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ON THE REISSNER-NORDSTRÖM-DE SITTER TYPE SPACETIMES

By

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Abstract. In the paper a family of curvature conditions of pseudosymmetry type is determined. We show that the curvature tensor of some Reissner-Nordström-de Sitter type spacetimes satisfy these conditions.[1]

1 Introduction

Let (M,g), $n \geq 3$, be a semi-Riemannian manifold. Let T be a $(0,4)$-tensor satisfying on M

$$T = \alpha\bar{A} + \beta g \wedge A + \gamma G, \tag{1}$$

where α, β, γ are functions on M and A a symmetric $(0,2)$-tensor on M. Clearly, T is a generalized curvature tensor. For precise definition of the symbols used we refer to Section 2 of this paper and [2]. It is known that if on M we have $T = \bar{A} + \gamma G$, where γ is a function on M and A a symmetric $(0,2)$-tensor on M, then

$$T \cdot T = Q(Ric(T), T) - (n-2)\gamma Q(g, Weyl(T))$$

on M ([18], Lemma 2.2). In section 3 we prove a generalization of this result (see Theorem 3.1). Namely, if (1) holds on M then at all points of M at which α is nonzero we have

$$(a) \quad T \cdot T = Q(Ric(T), T) + L_2 Q(g, Weyl(T)),$$

$$(b) \quad L_2 = (n-2)\left(\frac{\beta^2}{\alpha} - \gamma\right). \tag{2}$$

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In particular, if $A = Ric(T)$ then (1) takes the form

$$T = \alpha\,\overline{Ric(T)} + \beta g \wedge Ric(T) + \gamma G. \tag{3}$$

Now, at all points of M at which α is nonzero, (2) and (3) yield (see Theorem 3.1)

$$\begin{aligned} &(a) \quad T \cdot T = L_T Q(g, T), \\ &(b) \quad T \cdot Weyl(T) = L_T Q(g, Weyl(T)), \\ &(c) \quad L_T = (n-2)\left(\frac{\beta^2}{\alpha} - \gamma\right) - \frac{\beta}{\alpha}. \end{aligned} \tag{4}$$

Further, from (3) we get

$$\begin{aligned} &(a) \quad T \cdot Ric(T) = L_T Q(g, Ric(T)), \\ &(b) \quad Ric(T)^2 = (\kappa(T) - (n-2)\psi_2)\, Ric(T) + \frac{\psi_1}{\alpha} g, \\ &(c) \quad \psi_1 = (n-1)\gamma + \beta\kappa(T), \\ &(d) \quad \psi_2 = \frac{1-(n-2)\beta}{(n-2)\alpha}. \end{aligned} \tag{5}$$

Theorem 3.1 also states that (3) implies

$$\begin{aligned} &(a) \quad Weyl(T) \cdot Weyl(T) = L_1 Q(g, Weyl(T)), \\ &(b) \quad Weyl(T) \cdot Ric(T) = L_1 Q(g, Ric(T)), \\ &(c) \quad Weyl(T) \cdot T = L_1 Q(g, T), \\ &(d) \quad L_1 = \psi_2 - \psi_3, \\ &(e) \quad \psi_3 = \frac{\kappa(T)}{n-1} - L_T, \end{aligned} \tag{6}$$

on $U_{Weyl(T)} \subset M$. It is easy to see that if α vanishes at $x \in M$ then (1) implies $Weyl(T) = 0$. Similarly, if at $x \in M$ we have $A = \frac{tr(A)}{n} g$ then $T = \frac{\kappa(T)}{(n-1)n} G$ at this point. Therefore, we restrict to the set $U_A \cap U_{Weyl(T)} \subset M$ our considerations on tensors T satisfying (1). According to [8], a $(0,4)$-tensor T satisfying (1) on $U_A \cap U_{Weyl(T)} \subset M$ is said to be a *Roter type tensor*. Thus if a Roter type tensor satisfies (3) then (4) and (6) are fulfilled. Manifolds of dimension ≥ 4 with the curvature tensor R satisfying (3) on $U_S \cap U_C \subset M$, i.e.

$$R = \alpha \bar{S} + \beta g \wedge S + \gamma G, \tag{7}$$

where α, β, γ are some functions on $U_S \cap U_C$ and S is the Ricci tensor of (M, g), are called *Roter type manifolds* ([8]). We refer to [8], [11], [16], [22] and [24] for results related to Roter type manifolds. In Section 5 we present examples of Roter type manifolds.

We define on $\overline{M} = \{(t, r) \in \mathbf{R}^2 : r > 0\}$ the metric tensor $\bar{g}$ by

$$\bar{g}_{11} = -H, \quad \bar{g}_{12} = \bar{g}_{21} = 0, \quad \bar{g}_{22} = H^{-1}, \tag{8}$$

where $H = H(t, r)$ is a smooth positive (or negative) function on $\overline{M}$. The warped product $\overline{M} \times_F \tilde{N}$ of $(\overline{M}, \bar{g})$ and an $(n-2)$-dimensional semi-Riemannian space of constant curvature $(\tilde{N}, \tilde{g})$, $n \geq 4$, with the warping function $F = F(t, r)$, will be called a *Reissner-Nordström-de Sitter type spacetime*. If $H = H(r)$ and $F = F(r) = r^2$ then Reissner-Nordström-de Sitter type spacetimes are pseudosymmetric ([19], Example 1). Evidently, the Reissner-Nordström-de Sitter spacetime belongs to this class of manifolds (see Example 5.2(ii)). Certain Reissner-Nordström-de Sitter type spacetimes are non-Einsteinian and non-conformally flat manifolds, i.e. the set $U_S \cap U_C \subset \overline{M} \times_F \tilde{N}$ of that spacetimes is nonempty. Such spacetimes, in view of Theorem 4.1 of [16], satisfy (7) on $U_S \cap U_C$, i.e. they are Roter type manifolds ([8]). In Section 5 we present a suitable example (see Example 5.3).

