x) = 1, \quad 0 < x < \pi, \tag{1.2}$$

$\sigma(x) := c \sin^2 x$ *and* $b(x; p, q, r) := p \cos^2 x + (-q + \lambda r) \sin^2 x + c^2 \sin^3 x \cos x.$

Actually, we can write down the bounded solution of (1.2) explicitly. Thus we obtain the following corollary.

COROLLARY 1.1. *Under the same assumption of Theorem* 1.1, *we have*

$$N(\lambda) = \left(\sqrt{\frac{2\pi}{c^2 p(\infty)}} \int_0^{\infty} \frac{1}{\sqrt{x}} \exp\left[-\frac{1}{c^2}\left\{\frac{p(\infty)}{6}x^3 + 2(-q(\infty) + \lambda r(\infty))x\right\}\right] dx\right)^{-1}.$$

PROOF. By the proof of Lemma 4.2, the bounded solution u of (1.2) is given explicitly as $u(x) = 2S(x)\int_0^x dy/\sigma^2(y)S(y)$, where $S(x) = \exp[-2\int_{\pi/2}^x b(y; p, q, r)/\sigma^2(y)\, dy]$. From this expression, we obtain

$$S(x) = S(x; p, q, r) = \exp[(2/c^2)\{(p/3)\cot^3 x + (-q + \lambda r)\cot x\}]/\sin^2 x,$$

and here we can compute, by making change of variable twice,

$$\begin{aligned}
&\int_0^{\pi} u(v; p, q, r)\, dv \\
&= \frac{2}{c^2}\int_{-\infty}^{\infty} \exp\left[\frac{2}{c^2}\left\{\frac{p}{3}z^3 + (-q + \lambda r)z\right\}\right] dz \\
&\quad \times \int_z^{\infty} \exp\left[-\frac{2}{c^2}\left\{\frac{p}{3}y^3 + (-q + \lambda r)y\right\}\right] dy \\
&= \frac{2}{c^2}\int_0^{\infty} \exp\left[-\frac{1}{c^2}\left\{\frac{p}{6}x^3 + 2(-q + \lambda r)x\right\}\right] dx \times \int_{-\infty}^{\infty} \exp\left\{-\frac{2px}{c^2}\left(z + \frac{x}{2}\right)^2\right\} dz \\
&= \frac{2}{c^2}\int_0^{\infty} \exp\left[-\frac{1}{c^2}\left\{\frac{p}{6}x^3 + 2(-q + \lambda r)x\right\}\right] \sqrt{\frac{\pi c^2}{2px}}\, dx. \qquad \square
\end{aligned}$$

REMARK 1.1. When $p(t) = r(t) = 1$, $q(t) = 0$ and $c = 1$, we derive $N(\lambda)$ as given by Proposition 1.1. This is contained the above corollary.

In the remainder of this section we give a brief outline of this paper. In Section 2, we define the operator H_l rigorously. This argument is necessary since the Brownian motion $B(t)$ is not differentiable in t. We here follow Savchuk and Shkalikov [11] to define the Schrödinger operator

$$H := -\frac{1}{r(t)}\frac{d}{dt}\left(\frac{1}{p(t)}\frac{d}{dt}\right) + \frac{q(t)}{r(t)} + \frac{Q'(t)}{r(t)}$$

in $L^2((0,l); r(t)\,dt)$ for any $Q \in L^2_{loc}(\mathbf{R} : \mathbf{R})$ and $(p(t))$, $(q(t))$ and $(r(t)) \in C(\mathbf{R}; \mathbf{R})$. In fact introducing the *quasi derivative* $\phi^{[1]}(t) := \phi'(t)/p(t) - Q(t)\phi(t)$ as in [11], we can write

$$H\phi(t) = -\frac{1}{r(t)}(\phi^{[1]\prime}(t) + p(t)Q(t)\phi^{[1]}(t) + p(t)Q^2(t)\phi(t) - q(t)\phi(t)).$$

Since Q is a real function, H_l can be realized as a self-adjoint operator, whose domain is given by

$$D(H) = \{\varphi \in AC(0,l) \mid \varphi^{[1]} \in AC(0,l),\ \varphi \text{ satisfies } (b.c)_{\alpha,\beta}\},$$

where $AC(0,l)$ is the set of all absolutely continuous functions on $(0,l)$. The spectrum of H_l is discrete since H_l has a compact resolvent. Futhermore when Q is locally bounded, the self-adjoint operator is bounded from below. Two other definitions of the operator corresponding to the expression H_l have been known: Fukushima and Nakao [5] defined it as self-adjoint operators on $L^2(0,l)$ which is associated with a closed symmetric form. In [8], Minami defined it through formal integration by parts (1.1). One advantage of the method of introducing the *quasi derivative* is that it makes valid, with little modification, the classical proof of the Sturm-Liouville Oscillation theorem as given e.g. in [13], also for operators with singular potentials like our H_l. This will be verified in Section 3. In Section 4, we prove Theorem 1.1. As in [5], we introduce the phase function $\theta(t)$ of the solution ϕ of $H_l\phi = \lambda\phi$, $\phi(0) = \sin\alpha$, $\phi'(0)/p(0) = \cos\alpha$ by Prüffer transformation. The Sturm-Liouville Oscillation theorem implies $N(\lambda, l, \omega) = [(\theta(l,\lambda) - \beta)/\pi] + 1$. Therefore $N(\lambda) = \pi^{-1}\lim_{l\to\infty}\theta(l)/l$. Our proof follows the same line as in [12], but it is simplified in some technical points.

