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SMOOTHLY SYMMETRIZABLE COMPLEX SYSTEMS AND THE REAL REDUCED DIMENSION

By

Tatsuo NISHITANI and Jean VAILLANT

1 Introduction

Let L be a first order system

$$L(x,D)=\sum_{j=1}^{n}A_j(x)D_j$$

where $A_1=I$ is the identity matrix of order m and $A_j(x)$ are $m\times m$ complex valued smooth matrix functions. In this note we continue to study the question when we can smoothly symmetrize $L(x,D)$. In particular we discuss about the question whether we can smoothly reduce $L(x,D)$ to a hermitian system if $L(x,D)$, at every frozen x, is similar to hermitian system as a constant coefficient system. In [2], [3] the same question for real systems was studied. Let $L(x,\xi)$ be the symbol of $L(x,D)$: Let us denote

$$L(x,\xi)=(\theta_j^i(x,\xi))$$

which is a $m\times m$ complex valued matrix. We set

$$d(L(x,\cdot))=\dim \operatorname{span}_{\mathbf{R}}\{\operatorname{Re}\theta_j^i(x,\cdot),\operatorname{Im}\theta_j^i(x,\cdot)\}$$

which is called the real reduced dimension of L at x.

Our aim in this note is to prove

THEOREM 1.1. *Let $m\ge 2$. Assume that at every x near $\bar{x}$ there exists $S(x)$ which is possibly non smooth in x such that $S(x)^{-1}L(x,\xi)S(x)$ is hermitian for every ξ and the real reduced dimension of $L(\bar{x},\cdot)\ge m^2-m+2$. Then there is a smooth $T(x)$ defined near $\bar{x}$ such that*

$$T(x)^{-1}L(x,\xi)T(x)$$

is hermitian for any ξ and for any x near $\bar{x}$.

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REMARK. It is clear from the proof that $\xi = (\xi_1, \dots, \xi_n)$ is not necessary to be the covariables of $x = (x_1, \dots, x_n)$ and actually we prove the assertion for $\sum_{j=1}^{n} A_j(y)\xi_j$. Moreover $T(y)$ can be chosen as smooth as $A_j(y)$ which is also clear from the proof. This remark is available for applications of the result to quasi-linear systems $\sum_{j=1}^{n} A_j(x,u)D_j u$.

In a series of papers [4], [5] the second author proved that if $L(\xi)$ is diagonalizable with real eigenvalues for every ξ and the real reduced dimension $d(L) \geq m^2 - 2$ and $m \geq 3$ then there exists a constant matrix S such that $S^{-1}L(\xi)S$ is hermitian for every ξ. Combining the above theorem with this result we obtain

THEOREM 1.2. *Let $m \geq 3$. Assume that $L(x,\xi)$ is diagonalizable with real eigenvalues for every x near $\bar{x}$, every ξ and the real reduced dimension $d(L(x,\cdot)) \geq m^2 - 2$ for every x near $\bar{x}$. Then $L(x,\xi)$ is smoothly symmetrizable near $\bar{x}$ and hence $L(x,D)$ is strongly hyperbolic near $\bar{x}$.*

PROOF. If $m \geq 4$ and hence $m^2 - 2 \geq m^2 - m + 2$ the proof follows from Theorem 1.1. When $m = 3$ we will give a proof in §4. $\square$

REMARK. When $m = 2$ then Theorem 1.2 fails (see for example [6]).

2 Lemma

In this section we write $\theta_j^i(x,\xi) = \phi_j^i(x,\xi) + \sqrt{-1}\psi_j^i(x,\xi)$. Considering $S(\bar{x})^{-1}L(x,\xi)S(\bar{x})$ we may assume that $L(\bar{x},\xi)$ is hermitian for every ξ which will be assumed throughout the paper. Thus we have

$$\phi_j^i(\bar{x},\cdot) = \phi_i^j(\bar{x},\cdot), \quad \psi_j^i(\bar{x},\cdot) = -\psi_i^j(\bar{x},\cdot), \quad \psi_i^i(\bar{x},\cdot) = 0.$$

