TSUKUBA J. MATH.
Vol. 30 No. 1 (2006), 81–102

SPACE-LIKE ISOTHERMIC SURFACES AND GRASSMANNIAN SYSTEMS

By

Martha P. DUSSAN*

Abstract. We show that space-like isothermic surfaces in the pseudo-riemannian space $\mathbf{R}^{n-j,j}$ are associated to $O(n-j+1,j+1)/O(n-j,j)\times O(1,1)$-system, and that the action of a rational map with two simple poles on the space of local solutions of $O(n-j+1,j+1)/O(n-j,j)\times O(1,1)$-system correspond to Ribaucour and Darboux transformations to space-like surfaces in $\mathbf{R}^{n-j,j}$.

1. Introduction

It is known that there is a connection between the theory of submanifolds and the theory of solitons. Some examples are the well-known local correspondence between pseudospherical surface and the solutions of the Sine-Gordon equation $q_{xt}=\sin q$, and the recent reformulation of the theory of isothermic surfaces in $\mathbf{R}^3$ within the modern theory of completely integrable (soliton) systems, given in [4]. A key point to study this connection is the existence of a Lax Pair or a zero curvature representation which may give rise to an action of an infinite dimensional group on the space of local solutions of the equation, called the "dressing action" in the theory of soliton equations.

There are several excellent articles where is study that connection, specially in a recent work of Terng et al. ([9], [2]) is established a relationship between integrable systems and submanifolds geometry. In those articles it is considered a new integrable system the U/K-system and study the geometry associated to the particular $G_{m,n}=O(m+n)/O(m)\times O(n)$ and $G^1_{m,n}=O(m+n,1)/O(m)\times O(n,1)$-system. This study involved to find submanifolds in a certain symmetric space whose Gauss-Codazzi-Ricci equations are given by these systems, as well as

*Author's e-mail: dussan@ime.usp.br
Key Words: U/K-system, Dressing action, Space-like immersion.
AMS(2000): 53A05, 35Q51.
Received September 24, 2004.
Revised October 17, 2005.

the geometric transformations associated to the dressing action of certain simple elements.

The U/K-system is defined by Terng in ([9]) as the following PDE: Let U be a semi-simple Lie group, σ an involution on U and K the fixed point set of σ. Then U/K is a symmetric space. The Lie algebra $\mathscr{K}$ is fixed point set of the differential σ_* of σ at the identity, in others words, it is the $+1$ eigenspace of σ_*. Let $\mathscr{P}$ denote the -1 eigenspace of σ_*. Then we have $\mathscr{U} = \mathscr{K} \oplus \mathscr{P}$ and

$$[\mathscr{K},\mathscr{K}] \subset \mathscr{K}, \quad [\mathscr{K},\mathscr{P}] \subset \mathscr{P}, \quad [\mathscr{P},\mathscr{P}] \subset \mathscr{K}.$$

Let $\mathscr{A}$ be a maximal abelian subalgebra in $\mathscr{P}$, $a_1, a_2, \dots, a_n$ a basis for $\mathscr{A}$ and $\mathscr{A}^{\perp}$ the orthogonal complement of $\mathscr{A}$ in the algebra $\mathscr{U}$ with respect to the Killing form $\langle\,,\rangle$. The U/K-system is the following first order non-linear PDE for $v : \mathbf{R}^n \to \mathscr{P} \cap \mathscr{A}^{\perp}$

$$[a_i, v_{x_j}] - [a_j, v_{x_i}] = [[a_i, v], [a_j, v]], \quad 1 \le i \ne j \le n, \tag{1}$$

where $v_{x_j} = \frac{\partial v}{\partial x_j}$. It is not difficult to show that v is a solution of the U/K-system if and only if the connection 1-form

$$\theta_\lambda = \sum (a_i\lambda + [a_i, v])\, dx_i. \tag{2}$$

is flat for all $\lambda \in \mathbf{C}$, if and only if there exists an application E such that $E^{-1}\,dE = \theta_\lambda$. If θ_λ is flat for all $\lambda \in \mathbf{C}$ the $\theta_0 = \sum_i [a_i, v]\, dx_i$ is a $\mathscr{K}$-valued, flat connection and hence there exists $g : \mathbf{R}^n \to K$ such that $g^{-1}\,dg = \theta_0$. Suppose $K = K_1 \times K_2$ so we can write $g = (g_1, g_2) \in K_1 \times K_2$. The two new systems given by the flatness of the gauge transformation $g_1 * \theta_\lambda$ and $g_2 * \theta_\lambda$ are called the U/K-system I (II resp.).

In the study made in ([2]), it was obtained that the submanifolds geometries associated to the $G_{m,n}$ and $G^1_{m,n}$-systems, include submanifolds in space forms with constant sectional curvatures, submanifolds admitting principal curvature coordinates and isothermic surfaces in $\mathbf{R}^n$. Moreover, that the dressing action of simple elements on the space of solutions of these systems correspond to Backlund, Darboux and Ribaucour transformations for submanifolds.

In this note we are interested in to discuss the geometry of surfaces associated to the $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system as well as the geometric transformations corresponding to dressing actions. We show that in this case, the space-like isothermic surfaces in the pseudo-riemannian space $\mathbf{R}^{n-j,j}$ for any signature j are associated to $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system II, and that the Ribaucour and Darboux transformations to space-like surfaces correspond to the action of a rational map with two simples poles. In particular we obtain that the space-like 2-tuples in $\mathbf{R}^{n-j,j}$ of type $O(1,1)$ are in correspondence with the solutions of the $O(n-j+1,j+1)/O(n-j,j)\times O(1,1)$-system II, and that these correspond to an isothermic pair of space-like surfaces in $\mathbf{R}^{n-j,j}$. Some results of this note appeared initially in the research report No. 60 in 2002, see ([7]).

We should recall that the topic of isothermic surfaces has been of increasing interest to geometers because they can be reformulated within the soliton theory ([4]), or can be interpreted as the so-called curved flats in the symmetric space $O(4,1)/O(3)\times O(1,1)$ ([3]), and because of the relation between a 2-tuple in $\mathbf{R}^3$ of type $O(1,1)$ and an isothermic pair ([2]). So motivated by these relations and the general results in $\mathbf{R}^n$ ([1]), this note pay attention to the space-like isothermic surfaces and its relation with integrable systems. Finally, we observe that as in the classic situation, space-like minimal surfaces, space-like surfaces with constant mean curvature and space-like surfaces of revolution in $\mathbf{R}^{2,1}$, provide examples of space-like isothermic surfaces in the Lorenztian space ([8]).

2. The associated geometry

First of all, we will find one maximal abelian subalgebra in the subspace $\mathscr{P}$ for which we obtain elliptic Gauss equations, which is correct for space-like surfaces:

Let $U/K = O(n-j+1,j+1)/O(n-j,j)\times O(1,1)$, $n-j\geq 2$, where

$$O(n-j+1,j+1)=\left\{X\in GL(n+2)\,|\,X^t\begin{pmatrix} I_{n-j,j} & 0\\ 0 & I_{1,1}\end{pmatrix}X=\begin{pmatrix} I_{n-j,j} & 0\\ 0 & I_{1,1}\end{pmatrix}\right\},$$

$$I_{n-j,j}=\begin{pmatrix} I_{n-j} & 0\\ 0 & -I_j\end{pmatrix},\quad I_{1,1}=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}.$$

So, $O(n-j+1,j+1)$ is the Lie group of linear isomorphisms that leaves the following bilinear form on $\mathbf{R}^{n+2}$ invariant:

$$\begin{aligned}\langle x,y\rangle &= x_1y_1+x_2y_2+\cdots+x_{n-j}y_{n-j}-x_{n-j+1}y_{n-j+1}\\ &\quad -\cdots\cdot - x_ny_n + x_{n+1}y_{n+1} - x_{n+2}y_{n+2}.\end{aligned}$$

Let $\mathscr{U}=o(n-j+1,j+1)$ be the Lie algebra of U and $\sigma:\mathscr{U}\to\mathscr{U}$ be an involution defined by $\sigma(X)=I_{n,2}^{-1}XI_{n,2}$. Denote by $\mathscr{K}$, $\mathscr{P}$ the $+1$, -1 eigenspaces of σ respectively, i.e.,

$$\mathscr{K}=\left\{\begin{pmatrix} Y_1 & 0\\ 0 & Y_2\end{pmatrix}\middle| Y_1\in o(n-j,j),\, Y_2\in o(1,1)\right\}=o(n-j,j)\times o(1,1),$$

$$\mathscr{P}=\left\{\begin{pmatrix} 0 & \xi\\ -I_{1,1}\xi^t I_{n-j,j} & 0\end{pmatrix}\middle| \xi\in\mathscr{M}_{n\times 2}\right\}.$$

We define the matrices $a_1, a_2 \in \mathscr{M}_{(n+2)\times(n+2)}$, by

$$a_1=-e_{1,n+1}+e_{n+1,1},\quad a_2=e_{2,n+2}+e_{n+2,2},$$

where e_{ij} is the elementary $(n+2)\times(n+2)$ matrix, whose only non-zero entry is 1 in the ij^{th} place.