2. Schödinger Operator with Singular Potential

In this section, following [11], we define the Schrödinger operator of the type

$$H := -\frac{1}{r(t)}\frac{d}{dt}\left(\frac{1}{p(t)}\frac{d}{dt}\right) + \frac{q(t)}{r(t)} + \frac{Q'(t)}{r(t)}, \quad 0 \le t \le l,$$

with $Q \in L^2_{loc}(\mathbf{R})$ and continuous functions p, q and r, on the Hilbert space $L^2((0,l); r(t)\,dt)$, and show its self-adjointness. Let $Q \in L^2_{loc}(\mathbf{R}; \mathbf{R})$. For any absolutely continuous φ, we define the *quasi derivative* $\varphi^{[1]}$ of φ by

$$\varphi^{[1]} := \frac{\varphi'(t)}{p(t)} - Q(t)\varphi(t),$$

and we formally rewright H in the form,

$$H\varphi = -\frac{1}{r}\{(\varphi^{[1]})' + pQ\varphi^{[1]} + pQ^2\varphi - q\varphi\}. \tag{2.1}$$

We can express (2.1) without Q', so (2.1) is meaningful if φ and $\varphi^{[1]}$ are absolutely continuous function. Let us define the maximal operator H_M as follows:

$$D(H_M) := \{\varphi \in L^2([0,l]; r(t)\,dt) \mid \varphi, \varphi^{[1]} \in AC(0,l), h(\varphi) \in L^2([0,l]; r(t)\,dt)\},$$

$$H_M\varphi := -\frac{1}{r}\{(\varphi^{[1]})' + pQ\varphi^{[1]} + pQ^2\varphi - q\varphi\} \quad \text{for } \varphi \in D(H_M),$$

where $AC(0,l)$ is the set of all absolutely continuous functions on $(0,l)$. We also define the minimal operator H_m as the restriction of H_M to the domain

$$D(H_m) := \{\varphi \in D(H_M) \mid \varphi(0) = \varphi(l) = \varphi^{[1]}(0) = \varphi^{[1]}(l) = 0\}.$$

The following lemma is contained in Section 3.8 Problem 1 of [2] and Theorem 2.1 of [13].

LEMMA 2.1 (Savchuk and Shkalikov [11] Theorem 0). *Let f be in $L^1_{loc}(r(t)\,dt; \mathbf{C}^n)$ and A be in $L^1_{loc}(r(t)\,dt; \mathbf{C}^n \otimes \mathbf{C}^n)$. Then, for any $s \in [0,l]$ and $\xi \in \mathbf{C}^n$, an equation $y'(t) = A(t)y(t) + f(t)$, $y(s) = \xi$ has a unique solution in $AC(0,l)$.*

PROOF. We can verify the claim by successive approximation as follows.

$$\begin{cases} y_0(t) = \xi, \\ y_k(t) = \xi + \int_s^t A(x)y_{k-1}(x)\,dx + \int_s^t f(x)\,dx, \quad k \ge 1. \end{cases}$$

Then $(y_k)_k$ converges uniformly to the unique solution. □

Using Lemma 2.1, we define the solution of the equation

$$h(\varphi) = \lambda\varphi + f \tag{2.2}$$

for any $\lambda \in \mathbf{C}$, $f \in L^2_{loc}(r(t)\,dt; \mathbf{C})$ in the following way. We rewrite (2.2) as follows.

$$(\#)\quad \frac{d}{dt}\begin{pmatrix} \varphi \\ \varphi^{[1]} \end{pmatrix} = \begin{pmatrix} pQ & p \\ -pQ^2 - \lambda r + q & -pQ \end{pmatrix}\begin{pmatrix} \varphi \\ \varphi^{[1]} \end{pmatrix} + \begin{pmatrix} 0 \\ -rf \end{pmatrix}$$

Since p, q and r are continuous and $Q \in L^2_{loc}(\mathbf{R})$, each component of the coefficient matrix

$$\begin{pmatrix} pQ & p \\ -pQ^2 - \lambda r + q & -pQ \end{pmatrix}$$

is a locally integrable function. By Lemma 2.1, under a given initial condition the above normal system has a unique solution.

DEFINITION 2.1 (Savchuk and Shkalikov [11] Definition 1). *A square $r(t)$-integrable function φ on $\mathbf{R}$ is said to be a solution of* (2.2) *under a given initial condition if φ coincides with the first component of the solution of the system* (#) *under the same initial condition.*

We characterize the self-adjointness of H_l. To do so, we quote several lemmas.

LEMMA 2.2 (Lagrange formula [11] Lemma 1). *For any $\varphi \in D(H_M)$ and $\psi \in D(H_M)$,*

$$(H_M\varphi, \psi) = (\varphi, H_M\psi) + [\varphi, \psi]_0^l \tag{2.3}$$

where

$$[\varphi, \psi]_0^l := [-\varphi^{[1]}(t)\bar{\psi}(t) + \varphi(t)\overline{\psi^{[1]}}(t)]_{t=0}^{t=l}.$$

PROOF. See [11]. □

Using Lemma 2.2, we have the following lemma.

LEMMA 2.3 ([11] Lemmas 2, 3 and 4). (i) *$D(H_m)$ is dense in $L^2([0,l]; r(t)\,dt)$.*
(ii) *$H_M = H_m^*$ and $H_M^* = H_m$.*
(iii) *For any $\lambda \in \mathbf{C}$,* $\dim \operatorname{Ker}(H_M - \lambda) = 2$.
(iv) $\operatorname{Ran}(H_m) \perp \operatorname{Ker}(H_M)$.