Let $\phi_j^i(\bar{x},\cdot)$, $(i,j) \in \tilde{M}^R$, $\psi_j^i(\bar{x},\cdot)$, $(i,j) \in M^I$ be a maximal linearly independent set in $\{\phi_j^i(\bar{x},\cdot), \psi_j^i(\bar{x},\cdot) \mid i > j\}$. We add (i,i) to $\tilde{M}^R$ and denote by M^R thus obtained set so that

$$\phi_j^i(\bar{x},\cdot), (i,j) \in M^R, \quad \psi_j^i(\bar{x},\cdot), (i,j) \in M^I \tag{2.1}$$

is a maximal linearly independent set in $\{\phi_j^i(\bar{x},\cdot), i \geq j, \psi_j^i(\bar{x},\cdot), i > j\}$. Let us define the index set K:

$$K = \{(i,j) \mid i \geq j, (i,j) \notin M^R \cap M^I\}$$

and denote by $|K|$ the cardinal number of K. We denote

$$\check{M}^R = \{(i, j) \mid (i, j) \in M^R \text{ or } (j, i) \in M^R\}$$

and $\check{M}^I$, $\check{K}$ are defined in the same way.

For $1 \le p \le m$ we define $n(p)$ by

$$n(p) = |\{i \mid (p, i) \notin \check{M}^R\}| + |\{i \mid (p, i) \notin \check{M}^I\}|$$

that is, the number of $\phi_j^p(\bar{x}, \cdot)$, $\psi_i^p(\bar{x}, \cdot)$ on the p-th row which are linear combinations of (2.1). From the assumption we have

$$\sum_{p=1}^{m} n(p) \le 2(m-2). \tag{2.2}$$

Assume that $\check{K}$ contains $r \ge 0$ diagonal entries (i, i) then it is clear that

$$\sum_{p=1}^{m} n(p) \le 2(m-2) - r. \tag{2.3}$$

We start with

LEMMA 2.1. *Assume that $L(\bar{x}, \cdot)$ is hermitian and $\check{K}$ contains no diagonal entry and*

$$d(L(\bar{x}, \cdot)) \ge m^2 - m + 1. \tag{2.4}$$

Let $H(x) = (h_j^i(x) + \sqrt{-1} g_j^i(x))$ be a positive definite hermitian matrix such that

$$L(x, \xi)H(x) = H(x)L(x, \xi)^*, \quad \forall \xi \tag{2.5}$$

holds for every x near $\bar{x}$. Then $H(x)/h_m^m(x)$ is smooth near $\bar{x}$.

PROOF. Since $h_m^m(x) > 0$ then $H(x)/h_m^m(x)$ is again positive definite and verifies (2.5). Denoting $H(x)/h_m^m(x)$ by $H(x)$ again let us consider the real and the imaginary part of the (i, j)-th entry, $i < j$, of the equality (2.5):

$$\sum_{k=1}^{m} \phi_k^i h_j^k - \sum_{k=1}^{m} \psi_k^i g_j^k - \sum_{k=1}^{m} h_k^i \phi_k^j - \sum_{k=1}^{m} g_k^i \psi_k^j = 0 \tag{2.6}$$

$$\sum_{k=1}^{m} \phi_k^i g_j^k + \sum_{k=1}^{m} \psi_k^i h_j^k + \sum_{k=1}^{m} h_k^i \psi_k^j - \sum_{k=1}^{m} g_k^i \phi_k^j = 0. \tag{2.7}$$

Let us put

$$\hat{H}(x) = (h_1^1(x), h_2^2(x), \dots, h_{m-1}^{m-1}(x), h_2^1(x), g_2^1(x), \dots, h_m^1(x), g_m^1(x),$$
$$h_3^2(x), g_3^2(x), \dots, h_m^{m-1}(x), g_m^{m-1}(x)).$$

Then (2.6) and (2.7) with $1 \le i < j \le m$ yield $m(m-1)$ equations with $(m+1)(m-1)$ unknowns $\hat{H}(x)$:

$$G(x,\xi)\hat{H}(x) = F(x,\xi). \tag{2.8}$$

Recalling that $\phi_i^i(\bar{x},\cdot)$, $1 \le i \le m$ are linearly independent by the assumption we can choose $\xi^* \in \mathbf{R}^n$ so that

$$\phi_i^i(\bar{x},\xi^*) - \phi_j^j(\bar{x},\xi^*) \neq 0, \quad i > j$$
$$\phi_j^i(\bar{x},\xi^*) = 0, \quad (i,j) \in M^R,\ i > j$$
$$\psi_j^i(\bar{x},\xi^*) = 0, \quad (i,j) \in M^I,\ i > j.$$