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2 Preliminaries

Throughout this paper all manifolds are assumed to be connected paracompact manifolds of class C^∞. Let (M, g) be an n-dimensional, $n \geq 3$, semi-Riemannian manifold, ∇ its Levi-Civita connection and $\Xi(M)$ the Lie algebra of vector fields on M. On M we define the endomorphisms $X \wedge_A Y$ and $\mathscr{R}(X, Y)$ of $\Xi(M)$ by

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y,$$

$$\mathscr{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

respectively, where A is a symmetric $(0, 2)$-tensor on M and $X, Y, Z \in \Xi(M)$. The Ricci tensor S, the Ricci operator $\mathscr{S}$, the scalar curvature κ and the endomorphism $\mathscr{C}(X, Y)$ of (M, g) are defined by $S(X, Y) = tr\{Z \rightarrow \mathscr{R}(Z, X)Y\}$, $g(\mathscr{S}X, Y) = S(X, Y)$, $\kappa = tr\, \mathscr{S}$ and

$$\mathscr{C}(X,Y)Z = \mathscr{R}(X,Y)Z - \frac{1}{n-2}\Big(X \wedge_g \mathscr{S}Y + \mathscr{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y\Big)Z,$$

respectively. Now the $(0,4)$-tensor G, the Riemann-Christoffel curvature tensor R and the Weyl conformal curvature tensor C of (M,g) are defined by

$$G(X_1,X_2,X_3,X_4) = g((X_1 \wedge_g X_2)X_3,X_4),$$

$$R(X_1,X_2,X_3,X_4) = g(\mathscr{R}(X_1,X_2)X_3,X_4),$$

$$C(X_1,X_2,X_3,X_4) = g(\mathscr{C}(X_1,X_2)X_3,X_4),$$

respectively, where $X_1, X_2, \ldots \in \Xi(M)$. Let $\mathscr{T}(X,Y)$ be a skew-symmetric endomorphism of $\Xi(M)$. We define the $(0,4)$-tensor T by $T(X_1,X_2,X_3,X_4) = g(\mathscr{T}(X_1,X_2)X_3,X_4)$. The tensor T is said to be a *generalized curvature tensor* if

$$T(X_1,X_2,X_3,X_4) = T(X_3,X_4,X_1,X_2),$$

$$T(X_1,X_2,X_3,X_4) + T(X_2,X_3,X_1,X_4) + T(X_3,X_1,X_2,X_4) = 0.$$

For a generalized curvature tensor T, a symmetric $(0,2)$-tensor field A and a $(0,k)$-tensor field T_1, $k \geq 1$, we define the $(0,k+2)$-tensor fields $T \cdot T_1$, $Q(A,T)$ and $A \cdot T_1$ by

$$\begin{aligned}(T \cdot T_1)(X_1,\ldots,X_k;X,Y) &= (\mathscr{T}(X,Y)\cdot T_1)(X_1,\ldots,X_k)\\ &= -T_1(\mathscr{T}(X,Y)X_1,X_2,\ldots,X_k) - \cdots\\ &\quad - T_1(X_1,\ldots,X_{k-1},\mathscr{T}(X,Y)X_k),\\ Q(A,T_1)(X_1,\ldots,X_k;X,Y) &= ((X \wedge_A Y)\cdot T_1)(X_1,\ldots,X_k)\\ &= -T_1((X \wedge_A Y)X_1,X_2,\ldots,X_k) - \cdots\\ &\quad - T_1(X_1,\ldots,X_{k-1},(X \wedge_A Y)X_k),\end{aligned}$$

$$(A \cdot T_1)(X_1,\ldots,X_k) = -T_1(\mathscr{A}X_1,X_2,\ldots,X_k) - \cdots - T_1(X_1,X_2,\ldots,\mathscr{A}X_k),$$

respectively, where the endomorphism $\mathscr{A}$ is defined by $g(\mathscr{A}X,Y) = A(X,Y)$. Setting in the above formulas $\mathscr{T}(X,Y) = \mathscr{R}(X,Y)$ or $\mathscr{T}(X,Y) = \mathscr{C}(X,Y)$, $T_1 = R$, $T_1 = C$ or $T_1 = S$, $A = g$ or $A = S$, we obtain the tensors: $R \cdot R$, $R \cdot C$, $C \cdot R$, $C \cdot C$, $R \cdot S$, $C \cdot S$, $Q(g,R)$, $Q(g,C)$, $Q(S,R)$, $Q(S,C)$, $Q(g,S)$, $S \cdot R$ and $S \cdot C$. For symmetric $(0,2)$-tensors A and B we define their *Kulkarni-Nomizu product* $A \wedge B$ by

$$(A \wedge B)(X_1, X_2; X, Y) = A(X_1, Y)B(X_2, X) + A(X_2, X)B(X_1, Y) - A(X_1, X)B(X_2, Y) - A(X_2, Y)B(X_1, X).$$

In particular, for a symmetric $(0,2)$-tensor A we define the $(0,4)$-tensor $\bar{A}$ by $\bar{A} = \frac{1}{2} A \wedge A$. If T is a generalized curvature tensor then its Weyl curvature tensor $Weyl(T)$ is defined by

$$Weyl(T) = T - \frac{1}{n-2} g \wedge Ric(T) + \frac{\kappa(T)}{(n-2)(n-1)} G, \tag{9}$$

where $Ric(T)$ and $\kappa(T)$ is the Ricci tensor and the scalar curvature of T, respectively. If (3) holds on $U_{Ric(T)} \cap U_{Weyl(T)}$ then on this set we have

$$Weyl(T) = \alpha\, \overline{Ric(T)} + \left(\beta - \frac{1}{n-2}\right) g \wedge Ric(T) + \left(\gamma + \frac{\kappa(T)}{(n-2)(n-1)}\right) G. \tag{10}$$

Conversely, if on $U_{Ric(T)} \cap U_{Weyl(T)}$ we have

$$Weyl(T) = \alpha\, \overline{Ric(T)} + \beta g \wedge Ric(T) + \gamma G,$$

for some functions α, β, γ on $U_{Ric(T)} \cap U_{Weyl(T)}$, then

$$T = \alpha\, \overline{Ric(T)} + \left(\beta + \frac{1}{n-2}\right) g \wedge Ric(T) + \left(\gamma - \frac{\kappa(T)}{(n-2)(n-1)}\right) G.$$

In particular, the curvature tensor R of a semi-Riemannian manifold (M, g), $n \geq 4$, has a decomposition of the form (3) if and only if its Weyl tensor has a decomposition of this form.