PROOF. See [11]. □

Lemma 2.4. *Let* $Q \in L^2_{loc}(\mathbf{R};\mathbf{R})$ *and* H *be a self-adjoint extension of* H_m. *Then there are* w_1 *and* $w_2 \in D(H)\backslash D(H_m)$ *such that they are linearly independent and the domain of* H *is expressed as follows*:

$$D(H) = \{\varphi \in D(H_m^*) \mid \varphi = \psi_0 + \alpha_1 w_1 + \alpha_2 w_2 \text{ for some } \psi_0 \in D(H_m)\ \alpha_1, \alpha_2 \in \mathbf{C}\}.$$

Proof. See Reed and Simon [9] [Vol II Theorem X.2 (page 140)]. □

Lemma 2.5 ([4]). *Let* S *be a subspace of* $D(H_m^*)$ *which includes* $D(H_m)$. *Then the restriction of* H_m^* *to* S *is a self-adjoint extension of* H_m *if and only if* $S = S^*$, *where* $S^* := \{y \in D(H_m^*) \mid [y, \phi]_0^l = 0, \forall \phi \in S\}$.

Proof. See [4] (XII.4.16, Lemma 16 (b) page 1231). □

Then we have the following.

Proposition 2.1 (Savchuk and Shkalikov [11] Theorem 2). *Let* $Q \in L^2_{loc}(\mathbf{R};\mathbf{R})$. *Then a closed symmetric extension* H *of* H_m *is self-adjoint if and only if* H *has its domain as*

$$D(H) = \{\varphi \in D(H_m^*) \mid B_j(\varphi) = 0, j = 1, 2\},$$

where

$$B_j(\varphi) := a_{j1}\varphi(0) + a_{j2}\varphi^{[1]}(0) + b_{j1}\varphi(l) + b_{j2}\varphi^{[1]}(l), \quad j = 1, 2,$$

for some $a_{jk}, b_{jk} \in \mathbf{C}$, $(j, k = 1, 2)$ *such that*

$$a_{j1}\bar{a}_{k2} - a_{j2}\bar{a}_{k1} = b_{j1}\bar{b}_{k2} - b_{j2}\bar{b}_{k1}, \quad (j, k = 1, 2)$$

and that rank $A = 2$. *Here* A *is a matrix given by*

$$A := \begin{pmatrix} a_{12} & a_{22} \\ a_{11} & a_{21} \\ b_{12} & b_{22} \\ b_{11} & b_{21} \end{pmatrix}.$$

Proof. We follow Ahiezer and Glazman [1] (APPENDIX II.3) to prove the assertion. We suppose that H is a self-adjoint extension of H_m. Let $\varphi \in D(H_m^*)$. By Lemma 2.2 and Lemma 2.3, $\varphi \in D(H)$ is equivalent to saying $(H\psi, \varphi) = (\psi, H_m^*\varphi)$ for any $\psi \in D(H)$. This is, in turn, equivalent to saying $[\varphi, \psi]_0^l = 0$ for any $\psi \in D(H)$. By Lemma 2.4, there are $w_1, w_2 \in D(H)\backslash D(H_m)$ which are linearly independent, so that any element ψ of $D(H)$ is of the form $\psi = \psi_0 + \alpha_1 w_1 + \alpha_2 w_2$, for some $\psi_0 \in D(H_m)$, and $\alpha_1, \alpha_2 \in \mathbf{C}$. So, $[\varphi, \psi]_0^l = 0$ for any $\psi \in D(H)$ is equivalent to saying $\alpha_1[\varphi, w_1]_0^l + \alpha_2[\varphi, w_2]_0^l = 0$ for any $\alpha_1, \alpha_2 \in \mathbf{C}$, namely to saying $[\varphi, w_1]_0^l = [\varphi, w_2]_0^l = 0$. If we set

$$a_{j1} := \bar{w}_j^{[1]}(0), \quad a_{j2} := -\bar{w}_j(0), \quad b_{j1} := -\bar{w}_j^{[1]}(l), \quad b_{j2} := \bar{w}_j(l), \quad j = 1, 2,$$

then $B_j(\varphi) := -[\varphi, w_j]_0^l = 0$, $j = 1, 2$. Moreover $a_{j1}\bar{a}_{k2} - a_{j2}\bar{a}_{k1} = b_{j1}\bar{b}_{k2} - b_{j2}\bar{b}_{k1}$ for $j, k = 1, 2$ since $[w_j, w_k]_0^l = 0$ for $j, k = 1, 2$. Since w_1 is independent of w_2, we have $\operatorname{rank} A = 2$.