This gives that

$$\phi_j^i(\bar{x},\xi^*) = 0, \quad \psi_j^i(\bar{x},\xi^*) = 0, \quad \text{for all } i > j$$

since $\phi_j^i(\bar{x},\cdot)$, $(i,j) \notin M^R$, $\psi_j^i(\bar{x},\cdot)$, $(i,j) \notin M^I$, $i > j$ are linear combinations of $\phi_j^i(\bar{x},\cdot)$, $(i,j) \in M^R$, $\psi_j^i(\bar{x},\cdot)$, $(i,j) \in M^I$, $i > j$. In (2.8) choosing $\xi = \xi^*$ we have

$$G(x,\xi^*)\hat{H}(x) = F(x,\xi^*). \tag{2.9}$$

Here $G(\bar{x},\xi^*)$ has the form

$$G(\bar{x},\xi^*) = [O \ \vdots \ D]$$

where D is a non singular diagonal matrix of order $m(m-1)$ and O is the $m(m-1) \times (m-1)$ zero matrix. Let $N(x,\xi)$ be the submatrix of $G(x,\xi)$ consisting of $m(m-1)$ rows and the first $m-1$ columns. We show that with a suitable ξ^{**} one has

$$\operatorname{rank} N(\bar{x},\xi^{**}) = m-1 \tag{2.10}$$

so that we can find $i_1, i_2, \dots, i_{m-1}$ such that the submatrix $N(x)$, consisting of $m-1$ columns and $i_1, \dots, i_{m-1}$-th rows of $N(x,\xi^{**})$, is non singular at $x = \bar{x}$. We pick up these $i_1, \dots, i_{m-1}$-th equations from (2.8) with $\xi = \xi^{**}$ and add to (2.9) to get

$$G(x)\hat{H}(x) = F(x) \tag{2.11}$$

where

$$G(\bar{x}) = \begin{bmatrix} O & \vdots & D \\ \cdots & \cdots & \cdots \\ N(\bar{x}) & \vdots & * \end{bmatrix}.$$

Then noting $\det G(\bar{x}) \neq 0$ and hence $\det G(x) \neq 0$ near $\bar{x}$ we can conclude from (2.11) that $\hat{H}(x)$ is smooth near $\bar{x}$. This proves the assertion since $g_i^i(x) = 0$ and $h_m^m(x) = 1$.

We now show (2.10) proving that there is ξ such that $\operatorname{Ker} N(\bar{x}, \xi) = \{0\}$. Study

$$N(\bar{x}, \xi)\begin{pmatrix} y_1 \\ \vdots \\ y_{m-1} \end{pmatrix} = N(\bar{x}, \xi)y = 0$$

which is

$$\text{(2.12)} \qquad \begin{cases} \phi_j^i y_j - \phi_i^j y_i = 0,\ \psi_j^i y_j + \psi_i^j y_i = 0, \quad 1 \le i < j \le m-1 \\ -\phi_i^m y_i = 0,\ \psi_i^m y_i = 0, \quad 1 \le i \le m-1. \end{cases}$$

Consider $\phi_i^m(\bar{x}, \cdot)$, $\psi_i^m(\bar{x}, \cdot)$, $1 \le i \le m-1$. Assume that

$$\text{(2.13)} \qquad \phi_i^m(\bar{x}, \cdot)^2 + \psi_i^m(\bar{x}, \cdot)^2 \equiv 0, \quad i = i_1, i_2, \dots, i_p,$$

$$\text{(2.14)} \qquad \phi_i^m(\bar{x}, \cdot)^2 + \psi_i^m(\bar{x}, \cdot)^2 \not\equiv 0, \quad i = j_1, j_2, \dots, j_q$$

where $q = m - 1 - p$. If $p = 0$ then choosing ξ so that $\phi_i^m(\bar{x}, \xi)^2 + \psi_i^m(\bar{x}, \xi)^2 \neq 0$ for $i = 1, \dots, m-1$ we conclude that $y = 0$ by (2.12). Let $p \ge 1$. For $s = 1, \dots, q$ we set

$$v_s = |\{i \neq j_s \mid \phi_i^{j_s}(\bar{x}, \cdot)^2 + \psi_i^{j_s}(\bar{x}, \cdot)^2 \equiv 0\}|.$$

Then from the assumption it follows that

$$4p + 2\sum_{s=1}^{q} v_s \le 2(m-1)$$

which shows that

$$\sum_{s=1}^{q} v_s \le -m + 1 + 2q < q.$$

Therefore there is t such that $v_t = 0$. We now choose ξ so that

$$\phi_i^m(\bar{x},\xi)^2 + \psi_i^m(\bar{x},\xi)^2 \neq 0, \quad i = j_1, j_2, \dots, j_q,$$
$$\phi_i^{j_t}(\bar{x},\xi)^2 + \psi_i^{j_t}(\bar{x},\xi)^2 \neq 0, \quad i \neq j_t.$$