Remark 2.1. (i) From (3) and (10), by making use of (2)(b) and (6)(d), we get

$$T = \alpha \bar{A} - \frac{L_2}{n-2} G,$$

$$Weyl(T) = \alpha \overline{A_1} - \frac{L_1}{n-2} G,$$

on $U_{Ric(T)} \cap U_{Weyl(T)}$, where $A = Ric(T) + \frac{\beta}{\alpha} g$ and $A_1 = A - \frac{1}{(n-2)\alpha} g$. In Section 4 we consider tensors satisfying (3) on the subset of $U_{Ric(T)} \cap U_{Weyl(T)}$ of all points at which the functions L_1 and L_2 are nonzero.

(ii) Curvature properties of manifolds of dimension ≥ 4 whose curvature tensor R satisfies (3), with $\beta = \gamma = 0$ on $U_S \cap U_C \subset M$, were investigated in [24].

A semi-Riemannian manifold (M,g), $n \geq 3$, is said to be *pseudosymmetric* ([2], [7]) if at every point of M the tensors $R \cdot R$ and $Q(g,R)$ are linearly dependent. Thus the manifold (M,g) is pseudosymmetric if and only if

$$R \cdot R = L_R Q(g,R) \tag{11}$$

on $U_R = \{x \in M \mid R - \frac{\kappa}{n(n-1)} G \neq 0 \text{ at } x\}$, where L_R is some function on U_R. It is clear that every *semisymmetric* manifold ($R \cdot R = 0$) is pseudosymmetric. There exist pseudosymmetric manifolds which are non-semisymmetric (see e.g. [7], Section 3.6). We mention that certain spacetimes are pseudosymmetric, for instance: the Robertson-Walker spacetimes, the Schwarzschild spacetime, the Kottler spacetime, as well as the Reissner-Nordström spacetime ([4], [19]). The Reissner-Nordström-de Sitter spacetime is also pseudosymmetric (see Example 5.2(ii)). For more detailed information on the geometric motivation for the introduction of pseudosymmetric manifolds, and for a review of results on different aspects of pseudosymmetric manifolds, see [2], [7] and [27].

A semi-Riemannian manifold (M,g) is said to be *Ricci-pseudosymmetric* ([2], [7]) if at every point of M the tensors $R \cdot S$ and $Q(g,S)$ are linearly dependent. Thus the manifold (M,g) is Ricci-pseudosymmetric if and only if

$$R \cdot S = L_S Q(g,S) \tag{12}$$

on $U_S = \{x \in M \mid S - \frac{\kappa}{n} g \neq 0 \text{ at } x\}$, where L_S is some function on U_S. Note that $U_S \subset U_R$. Every pseudosymmetric manifold is Ricci-pseudosymmetric manifold. The converse statement is not true ([7], Section 8). Semi-Riemannian manifolds fulfilling (11) or (12) or other conditions of this kind are called *manifolds of pseudosymmetry type* ([7], [27]). We refer to [2] for a recent survey of results on pseudosymmetry type manifolds.

Let T_1 and T_2 be $(0,k)$-tensors on M. According to [5], we say that the tensors T_1 and T_2 are *pseudosymmetric related to a generalized curvature tensor T and a symmetric $(0,2)$-tensor A* if at every point of M the tensors $T \cdot T_1$ and $Q(A,T_2)$ are linearly dependent. This is equivalent to $T \cdot T_1 = LQ(A,T_2)$ on the subset $U \subset M$ of all points at which $Q(A,T_2)$ is nonzero, where L is some function on U. If $T_1 = T_2$, then we say that the tensor T_1 is *pseudosymmetric with respect to the tensors T and A*.

3 Roter Type Tensors

Let T be a generalized curvature tensor on a semi-Riemannian manifold (M,g), $n \geq 4$. We denote by $Ric(T)$, $Weyl(T)$ and $\kappa(T)$ its Ricci tensor, the Weyl tensor and the scalar curvature, respectively. The subsets U_T, $U_{Ric(T)}$ and

$U_{Weyl(T)}$ are defined in the same manner as the subsets U_R, U_S and U_C, respectively. Further, we assume that T is a generalized curvature tensor satisfying (1) on $U_{Ric(T)} \cap U_{Weyl(T)} \subset M$. Let U_A denote the subset of M consisting of all points at which the tensor A is not proportional to g. It is clear that $U_{Ric(T)} \cap U_{Weyl(T)} \subset U_A$. We have

LEMMA 3.1. *Let (M,g), $n \geq 4$, be a semi-Riemannian manifold admitting a generalized curvature tensor T satisfying* (1) *on M. If at $x \in U_A \subset M$ the tensor $Weyl(T)$ is nonzero then also α is nonzero at x.*

PROOF. We suppose that α vanishes at x. Now (1) reduces to $T = \beta g \wedge A + \gamma G$. From this, by standard calculations, we obtain $Weyl(T) = 0$, a contradiction.

LEMMA 3.2. *Any symmetric $(0,2)$-tensor on a semi-Riemannian manifold (M,g), $n \geq 4$, satisfies*

$$\begin{gathered} G \cdot G = 0, \quad \bar{A} \cdot G = 0, \quad (g \wedge A) \cdot G = 0, \quad G \cdot \bar{A} = Q(g, \bar{A}), \\ G \cdot (g \wedge A) = Q(g, g \wedge A), \quad \bar{A} \cdot \bar{A} = -Q(A^2, \bar{A}), \quad g \wedge Q(g, A) = Q(A, G), \\ (g \wedge A) \cdot A = Q(g, A^2), \quad \bar{A} \cdot (g \wedge A) + (g \wedge A) \cdot \bar{A} = -Q(A^2, g \wedge A), \\ (g \wedge A) \cdot (g \wedge A) = -Q(A^2, G), \quad Q(A, G) = -Q(g, g \wedge A), \\ Q(A, g \wedge A) = -Q(g, \bar{A}), \quad G \cdot A = Q(g, A), \quad \bar{A} \cdot A = Q(A, A^2). \end{gathered} \tag{13}$$

PROOF. The identities (13) are a consequence of suitable definitions.