Conversely suppose that the domain D of H is given as above. By (iii) of Lemma 2.3, we can take a basis $\{u_1, u_2\}$ of $\operatorname{Ker}(H_M)$. Let v_j, $j = 1, 2$, be the solutions of $H_M v_j = u_j$ such that $v_j(l) = v_j^{[1]}(l) = 0$, $j = 1, 2$. If we assume that $(v_1(0), v_1^{[1]}(0))$ and $(v_2(0), v_2^{[1]}(0))$ are not linearly independent, there exists α_1, α_2 such that $(\alpha_1, \alpha_2) \neq (0, 0)$ and $\alpha_1 v_1 + \alpha_2 v_2 \in D(H_m)$. Then $H_m(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 u_1 + \alpha_2 u_2$. The left hand side is an element of $\operatorname{Ran}(H_m)$ and not zero. On the other hand the right hand side belongs to $\operatorname{Ker}(H_M)$. This contradicts (iv) of Lemma 2.3. Thus we can take the suitable basis of $\operatorname{Ker}(H_M)$ such that v_1 and v_2 satisfy $(v_1(0), v_1^{[1]}(0)) = (1, 0)$, $(v_2(0), v_2^{[1]}(0)) = (0, 1)$. Similarly there exists v_3 and v_4 in $D(H_m^*)$ such that $(v_3(0), v_3^{[1]}(0), v_3(l), v_3^{[1]}(l)) = (0, 0, 1, 0)$ and $(v_4(0), v_4^{[1]}(0), v_4(l), v_4^{[1]}(l)) = (0, 0, 0, 1)$. We set $w_j := -\overline{a_{j2}}v_1 + \overline{a_{j1}}v_2 + \overline{b_{j2}}v_3 - \overline{b_{j1}}v_4$, $j = 1, 2$, then $w_j(0) = -\overline{a_{j2}}$, $w_j^{[1]}(0) = \overline{a_{j1}}$, $w_j(l) = \overline{b_{j2}}$, $w_j^{[1]}(l) = -\overline{b_{j1}}$, $j = 1, 2$. Since $\operatorname{rank} A = 2$ and v_1, v_2, v_3 and v_4 are linearly independent, w_1 and w_2 are linearly independent. Moreover $\operatorname{rank} A = 2$ implies that w_1 and $w_2 \notin D(H_m)$. Then $D = \{\phi \in D(H_m^*) \mid B_j(\phi) = 0, j = 1, 2\}$ and $D = D^*$. Hence the restriction of H_m^* to D is a self-adjoint extension of H_m by Lemma 2.5. □

REMARK 2.1. 1. Savchuk and Shkalikov [11] did not state the condition $\operatorname{rank} A = 2$. But H is not a self-adjoint operator unless $\operatorname{rank} A = 2$ in Proposition 2.1.

2. When the boundary condition that realizes a self-adjoint extention is $(b.c)_{\alpha,\beta}$, the corresponding matrix A in Proposition 2.1 is expressed as follows:

$$A = \begin{pmatrix} -\sin\alpha & 0 \\ \cos\alpha - Q(0)\sin\alpha & 0 \\ 0 & \sin\beta \\ 0 & \cos\beta - Q(l)\sin\beta \end{pmatrix},$$

and actually $\operatorname{rank} A = 2$.

Corollary 2.1. (i) *Let* $Q \in L^2_{loc}(\mathbf{R};\mathbf{R})$ *be a locally bounded function. Then the self-adjoint extensions of* H_m *are bounded from below.*

(ii) ([11] *Theorem* 3) *The spectrum of each self-adjoint extension of* H_m *is purely discrete.*

(iii) *For the sequence* $\{\lambda_n; n \geq 1\}$ *of the eigenvalues of the self-adjoint extension of* H_m, $\lambda_n \to \infty$ *as* $n \to \infty$.

Proof of (i). Since p, q, r and Q are bounded on $[0,l]$, it is easily seen that H_m is bounded from below. In fact

$$\begin{aligned}(H_m\varphi,\varphi) &= \int_0^l p(\varphi^{[1]})^2\,dt - \int_0^l pQ^2\varphi^2\,dt + \int_0^l q\varphi^2\,dt \\ &\geq -\int_0^l \frac{p}{r}Q^2\varphi^2 r\,dt - \int_0^l \frac{|q|}{r}\varphi^2 r\,dt.\end{aligned}$$

Therefore it follows from [9] (Vol II, X.3, Proposition, page 179) that any self-adjoint extension of H_m is also bounded from below since the deficiency indices of H_m are equal to $\{2,2\}$ by Lemma 2.3.

Proof of (ii), (iii). The deficiency indices of H_m are equal to $\{2,2\}$. Hence by [10] (Vol IV, page 117, Example 5), it suffices to show the assertion when the boundary condition which realizes self-adjoint extension is $(b.c)_{\alpha,\beta}$. In this case, it is well known that the H has compact resolvent (cf. see [1] APPENDIX II.6, THEOREM 2, page 182). Thus, by [10] (Theorem XIII.64, page 245), when the sequence of the eigenvalues of H is denoted by $\{\lambda_n; n \geq 1\}$, $\lambda_n \to \infty$ as $n \to \infty$. □

Remark 2.2. (ii) of Corollary 2.1 is same as Theorem 3 in [11], but the proof of (ii) of Corollary 2.1 is simpler than that of Theorem 3 in [11].

3. Oscillation Theorem

Using the *quasi derivative*, we can show the Sturm-Liouville Oscillation theorem for singular potentials by a minor modification of the classical argument ([13] Theorem 13.2, page 199). Let Q be a real valued bounded measurable function. Then from what we showed in Section 2, the associated self-adjoint operator $H = H_l$ with the boundary conditions

$$\begin{cases}\varphi(0)\cos\tilde{\alpha} - \varphi^{[1]}(0)\sin\tilde{\alpha} = 0, \\ \varphi(l)\cos\tilde{\beta} - \varphi^{[1]}(l)\sin\tilde{\beta} = 0\end{cases}$$

has eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n \to \infty$. Then we have the following:

PROPOSITION 3.1 ([13]). *Let Q be a real valued continuous function on* $[0, \infty)$. *Then the eigenfunction $\varphi_n = \varphi(*, \lambda_n)$ corresponding to λ_n has exactly $n-1$ zeros in* $(0, l)$.