This together with (2.12) proves that $y_{j_i} = 0$, $i = 1, 2, \dots, q$, $y_i = 0$, $i \neq j_t$ and hence $y = 0$. This concludes the proof. $\square$

A blocking of L and the corresponding blocking of H gives similar equalities as (2.5). We apply Lemma 2.1 to thus obtained equalities. To do so we introduce

DEFINITION. We say that $\check{K}$ is confined on $\{p_1, \dots, p_\ell\}$ if for every $i = 1, \dots, \ell$ we have

$$(p_i, p) \in \check{K} \Rightarrow p \in \{p_1, \dots, p_\ell\}.$$

If $n(p_j) = 0$ for $j = 1, \dots, \ell$ then, by definition, $\check{K}$ is confined on $\{p_1, \dots, p_\ell\}$. If $\check{K}$ is confined on $\{1, 2, \dots, \ell\}$ then $\phi_j^i(\bar{x},\cdot)$, $\psi_j^i(\bar{x},\cdot)$, $1 \le i \le \ell$, $\ell + 1 \le j$ are linearly independent.

For any permutation σ on $\{1, 2, \dots, m\}$ we define P_σ to be the matrix whose entries are zero except for $(j, \sigma(j))$-th entry which is 1 for $j = 1, \dots, m$. Note that P_σ is an orthogonal matrix. Let $L^\sigma(x,\xi) = P_\sigma^{-1} L(x,\xi) P_\sigma$ and $H^\sigma(x) = P_\sigma^{-1} H(x) P_\sigma$ then $L^\sigma(x,\xi)$ and $H^\sigma(x)$ verify (2.5). Moreover since $L(\bar{x},\cdot)$ is hermitian we see with $L^\sigma = (\tilde{\phi}_j^i + \sqrt{-1}\tilde{\psi}_j^i)$ that

$$\mathrm{span}\{\tilde{\phi}_j^i(\bar{x},\cdot), \tilde{\psi}_j^i(\bar{x},\cdot), i > j\} = \mathrm{span}\{\phi_j^i(\bar{x},\cdot), \psi_j^i(\bar{x},\cdot), i > j\}.$$

With

$$\check{M}_\sigma^R = \{(\sigma(i),\sigma(j)) \mid (i,j) \in \check{M}^R\}, \quad \check{M}_\sigma^I = \{(\sigma(i),\sigma(j)) \mid (i,j) \in \check{M}^I\}$$

we set

$$M_\sigma^R = \{(i,j) \in \check{M}_\sigma^R, i \ge j\}, \quad M_\sigma^I = \{(i,j) \in \check{M}_\sigma^I, i > j\}$$

then it is clear that $\tilde{\phi}_j^i$, $(i,j) \in M_\sigma^R$, $\tilde{\psi}_j^i$, $(i,j) \in M_\sigma^I$ is a maximal linearly independent set in $\{\tilde{\phi}_j^i, i \ge j, \tilde{\psi}_j^i, i > j\}$. It is also clear that

$$\{(i,j) \mid (i,j) \notin \check{M}_\sigma^R \cap \check{M}_\sigma^I\} = \{(\sigma(i),\sigma(j)) \mid (i,j) \in \check{K}\} = \check{K}_\sigma.$$

Note that if $\check{K}$ is confined on $\{p_1, \dots, p_\ell\}$ then $\check{K}_\sigma$ is confined on $\{\sigma(p_1), \dots, \sigma(p_\ell)\}$ and we have

$$n_\sigma(p) = n(p), \quad n_\sigma(p) = |\{i \mid (p,i) \notin \check{M}_\sigma^R\}| + |\{i \mid (p,i) \notin \check{M}_\sigma^I\}|.$$

Our aim in this section is to prove:

Lemma 2.2. *Assume that $\check{K}$ is confined on $\{p_1, p_2, \dots, p_\ell\}$, $(\ell \geq 2)$ where $(p_i, p_i) \notin K$ and*

$$\sum_{j=1}^{\ell} n(p_j) \leq 2(\ell - 1). \tag{2.15}$$

Assume that $L(\bar{x}, \cdot)$ is hermitian. Let $H(x)$ be a positive definite hermitian matrix such that (2.5) *holds for every x near $\bar{x}$. Then $H(x)/h_p^p(x)$ is smooth near $\bar{x}$ for some p.*

