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GRASSMANN GEOMETRY ON THE GROUPS OF RIGID MOTIONS ON THE EUCLIDEAN AND THE MINKOWSKI PLANES

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Abstract. We study the Grassmann geometries of surfaces when the ambient spaces are the Lie groups of rigid motions on the Euclidean and the Minkowski planes, furnished with left invariant metrics.

1. Introduction

Let M be an m-dimensional connected Riemannian manifold and r be an integer such that $1 \leq r \leq m$. Given a nonempty subset Σ in the Grassmann bundle $G^r(TM)$ over M which consists of all r-dimensional linear subspaces of the tangent spaces of M, an r-dimensional connected submanifold S of M is called a Σ-submanifold or simply an associated submanifold if all tangent spaces of S belong to Σ, and the collection of such the submanifolds is called a Σ-geometry. "Grassmann geometry" is a collected name for such a Σ-geometry. Let G denote the identity component of the isometry group of M. Then G acts on $G^r(TM)$ through the differentials of isometries and we have many G-orbits in $G^r(TM)$. If Σ is given by a G-orbit, the Σ-geometry is in particular called of orbit type. If M is a Riemannian homogeneous manifold, such a subset Σ is a subbundle of $G^r(TM)$ over M. In the study of Grassmann geometry, we shall first consider the existence of Σ-submanifolds for an arbitrary Σ-geometry, and next if they exist, we shall consider whether or not such the Σ-geometry has somewhat canonical Σ-submanifolds, *eg.*, totally geodesic submanifolds, minimal submanifolds, etc.

In the previous paper [2], from this view of points, we have studied the Grassmann geometry of surfaces, namely the case $r = 2$, in the 3-dimensional

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Heisenberg group with a left invariant metric. In this paper, we consider the cases when the ambient spaces M are the Lie groups of rigid motions on the Euclidean and the Minkowski planes, furnished with left invariant metrics. These spaces together with the Heisenberg case, are typical examples of 3-dimensional homogeneous Riemannian manifolds, and as Lie groups they are locally the only, not nilpotent, solvable Lie groups among the 3-dimensional unimodular Lie groups, while the Heisenberg group is locally the only, not commutative, nilpotent Lie group among them.

The aim of this paper is to determine the G-orbital Grassmann geometries of surfaces on these Riemannian manifolds which have the associated surfaces, and moreover to clarify geometric properties of their associated surfaces.

2. The Lie Groups of Rigid Motions on the Euclidean and the Minkowski Planes

Let $E(2)$ denote the Lie group of rigid motions on the Euclidean plane, which is a semi-direct product of the group $O(2)$ of orthogonal transformations and the vector group $\mathbf{R}^2$ of parallel translations. Moreover let $E(1,1)$ denote the Lie group of rigid motions on the Minkowski plane, which is a semi-direct product of the group $O(1,1)$ of Lorentz transformations and the vector group $\mathbf{R}^2$ of parallel translations. Hereafer we will consider only the connected components of the identity in these Lie groups. They are also denoted by the same notations $E(2)$ or $E(1,1)$. These Lie groups are solvable and together with any left invariant metric, become 3-dimensional homogeneous Riemannian manifolds, denoted by (M,g). Let $\mathfrak{l}$ be the Lie algebra of left invariant vector fields on the Lie group M and $\langle\,,\rangle$ an inner product on $\mathfrak{l}$ canonically induced from the Riemannian metric g. Then $\mathfrak{l}$ is identified with the tangent space T_eM at the unity e and the inner product $\langle\,,\rangle$ is equal to g_e.

More generally, let $\mathfrak{l}$ be a 3-dimensional unimodular Lie algebra with an inner product $\langle\,,\rangle$. Then the Lie bracket $[\,,]$ of $\mathfrak{l}$ induces a unique symmetric linear transformation L of $\mathfrak{l}$ such that $[u,v]=L(u\times v)$ for $u,v\in\mathfrak{l}$ where $\times$ denotes the cross product on $\mathfrak{l}$ with respect to a fixed orientation. Moreover, taking the eigenvalues λ_1, λ_2, λ_3 of L and their positively oriented, unit eigenvectors E_1, E_2, E_3, we have the following relations:

$$[E_2,E_3]=\lambda_1E_1,\quad [E_3,E_1]=\lambda_2E_2,\quad [E_1,E_2]=\lambda_3E_3. \tag{2.1}$$