OUTLINE OF PROOF. For $\lambda \in \mathbf{R}$, let $\varphi(t, \lambda)$ be the (real) solution of the equations

$$H_l\varphi = \lambda\varphi, \quad \varphi(0) = \sin\tilde{\alpha}, \quad \varphi^{[1]}(0) = \cos\tilde{\alpha}.$$

We introduce the variables ξ and η through the following Prüffer transformation:

$$(P.t)_{\tilde{\alpha}} \quad \begin{cases} \varphi(t, \lambda) = \eta(t, \lambda) \sin \xi(t, \lambda), \\ \varphi^{[1]}(t, \lambda) = \eta(t, \lambda) \cos \xi(t, \lambda), \\ \xi(0, \lambda) = \tilde{\alpha}, \end{cases}$$

where $\xi(t, \lambda)$ can be defined as a cotinuous function in t. We may restrict to $0 \le \tilde{\alpha} < \pi$, $0 < \tilde{\beta} \le \pi$ without loss of generality. $(P.t)_{\tilde{\alpha}}$ implies that $\xi(t, \lambda)$ satisfies the equation

$$\xi(t, \lambda) - \xi(0, \lambda) = \int_0^t pQ \sin 2\xi \, ds + \int_0^t p \, ds + \int_0^t \{-p + pQ^2 + \lambda r - q\} \sin^2 \xi \, ds,$$

that is

$$(3.1) \quad \frac{d}{dt}\xi(t, \lambda) = pQ \sin 2\xi(t, \lambda) + p(t) + (-p + pQ^2 + \lambda r - q) \sin^2 \xi(t, \lambda).$$

Since the equation (3.1) and Corollary 2.1 hold, we can verify the following assertions:

(i) if there exists $j \in \mathbf{N}$, $t_0 > 0$ such that $\xi(t_0, \lambda) = j\pi$ then $\xi(t, \lambda) \ge j\pi$ for $t \ge t_0$,

(ii) the function $\xi(t, \lambda)$ is increasing in λ, and $\lim_{\lambda \downarrow -\infty} \xi(t, \lambda) = 0$, $\lim_{\lambda \uparrow \infty} \xi(t, \lambda) = \infty$. $(0 < t \le l)$.

Thus the remainder of the proof is same as Weidmann [13]. □

4. Proof of the Main Result

In this section, we prove Theorem 1.1. We define the IDS, $N(\lambda)$ as follows:

$$N(\lambda) := \lim_{l \to \infty} \frac{N(l, \lambda, \omega)}{l},$$

where $N(l, \lambda, \omega) = N_{\alpha\beta}(l, \lambda, \omega)$ is the number of eigenvalues which are less than or equal to λ of the operator H_l with the boundary conditions $(b.c)_{\alpha,\beta}$. To find this, let φ be the solution of the equation $H_l\varphi = \lambda\varphi$, $\varphi(0) = \sin\alpha$, $\varphi'(0)/p(0) = \cos\alpha$. Then we introduce the new functions $\theta(t,\lambda)$, $\rho(t,\lambda)$ which are defined by

$$(P.t)\quad \begin{cases} \varphi(t,\lambda) = \rho(t,\lambda)\sin\theta(t,\lambda), \\ \varphi'(t,\lambda) = p(t)\rho(t,\lambda)\cos\theta(t,\lambda). \end{cases}$$

$\theta(t)$ satisfies the following stochastic differential equation;

$$d\theta(t) = -\sigma(\theta(t))\,dB(t) + b(\theta(t); p(t), q(t), r(t))\,dt, \tag{4.1}$$

where $\sigma(x) := c\sin^2 x$ and $b(x; p, q, r) := p\cos^2 x + (-q + \lambda r)\sin^2 x + c^2\sin^3 x\cos x$.

Proposition 3.1 (the Oscillation theorem) and its proof imply the following Lemma.

Lemma 4.1.

$$N_{\alpha\beta}(l,\lambda,\omega) = \left[\frac{\theta(l,\lambda) - \beta}{\pi}\right] + 1,$$

where $[x]$ *denotes the integer part of* $x \in \mathbf{R}$.

Proof. By the definition of $N_{\alpha\beta}(l,\lambda,\omega)$, $N_{\alpha\beta}(l,\lambda,\omega) = n$ if and only if $\lambda_n \leq \lambda < \lambda_{n+1}$. The proof of Proposition 3.1 implies that $\xi(l,\lambda_m) = (m-1)\pi + \tilde{\beta}$, for $m \in \mathbf{N}$, and $\xi(l,\lambda)$ is increasing in λ. Hence $\lambda_n \leq \lambda < \lambda_{n+1}$ is equivalent to $(n-1)\pi + \tilde{\beta} \leq \xi(l,\lambda) < n\pi + \tilde{\beta}$. Since $\theta(t)$ satisfies (4.1) and $\theta(t) \equiv 0$, $(\mathrm{mod}\,\pi)$, $\theta(t)$ is differentiable in t at the zeros of φ and $d\theta(t)/dt$ is positive there. Moreover $d\xi(t)/dt$ is also positive at zeros of φ by the proof of Propositon 3.1. Thus if $m\pi \leq \xi(l,\lambda_n) < (m+1)\pi$, for each $m \in \mathbf{N}$, then $m\pi \leq \theta(l,\lambda_n) < (m+1)\pi$.