Proof. As observed above, considering $P_\sigma^{-1} L(x, \xi) P_\sigma$ with a suitable permutation matrix P_σ we may assume that $\check{K}$ is confined on $\{1, \dots, \ell\}$ and $(j, j) \notin K$ for $j = 1, \dots, \ell$. Let us write

$$L(x, \xi) = \begin{pmatrix} L_{11}(x, \xi) & L_{12}(x, \xi) \\ L_{21}(x, \xi) & L_{22}(x, \xi) \end{pmatrix} \tag{2.16}$$

where $L_{11}(x, \xi)$ is the $\ell \times \ell$ submatrix consisting of the first ℓ rows and the first ℓ columns of $L(x, \xi)$. Let us write

$$H(x) = \begin{pmatrix} H_{11}(x) & H_{12}(x) \\ H_{21}(x) & H_{22}(x) \end{pmatrix} \tag{2.17}$$

where the blocking corresponds to that of (2.16). Then from (2.5) we have

$$L_{11} H_{11} + L_{12} H_{21} = H_{11} L_{11}^* + H_{12} L_{12}^*. \tag{2.18}$$

Since $\phi_j^i(\bar{x}, \cdot)$, $\psi_j^i(\bar{x}, \cdot)$, $1 \leq i \leq \ell$, $j \geq \ell + 1$ are linearly independent one can solve the equation $L_{12}(x, \xi) = 0$ near $\bar{x}$ so that $\xi_b = (\xi_{i_1}, \dots, \xi_{i_N})$, $N = 2\ell(m - \ell)$ are linear combinations of the other $\xi_a = (\xi_{j_1}, \dots, \xi_{j_M})$ with coefficients which are smooth functions in x near $\bar{x}$ where $\xi = (\xi_a, \xi_b)$ is some partition of the variables ξ. Inserting these ξ_b into $L(x, \xi)$ the equation (2.18) becomes

$$L_{11}(x, \xi_a) H_{11}(x) = H_{11}(x) L_{11}(x, \xi_a)^*.$$

Note that

$$d(L_{11}(\bar{x}, \xi_a)) \geq \ell^2 - (\ell - 1)$$

by assumption. By Lemma 2.1 we conclude that $H_{11}(x)/h_\ell^\ell(x)$ is smooth near $\bar{x}$. Denoting $H(x)/h_\ell^\ell(x)$ by $H(x)$ we have (2.18) again. Since $H_{11}(x)$ is smooth we write

(2.19) $$L_{12}H_{21} - H_{12}L_{12}^* = H_{11}L_{11}^* - L_{11}H_{11}.$$

Let $1 \le p \le \ell$. Take $1 \le r \le \ell$ with $r \ne p$ (note that $\ell \ge 2$). Consider the real part of the (p,r)-th entry of (2.19):

(2.20) $$\sum_{k=\ell+1}^m \phi_k^p h_r^k - \sum_{k=\ell+1}^m \psi_k^p g_r^k - \sum_{k=\ell+1}^m \phi_k^r h_k^p - \sum_{k=\ell+1}^m \psi_k^r g_k^p = \text{smooth}.$$

Recall that $\phi_k^p(\bar{x},\cdot)$, $\psi_k^p(\bar{x},\cdot)$, $\phi_k^r(\bar{x},\cdot)$, $\psi_k^r(\bar{x},\cdot)$, $\ell+1 \le k \le m$ are linearly independent. Let $q \ge \ell+1$ and solve the equations:

$$\phi_k^p(x,\xi) = 0, \quad \psi_k^p(x,\xi) = 0, \quad \ell+1 \le k \le m$$

$$\phi_k^r(x,\xi) = 0, \quad \ell+1 \le k \le m,\ k \ne q$$

$$\psi_k^r(x,\xi) = 0, \quad \ell+1 \le k \le m.$$

Then we find $\xi_b = (\xi_{i_1},\dots,\xi_{i_{N'}})$, $N' = 4(m-\ell)-1$ so that ξ_b are linear combinations of the other $\xi_a = (\xi_{j_1},\dots,\xi_{j_{M'}})$ with coefficients which are smooth in x near $\bar{x}$. Inserting these ξ_b into (2.20) we have

$$\phi_q^r(x,\xi_a)h_q^p(x) = \text{smooth}.$$

Since $\phi_q^r(x,\xi_a) \not\equiv 0$ we conclude that $h_q^p(x)$ is smooth. The same argument shows that $g_q^p(x)$ is smooth. Since $1 \le p \le \ell$, $q \ge \ell+1$ are arbitrary we conclude that $H_{12}(x)$ is smooth.