It is easy to see that the subalgebra $\mathscr{A}=\langle a_1,a_2\rangle$ is maximal abelian in $\mathscr{P}$. Then using this basis $\{a_1,a_2\}$, the U/K-system (1) for this symmetric space is the following PDE for

$$\xi=\begin{pmatrix} 0 & \xi_1\\ \xi_2 & 0\\ r_{1,1} & r_{1,2}\\ \vdots & \vdots\\ r_{n-2,1} & r_{n-2,2}\end{pmatrix}:\mathbf{R}^2\to\mathscr{M}_{n\times 2},$$

$$\begin{cases}(r_{i,2})_{x_1}=r_{i,1}\xi_1, & i=1,\dots,n-2\\ (r_{i,1})_{x_2}=r_{i,2}\xi_2, & i=1,\dots,n-2\\ (\xi_1)_{x_1}+(\xi_2)_{x_2}=-\sum_{i=1}^{n-j-2}r_{i,1}r_{i,2}+\sum_{i=n-j-1}^{n-2}r_{i,1}r_{i,2}, & \\ (\xi_2)_{x_1}=(\xi_1)_{x_2}. & \end{cases}\tag{3}$$

We now denote the entries of ξ by:

$$\begin{pmatrix} 0 & \xi_1\\ \xi_2 & 0\end{pmatrix}=F\quad\text{and}\quad\begin{pmatrix} r_{1,1} & r_{1,2}\\ \vdots & \vdots\\ r_{n-2,1} & r_{n-2,2}\end{pmatrix}=G.$$

The $O(n-j+1,j+1)/O(n-j,j)\times O(1,1)$-system II is the PDE for $(F,G,B):\mathbf{R}^2\to gl_*(2)\times\mathscr{M}_{(n-2)\times 2}\times O(1,1)$, where $gl_*(2)$ is the set of matrices 2×2 with diagonal elements zero,

$$\begin{cases}(r_{i,2})_{x_1}=r_{i,1}\xi_1, & i=1,\dots,n-2\\ (r_{i,1})_{x_2}=r_{i,2}\xi_2, & i=1,\dots,n-2\\ (\xi_1)_{x_1}+(\xi_2)_{x_2}=-\sum_{i=1}^{n-j-2}r_{i,1}r_{i,2}+\sum_{i=n-j-1}^{n-2}r_{i,1}r_{i,2} & \\ (b_{ij})_{x_k}=\xi_k b_{ik}, & k\neq j,\end{cases}\tag{4}$$

where the matrix $B = (b_{ij}) \in O(1,1)$. Now we take $B = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix}$, and use the fact that

$$B^{-1}\,dB = \begin{pmatrix} 0 & \xi_1\,dx_1 + \xi_2\,dx_2 \\ \xi_1\,dx_1 + \xi_2\,dx_2 & 0 \end{pmatrix},$$

to have $\xi_1 = u_{x_1}$, $\xi_2 = u_{x_2}$ and that $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system II becomes in the set of partial differential equations for $(u, r_{1,1}, r_{1,2}, \dots, r_{n-2,1}, r_{n-2,2})$:

$$\begin{cases} (r_{i,2})_{x_1} = r_{i,1}u_{x_1}, & i = 1, \dots, n-2 \\ (r_{i,1})_{x_2} = r_{i,2}u_{x_2}, & i = 1, \dots, n-2 \\ (u_{x_1})_{x_1} + (u_{x_2})_{x_2} = -\sum_{i=1}^{n-j-2} r_{i,1}r_{i,2} + \sum_{i=n-j-1}^{n-2} r_{i,1}r_{i,2}. \end{cases} \tag{5}$$

Next we identify the geometries of space-like surfaces corresponding to the $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system II. We start by defining the space-like 2-tuples in the pseudo-riemannian space $\mathbf{R}^{n-j,j}$ of type $O(1,1)$:

DEFINITION 2.1. *Let $\mathcal{O}$ be a domain in $\mathbf{R}^2$ and $X_i : \mathcal{O} \to \mathbf{R}^{n-j,j}$ an immersion with flat and non-degenerate normal bundle for $i = 1, 2$. (X_1, X_2) is called a space-like 2-tuple in $\mathbf{R}^{n-j,j}$ of type $O(1,1)$ if:*

(i) *The normal plane of $X_1(x)$ is parallel to the normal plane of $X_2(x)$ for any $x \in \mathcal{O}$,*

(ii) *there exists a common parallel normal frame $\{e_3, \dots, e_n\}$, where $\{e_\alpha\}_3^{n-j}$ and $\{e_\alpha\}_{n-j+1}^{n}$ are space-like and time-like vectors resp.*

(iii) *$x \in \mathcal{O}$ is a hyperbolic line of curvature coordinate system with respect to $\{e_3, \dots, e_n\}$ for each X_k such that the fundamental forms of X_k are:*

$$I_k = b_{k1}^2\,dx_1^2 + b_{k2}^2\,dx_2^2,$$

$$II_k = \varepsilon_k \sum_{i=1}^{n-2} (b_{k1}g_{i1}\,dx_1^2 + b_{k2}g_{i2}\,dx_2^2)e_{i+2}, \quad \varepsilon_1 = -\varepsilon_2 = 1, \tag{6}$$

for some $O(1,1)$-valued map $B = (b_{ij})$ and a $\mathcal{M}_{(n-2)\times 2}$-valued map $G = (g_{ij})$.

Our first result, which gives us the relationship between the space-like 2-tuples and the solutions of the $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system II (5), is the following:

THEOREM 2.1. *Suppose $(u, r_{1,1}, r_{1,2}, \dots, r_{n-2,1}, r_{n-2,2})$ is solution of (5) and F, B are given by*

$$F = \begin{pmatrix} 0 & \xi_1 \\ \xi_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & u_{x_1} \\ u_{x_2} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix}.$$

Then: (*a*)

$$\omega = \begin{pmatrix} 0 & -\xi_1\, dx_2 + \xi_2\, dx_1 & r_{1,1}\, dx_1 & \cdots & r_{n-j-2,1}\, dx_1 & -r_{n-j-1,1}\, dx_1 & \cdots & -r_{n-2,1}\, dx_1 \\ \xi_1\, dx_2 - \xi_2\, dx_1 & 0 & r_{1,2}\, dx_2 & \cdots & r_{n-j-2,2}\, dx_2 & -r_{n-j-1,2}\, dx_2 & \cdots & -r_{n-2,2}\, dx_2 \\ -r_{1,1}\, dx_1 & -r_{1,2}\, dx_2 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 & \cdots & \vdots & \vdots \\ -r_{n-2,1}\, dx_1 & -r_{n-2,2}\, dx_2 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}$$

is a flat $o(n-j,j)$*-valued connection* 1*-form. Hence there exists* $A : \mathbf{R}^2 \to O(n-j,j)$ *such that*

$$A^{-1}\, dA = \omega, \tag{8}$$

where ω *is given by* (7).

(*b*) $A\begin{pmatrix} -dx_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & dx_2 & 0 & \cdots & 0 & 0 \end{pmatrix}^t B^{-1}$ *is exact. So there exists a map* $X : \mathbf{R}^2 \to \mathscr{M}_{n\times 2}$ *such that*

$$dX = A\begin{pmatrix} -dx_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & dx_2 & 0 & \cdots & 0 & 0 \end{pmatrix}^t B^{-1}. \tag{9}$$

(*c*) *Suppose that all the entries of* B *are non-zero. Let* $X_j : \mathbf{R}^2 \to \mathbf{R}^{n-j,j}$ *denote the* j*-th column of* X *(solution of* (9)*) and* e_i *denotes the* i*-th column of* A*. Then* (X_1, X_2) *is a space-like* 2*-tuple in* $\mathbf{R}^{n-j,j}$ *of type* $O(1,1)$*. In fact,*

(1) e_1, e_2 *are space-like tangent vectors to* X_1 *and* X_2*, i.e., the tangent planes of* X_1, X_2 *are parallel.*

(2) $\{e_3, \ldots, e_n\}$ *is a parallel normal frame for* X_1 *and* X_2*, with* $\{e_3, \ldots, e_{n-j}\}$ *and* $\{e_{n-j+1}, \ldots, e_n\}$ *being resp. space-like and time-like vectors.*

(3) *the two fundamental forms for the immersions* X_k *are*:

$$\begin{cases} I_1 = \cosh^2 u\, dx_1^2 + \sinh^2 u\, dx_2^2 \\ II_1 = \sum_{i=1}^{n-2} (r_{i,1} \cosh u\, dx_1^2 + r_{i,2} \sinh u\, dx_2^2) e_{i+2} \\ I_2 = \sinh^2 u\, dx_1^2 + \cosh^2 u\, dx_2^2 \\ II_2 = -\sum_{i=1}^{n-2} (r_{i,1} \sinh u\, dx_1^2 + r_{i,2} \cosh u\, dx_2^2) e_{i+2}. \end{cases}$$

PROOF OF THEOREM 2.1. The proof follows from an argument similar to those for Theorem 6.8 or 7.4 in [2]. ■

REMARK 2.1. *We observe that, taking a generic* $b = (b_{ij}) \in O(1,1)$*, Theorem* (2.1) *shows that* I_j, II_j *are given by* (6) *and* dX_j *are*:

$$dX_1 = -(b_{11}\,dx_1 e_1 + b_{12}\,dx_2 e_2) \quad dX_2 = (b_{21}\,dx_1 e_1 + b_{22}\,dx_2 e_2).$$

We also note that Theorem (2.1) *can be stated for a generic* (F, G, B) *solution of the system* (4).

Now we have the converse theorem.