By the comparison theorem ([6]), $\theta(t,\lambda)$ is also increasing in λ. For the eigenvalues λ_m, $m \in \mathbf{N}$, of H_l, $\theta(l,\lambda_m) \equiv \beta$ $(\mathrm{mod}\,\pi)$. So, $(n-1)\pi + \tilde{\beta} \leq \xi(l,\lambda) < n\pi + \tilde{\beta}$ is equivalent to saying $(n-1)\pi + \beta \leq \theta(l,\lambda) < n\pi + \beta$, namely to saying $[(\theta(l,\lambda) - \beta)/\pi] = n - 1$. □

Therefore it suffices to prove the existence of

$$N(\lambda) = \frac{1}{\pi}\lim_{l\to\infty}\frac{\theta(l,\lambda)}{l}.$$

We prepare several lemmas to prove Theorem 1.1.

Lemma 4.2. *The function* u *in the Theorem* 1.1 *is extended as a continuous periodic function on* $\mathbf{R}$ *with period* π.

PROOF. Since the function u is the bounded solution of the first order differetial equation, u is represented explicitly as follows:

$$u(x; p, q, r) = 2S(x) \int_0^x \frac{dy}{\sigma^2(y)S(y)}, \quad 0 < x < \pi,$$

where

$$S(x) = S(x; p, q, r) = \exp\left\{ -2 \int_{\pi/2}^x \frac{b(y; p, q, r)}{\sigma^2(y)}\, dy \right\}.$$

By *de l' Hôpital theorem,* it can be verified $u(0+) = u(\pi-) = 1/p$. Therefore we can extend u as a continuous periodic function on $\mathbf{R}$ with period π. □

LEMMA 4.3. *Let* $\tilde{b}(x; p, q, r)$ *be* $b(x; p, q, r)$ *or* $b(x; p, q, r) + 2c^2 \sin^3 x \cos x$. *Let* $h(x; p, q, r)$ *be bounded, periodic in* x *with period* π*, and Lipschitz continuous in* (p, q, r) *with a Lipschitz constant independent of* x. *Then a bounded solution* v *of the equation*

$$\frac{1}{2}\sigma^2(x)v'(x) + \tilde{b}(x; p, q, r)v(x) = h(x; p, q, r)$$

is also a Lipschitz continuous function of (p, q, r) *and its Lipschitz constant is independent of* x. *Moreover* v *is jointly continuous at* $(0, p, q, r)$.

PROOF. Suppose $(p, q, r) \neq (p', q', r')$ and let $\tilde{v}(x) := v(x; p, q, r) - v(x; p', q', r')$. Then $\tilde{v}$ satisfies the equation

$$\begin{aligned}
&\frac{1}{2}\sigma^2(x)\tilde{v}'(x) + \tilde{b}(x; p, q, r)\tilde{v}(x) \\
&\quad = \{\tilde{b}(x; p', q', r') - \tilde{b}(x; p, q, r)\}v(x; p', q', r') + h(x; p, q, r) - h(x; p', q', r') \\
&\quad =: H(x).
\end{aligned}$$

We can solve this equation explicitly as follows.

$$\tilde{v}(x) = 2S(x; p, q, r) \int_0^x \frac{H(y)}{\sigma^2(y)S(y; p, q, r)}\, dy,$$

where $S(x; p, q, r)$ is given in Lemma 4.2 with $\tilde{b}$ instead of b. By the assumption,

$$|H(x)| \leq C(|p - p'| + |q - q'| + |r - r'|)$$

for some constant C independent of x.

Hence v is a Lipschitz continuous function in (p, q, r). Then

$$
\begin{aligned}
(4.2)\quad & |v(x_n; p_n, q_n, r_n) - v(0; p, q, r)| \\
& \leq |v(x_n; p_n, q_n, r_n) - v(x_n; p, q, r)| + |v(x_n; p, q, r) - v(0; p, q, r)| \\
& \leq C(|p_n - p| + |q_n - q| + |r_n - r|) + |v(x_n; p, q, r) - v(0; p, q, r)|.
\end{aligned}
$$

Since v is continuous at $x = 0$, v is continuous at $(0, p, q, r)$ as a four-variable function. □

Lemma 4.4. *We set*

$$g(\theta, p, q, r) := \int_0^{\theta} u(x; p, q, r)\, dx.$$

Then g is a C^2-class function in (θ, p, q, r).

Proof. It is sufficient to prove that $g(\theta, p, q, r)$ is a C^2-class function on $[0, \pi] \times (c_1, c_2) \times (-c_3, c_3) \times (c_1, c_2)$ since $u(x; p, q, r)$ is periodic in x with period π. Here the constants c_1, c_2 and c_3 appeared in the assumption (A.2). Lemma 4.3 implies u is bounded and periodic in x with period π. Moreover u is Lipschitz continuous in (p, q, r) and its Lipschitz constant is independent of x by Lemma 4.3. By differentiating the equation (1.2) in Theorem 1.1 with respect to p, $\partial_p u$ satisfies

$$(4.3)\quad \frac{1}{2}\sigma^2(x)(\partial_p u)'(x) + b(x; p, q, r)(\partial_p u)(x) = -u(x) \cos^2 x, \quad 0 < x < \pi,$$

where $\partial_p := \partial/\partial p$. Thus $\partial_p u$ is bounded and periodic in x with period π, and

$$\partial_p u(0+; p, q, r) = \partial_p u(\pi-; p, q, r) = -\frac{1}{p^2},$$

by *de l' Hôpital Theorem* as in proof of Lemma 4.2. By Lemma 4.3, $\partial_p u$ is a Lipschitz continuous function in (p, q, r), and its Lipschitz constant is independent of x. Moreover $\partial_p u$ is jointly continuous at $(0, p, q, r)$.