LEMMA 3.3. *Let on a semi-Riemannian manifold (M,g), $n \geq 4$, be given a generalized curvature tensor T satisfying* (1). *Then at all points at which α is nonzero we have*

$$\begin{aligned} A^2 &= \frac{1}{\alpha}((\alpha\, tr(A) + (n-2)\beta)A + (\beta\, tr\, A + (n-1)\gamma)g - Ric(T)), \\ T \cdot A &= (n-2)\left(\frac{\beta^2}{\alpha} - \gamma\right) Q(g, A) - Q(A, Ric(T)) - \frac{\beta}{\alpha} Q(g, Ric(T)). \end{aligned} \tag{14}$$

A consequence of the above lemma is the following

COROLLARY 3.1. *Let on a semi-Riemannian manifold (M,g), $n \geq 4$, be given a generalized curvature tensor T satisfying* (1) *on $U_{Ric(T)} \cap U_{Weyl(T)} \subset M$ and let*

L_T and L_1 be the functions on $U_{Ric(T)} \cap U_{Weyl(T)}$ defined by (4)(c) *and* (6)(c), *respectively.*

(i) *If $A = Ric(T)$ then $T \cdot Ric(T) = L_T Q(g, Ric(T))$ on $U_{Ric(T)} \cap U_{Weyl(T)}$.*

(ii) *If $T = R$ then $R \cdot S = L_T Q(g, S)$ and $C \cdot S = L_1 Q(g, S)$ on $U_S \cap U_C \subset M$.*

Using the above lemmas we can prove the following generalization of Lemma 2.2 of [18].

THEOREM 3.1. *Let (M, g), $n \geq 3$, be a semi-Riemannian manifold admitting a generalized curvature tensor T satisfying* (1) *on M.*

(i) *At all points of M at which α is nonzero we have* (2). *In addition, if $A = Ric(T)$ then* (4) *is fulfilled.*

(ii) *On $U_{Weyl(T)} \subset M$ we have* (6), *provided that $n \geq 4$.*

PROOF. (i) First of all we note that for any generalized curvature T and any function γ on M the following identity is satisfied

$$(T - \gamma G) \cdot (T - \gamma G) = T \cdot T - \gamma Q(g, T). \tag{15}$$

Further, if T satisfies (1) then we have

$$\begin{aligned}(T - \gamma G) \cdot (T - \gamma G) &= (\alpha \bar{A} + \beta g \wedge A) \cdot (\alpha \bar{A} + \beta g \wedge A) \\ &= \alpha^2 \bar{A} \cdot \bar{A} + \alpha\beta((g \wedge A) \cdot \bar{A} + \bar{A} \cdot (g \wedge A)) \\ &\quad + \beta^2 (g \wedge A) \cdot (g \wedge A).\end{aligned} \tag{16}$$

In addition, let x be a point of M at which α is nonzero. Now (16), in view of Lemma 3.2, (14) and (15), yields

$$\begin{aligned}T \cdot T &= \gamma Q(g, T) - Q(\alpha A^2, \alpha \bar{A}) - Q(\alpha A^2, \beta g \wedge A) - \frac{\beta^2}{\alpha} Q(\alpha A^2, G) \\ &= \gamma Q(g, T) - Q((\beta\, tr(A) + (n-1)\gamma) g, \alpha \bar{A}) + Q(Ric(T), \alpha \bar{A}) \\ &\quad - Q((\alpha\, tr(A) + (n-2)\beta) A, \beta g \wedge A) - Q((\beta\, tr(A) + (n-1)\gamma) g, \beta g \wedge A) \\ &\quad + Q(Ric(T), \beta g \wedge A) - \frac{\beta^2}{\alpha} Q((\alpha\, tr(A) + (n-2)\beta) A, G) \\ &\quad + \frac{\beta^2}{\alpha} Q(Ric(T), G) + Q(Ric(T), \gamma G) - \gamma Q(Ric(T), G)\end{aligned}$$

$$= Q(Ric(T), T) + \gamma Q(g, T) - \left(\frac{\beta^2}{\alpha} - \gamma\right) Q(g, g \wedge Ric(T)) - (n-1)\gamma Q(g, \alpha \bar{A})$$

$$+ \frac{(n-2)\beta^2}{\alpha} Q(g, \alpha \bar{A}) - (n-1)\gamma Q(g, \beta g \wedge A) + \frac{(n-2)\beta^2}{\alpha} Q(g, \beta g \wedge A)$$

$$= Q(Ric(T), T) + L_2 Q(g, Weyl(T)).$$

Thus (2) is proved. Let $A = Ric(T)$. We have

$$Q(Ric(T), T) + L_2 Q(g, Weyl(T))$$

$$= Q(Ric(T), \beta g \wedge Ric(T) + \gamma G) + L_2 Q(g, T) - \frac{L_2}{n-2} Q(g, g \wedge Ric(T))$$

$$= L_2 Q(g, T) - \frac{\beta}{\alpha} Q(g, \alpha\, \overline{Ric(T)}) + \gamma Q(Ric(T), G) - \frac{\beta}{\alpha} Q(g, \beta g \wedge Ric(T))$$

$$+ \gamma Q(g, g \wedge Ric(T)) = \left(L_2 - \frac{\beta}{\alpha}\right) Q(g, T).$$

This, together with (2), leads to (4)(a). Note that (5)(a) is an immediate consequence of (4)(a). Further, (4)(a) and (5)(a), together with (9), imply (4)(b).