THEOREM 2.2. *Let* (X_1, X_2) *be a space-like 2-tuple in* $\mathbf{R}^{n-j,j}$ *of type* $O(1,1)$, $\{e_3, \dots, e_n\}$ *a common parallel normal frame and* (x_1, x_2) *a common hyperbolic line of curvature coordinates for* X_1 *and* X_2, *such that the two fundamental forms* I_k, II_k *for* X_k *are given by* (6). *Set* $f_{ij} = \frac{(b_{1j})_{x_i}}{b_{1i}}$ *if* $i \neq j$, $f_{ii} = 0$ *and* $F = (f_{ij})_{2\times 2}$. *If all entries of* G *are non-zero then* (F, G, B) *is a solution of* (4).

PROOF. From the definition of space-like 2-tuples in $\mathbf{R}^{n-j,j}$, we have

$$\omega_1^{(k)} = -\varepsilon_k b_{k1}\,dx_1, \quad \omega_2^{(k)} = -\varepsilon_k b_{k2}\,dx_2$$

is a dual 1-frame for X_k and $\omega_{1\alpha}^{(k)} = \sigma_\alpha r_{\alpha-2,1}\,dx_1$, $\omega_{2\alpha}^{(k)} = \sigma_\alpha r_{\alpha-2,2}\,dx_2$ for each X_k, where $\sigma_\alpha = 1$ if $\alpha = 3, \dots, n-j$ and $\sigma_\alpha = -1$ if $\alpha = n-j+1, \dots, n$. We observe that $\omega_{i\alpha}^{(k)}$, $i = 1, 2$, $\alpha = 3, \dots, n$ are independent of k. We have the Levi-Civita connection 1-form for the metric I_k is:

$$\omega_{12}^{(k)} = \frac{(b_{k1})_{x_2}}{b_{k2}}\,dx_1 - \frac{(b_{k2})_{x_1}}{b_{k1}}\,dx_2 = f_{21}^{(k)}\,dx_1 - f_{12}^{(k)}\,dx_2,$$

where $f_{21}^{(k)} = \frac{(b_{k1})_{x_2}}{b_{k2}}$, $f_{12}^{(k)} = \frac{(b_{k2})_{x_1}}{b_{k1}}$. Since $d\omega_{1\alpha}^{(k)} = -\omega_{12}^{(k)} \wedge \omega_{2\alpha}^{(k)}$ and $d\omega_{2\alpha}^{(k)} = -\omega_{21}^{(k)} \wedge \omega_{1\alpha}^{(k)}$ (see appendix), we get

$$\frac{(b_{k2})_{x_1}}{b_{k1}} = \frac{(r_{\alpha-2,2})_{x_1}}{r_{\alpha-2,1}}, \quad \frac{(b_{k1})_{x_2}}{b_{k2}} = \frac{(r_{\alpha-2,1})_{x_2}}{r_{\alpha-2,2}},$$

so $f_{21}^{(k)}$, $f_{12}^{(k)}$ are independent of k. Hence $\omega_{12}^{(k)} = \omega_{12}^{(1)} = \frac{(b_{11})_{x_2}}{b_{12}}\,dx_1 - \frac{(b_{12})_{x_1}}{b_{11}}\,dx_2 = f_{21}\,dx_1 - f_{12}\,dx_2 = \xi_2\,dx_1 - \xi_1\,dx_2$. So the structure equations and the Gauss-Codazzi equations for X_1, X_2 imply that (F, G, B) is a solution of system (4). ∎

THEOREM 2.3. *The* $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$*-system II* (4) *is the Gauss-Codazzi equations for a space-like surface in* $\mathbf{R}^{n-j,j}$ *such that*:

$$I = \sinh^2 u\,dx_1^2 + \cosh^2 u\,dx_2^2, \quad II = -\sum_{i=1}^{n-2}(r_{i,1}\sinh u\,dx_1^2 + r_{i,2}\cosh u\,dx_2^2)e_{i+2}.$$

PROOF. From the form of I and II, we have:

$$\omega_1 = \sinh u\, dx_1, \quad \omega_2 = \cosh u\, dx_2, \quad \omega_{1\alpha} = \sigma_\alpha r_{\alpha-2,1}\, dx_1, \quad \omega_{2\alpha} = \sigma_\alpha r_{\alpha-2,2}\, dx_2,$$

where $\sigma_\alpha = 1$ if $\alpha = 3, \dots, n-j$ and $\sigma_\alpha = -1$ if $\alpha = n-j+1, \dots, n$. Now use the structure equation to obtain: $\omega_{12} = u_{x_2}\, dx_1 - u_{x_1}\, dx_2$. Using the Gauss-Codazzi equation (see appendix), we obtain that these are the following system for $(u, r_{1,1}, r_{1,2}, \dots, r_{n-2,1}, r_{n-2,2})$:

$$\begin{cases} (r_{i,2})_{x_1} = r_{i,1} u_{x_1}, & i = 1, \dots, n-2 \\ (r_{i,1})_{x_2} = r_{i,2} u_{x_2}, & i = 1, \dots, n-2 \\ u_{x_1x_1} + u_{x_2x_2} = -\sum_{i=1}^{n-j-2} r_{i,1} r_{i,2} + \sum_{i=n-j-1}^{n-2} r_{i,1} r_{i,2}. \end{cases} \tag{10}$$

Hence if we put

$$B = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix}, \quad F = \begin{pmatrix} 0 & u_{x_1} \\ u_{x_2} & 0 \end{pmatrix}, \quad G = \begin{pmatrix} r_{1,1} & r_{1,2} \\ \vdots & \vdots \\ r_{n-2,1} & r_{n-2,2} \end{pmatrix}, \tag{11}$$

we see (F, G, B) is solution of the system (4). Conversely, if (F, G, B) is solution of the system (4), and we assume B is as in (11), then from the fourth equation of system (4) we get $\xi_1 = u_{x_1}$, $\xi_2 = u_{x_2}$, i.e., (F, G, B) is of the form (11). Finally writing the $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system II for this (F, G, B), in terms of u and r_{ij} we get equation (10). ■

Combining Theorems 2.1 and 2.3 we get

THEOREM 2.4. *Let $\mathcal{O}$ be a domain of $\mathbf{R}^2$, and $X_2 : \mathcal{O} \to \mathbf{R}^{n-j,j}$ an immersion with flat normal bundle and $(x, y) \in \mathcal{O}$ a hyperbolic line of curvature coordinates system with respect to a parallel normal frame $\{e_3, \dots, e_n\}$. Then there exists an immersion X_1, unique up to translation, such that (X_1, X_2) is a space-like 2-tuple in $\mathbf{R}^{n-j,j}$ of type $O(1,1)$. Moreover, the fundamental forms of X_1, X_2 are respectively*:

$$\begin{cases} I_1 = \cosh^2 u\, dx_1^2 + \sinh^2 u\, dx_2^2 \\ II_1 = \sum_{i=1}^{n-2} (r_{i,1} \cosh u\, dx_1^2 + r_{i,2} \sinh u\, dx_2^2) e_{i+2} \\ I_2 = \sinh^2 u\, dx_1^2 + \cosh^2 u\, dx_2^2 \\ II_2 = -\sum_{i=1}^{n-2} (r_{i,1} \sinh u\, dx_1^2 + r_{i,2} \cosh u\, dx_2^2) e_{i+2}. \end{cases} \tag{12}$$

EXAMPLE 2.1. *Recall that given a space-like surface in $\mathbf{R}^{2,1}$ with curvature -1 and free of umbilic points, there exists a local coordinates system x_1, x_2 such that the two fundamental forms are* ([11]):

$$I = \cosh^2 u\, dx_1^2 + \sinh^2 u\, dx_2^2, \quad II = \cosh u \sinh u (dx_1^2 + dx_2^2).$$

With respect to this coordinates system, the Gauss-Codazzi equation of the surface is written in the following form (Elliptic sinh-Gordon equation):

$$u_{x_1x_1} + u_{x_2x_2} = \sinh u \cosh u.$$

This implies that $(u, \sinh u, \cosh u)$ *is a solution of* $O(3,2)/O(2,1) \times O(1,1)$*-system II (5). Let* $X(x_1, x_2)$ *denote the immersion of* M *and* e_3 *the unit normal of* M*. Then* (X, e_3) *is a space-like 2-tuple in* $\mathbf{R}^{2,1}$ *of type* $O(1,1)$*, where* e_3 *is a parametrization of an open subset of pseudo-hyperbolic space* $H^2(1) = \{q \in \mathbf{R}^{2,1} \mid \langle q, q\rangle = -1\}$.

Now we are interested in finding the connection between the space-like isothermic surfaces in $\mathbf{R}^{n-j,j}$ and the solutions of the $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system II. We begin by defining space-like isothermic surfaces in $\mathbf{R}^{n-j,j}$ just as in the classic situation of isothermic surfaces in $\mathbf{R}^n$ ([1]).

Definition 2.2 (Space-like isothermic surface). *Let* $\mathcal{O}$ *be a domain in* $\mathbf{R}^2$*. An immersion* $X : \mathcal{O} \to \mathbf{R}^{n-j,j}$ *is called a space-like isothermic surface if it has flat normal bundle and the two fundamental forms are:*

$$I = e^{2u}(dx_1^2 + dx_2^2), \quad II = e^u \sum_{i=1}^{n-2} (g_{i,1}\, dx_1^2 + g_{i,2}\, dx_2^2) e_{i+2},$$

with respect to some parallel normal frame $\{e_\alpha\}$*. Or equivalently* $(x_1, x_2) \in \mathcal{O}$ *is conformal and line of curvature coordinate system for* X*.*

It is not difficult to see that The Gauss-Codazzi-Ricci equation for space-like isothermic surfaces in $\mathbf{R}^{n-j,j}$ is (10).

Our next result establishes the relation between space-like isothermic surfaces and the space-like 2-tuples in $\mathbf{R}^{n-j,j}$ of type $O(1,1)$.