By differentiating the equation (4.3) with respect to p, we can also show that

$$\partial_p^2 u(0+; p, q, r) = \partial_p^2 u(\pi-; p, q, r) = \frac{2}{p^3},$$

and $\partial_p^2 u$ is jointly continuous at $(0, p, q, r)$ in a similar way. Similarly we can prove that

$$\partial_x^{n_1}\partial_p^{n_2}\partial_q^{n_3}\partial_r^{n_4}u(0+;p,q,r) = \partial_x^{n_1}\partial_p^{n_2}\partial_q^{n_3}\partial_r^{n_4}u(\pi-;p,q,r)$$

for $0 \le n_1+n_2+n_3+n_4 \le 2$, $0 \le n_1 \le 1$, $0 \le n_2, n_3, n_4 \le 2$, where $\partial_x := \partial/\partial x$, $\partial_p := \partial/\partial p$, $\partial_q := \partial/\partial q$, $\partial_r := \partial/\partial r$, and they are jointly continuous at $(0,p,q,r)$. Hence the lemma is proved. □

REMARK 4.1. Thompson was not aware that g is actually of C^2-*class*.

PROOF OF THEOREM 1.1. For notational brevity, we set $p_1(t) := p(t)$, $p_2(t) := q(t)$, $p_3(t) := r(t)$. Then

$$g(\theta,p_1,p_2,p_3) = \int_0^\theta u(x;p_1,p_2,p_3)\,dx,$$

$$\frac{\theta(l)}{l} = \frac{g(\theta(l),p_1(l),p_2(l),p_3(l))}{l} \times \frac{\theta(l)}{g(\theta(l),p_1(l),p_2(l),p_3(l))}. \tag{4.4}$$

By Lemma 4.3, $g(\theta,p,q,r)$ is of C^2-*class* in (θ,p,q,r). We can apply Itô formula, to obtain

$$\begin{aligned} &g(\theta(l),p_1(l),p_2(l),p_3(l)) \\ &= g(\theta(0),p_1(0),p_2(0),p_3(0)) + \int_0^l Lg(\theta(s),p_1(s),p_2(s),p_3(s))\,ds \\ &+ \int_0^l g_\theta(\theta(s),p_1(s),p_2(s),p_3(s))\sigma(\theta(s))\,dB(s) \\ &+ \sum_{j=1}^3 \int_0^l g_j(\theta(s),p_1(s),p_2(s),p_3(s))\,dM^j(s) \\ &+ \sum_{j=1}^3 \int_0^l g_j(\theta(s),p_1(s),p_2(s),p_3(s))\,dA^j(s) \\ &+ \sum_{j=1}^3 \int_0^l g_{\theta j}(\theta(s),p_1(s),p_2(s),p_3(s))\,d\langle N,M^j\rangle(s) \\ &+ \frac{1}{2}\sum_{j,k=1}^3 \int_0^l g_{jk}(\theta(s),p_1(s),p_2(s),p_3(s))\,d\langle M^j,M^k\rangle(s) \\ &=: I_1+I_2+I_3+I_4+I_5+I_6+I_7, \end{aligned} \tag{4.5}$$

where we have set $L := \frac{1}{2}\sigma^2(\theta)\partial^2/\partial\theta^2 + b(\theta; p_1, p_2, p_3)\partial/\partial\theta$, $N(t) := \int_0^t \sigma(\theta(s))\, dB(s)$, $g_\theta := \partial g/\partial\theta$, $g_j := \partial g/\partial p_j$, $g_{\theta j} := \partial^2 g/(\partial\theta\partial p_j)$, and $g_{jk} := \partial^2 g/(\partial p_j \partial p_k)$, $j, k = 1, 2, 3$.

Now we claim

$$\lim_{l \to \infty} \frac{g(\theta(l), p_1(l), p_2(l), p_3(l))}{l} = 1. \tag{4.6}$$

Let us estimate I_i, $1 \leq i \leq 7$, separatery.

It is clear that $|I_1| = |g(\theta(0), p_1(0), p_2(0), p_3(0))| = o(l)$ as $l \to \infty$. By the definition of u, $|I_2| = l$, and $|I_3| = o(l)$ as $l \to \infty$. $|I_4| = O(l^{\delta(1/2+\varepsilon)}) = o(l)$ as $l \to \infty$. Indeed, $\theta(t) = O(t)$ as $t \to \infty$ and $g_j = \int_0^\theta \partial u/\partial p_j = O(\theta)$ as $\theta \to \infty$. Thus if we set $m_j(l) := \int_0^l g_j\, dM^j$ then by the assumption (A.3),

$$\langle m_j \rangle(l) = \int_0^l g_j^2\, d\langle M^j \rangle \leq const. \int_0^l t^2\, d\langle M^j \rangle = O(l^\delta)$$