(ii) The relations (3) and (9) give

$$Weyl(T) = \alpha\, \overline{Ric(T)} + \left(\beta - \frac{1}{n-2}\right) g \wedge Ric(T) + \left(\gamma + \frac{\kappa(T)}{(n-2)(n-1)}\right) G.$$

We note that $Ric(Weyl(T)) = 0$. Now, in view of Theorem 3.1(i), we get

$$Weyl(T) \cdot Weyl(T)$$

$$= (n-2)\left(\frac{1}{\alpha}\left(\beta - \frac{1}{n-2}\right)^2 - \gamma - \frac{\kappa(T)}{(n-2)(n-1)}\right) Q(g, Weyl(T))$$

$$= \left((n-2)\left(\frac{\beta^2}{\alpha} - \gamma\right) - \frac{\beta}{\alpha} + \frac{1-(n-2)\beta}{(n-2)\alpha} - \frac{\kappa(T)}{n-1}\right) Q(g, Weyl(T)),$$

i.e. (6)(a). Now we prove that (6)(b) and (6)(c) are satisfied. From (6)(a) and (9) we obtain

$$Weyl(T) \cdot \left(T - \frac{1}{n-2} g \wedge Ric(T)\right) = L_1 Q(g, Weyl(T)),$$

whence

$$Weyl(T)\cdot T = \frac{1}{n-2} g\wedge (Weyl(T)\cdot Ric(T)) + L_1 Q(g,T) - \frac{L_1}{n-2} Q(g, g\wedge Ric(T)). \quad (17)$$

Further, applying (5), (6)(d), (9) and Lemma 3.2 into (17) we find

$$\begin{aligned} &Weyl(T)\cdot Ric(T)\\ &= \left(T - \frac{1}{n-2} g\wedge Ric(T) + \frac{\kappa(T)}{(n-2)(n-1)} G\right)\cdot Ric(T)\\ &= T\cdot Ric(T) - \frac{1}{n-2}(g\wedge Ric(T))\cdot Ric(T) + \frac{\kappa(T)}{(n-2)(n-1)} Q(g, Ric(T))\\ &= L_T Q(g, Ric(T)) - \frac{1}{n-2} Q(g, Ric(T)^2) + \frac{\kappa(T)}{(n-2)(n-1)} Q(g, Ric(T))\\ &= \left(L_T + \frac{\kappa(T)}{(n-2)(n-1)} - \frac{1}{n-2}\left(\kappa(T) + \frac{(n-2)\beta - 1}{\alpha}\right)\right) Q(g, Ric(T)). \end{aligned}$$

This, by (6)(d), yields (6)(b). Finally, (6)(b) together with (17) and the identity (see Lemma 3.2)

$$g\wedge Q(g, Ric(T)) = Q(Ric(T), G),$$

leads to (6)(c), completing the proof.

From Theorem 3.1 it follows

Corollary 3.2 (cf. [13], Theorem 4.2; [22]). *If the curvature tensor R of a semi-Riemannian manifold (M,g), $n\geq 4$, satisfies (1) on $U_S\cap U_C\subset M$, with $A = S$, then on this set we have*

$$\begin{aligned} &R\cdot R = L_R Q(g,R),\quad R\cdot S = L_R Q(g,S),\quad R\cdot C = L_R Q(g,C),\\ &R\cdot R = Q(S,R) + \left(L_R + \frac{\beta}{\alpha}\right) Q(g,C),\\ &C\cdot C = L_C Q(g,C),\quad L_C = L_R + \frac{1-(n-2)\beta}{(n-2)\alpha} - \frac{\kappa}{n-1},\\ &C\cdot R = L_C Q(g,R),\\ &\quad S^2 = \left(\frac{(n-2)\beta - 1}{\alpha} + \kappa\right) S + \frac{(n-1)\gamma + \beta\kappa(T)}{\alpha} g. \end{aligned}$$

We have also the following

Proposition 3.1 ([10], Proposition 6.5). *Let (M,g), $n \geq 4$, be a semi-Riemannian manifold admitting a generalized curvature tensor T and let the conditions*:

$$T \cdot T = L_T Q(g,T) \quad \text{and} \quad T \cdot T = Q(Ric(T),T) + LQ(g, Weyl(T))$$

be fulfilled on $U_{Ric(T)} \cap U_{Weyl(T)} \subset M$. Then on this set we have

$$Q\left(Ric(T) - (L_T - L)g, T - \frac{L}{n-2} G\right) = 0.$$

Proof. From our assumptions it follows that

$$Q(Ric(T),T) + LQ(g, Weyl(T)) = L_T Q(g,T),$$

hence

$$Q(Ric(T),T) - \frac{L}{n-2} Q(g, g \wedge Ric(T)) = (L_T - L)Q(g,T).$$

This, by the identity (see Lemma 3.2)

$$Q(g, g \wedge Ric(T)) = -Q(Ric(T), G), \tag{18}$$

turns into

$$\frac{L}{n-2} Q(Ric(T), G) = Q((L_T - L)g - Ric(T), T),$$

which yields (4), completing the proof.

The last proposition, together with Lemma 3.4 of [13], implies

Corollary 3.3 ([10], Corollary 6.1). *Let (M,g), $n \geq 4$, be a semi-Riemannian manifold admitting a generalized curvature tensor T and let the conditions*:

$$T \cdot T = L_T Q(g,T) \quad \text{and} \quad T \cdot T = Q(Ric(T),T) + LQ(g, Weyl(T))$$

be satisfied on $U_{Ric(T)} \cap U_{Weyl(T)} \subset M$. If at every point of this set the tensor $Ric(T)$ has no a decomposition in a metrical part and a part of rank at most one then (3) holds on $U_{Ric(T)} \cap U_{Weyl(T)}$.