Proposition 2.1. *Suppose that* (X_1, X_2) *is a space-like 2-tuple in* $\mathbf{R}^{n-j,j}$ *of type* $O(1,1)$*. Let* $Z_1 = X_1 - X_2$ *and* $Z_2 = X_1 + X_2$*. Then both* Z_1 *and* Z_2 *are space-like isothermic.*

Proof. Let $(u, r_{1,1}, r_{1,2}, \ldots, r_{n-2,1}, r_{n-2,2})$ be a solution of (10) associated to (X_1, X_2). Set $X = (X_1, X_2)$. Then by (9), we have:

$$\begin{cases} dX_1 = -(\cosh u\, dx_1 e_1 + \sinh u\, dx_2 e_2), \\ dX_2 = \sinh u\, dx_1 e_1 + \cosh u\, dx_2 e_2. \end{cases}$$

We note that $F = \begin{pmatrix} 0 & u_{x_1} \\ u_{x_2} & 0 \end{pmatrix}$, $w_{12} = u_{x_2}\, dx_1 - u_{x_1}\, dx_2$ and $w_{i\alpha} = \sigma_\alpha r_{\alpha-2,i}\, dx_i$ where $\sigma_\alpha = 1$, $\alpha = 3, \dots, n-j$ and $\sigma_\alpha = -1$, $\alpha = n-j+1, \dots, n$. Now using (8), we have that for $\alpha = 3, \dots, n$, $de_\alpha = \sigma_\alpha(r_{\alpha-2,1}\, dx_1 e_1 + r_{\alpha-2,2}\, dx_2 e_2)$. We compute that

$$\begin{cases} dZ_1 = dX_1 - dX_2 = -e^u(dx_1 e_1 + dx_2 e_2), \\ dZ_2 = dX_1 + dX_2 = -e^{-u}(dx_1 e_1 - dx_2 e_2). \end{cases}$$

Hence the induced metric and the second fundamental form for Z_1 and Z_2 are, respectively:

$$\begin{cases} I_1 = e^{2u}(dx_1^2 + dx_2^2) \\ II_1 = e^u \sum_{i=1}^{n-2} (r_{i,1}\, dx_1^2 + r_{i,2}\, dx_2^2) e_{i+2} \\ I_2 = e^{-2u}(dx_1^2 + dx_2^2) \\ II_2 = e^{-u} \sum_{i=1}^{n-2} (r_{i,1}\, dx_1^2 - r_{i,2}\, dx_2^2) e_{i+2}. \end{cases} \qquad \blacksquare$$

We observe that the two immersions Z_1 and Z_2 given in proposition above are isothermic dual surfaces, which we called a space-like isothermic pair in $\mathbf{R}^{n-j,j}$.

Now we will study the dressing action of a rational map with two simple poles on the space of solutions of the $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system II.

Let $O(n-j+1, j+1) \otimes \mathbf{C} = O(n-j+1, j+1; \mathbf{C})$ the complexified Lie group. The symmetric space $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$ is determined by two involutions, namely $\tau, \sigma : O(n-j+1, j+1; \mathbf{C}) \to O(n-j+1, j+1; \mathbf{C})$ defined by: $X \mapsto \tau(X) = \bar{X}$ and $X \mapsto \sigma(X) = I_{n,2}^{-1} X I_{n,2}$, resp. Then $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-reality condition is:

$$\begin{cases} \overline{g(\bar{\lambda})} = g(\lambda) \\ I_{n,2}\, g(-\lambda) I_{n,2} = g(\lambda) \\ g(\lambda)^t \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & I_{1,1} \end{pmatrix} g(\lambda) = \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & I_{1,1} \end{pmatrix} \end{cases} \tag{13}$$

for a map $g : \mathbf{C} \to U_{\mathbf{C}} = O(n-j+1, j+1; \mathbf{C})$.

We recall that a **frame** for a solution v of the U/K-system (II) is a trivialization of the corresponding Lax connection θ_λ (θ_λ^{II}) that satisfies the U/K-reality condition.

Let

$G_+ = \{g : \mathbf{C} \to O(n-j+1, j+1; \mathbf{C}) \mid g$ is holomorphic and satisfies the reality condition (13)$\}$

$G_- = \{g : S^2 \to O(n-j+1, j+1; \mathbf{C}) \mid g$ is meromorphic, $g(\infty) = I$ and satisfies the reality condition (13)$\}$.

Now we find certain simple elements in G_- explicitly. Let $W = (w_1, \dots, w_n)^t \in \mathbf{R}^{n-j,j}$, $Z = (z_1, z_2)^t \in \mathbf{R}^{1,1}$ unit vectors and $\mathbf{C}^{n+2}$ be equipped with the bi-linear form:

$$\langle U, V \rangle_2 = \bar{U}^t \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & I_{1,1} \end{pmatrix} V = \bar{u}_1 v_1 + \cdots + \bar{u}_{n-j} v_{n-j} - \bar{u}_{n-j+1} v_{n-j+1}$$

$$- \cdots - \bar{u}_n v_n + \bar{u}_{n+1} v_{n+1} - \bar{u}_{n+2} v_{n+2}.$$

Let π the orthogonal projection of $\mathbf{C}^{n+2}$ onto the span of $\begin{pmatrix} W \\ iZ \end{pmatrix}$ with respect to $\langle\, , \rangle_2$. So

$$\pi = \frac{1}{2} \begin{pmatrix} WW^t & -iWZ^t \\ iZW^t & ZZ^t \end{pmatrix} \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & I_{1,1} \end{pmatrix}. \tag{14}$$

$\bar{\pi}$ is the projection onto $\mathbf{C}\begin{pmatrix} -W \\ iZ \end{pmatrix}$, which is perpendicular to $\begin{pmatrix} W \\ iZ \end{pmatrix}$. So $\bar{\pi}\pi = \pi\bar{\pi} = 0$. Let $s \in \mathbf{R}$, $s \neq 0$, and it defines

$$g_{s,\pi}(\lambda) = \left(\pi + \frac{\lambda - is}{\lambda + is}(I - \pi)\right)\left(\bar{\pi} + \frac{\lambda + is}{\lambda - is}(I - \bar{\pi})\right).$$

So substituting (14) to $g_{s,\pi}$, we get

$$g_{s,\pi}(\lambda) = \frac{1}{\lambda^2 + s^2}\left[\lambda^2 I + s^2 \begin{pmatrix} I - 2WW^t I_{n-j,j} & 0 \\ 0 & I - 2ZZ^t I_{1,1} \end{pmatrix}\right.$$

$$\left. + 2s\lambda \begin{pmatrix} 0 & WZ^t I_{1,1} \\ -ZW^t I_{n-j,j} & 0 \end{pmatrix}\right]. \tag{15}$$

One can see that $g_{s,\pi}(\lambda)$ (15) satisfies the $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-reality condition (13), therefore the element $g_{s,\pi} \in G_-$.

Now we get an explicit construction of the action of $g_{s,\pi}$ on the space of solutions of the $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system.

THEOREM 2.5. *Let $\xi : \mathbf{R}^2 \to \mathscr{M}_{n\times 2}$ be a solution of the $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system* (3), *and $E(x, \lambda)$ a frame of ξ such that $E(x, \lambda)$ is* *holomorphic for* $\lambda \in \mathbf{C}$. *Let* W *and* Z *be unit vectors in* $\mathbf{R}^{n-j,j}$, $\mathbf{R}^{1,1}$ *respectively,* π *the orthogonal projection onto* $\mathbf{C}\begin{pmatrix} W \\ iZ \end{pmatrix}$ *with respect to* $\langle\,,\rangle_2$ *and* $g_{s,\pi}$ *the map defined by* (15). *Let* $\tilde{\pi}(x)$ *denote the orthogonal projection onto* $\mathbf{C}\begin{pmatrix} \tilde{W} \\ i\tilde{Z} \end{pmatrix}(x)$ *with respect to* $\langle\,,\rangle_2$, *where*

$$\begin{pmatrix} \tilde{W} \\ i\tilde{Z} \end{pmatrix}(x) = E(x,-is)^{-1}\begin{pmatrix} W \\ iZ \end{pmatrix}. \tag{16}$$

Let $\hat{W} = \frac{\tilde{W}}{\|\tilde{W}\|_{n-j,j}}$ *and* $\hat{Z} = \frac{\tilde{Z}}{\|\tilde{Z}\|_{1,1}}$, $\tilde{E}(x,\lambda) = g_{s,\pi}(\lambda)E(x,\lambda)g_{s,\tilde{\pi}(x)}(\lambda)^{-1}$,

$$\tilde{\xi} = \xi - 2s(\hat{W}\hat{Z}^t I_{1,1})_*, \tag{17}$$

where $(\vartheta_*)_{ij} = \vartheta_{ij}$ *if* $i \neq j$, *and* $(\vartheta_*)_{ii} = 0$, $1 \leq i \leq 2$. *Let*

$$\tilde{E}^{\#}(x,\lambda) = E(x,\lambda)g_{s,\tilde{\pi}}^{-1}. \tag{18}$$

Then

(a) $\tilde{\xi}$ *is a new solution of system* (3).

(b) $\tilde{E}^{\#}$ *is a frame for* $\tilde{\xi}$.