Finally we study $H_{22}(x)$. From (2.5) we have

$$L_{12}H_{22} = H_{11}L_{21}^* + H_{12}L_{22}^* - L_{11}H_{12}.$$

Since $H_{11}(x)$ and $H_{12}(x)$ are smooth and $\phi_j^i(\bar{x},\cdot)$, $\psi_j^i(\bar{x},\cdot)$, $1 \le i \le \ell$, $j \ge \ell+1$ are linearly independent, the same argument proves that $H_{22}(x)$ is smooth and hence the result. $\square$

3 Proof of Theorem 1.1

Let $|\check{K} \cap \{\text{diag}\}| = r \ge 1$. Considering $P_\sigma^{-1}L(x,\xi)P_\sigma$ with a suitable permutation matrix P_σ, we may assume that

$$\check{K} \cap \{\text{diag}\} = \{(1,1),(2,2),\dots,(r,r)\}.$$

Let us set $I = \{r+1,\dots,m\}$. Recall that $L(\bar{x},\cdot)$ is hermitian and then

$$(q,p) \in \check{K} \quad \text{if} \quad (p,q) \in \check{K}.$$

A key to the proof of Theorem 1.1 is

LEMMA 3.1. *Assume that*

$$d(L(\bar{x},\cdot)) \geq m^2 - m + 2$$

and $\check{K} \cap \{\mathrm{diag}\} = \{(1,1),\dots,(r,r)\}$. *Then there exists* $\{j_1, j_2, \dots, j_q\} \subset I$ *such that* $\check{K}$ *is confined on* $\{j_1,\dots,j_q\}$ *and*

$$\sum_{i=1}^{q} n(j_i) \leq 2(q-1). \tag{3.1}$$

PROOF. Let us set $I_1 = I$. If there is no (i,j), $i \in I_1$, $j \leq r$ belonging to $\check{K}$ then $\check{K}$ is confined on I_1 and

$$\sum_{j\in I_1} n(j) \leq 2(m-2) - r - r = 2(|I_1| - 2).$$

Hence I_1 is a desired index set. If not there exists $(i,j) \in \check{K}$ with $i \in I_1$, $j \leq r$. Considering $P_\sigma^{-1}L(x,\xi)P_\sigma$ with a suitable permutation σ on I_1 we may assume that $(r+1,j) \in \check{K}$ with $j \leq r$. With $I_2 = \{r+2,\dots,m\}$ note that

$$\sum_{j\in I_2} n(j) \leq 2(m-2) - r - (r+2) = 2(|I_2| - 2) \tag{3.2}$$

because we have $\sum_{j\in\{1,\dots,r,r+1\}} n(j) \geq r+2$. If no (i,j), $i \in I_2$, $j \leq r+1$ belongs to $\check{K}$ then $\check{K}$ is confined on I_2 and hence I_2 is a desired index set thanks to (3.2). Otherwise considering $P_\sigma^{-1}L(x,\xi)P_\sigma$ with a suitable permutation σ on I_2, which is identity on $\{1,\dots,r,r+1\}$, we may assume that $(r+2,j) \in \check{K}$ with $j \leq r+1$. Note again with $I_3 = \{r+3,\dots,m\}$ that

$$\sum_{j\in I_3} n(j) \leq 2(|I_3| - 2).$$

Repeating this argument either we find a desired index set or arrive at the case

$$\sum_{j\in\{m-k,\dots,m\}} n(j) = 0$$

with some $k \geq 2$. In this case $\check{K}$ is confined on $\{m-k,\dots,m\}$. This ends the proof. $\square$