Remark 3.1. As it was stated above, if T be a generalized curvature tensor on a semi-Riemannian manifold (M, g), $n \geq 4$, then (18) holds on M. We define now on M the following $(0,6)$-tensors:

$$Q(g, T), \quad Q(g, \overline{Ric(T)}) = -Q(Ric(T), g \wedge Ric(T)),$$
$$Q(g, g \wedge Ric(T)) = -Q(Ric(T), G), \quad Q(g, G) = 0,$$
$$Q(Ric(T), T), \quad Q(Ric(T), \overline{Ric(T)}) = 0. \tag{19}$$

Now we assume that (3) holds on $U_{Ric(T)} \cap U_{Weyl(T)} \subset M$. Applying (3) into (19) we obtain (cf. [11], p. 162)

$$Q(g, \overline{Ric(T)}) = \frac{1}{\alpha} Q(g, T) + \frac{\beta}{\alpha} Q(Ric(T), G),$$
$$Q(Ric(T), T) = -\frac{\beta}{\alpha} Q(g, T) + \left(\gamma - \frac{\beta^2}{\alpha}\right) Q(Ric(T), G), \tag{20}$$

Using (4)(c), (9), (19) and (20) we also obtain

$$Q(Ric(T), Weyl(T)) = \psi_2 Q(g, T) + \frac{\psi_3}{n-2} Q(Ric(T), G),$$
$$Q(g, Weyl(T)) = Q(g, T) + \frac{1}{n-2} Q(Ric(T), G).$$

4 New Curvature Conditions of Pseudosymmetry Type

In this section we present a family of new curvature conditions of pseudosymmetry type. Such conditions are fulfilled on a semi-Riemannian manifolds (M, g), $n \geq 4$, admitting a generalized curvature tensor T such that (3) holds on $U_{Ric(T)} \cap U_{Weyl(T)} \subset M$. Namely, using results from previous sections we can prove

Proposition 4.1. *Let (M, g), $n \geq 4$, be a semi-Riemannian manifold admitting a generalized curvature tensor T satisfying* (3) *on $U_{Ric(T)} \cap U_{Weyl(T)} \subset M$. Then on some open subset V of this set we have*: (2), (4), (6) *and*

$$T \cdot T = L_3 Q(Ric(T), Weyl(T)) + L_4 Q(Ric(T), T), \tag{21}$$

$$L_3 = -\frac{(n-1)\alpha L_2 L_T}{\psi_1}, \quad L_4 = \frac{(n-1)\alpha L_T \psi_3}{\psi_1}, \tag{22}$$

$$T \cdot T = L_5 Q(Ric(T), Weyl(T)) + L_6 Q(g, Weyl(T)), \tag{23}$$

$$L_5 = -\frac{L_T}{L_1}, \quad L_6 = \left(\frac{\psi_2}{L_1} - 1\right) L_T, \tag{24}$$

$$Weyl(T) \cdot Weyl(T) = L_7 Q(g,T) + L_8 Q(Ric(T),T), \tag{25}$$

$$L_7 = \frac{L_1 L_T}{L_2}, \quad L_8 = -\frac{L_1}{L_2}, \tag{26}$$

$$Weyl(T) \cdot Weyl(T) = L_9 Q(g,T) + L_{10} Q(Ric(T), Weyl(T)), \tag{27}$$

$$L_9 = L_1\left(1 - \frac{\psi_2}{\psi_3}\right), \quad L_{10} = \frac{L_1}{\psi_3}, \tag{28}$$

$$Weyl(T) \cdot Weyl(T) = L_{11} Q(Ric(T),T) + L_{12} Q(Ric(T), Weyl(T)), \tag{29}$$

$$L_{11} = \frac{\alpha}{\beta} L_1\left(\frac{(n-1)\alpha\beta\psi_2 L_T}{\psi_1} - 1\right), \quad L_{12} = -\frac{(n-1)\alpha L_1 L_T}{\psi_1}, \tag{30}$$

$$T \cdot Weyl(T) = L_{13} Q(g,T) + L_{14} Q(Ric(T),T), \tag{31}$$

$$L_{13} = -L_T L_{14}, \quad L_{14} = -\frac{L_T}{L_2}, \tag{32}$$

$$T \cdot Weyl(T) = L_{15} Q(g,T) + L_{16} Q(Ric(T), Weyl(T)), \tag{33}$$

$$L_{15} = L_T\left(1 - \frac{\psi_2}{\psi_3}\right), \quad L_{16} = \frac{L_T}{\psi_3}, \tag{34}$$

$$T \cdot Weyl(T) = L_{17} Q(Ric(T),T) + L_{18} Q(Ric(T), Weyl(T)), \tag{35}$$

$$L_{17} = -\frac{\alpha}{\beta} L_T\left(\frac{(n-1)\alpha\beta\psi_2 L_T}{\psi_1} + 1\right), \quad L_{18} = -\frac{(n-1)\alpha L_T^2}{\psi_1}, \tag{36}$$

$$Weyl(T) \cdot T = Q(Ric(T), Weyl(T)) + L_{19} Q(g, Weyl(T)), \tag{37}$$

$$L_{19} = -\psi_3, \tag{38}$$

$$Weyl(T) \cdot T = L_{20} Q(Ric(T),T) + L_{21} Q(Ric(T), Weyl(T)), \tag{39}$$

$$L_{20} = -\frac{\alpha}{\beta} L_1\left(\frac{(n-1)\alpha\beta L_2}{\psi_1} + 1\right), \quad L_{21} = -\frac{(n-1)\alpha L_1 L_2}{\psi_1}, \tag{40}$$

$$Weyl(T) \cdot T = -L_5 Q(Ric(T),T) + L_{22} Q(g, Weyl(T)), \tag{41}$$

$$L_{22} = \frac{L_T L_2}{L_1}, \tag{42}$$

provided that the functions β, ψ_1, ψ_3, L_1 *and* L_2 *are nonzero at every point of* V.

COROLLARY 4.1. *Let* (M, g), $n \geq 4$, *be a semi-Riemannian manifold admitting a generalized curvature tensor* T *satisfying* (3) *on* $U_{Ric(T)} \cap U_{Weyl(T)} \subset M$. *Using* (2), (4), (6) *and* (21)–(42) *we can state that on a certain subset of* $U_{Ric(T)} \cap U_{Weyl(T)}$ *any linear combination of the tensors*: $T \cdot T$, $T \cdot Weyl(T)$, $Weyl(T) \cdot T$ *and* $Weyl(T) \cdot Weyl(T)$ *is equal to some linear combination of the tensors*: $Q(g, T)$, $Q(Ric(T), T)$, $Q(g, Weyl(T))$ *and* $Q(Ric(T), Weyl(T))$.