(c)

$$E(x,0) = \begin{pmatrix} A(x) & 0 \\ 0 & B(x) \end{pmatrix}, \quad \tilde{E}^{\#}(x,0) = \begin{pmatrix} \tilde{A}^{\#}(x) & 0 \\ 0 & \tilde{B}^{\#}(x) \end{pmatrix},$$

for some A, B, $\tilde{A}^{\#}(x)$, $\tilde{B}^{\#}(x)$, *and*

$$\tilde{A}^{\#} = A(I - 2\hat{W}\hat{W}^t I_{n-j,j}), \quad \tilde{B}^{\#} = B(I - 2\hat{Z}\hat{Z}^t I_{1,1}). \tag{19}$$

(d) $(\tilde{W},\tilde{Z})$ *is a solution of the system*:

$$d\begin{pmatrix} \tilde{W} \\ i\tilde{Z} \end{pmatrix}(x) = -\theta_{-is}\begin{pmatrix} \tilde{W} \\ i\tilde{Z} \end{pmatrix}(x)$$

where θ_λ *is the Lax connection.*

For proving this theorem we have the following lemma whose proof is quite similar to the proof of Lemma 9.4 in [2] and which we omit.

LEMMA 2.1. *With the same conditions as in theorem above, we get*

(i) $\tilde{W}(x) \in \mathbf{R}^n$, $\tilde{Z}(x) \in \mathbf{R}^2$.

(ii) $\|\tilde{W}(x)\|_{n-j,j} = \|\tilde{Z}(x)\|_{1,1}$ $\forall x$ *and* $g_{s,\tilde{\pi}}$ *satisfies the* $O(n-j+1,j+1)/O(n-j,j) \times O(1,1)$*-reality conditions* (13), *i.e.* $g_{s,\tilde{\pi}} \in G_-$.

(iii) $\tilde{E}(x,\lambda)$ *is holomorphic in* $\lambda \in \mathbf{C}$.

Proof of Theorem 2.5. The computations give us

$$\tilde{E}^{-1}\,d\tilde{E} = g_{s,\tilde{\pi}}E^{-1}\,dEg_{s,\tilde{\pi}}^{-1} - dg_{s,\tilde{\pi}}g_{s,\tilde{\pi}}^{-1}. \tag{20}$$

But $E^{-1}\,dE = \sum_{i=1}^{2}(a_i\lambda + [a_i, v])\,dx_i$, $\tilde{E}(x,\lambda)$ is holomorphic in $\lambda \in \mathbf{C}$ and $g_{s,\tilde{\pi}}(\lambda)$ is holomorphic at $\lambda = \infty$. So $\tilde{E}^{-1}\,d\tilde{E}$ must be of the form: $\sum_{i=1}^{2}(a_i\lambda + \mu_i)\,dx_i$. Now we write

$$g_{s,\tilde{\pi}}(\lambda) = I + \frac{m_1(x)}{\lambda} + \frac{m_2(x)}{\lambda^2} + \cdots$$

then $m_1(x) = 2s\begin{pmatrix} 0 & \hat{W}\hat{Z}^t I_{1,1} \\ -\hat{Z}\hat{W}^t I_{n-j,j} & 0 \end{pmatrix}$ and $m_1(x) \in \mathscr{P}$. In fact, one sees that the element

$$\begin{aligned} g_{s,\tilde{\pi}(x)}(\lambda) &= \left(\tilde{\pi} + \frac{\lambda - is}{\lambda + is}(I - \tilde{\pi})\right)\left(\bar{\tilde{\pi}} + \frac{\lambda + is}{\lambda - is}(I - \bar{\tilde{\pi}})\right) \\ &= \frac{\lambda + is}{\lambda - is}\tilde{\pi} + \frac{\lambda - is}{\lambda + is}\bar{\tilde{\pi}} + (I - \tilde{\pi})(I - \bar{\tilde{\pi}}), \end{aligned} \tag{21}$$

and now we make the expansion of terms $\frac{\lambda+is}{\lambda-is}$ and $\frac{\lambda-is}{\lambda+is}$ around $\lambda = \infty$. Substituting this in (21) we have

$$\begin{aligned} g_{s,\tilde{\pi}(x)}(\lambda) = {} & \left(\tilde{\pi} + \frac{2is}{\lambda}\tilde{\pi} + 2is\left[\frac{is}{\lambda^2} - \frac{s^2}{\lambda^3} + \cdots\right]\tilde{\pi}\right) + \left(\bar{\tilde{\pi}} - \frac{2is}{\lambda}\bar{\tilde{\pi}} + \frac{2i^2s^2}{\lambda^2}\bar{\tilde{\pi}} + \cdots\right) \\ & + (I - \tilde{\pi})(I - \bar{\tilde{\pi}}). \end{aligned}$$

Hence we obtain that $m_1(x) = 2is(\tilde{\pi} - \bar{\tilde{\pi}})$, i.e., $m_1(x)$ is as we had claimed.

Now multiply (20) by $g_{s,\tilde{\pi}}$ on the right side and equate the constant term of equation which results of that operation, to obtain

$$\mu_i = [a_i, v - m_1] = [a_i, v - p_o(m_1)],$$

where p_o is the projection from $\mathscr{P}$ onto $\mathscr{P} \cap \mathscr{A}^{\perp}$. Therefore $\tilde{v} = v - p_o(m_1)$ is a solution of $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system. More specifically, writing $v = \begin{pmatrix} 0 & \xi \\ -I_{1,1}\xi^t I_{n-j,j} & 0 \end{pmatrix}$ and $\tilde{v} = \begin{pmatrix} 0 & \tilde{\xi} \\ -I_{1,1}\tilde{\xi}^t I_{n-j,j} & 0 \end{pmatrix}$, $\tilde{\xi} = \xi - 2s(\hat{W}\hat{Z}^t I_{1,1})_*$ is a solution of $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system (3).

For item (b) and (c), since $g_{s,\tilde{\pi}} \in O(n-j+1, j+1; \mathbf{C})$, (18) becomes in

$$\tilde{E}^{\#}(x,\lambda) = E(x,\lambda)\begin{pmatrix} I_{n-j,j} & 0 \\ 0 & I_{1,1} \end{pmatrix} \frac{1}{\lambda^2+s^2}$$
$$\times \left[\lambda^2 I + s^2 \begin{pmatrix} I - 2I_{n-j,j}\hat{W}\hat{W}^t & 0 \\ 0 & I - 2I_{1,1}\hat{Z}\hat{Z}^t \end{pmatrix}\right.$$
$$\left. + 2s\lambda \begin{pmatrix} 0 & -I_{n-j,j}\hat{W}\hat{Z}^t \\ I_{1,1}\hat{Z}\hat{W}^t & 0 \end{pmatrix}\right] \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & I_{1,1} \end{pmatrix}. \tag{22}$$

We note that $\tilde{E}^{\#}$ is a frame of $\tilde{\xi}$ and that $\tilde{E}^{\#}(x,.)$ is not in G_+. The reality condition (13) implies that both $E(x,0)$ and $\tilde{E}^{\#}(x,0)$ are in $O(n-j,j) \times O(1,1)$. So, now we write:

$$E(x,0) = \begin{pmatrix} A(x) & 0 \\ 0 & B(x) \end{pmatrix}, \quad \tilde{E}^{\#}(x,0) = \begin{pmatrix} \tilde{A}^{\#}(x) & 0 \\ 0 & \tilde{B}^{\#}(x) \end{pmatrix}.$$

Finally, from (18) we have $\tilde{E}^{\#}(x,0) = E(x,0)g_{s,\tilde{\pi}(x)}(0)^{-1}$, this means

$$\begin{pmatrix} \tilde{A}^{\#} & 0 \\ 0 & \tilde{B}^{\#} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I_{n-j,j} & 0 \\ 0 & I_{1,1} \end{pmatrix}\begin{pmatrix} I - 2I_{n-j,j}\hat{W}\hat{W}^t & 0 \\ 0 & I - 2I_{1,1}\hat{Z}\hat{Z}^t \end{pmatrix}$$
$$\times \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & I_{1,1} \end{pmatrix}$$
$$= \begin{pmatrix} A(I - 2\hat{W}\hat{W}^t I_{n-j,j}) & 0 \\ 0 & B(I - 2\hat{Z}\hat{Z}^t I_{1,1}) \end{pmatrix},$$

from which follows $\tilde{A}^{\#} = A(I - 2\hat{W}\hat{W}^t I_{n-j,j})$, $\tilde{B}^{\#} = B(I - 2\hat{Z}\hat{Z}^t I_{1,1})$.

(d) Follows directly taking the differential of (16). ■

In the next statement we will use the notation $g_{s,\pi}\#\xi$, for denoting the dressing action of $g_{s,\pi}$ on ξ, and $\tilde{g}_-\#\xi$ for the dressing action of $\tilde{g}_-$ on ξ (see ([2])).

COROLLARY 2.1. *Suppose E is a frame of the solution ξ of the $O(n-j+1,j+1)/O(n-j,j) \times O(1,1)$-system* (3) *such that $E(x,\lambda)$ is holomorphic for $\lambda \in \mathbf{C}$.*

(i) *If $E(0,\lambda) = I$, then $\tilde{\xi}$ obtained in Theorem* 2.5, *is $g_{s,\pi}\#\xi$ and $\tilde{E}$ is the frame of $\tilde{\xi}$ with $\tilde{E}(0,\lambda) = I$.*

(ii) *Let $g_+(\lambda) = E(0,\lambda)$ and $\tilde{\xi}$ the new solution of* (3) *obtained in Theorem* 2.5. *Then $g_+ \in G_+$ and $\tilde{\xi} = \tilde{g}_-\#\xi$, where $\tilde{g}_-$ is obtained by factoring $g_{s,\pi}g_+ = \tilde{g}_+\tilde{g}_-$ with $\tilde{g}_\pm \in G_\pm$.*

Now writing $\xi = \begin{pmatrix} F \\ G \end{pmatrix}$, $\tilde{\xi} = \begin{pmatrix} \tilde{F} \\ \tilde{G} \end{pmatrix}$, we rewrite the new solution $\tilde{\xi}$ given by Theorem 2.5 as

$$\begin{pmatrix} \tilde{F} \\ \tilde{G} \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix} - 2s(\hat{W}\hat{Z}^t I_{1,1})_*. \tag{23}$$

So (F, G, B) and $(\tilde{F}, \tilde{G}, \tilde{B}^\#)$ are solutions of the $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system II (4).