If the Lie algebra $\mathfrak{l}$ with $\langle\,,\rangle$ is associated with a Lie group M with a left invariant metric g, by these relations the Levi-Civita connection ∇ of (M,g) is expressed as follows:

$$\nabla_{E_i}E_j = (1/2)\sum_k(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij})E_k \tag{2.2}$$

where $\alpha_{ijk} = \langle [E_i, E_j], E_k\rangle$ and $i = 1,2,3$. Moreover various curvatures of (M, g) at the unity e can be calculated as follows: The Ricci quadratic form r is diagonalized by the eigenvectors E_1, E_2, E_3, together with its principal Ricci curvatures given by

$$r(E_1) = 2\mu_2\mu_3, \quad r(E_2) = 2\mu_3\mu_1, \quad r(E_3) = 2\mu_1\mu_2, \tag{2.3}$$

where $\lambda_i = \mu_1 + \mu_2 + \mu_3 - \mu_i$ for $i = 1,2,3$. In particular, the scalar curvature ρ is given by the equation $\rho = 2(\mu_2\mu_3 + \mu_3\mu_1 + \mu_1\mu_2)$. Also, the sectional curvature $\kappa(u, v)$ of the plane generated by vectors u, v can be explicitly calculated by the general formula

$$\kappa(u, v) = \|u \times v\|^2\rho/2 - r(u \times v) \tag{2.4}$$

for any 3-dimensional Riemannian manifold. Refer to [3] for the details of these.

Let (M, g) be $E(2)$ with any left invariant metric g. Then the triple $\{\lambda_1, \lambda_2, \lambda_3\}$, determined as above, consists of one zero and two positive constants. We may here suppose $\lambda_3 = 0$. Moreover the set of isometry classes of left invariant metrics on $E(2)$ is characterized by the set

$$\Lambda(E(2)) = \{(\lambda_1, \lambda_2) \in \mathbf{R}^2 : \lambda_1 > \lambda_2 > 0 \text{ or } \lambda_1 = \lambda_2 = 1\},$$

where the case that $\lambda_1 = \lambda_2 = 1$ corresponds to the local Euclidean metric. Next let (M, g) be $E(1,1)$ with any left invariant metric g. Then the triple $\{\lambda_1, \lambda_2, \lambda_3\}$ consists of zero, positive, and negative constants. We may suppose $\lambda_1 > 0$, $\lambda_2 < 0$, and $\lambda_3 = 0$. Moreover the set of isometry classes of left invariant metrics on $E(1,1)$ is characterized by the set

$$\Lambda(E(1,1)) = \{(\lambda_1, \lambda_2) \in \mathbf{R}^2 : \lambda_1 > 0 > \lambda_2 \geq -\lambda_1\}.$$

Refer to [4] for these. In these cases the principal Ricci curvatures of the left invariant metric g corresponding to $(\lambda_1, \lambda_2) \in \Lambda(E(2))$ (or $\Lambda(E(1,1))$) are calculated as follows:

$$r(E_1) = (\lambda_1^2 - \lambda_2^2)/2, \quad r(E_2) = (\lambda_2^2 - \lambda_1^2)/2, \quad r(E_3) = -(\lambda_1 - \lambda_2)^2/2. \tag{2.5}$$

Here we note that the signature of Ricci quadratic form r is $(+,-,-)$ except for the case of $E(2)$ that $\lambda_1 = \lambda_2 = 1$ and the case of $E(1,1)$ that $\lambda_1 = -\lambda_2$. Also, it is known that for any left invariant metric of these cases the isometry group of (M, g) has three dimension except for the local Euclidean case of $E(2)$. Hence, for any left invariant metric but for that case, the connected component of the identity in the isometry group is equal to the group of all the left translations, thus, it is isomorphic to $E(2)$ or $E(1,1)$.