REMARK 4.1. From the above statement it follows that on some subset of $U_{Ric(T)} \cap U_{Weyl(T)} \subset M$ the tensor $T \cdot Weyl(T) - Weyl(T) \cdot T$ is expressed by a linear combination of the tensors $Q(g, T)$, $Q(Ric(T), T)$, $Q(g, Weyl(T))$ and $Q(Ric(T), Weyl(T))$. Recently manifolds with the tensor $R \cdot C - C \cdot R$ expressed by a linear combination of the tensors $Q(g, R)$, $Q(g, C)$, $Q(S, R)$ and $Q(S, C)$ were investigated among others in [9], [14] and [15] (see also [10], Section 5).

5 Examples

It is known that certain spacetimes are pseudosymmetric. Such spacetimes were investigated in [4], [12] and [19]. For instance, in [19] it was stated that every Robertson-Walker, the Schwarzschild, the Kottler and the Reissner-Nordström spacetimes are pseudosymmetric. There are also spacetimes satisfying other conditions of pseudosymmetric type (see e.g. [17] and references therein). In this section we give an example of a family of warped product spacetimes satisfying (3) for $T = R$.

EXAMPLE 5.1. We recall that the warped product $\overline{M} \times_F \tilde{N}$, of a 1-dimensional manifold $(\overline{M}, \bar{g})$, $\bar{g}_{11} = -1$, with a warping function F and a 3-dimensional Riemannian manifold $(\tilde{N}, \tilde{g})$ is said to be a *generalized Robertson-Walker spacetime* ([1], [20]). Generalized Robertson-Walker spacetimes were investigated among others in [25]. In particular, if $(\tilde{N}, \tilde{g})$ is a Riemannian space of constant curvature then $\overline{M} \times_F \tilde{N}$ is called a *Robertson-Walker spacetime*. It is well-known that such spacetimes are conformally flat. Every Robertson-Walker spacetime is pseudosymmetric ([7], Section 6). In [3] it was shown that at every point of a generalized Robertson-Walker spacetime $\overline{M} \times_F N$ the following condition is satisfied: the tensors $R \cdot R - Q(S, R)$ and $Q(g, C)$ are linearly dependent. This is equivalent to $R \cdot R - Q(S, R) = LQ(g, C)$ on $U_C \subset M$, where L is some function on U_C. Generalized-Robertson Walker spacetimes satisfying some curvature condition of pseudosymmetry type were investigated in [17].

EXAMPLE 5.2. (i) Let $\overline{M} = \{(t,r) \in \mathbf{R}^2 : r > 0\}$ be on an open connected nonempty subset of $\mathbf{R}^2$ and let on $\overline{M}$ be defined the metric tensor $\bar{g}$ as in (8). We consider the warped product $\overline{M} \times_F \tilde{N}$ of the manifold $(\overline{M}, \bar{g})$ and the 2-dimensional unit standard sphere $(\tilde{N}, \tilde{g})$, with the warping function $F = F(r) = r^2$.

(ii) According to [23] the warped product $\overline{M} \times_F \tilde{N}$ defined in (i) is said to be the *Reissner-Nordström-de Sitter spacetime* if $H(r) = 1 - \frac{2m}{r} + \frac{e^2}{r^2} - \frac{1}{3}\Lambda r^2$, where $m = const. > 0$, $e = const.$ and $\Lambda = const.$ In particular, if $e \neq 0$ and $\Lambda = 0$, or $e = 0$ and $\Lambda \neq 0$, or $e = 0$ and $\Lambda = 0$, then the Reissner-Nordström-de Sitter spacetime is called the *Reissner-Nordström spacetime*, the *Kottler spacetime* or the *Schwarzschild spacetime*, respectively [26] (Section 13). These spacetimes are non-semisymmetric pseudosymmetric manifolds ([19], Example 1). It is well-known that the Kottler spacetime is a non-Ricci flat Einstein manifold. The Schwarzschild spacetime is a Ricci flat manifold.

(iii) If $H(t,r) = 1 - \frac{2m(t)}{r}$ then the warped product $\overline{M} \times_F \tilde{N}$ is called the *Vaidya spacetime*. The Ricci tensor S of the Vaidya spacetime satisfies $\operatorname{rank} S \leq 1$, which means that this spacetime is a special quasi-Einstein manifold. We can check that the Vaidya spacetime is a non-pseudosymmetric manifold satisfying

$$R \cdot R - Q(S,R) = -\frac{\rho_1}{\rho_2} Q(g,C),$$

$$\rho_1 = 2(8m^3 m''(-5r + 2m) + 2r^2 m(2 + 2m'^2 - 7rm'') + r^3(-m'^2 + 2rm'') + rm(-5 - 4m' + 36rm'')),$$

$$\rho_2 = r(r - 2m)(2m(-3r^2 + 6rm - 4m^2)(1 + rm'') + r^3(2 + rm'')),$$

at all points at which ρ_2 is nonzero, where $m'' = \frac{dm'}{dt}$ and $m' = \frac{dm}{dt}$.

EXAMPLE 5.3. Let $\overline{M} \times_F \tilde{N}$ be the spacetime defined in Example 5.2 with the warping function $F = F(r) = r^2$. Let $\bar{\kappa}$ and κ be the scalar curvature of $(\overline{M}, \bar{g})$ and $\overline{M} \times_F \tilde{N}$, respectively. We have

$$\bar{\kappa} = (2H'^2 - HH'')H^{-3},$$

$$\kappa = (2H^2 + 2H^3 + 2r^2 H'^2 - rH(4H' + rH''))r^{-2}H^{-3},$$

where $H'' = \frac{dH'}{dr}$ and $H' = \frac{dH}{dr}$. In addition, we set

$$\tau = 2H^2 + 2H^3 - 2r^2 H'^2 + r^2 HH''.$$

For the Reissner-Nordström-de Sitter spacetime the last three formulas turn into

$$\bar{\kappa} = \frac{18(3e^4 + \Lambda r^6(1 + \Lambda r^2) + 3e^2r^2(-3 + 4\Lambda r^2) + 6r^3m(1 - 2\Lambda r^2))}{(3e^2 - 6mr + 3r^2 - \Lambda r^4)^3},$$