We note that if we write $F = (f_{ij})_{2\times 2}$, $G = (r_{ij})_{(n-2)\times 2}$, $\tilde{F} = (\tilde{f}_{ij})_{2\times 2}$, $\tilde{G} = (\tilde{r}_{ij})_{(n-2)\times 2}$, then (23) for $\tilde{\xi}$ is

$$\begin{cases} \tilde{f}_{ij} = f_{ij} - 2s\hat{w}_i\hat{z}_j\varepsilon_j, & i, j = 1, 2 \\ \tilde{r}_{ij} = r_{ij} - 2s\hat{w}_{2+i}\hat{z}_j\varepsilon_j, & i = 1, \dots, n-2,\ j = 1, 2 \end{cases} \tag{24}$$

where $\varepsilon_1 = -\varepsilon_2 = 1$.

Now let $\tilde{E}^\#$ frame of $\tilde{\xi}$, E^{II} of (F, G, B) and $\widetilde{E^\#}^{II}$ of $(\tilde{F}, \tilde{G}, \widetilde{B^\#})$. Then we get

$$\widetilde{E^\#}^{II}(x,\lambda) = \widetilde{E^\#}(x,\lambda)\begin{pmatrix} I_{n-j,j} & 0 \\ 0 & I_{1,1} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & \widetilde{B^\#}^t \end{pmatrix}\begin{pmatrix} I_{n-j,j} & 0 \\ 0 & I_{1,1} \end{pmatrix},$$

now, using (22), we get that the frames E^{II} and $\widetilde{E^\#}^{II}$ are related by

$$\widetilde{E^\#}^{II}(x,\lambda) = E^{II}(x,\lambda)\left[I - \frac{2}{\lambda^2 + s^2}\begin{pmatrix} s^2\hat{W}\hat{W}^t I_{n-j,j} & -s\lambda\hat{W}\hat{Z}^t B^t I_{1,1} \\ -s\lambda B\hat{Z}\hat{W}^t I_{n-j,j} & \lambda^2 B\hat{Z}\hat{Z}^t B^t I_{1,1} \end{pmatrix}\right]. \tag{25}$$

In the next we will use the notation

$$(\tilde{\xi}, \tilde{E}^\#) = g_{s,\pi}\cdot(\xi, E), \quad \tilde{A}^\# = g_{s,\pi}\cdot A, \quad \tilde{B}^\# = g_{s,\pi}\cdot B,$$

$$(\tilde{F}, \tilde{G}, \tilde{B}^\#, \widetilde{E^\#}^{II}) = g_{s,\pi}\cdot(F, G, B, E^{II}),$$

to understand the result obtained after the action of the element $g_{s,\pi}$ over the solutions given.

3. Associated Geometric Transformations

Here we describe the corresponding geometric transformations on surfaces in the pseudo-riemannian space $\mathbf{R}^{n-j,j}$ corresponding to the action of $g_{s,\pi}$, given in (15), on the space of local solution of the $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system II (4).

We start with the definition of Ribaucour transformation given by Dajczer-Tojeiro in [6]. Let $\mathbf{R}_s^{n+p}$ be the standard flat pseudo-riemannian space form of index s, $0 \le s \le p$. For $x \in \mathbf{R}_s^{n+p}$ and $v \in (T\mathbf{R}_s^{n+p})_x$, where let $\gamma_{x,v}(t) = exp(tv)$ denote the geodesic.

DEFINITION 3.1. *Let M^n and $\tilde{M}^n$ be riemannian submanifolds of the pseudo-riemannian space $\mathbf{R}_s^{n+p}$, $0 \leq s \leq p$. A* **sphere congruence** *is a vector bundle isomorphism $P : \mathscr{V}(M) \to \mathscr{V}(\tilde{M})$ that covers a diffeomorphism $l : M \to \tilde{M}$ with the following conditions:*

(1) *If ξ is a parallel normal vector field of M, then $P \circ \xi \circ l^{-1}$ is a parallel normal field of $\tilde{M}$.*

(2) *For any nonnull vector $\xi \in \mathscr{V}_x(M)$, the geodesics $\gamma_{x,\xi}$ and $\gamma_{l(x),P(\xi)}$ intersect at a point that is equidistant from x and $l(x)$ (the distance depends on x).*

DEFINITION 3.2. *A sphere congruence $P : \mathscr{V}(M) \to \mathscr{V}(\tilde{M})$ that covers $l : M \to \tilde{M}$ is called a* **Ribaucour transformation** *if it satisfies the following additional properties:*

(1) *If e is an eigenvector of the shape operator A_ξ of M, then $l_*(e)$ is an eigenvector of the shape operator $A_{P(\xi)}$ of $\tilde{M}$.*

(2) *The geodesics $\gamma_{x,e}$ and $\gamma_{l(x),l_*(e)}$ intersect at a point that is equidistant to x and $l(x)$.*

THEOREM 3.1. *Let $\xi = \begin{pmatrix} F \\ G \end{pmatrix}$ solution of (3), E frame of ξ, $E(x,0) = \begin{pmatrix} A(x) & 0 \\ 0 & B(x) \end{pmatrix}$, (F, G, B) a solution corresponding to $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system II (4), and*

$$(\tilde{F}, \tilde{G}, \tilde{B}^\#, \widetilde{E^{\#II}}) = g_{s,\pi}\cdot(F, G, B, E^{II}), \quad \tilde{A}^\# = g_{s,\pi}\cdot A.$$

Let e_i, $\tilde{e}_i$ denote the i-th column of A and $\tilde{A}^\#$ resp. Then we have

(i)

$$\frac{\partial E}{\partial \lambda}(x,0)E^{-1}(x,0) = \begin{pmatrix} 0 & X \\ -I_{1,1}X^t I_{n-j,j} & 0 \end{pmatrix},$$

$$\frac{\partial \widetilde{E^\#}}{\partial \lambda}(x,0)\widetilde{E^\#}^{-1}(x,0) = \begin{pmatrix} 0 & \tilde{X} \\ -I_{1,1}\tilde{X}^t I_{n-j,j} & 0 \end{pmatrix}$$

for some X and $\tilde{X}$.

(ii) *$X = (X_1, X_2)$ and $\tilde{X} = (\tilde{X}_1, \tilde{X}_2)$ are space-like 2-tuples in $\mathbf{R}^{n-j,j}$ of type $O(1,1)$ such that $\{e_\alpha\}$ and $\{\tilde{e}_\alpha\}$ are resp. parallel normal frames for X_j and $\tilde{X}_j$ for $j = 1, 2$, with indices $\alpha = 3, \ldots, n-j$ and $\alpha = n-j+1, \ldots, n$ corresponding to space-like and time-like vectors resp.*

(iii) *The solutions of $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system II (4) corresponding to X and $\tilde{X}$ as given in Theorem 2.2 are (F, G, B) and $(\tilde{F}, \tilde{G}, \tilde{B}^\#)$ resp.*

(iv) *The bundle morphism* $P(e_k(x)) = \tilde{e}_k(x)$, $k = 3, \dots, n$ *is a Ribaucour Transformation covering the map* $X_j(x) \mapsto \tilde{X}_j(x)$ *for each* $1 \le j \le 2$.

(v) *There exist smooth functions* ϕ_{ij} *such that* $X_j + \phi_{ij} e_i = \tilde{X}_j + \phi_{ij} \tilde{e}_i$ *for* $1 \le j \le 2$ *and* $1 \le i \le n$.

For the proof we will need the following result whose proof is quite similar to proof of Corollary (6.11) in [2].

PROPOSITION 3.1. *Let* $E(x, \lambda)$ *be a frame for the solution* ξ *of system* (3), *and* $Y(x) = \frac{\partial E}{\partial \lambda}(x, 0) E^{-1}(x, 0)$. *Then we have*

(i)

$$Y = \begin{pmatrix} 0 & X \\ -I_{1,1} X^t I_{n-j,j} & 0 \end{pmatrix} \quad \textit{for some } X \in \mathcal{M}_{n \times 2}.$$

(ii) $X = (X_1, X_2)$ *is a space-like* 2-*tuple in* $\mathbf{R}^{n-j,j}$ *of type* $O(1,1)$.

(iii) $dX = A \begin{pmatrix} -dx_1 & 0 & 0 & \cdots & 0 \\ 0 & dx_2 & 0 & \cdots & 0 \end{pmatrix}^t B^{-1}$. *This means* X *satisfies* (9).