PROOF OF THEOREM 1.1. By the assumption for any x there is a $S(x)$ such that

$$S(x)^{-1}L(x,\xi)S(x)$$

is hermitian for any ξ. Taking $S(\bar{x})^{-1}L(x,\xi)S(\bar{x})$ instead of $L(x,\xi)$ we may assume that $L(\bar{x},\xi)$ is hermitian for every ξ. Let us set

$$H(x) = S(x)S(x)^*$$

which is positive definite hermitian matrix and satisfies

$$L(x,\xi)H(x) = H(x)L(x,\xi)^*.$$

If $|\check{K} \cap \{\mathrm{diag}\}| = 0$ then we can apply Lemma 2.1 to conclude that $H(x)/h_m^m(x)$ is smooth near $\bar{x}$ since $m^2 - m + 2 \geq m^2 - m + 1$. Let $|\check{K} \cap \{\mathrm{diag}\}| \geq 1$. Then one can apply Lemma 3.1 and Lemma 2.2 to conclude that $\tilde{H}(x) = H(x)/h_p^p(x)$ is smooth near $\bar{x}$ with some p. Then $T(x) = \tilde{H}(x)^{1/2}$ is a desired one. $\square$

4 Case $m = 3$ in Theorem 1.2

In this section we give a proof of Theorem 1.2 for the case $m = 3$. From the assumptions it follows that for every x near $\bar{x}$ there is a positive definite hermitian matrix $H(x)$ such that

$$L(x,\xi)H(x) = H(x)L(x,\xi)^*. \tag{4.1}$$

We assume

$$d(L(\bar{x},\cdot)) \geq m^2 - 2 = 7.$$

If $|K \cap \{\mathrm{diag}\}| = 0$, since $m^2 - 2 = m^2 - m + 1 = 7$, one can apply Lemma 2.1 to conclude the assertion. Let us turn to the case $|K \cap \{\mathrm{diag}\}| = 1$. Considering $P_\sigma^{-1}LP_\sigma$ with a suitable permutation matrix P_σ we may assume that either $K = \{(1,1),(2,1)\}$ or $K = \{(1,1),(3,2)\}$. If the latter case occurs then $\check{K}$ is confined on $\{2,3\}$ and

$$\sum_{j=1}^{2} n(p_j) = 2 \leq 2(2-1)$$

so that one can apply Lemma 2.2 to conclude Theorem 1.1. It remains the former case. Considering $H(x)/h_1^1(x)$ instead of $H(x)$ we may assume $h_1^1(x) = 1$ in (4.1). Let us set

$$\hat{h} = {}^t(h_2^2, h_3^3, h_1^2, g_1^2, h_1^3, g_1^3, h_2^3, g_2^3).$$

Taking the real and imaginary part of the (2,1)-th, (3,1)-th and (3,2)-th entry of (4.1) in this order we get

$$G(x,\xi)\hat{h}(x) = F(x,\xi) \tag{4.2}$$

where $F(x,\xi)$ is a linear function in ξ with coefficients which are smooth in x. At $x = \bar{x}$, $G(x,\xi)$ has the form

$$\begin{bmatrix} -\phi_2^1 & 0 & \phi_2^2 - \phi_1^1 & -\psi_2^2 - \psi_1^1 & \phi_3^2 & -\psi_3^2 & -\phi_3^1 & -\psi_3^1 \\ \psi_2^1 & 0 & \psi_2^2 + \psi_1^1 & \phi_2^2 - \phi_1^1 & \psi_3^2 & \phi_3^2 & \psi_3^1 & -\phi_3^1 \\ 0 & -\phi_3^1 & \phi_2^3 & -\psi_2^3 & \phi_3^3 - \phi_1^1 & -\psi_3^3 - \psi_1^1 & -\phi_2^1 & -\psi_2^1 \\ 0 & \psi_3^1 & \psi_2^3 & \phi_2^3 & \psi_3^3 + \psi_1^1 & \phi_3^3 - \phi_1^1 & \psi_2^1 & -\phi_2^1 \\ \phi_2^3 & -\phi_3^2 & \phi_1^3 & \psi_1^3 & -\phi_1^2 & -\psi_1^2 & \phi_3^3 - \phi_2^2 & -\psi_3^3 - \psi_2^2 \\ \psi_2^3 & \psi_3^2 & \psi_1^3 & -\phi_1^3 & \psi_1^2 & -\phi_1^2 & \psi_3^3 + \psi_2^2 & \phi_3^3 - \phi_2^2 \end{bmatrix}.$$

From the assumption $\phi_1^2(\bar{x},\cdot)$ or $\psi_1^2(\bar{x},\cdot)$ is a linear combination of the other $\phi_j^i(\bar{x},\cdot)$, $\psi_j^i(\bar{x},\cdot)$, $i > j$. Since the argument is similar we may assume that $\phi_1^2(\bar{x},\cdot)$ is a linear combination of the others. Take ξ so that

$$\psi_1^2(\bar{x},\xi) = 0, \quad \phi_1^3(\bar{x},\xi) = 0, \quad \psi_1^3(\bar{x},\xi) = 0,$$

$$\phi_2^3(\bar{x},\xi) = 0, \quad \psi_2^3(\bar{x},\xi) = 0, \quad \phi_3^3(\bar{x},\xi) - \phi_2^2(\bar{x},\xi) = 1.$$