for some $0 < \delta < 2$. For a continuous local martingale there exists a Brownian motion $\tilde{B}$ such that $m_j(t) = \tilde{B}(\langle m_j \rangle(t))$. By the law of iterated logarithm, for any $\varepsilon > 0$, $\tilde{B}(t) = O(t^{1/2+\varepsilon})$ as $t \to \infty$. Thus, for $0 < \varepsilon < (2-\delta)/2\delta$,

$$m_j(l) = O(\langle m_j^{1/2+\varepsilon}(l) \rangle) = O(l^{\delta(1/2+\varepsilon)}) = o(l).$$

$$\begin{aligned} |I_5| &\leq \sum_{j=1}^3 \left| \int_0^l g_j\, dA^j(t) \right| \\ &\leq const. \sum_{j=1}^3 \int_0^l |\theta(t)|\, |dA^j(t)| \\ &\leq const. \sum_{j=1}^3 \int_0^l t |dA^j(t)| \\ &= o(l) \quad as\ l \to \infty. \end{aligned}$$

By Propositon 3.2.14 of [7],

$$\begin{aligned} |I_6| &\leq \sum_{j=1}^3 \left| \int_0^l g_{\theta j}\, d\langle N, M^j \rangle \right| \\ &= \sum_{j=1}^3 \left| \int_0^l u_j \sigma(\theta(s))\, d\langle B, M^j \rangle(s) \right| \end{aligned}$$

$$
\begin{aligned}
&\le const.\ \sum_{j=1}^{3}\int_0^l |d\langle B, M^j\rangle(s)| \\
&\le const.\ \sum_{j=1}^{3}\sqrt{\langle B\rangle(l)}\sqrt{\langle M^j\rangle(l)} \\
&\le const.\ \sum_{j=1}^{3}\sqrt{l}\sqrt{\langle M^j\rangle(\infty)} \\
&= o(l) \quad as\ l\to\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
|I_7| &\le \sum_{j,k=1}^{3}\left|\int_0^l g_{jk}\, d\langle M^j, M^k\rangle\right| \\
&\le \sum_{j,k=1}^{3}\sqrt{\int_0^l g_{jk}^2\, d\langle M^j\rangle}\sqrt{\int_o^l 1\, d\langle M^k\rangle} \\
&\le const.\ \sum_{j,k=1}^{3}\sqrt{\int_0^l |\theta(t)|^2\, d\langle M^j\rangle(t)}\sqrt{\langle M^k\rangle(l)} \\
&\le const.\ \sum_{j=1}^{3}\sqrt{\int_0^l t^2\, d\langle M^j\rangle(t)}\sqrt{\langle M^j\rangle(\infty)} \\
&\le O(l^{\delta/2})\sum_{j=1}^{3}\sqrt{\langle M^j\rangle(\infty)} \\
&= o(l) \quad as\ l\to\infty.
\end{aligned}
$$

Thus we obtain (4.6). Hence

$$
\lim_{l\to\infty}\frac{\theta(l)}{l} = \lim_{l\to\infty}\frac{\theta(l)}{g(\theta(l), p(l), q(l), r(l))}. \tag{4.7}
$$

In order to get the right hand side of (4.7), we claim the following:

$$
\theta(l)\to\infty \quad \text{as } l\to\infty \tag{4.8}
$$

and

$$
|g(\theta, p, q, r) - g(\theta, \tilde{p}, \tilde{q}, \tilde{r})| \le C(|p-\tilde{p}| + |q-\tilde{q}| + |r-\tilde{r}|). \tag{4.9}
$$

Proof of (4.8). The boundedness of u and (4.6) implies $\lim_{l\to\infty} \theta(l) = \infty$. In fact for any $\theta > 0$,

$$\begin{aligned}|g(\theta, p, q, r)| &\le \int_0^\theta |u(x : p, q, r)|\, dx \\ &\le C\theta.\end{aligned}$$

where $C > 0$ is independent of θ.

$$\frac{\theta(l)}{l} \ge \frac{1}{C}\frac{g(\theta(l), p(l), q(l), r(l))}{l} \to \frac{1}{C} > 0 \quad (l \to \infty).$$

Hence

$$\lim_{l\to\infty} \theta(l) = \infty.$$

Proof of (4.9). By Lemma 4.3, u is a uniformly Lipschitz continuous function in (p, q, r). Thus g satisfies the inequality (4.9).

The existence of $p(\infty) = \lim_{t\to\infty} p(t)$, $q(\infty) = \lim_{t\to\infty} q(t)$ and $r(\infty) = \lim_{t\to\infty} r(t)$ in the assumption (A.1), the inequality (4.9) and (4.8) imply

$$\begin{aligned}\lim_{l\to\infty} \frac{g(\theta(l), p(l), q(l), r(l))}{\theta(l)} &= \lim_{l\to\infty} \frac{g(\theta(l), p(\infty), q(\infty), r(\infty))}{\theta(l)} \\ &= \lim_{\theta\to\infty} \frac{g(\theta, p(\infty), q(\infty), r(\infty))}{\theta}.\end{aligned}$$

By the periodicity of u in x with period π,

$$\lim_{l\to\infty} \frac{g(\theta(l), p(l), q(l), r(l))}{\theta(l)} = \frac{1}{\pi}\int_0^\pi u(x : p(\infty), q(\infty), r(\infty))\, dx. \tag{4.10}$$

Therefore we obtain by (4.7) and (4.10) that

$$N(\lambda) = \left(\int_0^\pi u(x : p(\infty), q(\infty), r(\infty))\, dx\right)^{-1}. \qquad \square$$

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