$$\begin{aligned} \kappa = {} & -2\Lambda^3 r^{12} - 18\Lambda^2 r^{10} - 36m\Lambda^2 r^9 + 18\Lambda(6 - e^2\Lambda)r^8 - 432\Lambda m r^7 \\ & - 72\Lambda(3m^2 - 4e^2)r^6 + 216e^2 m\Lambda r^5 - 54e^4\Lambda r^4 \\ & - 216m(e^2 - 2m^2)r^3 + 162e^2(4m^2 + e^2)r^2 - 324e^4 mr + 54e^6, \end{aligned}$$

$$\begin{aligned} \tau = {} & -2\Lambda^3 r^{12} - 30\Lambda^2 r^{10} - 36m\Lambda^2 r^9 + 18e^2\Lambda^2 r^8 + 72m\Lambda r^7 - 72\Lambda(3m^2 + 2e^2)r^6 \\ & + 216e^2 m\Lambda r^5 + (-108e^2 - 54e^4\Lambda - 432m^2)r^4 + 432m(e^2 - m^2)r^3 \\ & + 162e^2(4m^2 - e^2)r^2 - 324e^4 mr + 54e^6. \end{aligned}$$

We can check that the tensor $S - \frac{\kappa}{4}g$ of $\overline{M} \times_F \tilde{N}$ is a zero tensor if and only if

$$\tau = 0, \tag{43}$$

holds on $\overline{M}$. Further, the tensor C of $\overline{M} \times_F \tilde{N}$ is a zero tensor if and only if on $\overline{M}$ we have

$$2H^2 + 2H^3 + 2r^2H'^2 + rH(2H' - rH'') = 0. \tag{44}$$

For the Reissner-Nordström-de Sitter spacetime the left-hand side of (44) has the form

$$\begin{aligned} & 2H^2 + 2H^3 + 2r^2H'^2 + rH(2H' - rH'') \\ & \quad = -\Lambda^3 r^{12} + 9\Lambda^2 r^{10} - 18m\Lambda^2 r^9 + 9e^2\Lambda^2 r^8 - 162m\Lambda r^7 - 36\Lambda(3m^2 - 4e^2)r^6 \\ & \qquad - 54m(2e^2 + 3)r^5 + (162e^2 - 27e^4\Lambda - 324m^2)r^4 + 54m(7e^2 - 4m^2)r^3 \\ & \qquad + 81e^2(4m^2 - e^2)r^2 - 162e^4 mr + 27e^6. \end{aligned}$$

From the above considerations it follows that $x \in U_S \cap U_C \subset \overline{M} \times_F \tilde{N}$ if and only if the left-hand sides of (43) and (44) are nonzero at $\pi_1(x)$, where $\pi_1 : \overline{M} \times \tilde{N} \to \overline{M}$ denotes the natural projection. The curvature tensor R of $\overline{M} \times_F \tilde{N}$ satisfies (7) on $U_S \cap U_C$ with

$$\alpha = (r^2H^3(2H^2 + 2H^3 + 2r^2H'^2 + rH(2H' - rH'')) \tau^{-2}, \tag{45}$$

$$\begin{aligned} \beta = {} & (rH(2r^2H'^3 - rHH'(4H' + rH'') + 2H^3(H' + 2rH'') \\ & + 2H^2(H' - 4rH'^2 + 2rH'')))\tau^{-2}, \end{aligned} \tag{46}$$

$$\gamma = 4r^2H'^4 + 4H^4(H'^2 - H'') - 2H'^5H'' + 4rHH'^2(-2H' + rH'^2 - rH'')$$
$$+ H^2(-12rH'^3 + 6rH'H'' + r^2H''^2 + 4H'^2(1 - r^2H''))$$
$$+ H^3(8H'^2 + 6rH'H'' + H''(-2 + r^2H''))\tau^{-2}. \tag{47}$$

In addition, on $U_S \cap U_C$ we have

$$\psi_1 = ((H + H^2 - rH')(-4r^3H'^4 + 4r^3HH'^2H'' - r^3H^2H''^2$$
$$+ 2H^4(2H' + rH'') + H^3(4H' - 4rH'^2 + 2rH'')))\tau^{-2}r^{-1}H^{-2},$$
$$\psi_2 = \frac{\kappa}{4},$$
$$\psi_3 = (4H^2 + 4H^3 + 4r^2H'^2 - rH(5H' + 2rH''))6r^{-2}H^{-3},$$
$$L_1 = (-2H^2 - 2H^3 - 2r^2H'^2 + rH(-2H' + rH''))12r^{-2}H^{-3},$$
$$L_2 = (-(3 + 4H)H'^2 + 2(1 + H)HH'') - 12r^{-2}H^{-4}L_1^{-1}, \tag{48}$$

$$R \cdot R = -\frac{H'}{2rH^2}Q(g, R). \tag{49}$$

Remark 5.1. Warped products $\overline{M} \times_F \tilde{N}$ of semi-Riemannian spaces of constant curvature $(\overline{M}, \bar{g})$, $p \geq 2$, and $(\tilde{N}, \tilde{g})$, $n - p \geq 2$, satisfying (7) were investigated in [16]. In that paper (see [16], Example 4.1) an example of such warped product is given. That warped product can be locally realized as hypersurfaces immersed isometrically in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, with signature $(s, n+1-s)$.

Remark 5.2. Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. On M we have ([18])

$$R \cdot R - Q(S, R) = -\frac{(n-2)\rho}{n(n+1)}Q(g, C), \tag{50}$$

where ρ is the scalar curvature of the ambient space. We assume that M is a pseudosymmetric manifold. Thus (11) holds on $U_R \subset M$. From (11) and (50) it follows that

$$Q\left(A, R - \frac{\rho}{n(n+1)}G\right) = 0 \tag{51}$$

holds on U_R ([6]), where $A = S - \left(L_R + \frac{(n-2)\rho}{n(n+1)}\right)g$. In addition, we assume that $\operatorname{rank} A \geq 2$ at $x \in U_R \cap U_S$. Applying now Lemma 3.4 of [13] to (51), we obtain

$R = \frac{\phi}{2}\bar{A}$ on some neighbourhood $U \subset U_R \cap U_S$ of x, where ϕ is some function on U. Thus on U the tensor R satisfies (3).

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