Since $\phi_3^3(\bar{x},\xi) - \phi_1^1(\bar{x},\xi) \neq 0$ or $\phi_2^2(\bar{x},\xi) - \phi_1^1(\bar{x},\xi) \neq 0$, without restrictions we may assume that

$$\phi_3^3(\bar{x},\xi) - \phi_1^1(\bar{x},\xi) = a \neq 0.$$

We pick up the last four equations of (4.2), then the coefficient matrix at $x = \bar{x}$ is:

$$\left[\begin{array}{c|cccc} & a & 0 & 0 & 0 \\ O & 0 & a & 0 & 0 \\ & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \end{array}\right] = [O \vdots E]. \tag{4.3}$$

Choosing ξ so that

$$\psi_1^2(\bar{x},\xi) = 1, \quad \phi_1^3(\bar{x},\xi) = 1, \quad \psi_1^3(\bar{x},\xi) = 0, \quad \phi_2^3(\bar{x},\xi) = 0, \quad \psi_2^3(\bar{x},\xi) = 0$$

and we pick up the second, third, fifth and sixth equations in (4.2). Thus we have 8 equations which yield

$$\hat{G}(x)\hat{h}(x) = \hat{f}(x)$$

where

$$\hat{G}(\bar{x}) = \begin{bmatrix} G_1(\bar{x}) & \vdots & * \\ \cdots & \cdots & \cdots \\ O & \vdots & E \end{bmatrix}, \quad G_1 = \begin{bmatrix} 1 & & & * \\ & -1 & & \\ & & 1 & \\ O & & & -1 \end{bmatrix}$$

is non singular and $\hat{f}(x)$ is smooth. Then we can conclude that $\hat{h}(x)$ is smooth near $\bar{x}$.

We finally study the case $|K \cap \{\mathrm{diag}\}| = 2$. As before without restrictions we may assume that $K = \{(2,2),(3,3)\}$. We first assume that there is ξ such that $\phi_j^i(\bar{x},\xi) = 0,\ i > j,\ \psi_j^i(x,\xi) = 0,\ i > j$ and two of

$$\phi_i^i(\bar{x},\xi) - \phi_j^j(\bar{x},\xi), \quad i > j \tag{4.4}$$

are different from zero. In this case the same reasoning as before shows the assertion. We then treat the case that no two of (4.4) are different from zero for any ξ with $\phi_j^i(\bar{x},\xi) = 0,\ \psi_j^i(\bar{x},\xi) = 0,\ i > j$. This implies that $\phi_i^i(\bar{x},\cdot) - \phi_j^j(\bar{x},\cdot)$, $i > j$ are linear combinations of $\{\phi_j^i(\bar{x},\cdot), \psi_j^i(\bar{x},\cdot), i > j\}$. Take ξ so that

$$\phi_i^3(\bar{x},\xi) = 0,\ \psi_i^3(\bar{x},\xi) = 0, \quad i = 1,2$$

and let us denote by G_2 the submatrix consisting of the last 4 columns and rows of G. With $x = -\phi_1^2(\bar{x},\xi)$, $y = \psi_1^2(\bar{x},\xi)$, $\alpha = \phi_3^3(\bar{x},\xi) - \phi_1^1(\bar{x},\xi)$, $\beta = \phi_3^3(\bar{x},\xi) - \phi_2^2(\bar{x},\xi)$ one has

$$|G_2(\bar{x},\xi)| = (x^2 + y^2 - \alpha\beta)^2, \quad G_2 = \begin{bmatrix} \alpha & 0 & x & y \\ 0 & \alpha & -y & -x \\ x & -y & \beta & 0 \\ y & x & 0 & y \end{bmatrix}.$$

Since α, β are real linear forms in x, y then there is a ξ such that $|G_2(\bar{x},\xi)| \neq 0$. The rest of the proof is a repetition of the preceding arguments.

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Department of Mathematics
Graduate School of Science
Osaka University, Machikaneyama 1-16
Toyonaka Osaka 560-0043, Japan

Université Pierre et Marie Curie
Mathématiques, BC 172
4, Place Jussieu 75252 Paris, France