PROOF OF THEOREM 3.1. From (18), Theorem 2.1, proposition above, we get

$$\begin{aligned}
\begin{pmatrix} 0 & \tilde{X} \\ -I_{1,1} \tilde{X}^t I_{n-j,j} & 0 \end{pmatrix} &= \frac{\partial \widetilde{E^{\#}}}{\partial \lambda}(x, 0) \widetilde{E^{\#}}^{-1}(x, 0) \\
&= \frac{\partial E}{\partial \lambda}(x, 0) E^{-1}(x, 0) + \frac{2}{s} E(x, 0) \begin{pmatrix} 0 & \hat{W} \hat{Z}^t I_{1,1} \\ -\hat{Z} \hat{W}^t I_{n-j,j} & 0 \end{pmatrix} E(x, 0)^{-1} \\
&= \begin{pmatrix} 0 & X \\ -I_{1,1} X^t I_{n-j,j} & 0 \end{pmatrix} + \frac{2}{s} \begin{pmatrix} 0 & A \hat{W} \hat{Z}^t B^t I_{1,1} \\ -B \hat{Z} \hat{W}^t A^t I_{n-j,j} & 0 \end{pmatrix}.
\end{aligned}$$

Hence

$$\tilde{X} = X + \frac{2}{s} A \hat{W} \hat{Z}^t B^t I_{1,1}.$$

Let now $\eta = \sum_{j=1}^n \hat{w}_j e_j$, then the i-th column of $\tilde{X}$ is given by

$$\tilde{X}_i = X_i + \frac{2}{s} \varepsilon_i \sum_{j=1}^{2} (\hat{z}_j b_{ij}) \eta, \tag{26}$$

where $\varepsilon_1 = -\varepsilon_2 = 1$. Now from the relation $\tilde{A}^{\#} = A(I - 2\hat{W}\hat{W}^t I_{n-j,j})$ we get $\tilde{e}_i = e_i - 2\hat{w}_i \eta \varepsilon_i$ with $\varepsilon_i = 1$, $i = 1, \dots, n-j$ and $\varepsilon_i = -1$, $i = n-j+1, \dots, n$. So using this last relation, we have

$$X_j + \phi_{ij} e_i = \tilde{X}_j + \phi_{ij} \tilde{e}_i, \tag{27}$$

where

$$\phi_{ij} = \frac{\varepsilon_i \varepsilon_j}{s \hat{w}_i} \sum_{k=1}^{2} \hat{z}_k b_{jk}, \quad \text{for } j = 1,2,\ i = 1,\dots,n.$$

This proves that $P : \mathscr{V}(X_j) \to \mathscr{V}(\tilde{X}_j)$ given by $P(e_k(x)) = \tilde{e}_k(x)$, $k = 3,\dots,n$ is a Ribaucour transformation covering the map $l : X_j \to \tilde{X}_j$, $l(X_j(x)) = \tilde{X}_j(x)$ for each $1 \le j \le 2$. ∎

The next result shows that the transformation constructed in Theorem 3.1, for space-like 2-tuples in $\mathbf{R}^{n-j,j}$ of type $O(1,1)$, is a Darboux transformation for spacelike isothermic surfaces.

DEFINITION 3.3. *Let M, $\tilde{M}$ be two space-like surfaces in $\mathbf{R}^{n-j,j}$ with flat and non-degenerate normal bundle and $P : \mathscr{V}(M) \to \mathscr{V}(\tilde{M})$ a Ribaucour transformation that covers the map $l : M \to \tilde{M}$. If l is a conformal diffeomorphism, then P is called a Darboux transformation.*

THEOREM 3.2. *Let (Y_1, Y_2) be a space-like isothermic pair in $\mathbf{R}^{n-j,j}$ corresponding to the solution (u, G) of the system* (10), *and let $\xi = \begin{pmatrix} F \\ G \end{pmatrix}$ the corresponding solution of the system* (3), *where $F = \begin{pmatrix} 0 & u_{x_1} \\ u_{x_2} & 0 \end{pmatrix}$. Let also $s \in \mathbf{R}$ different of zero, π a projection on $\mathbf{C}\begin{pmatrix} W \\ iZ \end{pmatrix}$, $g_{s,\pi}$ the rational element defined in* (15), *and $\hat{W}$, $\hat{Z}$ as in Theorem* 2.5, *for the solution ξ of the system* (3). *Let*

$$B = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix}, \quad (\widetilde{E^{\#}}^{II}, \tilde{A}^{\#}, \tilde{B}^{\#}) = g_{s,\pi} \cdot (E^{II}, A, B).$$

Write $A = (e_1, \dots, e_n)$ and $\tilde{A}^{\#} = (\tilde{e}_1, \dots, \tilde{e}_n)$. Set

$$\begin{cases} \tilde{Y}_1 = Y_1 + \frac{2}{s}(\hat{z}_1 + \hat{z}_2) e^{u} \sum_{i=1}^{n} \hat{w}_i e_i, \\ \tilde{Y}_2 = Y_2 + \frac{2}{s}(\hat{z}_1 - \hat{z}_2) e^{-u} \sum_{i=1}^{n} \hat{w}_i e_i. \end{cases} \tag{28}$$

Then

(i) *$(\tilde{Y}_1, \tilde{Y}_2)$ is also a space-like isothermic pair in $\mathbf{R}^{n-j,j}$.*

(ii) *$(\tilde{u}, \tilde{G})$ is the solution of system* (10), *corresponding to $(\tilde{Y}_1, \tilde{Y}_2)$, where $\tilde{u} = 2\alpha - u$, $\sinh \alpha = -\hat{z}_2$ and $\tilde{G} = (\tilde{r}_{ij})$ is defined by* (24).

(iii) *The fundamental forms of pair* $(\tilde{Y}_1, \tilde{Y}_2)$ *are respectively*

$$\begin{cases} \tilde{I}_1 = e^{-2\tilde{u}}(dx_1^2 + dx_2^2) \\ \tilde{II}_1 = e^{-\tilde{u}} \sum_{\alpha=3}^n (-\tilde{r}_{\alpha-2,1}\, dx_1^2 + \tilde{r}_{\alpha-2,2}\, dx_2^2)\tilde{e}_\alpha \\ \tilde{I}_2 = e^{2\tilde{u}}(dx_1^2 + dx_2^2) \\ \tilde{II}_2 = -e^{\tilde{u}} \sum_{\alpha=3}^n (\tilde{r}_{\alpha-2,1}\, dx_1^2 + \tilde{r}_{\alpha-2,2}\, dx_2^2)\tilde{e}_\alpha. \end{cases}$$

(iv) *The bundle morphism* $P(e_k(x)) = \tilde{e}_k(x)$, $k = 3, \dots, n$ *covering the map* $Y_i \to \tilde{Y}_i$ *is a Darboux transformation for each* $i = 1, 2$.

Proof. It follows from Proposition 2.1 and Theorem 2.3 that (F, G, B) is a solution of $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system II (5), and $X = (X_1, X_2) = \left(\frac{Y_1+Y_2}{2}, \frac{Y_2-Y_1}{2}\right)$ is the corresponding space-like 2-tuple in $\mathbf{R}^{n-j,j}$ of type $O(1,1)$. Now let $\tilde{X} = (\tilde{X}_1, \tilde{X}_2)$ be as in Theorem 3.1 and let $\tilde{Y}_1 = \tilde{X}_1 - \tilde{X}_2$, $\tilde{Y}_2 = \tilde{X}_1 + \tilde{X}_2$. Then using (26) we get that $\tilde{Y}_1$ and $\tilde{Y}_2$ are given by (28). Since $\hat{z}_1^2 - \hat{z}_2^2 = 1$, there exists a function $\alpha : \mathbf{R}^2 \to \mathbf{R}$ such that $\hat{z}_1 = \cosh\alpha$, $\hat{z}_2 = -\sinh\alpha$. It follows from $\tilde{B}^{\#} = B(I - 2\hat{Z}\hat{Z}^t I_{1,1})$ that

$$\tilde{B}^{\#} = \begin{pmatrix} -\cosh(2\alpha - u) & -\sinh(2\alpha - u) \\ \sinh(2\alpha - u) & \cosh(2\alpha - u) \end{pmatrix} = \begin{pmatrix} -\cosh\tilde{u} & -\sinh\tilde{u} \\ \sinh\tilde{u} & \cosh\tilde{u} \end{pmatrix}.$$

Since $d\tilde{X}_1 = \cosh\tilde{u}\, dx_1\tilde{e}_1 + \sinh\tilde{u}\, dx_2\tilde{e}_2$, and $d\tilde{X}_2 = \sinh\tilde{u}\, dx_1\tilde{e}_1 + \cosh\tilde{u}\, dx_2\tilde{e}_2$, it follows that

$$d\tilde{Y}_1 = d\tilde{X}_1 - d\tilde{X}_2 = e^{-\tilde{u}}(dx_1\tilde{e}_1 - dx_2\tilde{e}_2), \quad d\tilde{Y}_2 = d\tilde{X}_1 + d\tilde{X}_2 = e^{\tilde{u}}(dx_1\tilde{e}_1 + dx_2\tilde{e}_2).$$

So we get the claim (i)–(iii).

For (iv) we observe that the map $l : Y_i \to \tilde{Y}_i$ is conformal because the coordinates (x_1, x_2) are isothermic for Y_i and $\tilde{Y}_i$. Now we need to prove that $P : \mathscr{V}(Y_i) \to \mathscr{V}(\tilde{Y}_i)$ given by $e_k(x) \to \tilde{e}_k(x)$, $k = 3, \dots, n$ is a Ribaucour transformation. For that, we use the fact that there exist smooth functions ϕ_{ij} such that $X_j + \phi_{ij}e_i = \tilde{X}_j + \phi_{ij}\tilde{e}_i$ for $1 \le i \le n$ and $j = 1, 2$, (Theorem 3.1 (v)), and so

$$Y_1 + (\phi_{i1} - \phi_{i2})e_i = \tilde{Y}_1 + (\phi_{i1} - \phi_{i2})\tilde{e}_i, \quad Y_2 + (\phi_{i1} + \phi_{i2})e_i = \tilde{Y}_2 + (\phi_{i1} + \phi_{i2})\tilde{e}_i,$$

for each $1 \le i \le n$. Hence the map P is a Darboux transformation. ■

Example 3.1. *Let* $n = 3$, $j = 1$, *then we have the* $O(3,2)/O(2,1) \times O(1,1)$-*system. Let* $(u, r_{11}, r_{12}) = (0,0,0)$ *be a trivial solution of* (10), *then* $F = 0$, $G = 0$, $B = I$. *So a space-like* 2-*tuple* X *in* $\mathbf{R}^{2,1}$ *of type* $O(1,1)$ *and the frame* $E(x, y, \lambda)$ *associated to trivial solution are*:

$$X=\begin{pmatrix}-x & 0\\ 0 & y\\ 0 & 0\end{pmatrix},\quad E(x,y,\lambda)=\begin{pmatrix}\cos(\lambda x) & 0 & 0 & -\sin(\lambda x) & 0\\ 0 & \cosh(\lambda y) & 0 & 0 & \sinh(\lambda y)\\ 0 & 0 & 1 & 0 & 0\\ \sin(\lambda x) & 0 & 0 & \cos(\lambda x) & 0\\ 0 & \sinh(\lambda y) & 0 & 0 & \cosh(\lambda y)\end{pmatrix}$$

Then from (16), *we obtain that*

$$\begin{pmatrix}\tilde{w}_1\\ \tilde{w}_2\\ \tilde{w}_3\\ \tilde{z}_1\\ \tilde{z}_2\end{pmatrix}=\begin{pmatrix}w_1\cosh sx+z_1\sinh sx\\ w_2\cos sy-z_2\sin sy\\ w_3\\ w_1\sinh sx+z_1\cosh sx\\ w_2\sin sy+z_2\cos sy\end{pmatrix}.$$

From (26), *we get that the 2-tuple in* $\mathbf{R}^{2,1}$ *of type* $O(1,1)$ *constructed by applying the Ribaucour transformation to the trivial solution is*:

$$\tilde{X}_1=X_1+\frac{2}{s}\hat{z}_1\sum_{i=1}^{3}\hat{w}_ie_i,\quad \tilde{X}_2=X_2-\frac{2}{s}\hat{z}_2\sum_{i=1}^{3}\hat{w}_ie_i,$$

i.e.,

$$\tilde{X}_1=\begin{pmatrix}-x\\ 0\\ 0\end{pmatrix}+\frac{2}{s}\frac{w_1\sinh sx+z_1\cosh sx}{(w_1\sinh sx+z_1\cosh sx)^2-(w_2\sin sy+z_2\cos sy)^2}$$

$$\times\begin{pmatrix}w_1\cosh sx+z_1\sinh sx\\ w_2\cos sy-z_2\sin sy\\ w_3\end{pmatrix}$$

$$\tilde{X}_2=\begin{pmatrix}0\\ y\\ 0\end{pmatrix}-\frac{2}{s}\frac{w_2\sin sy+z_2\cos sy}{(w_1\sinh sx+z_1\cosh sx)^2-(w_2\sin sy+z_2\cos sy)^2}$$

$$\times\begin{pmatrix}w_1\cosh sx+z_1\sinh sx\\ w_2\cos sy-z_2\sin sy\\ w_3\end{pmatrix}.$$

EXAMPLE 3.2. *A space-like plane in* $\mathbf{R}^{2,1}$ *is an isothermic surface corresponding to trivial solution* $(0,0,0)$ *of* (10). *Then the space-like isothermic pair associated to the trivial solution is*:

$$Y_1 = \begin{pmatrix} -x \\ -y \\ 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} -x \\ y \\ 0 \end{pmatrix}.$$

So the isothermic pair obtained by applying the Darboux transformations to the trivial solution, given by (28) *is*:

$$\tilde{Y}_1 = Y_1 + \frac{2}{s}(\hat{z}_1 + \hat{z}_2)\sum_{i=1}^{3} \hat{w}_i e_i, \quad \tilde{Y}_2 = Y_2 + \frac{2}{s}(\hat{z}_1 - \hat{z}_2)\sum_{i=1}^{3} \hat{w}_i e_i,$$

i.e.,

$$\tilde{Y}_1 = \begin{pmatrix} -x \\ -y \\ 0 \end{pmatrix} + \frac{2}{s}\frac{w_1 \sinh sx + z_1 \cosh sx + w_2 \sin sy + z_2 \cos sy}{(w_1 \sinh sx + z_1 \cosh sx)^2 - (w_2 \sin sy + z_2 \cos sy)^2}$$

$$\times \begin{pmatrix} w_1 \cosh sx + z_1 \sinh sx \\ w_2 \cos sy - z_2 \sin sy \\ w_3 \end{pmatrix}$$

$$\tilde{Y}_2 = \begin{pmatrix} -x \\ y \\ 0 \end{pmatrix} + \frac{2}{s}\frac{w_1 \sinh sx + z_1 \cosh sx - w_2 \sin sy - z_2 \cos sy}{(w_1 \sinh sx + z_1 \cosh sx)^2 - (w_2 \sin sy + z_2 \cos sy)^2}$$

$$\times \begin{pmatrix} w_1 \cosh sx + z_1 \sinh sx \\ w_2 \cos sy - z_2 \sin sy \\ w_3 \end{pmatrix}$$

4. Appendix: Moving Frames

We review the method of moving frames for space-like surfaces in the Lorentz space $\mathbf{R}^{n-j,j}$: Set

$$e_A \cdot e_B = \sigma_{AB} = I_{n-j,j} = \begin{pmatrix} I_{n-j} & 0 \\ 0 & -I_j \end{pmatrix}.$$

We also let $\sigma_i := \sigma_{ii}$. For the space-like immersion X set $dX = \omega_1 e_1 + \omega_2 e_2$, with e_1, e_2 space-like unit tangent vectors to the surface and the normal space is spanned by e_β, for $3 \le \beta \le n$. Define

$$de_B = \sum_A \omega_{AB} e_A. \tag{29}$$

This gives $\omega_{AB} = \sigma_A e_A \cdot de_B$ and $\omega_{AB}\sigma_A + \omega_{BA}\sigma_B = 0$.

From $d(dX)=0$ we get:

$$d\omega_1=\omega_2\wedge\omega_{12},\quad d\omega_2=\omega_1\wedge\omega_{21},\quad \omega_1\wedge\omega_{\beta 1}+\omega_2\wedge\omega_{\beta 2}=0,$$

for β as above.

In addition, by Cartan's Lemma we have:

$$\omega_{1\beta}=h_{11}^{\beta}\omega_1+h_{12}^{\beta}\omega_2,\quad \omega_{2\beta}=h_{21}^{\beta}\omega_1+h_{22}^{\beta}\omega_2,$$

this makes the first and second fundamental form:

$$I:\omega_1^2+\omega_2^2,\quad II:-\sum_{k=1,2}\sum_{\alpha}\omega_{k\alpha}\omega_k\sigma_\alpha e_\alpha \tag{30}$$

We also have: $d\omega_{CA}=-\sum_B\omega_{CB}\wedge\omega_{BA}$, which yield the Gauss and Codazzi equations. The Gauss equation comes from examining $d\omega_{12}$, while the Codazzi equations are from $d\omega_{1\beta}$ and $d\omega_{2\beta}$.

References

[1] Burstall, F., Isothermic surfaces: conformal geometry, Clifford algebras and integrable systems, preprint, math-dg/0003096.

[2] Brück, M., Du, X., Park, J., Terng, C.-L., The Submanifolds Geometries associated to Grassmannian Systems, Memoirs of A.M.S. no. **735** (2002).

[3] Burstall, F., Hertrich, J. U., Pedit, F., Pinkall, U., Curved flats and isothermic surfaces, Math. Z. no. 2, **225** (1997).

[4] Cieslinski, J., Goldstein, P., Sym, A., Isothermic surfaces in $\mathbf{R}^3$ as soliton surfaces, Phys. Lett. A. **205** (1995), 37–43.

[5] Cieslinski, J., The Darboux-Bianchi transformation for isothermic surfaces, Isothermic surfaces, Diff. Geom. Appl. **7** (1997), 1–28.

[6] Dajczer, M., Tojeiro, R., Commuting Codazzi tensors and the Ribaucour transformation for submanifolds, Results in Mathematics. **44** (2003), 258–278.

[7] Dussan, M. P., Geometry of surfaces corresponding to $O(3,2)/O(2,1)\times O(1,1)$-system, Relatorio de Pesquisa, Outubro, **60** (2002). Universidade Estadual de Campinas, Brazil.

[8] Fujioka, A., Inoguchi, J., Spacelike surfaces and Harmonic Inverse Mean Curvature, J. Math. sci. Univ. Tokyo. **7** (2000), 657–698.

[9] Terng, C.-L., Solitons equations and Differential Geometry, J. Differential Geom. **45** (1997), no. 2, 407–445.

[10] Terng, C.-L., Uhlenbeck, K., Backlund transformations and loop group actions, Comm. Pure. Appl. Math. **53** (2000), 1–75.

[11] Tian, C., Backlund transformation on surfaces with $K=-1$ in $\mathbf{R}^{2,1}$, J. Geometry and Physics, **22** (1997), 212–218.

Author's address: IME—USP, C.P. 66281
CEP 05315-970, São Paulo, SP. Brazil