3. Grassmann Geometry on the Lie Group $E(2)$

In this section we consider the orbital Grassmann geometry of surfaces in $E(2)$, denoted by M, with any left invariant metric g but for the local Euclidean case. Let G denote the connected component of the identity in the isometry group of (M,g). Then, since G is the group of left translations, a G-orbit in the Grassmann bundle $G^2(TM)$ is a homogeneous subbundle of 2-planes on which G acts simply and transitively. Hence, the orbit space $\mathscr{S}(G)$, the set of all the G-orbits, is identified with the Grassmann manifolds $G^2(T_eM)$ of linear planes in T_eM, thus, the Grassmann manifold $G^2(\mathfrak{l})$ where $\mathfrak{l}$ denotes the Lie algebra of M. This is also bijective to the real projective plane of all the linear lines in $\mathfrak{l}$.

Let $S^2(\mathfrak{l})$ be the unit sphere in $\mathfrak{l}$ centered at the origin and for $W \in S^2(\mathfrak{l})$, let $P(W)$ denote the linear plane orthogonal to W. Then, by assigning to W the sectional curvature $\kappa(P(W))$ of the plane $P(W)$, we can induce a *curvature function* $\kappa(W)$ on $S^2(\mathfrak{l})$ such that $\kappa(W) = \kappa(-W)$. Let (λ_1, λ_2) be the pair in $\Lambda(E(2))$ corresponding to g and $\{E_1, E_2, E_3\}$ be the orthonormal basis of $\mathfrak{l}$ given in (2.1). Moreover identify a vector W in $S^2(\mathfrak{l})$ with an element (w_1, w_2, w_3) in $\mathbf{R}^3$ by the relation $W = w_1E_1 + w_2E_2 + w_3E_3$. Then, by the formula (2.4), it follows

$$\kappa(W) = -\frac{(\lambda_1-\lambda_2)^2}{4} - \frac{\lambda_1^2-\lambda_2^2}{2}w_1^2 + \frac{\lambda_1^2-\lambda_2^2}{2}w_2^2 + \frac{(\lambda_1-\lambda_2)^2}{2}w_3^2. \tag{3.6}$$

Next for $W \in S^2(\mathfrak{l})$ let $\mathcal{O}(W)$ be the G-orbit containing the 2-plane $P(W)$. We note that $\mathcal{O}(W) = \mathcal{O}(-W)$. Then we have the following.

Proposition 3.1. *There exists an $\mathcal{O}(W)$-surface if and only if $W = (0,0,\pm 1)$, thus, $P(W)$ is the w_1w_2-plane.*

Proof. Note that $\mathcal{O}(W)$ can be regarded as a distribution of 2-planes on M. Then there exists an $\mathcal{O}(W)$-surface if and only if the distribution $\mathcal{O}(W)$ is involutive. Since $\mathcal{O}(W)$ is invariant by the left translations, we may take a suitable basis $\{U, V\}$ of $P(W)$ and see whether or not it holds $[U,V] \in P(W)$, equivalently, $\langle [U,V], W\rangle = 0$.

Let $W = (w_1, w_2, w_3)$. We first suppose that it holds that $w_1 = w_2 = w_3$. In this case it follows $W = \pm(1/\sqrt{3})(1,1,1)$. Put $U = (1,-1,0)$ and $V = (1,0,-1)$. Then, the pair $\{U, V\}$ are an orthogonal basis of $P(W)$ and moreover it follows

$\langle [U,V],W\rangle = \pm(\lambda_1+\lambda_2)/\sqrt{3}$. Since λ_1 and λ_2 are positive, this value is not zero.

We next suppose that it doesn't hold that $w_1=w_2=w_3$. In this case we put

$$U=(w_2-w_3,w_3-w_1,w_1-w_2),\quad V=W\times U.$$

Then, U and V are orthogonal to W, and moreover since $U\neq 0$, they are linearly independent. Hence, the pair $\{U,V\}$ is a basis of $P(W)$. Moreover it follows

$$\langle [U,V],W\rangle = 2(1-w_1w_2-w_1w_3-w_2w_3)(\lambda_1w_1^2+\lambda_2w_2^2).$$

If $\lambda_1w_1^2+\lambda_2w_2^2\neq 0$, it holds $\langle [U,V],W\rangle=0$ if and only if $w_1w_2+w_1w_3+w_2w_3=1$. This, together with the equation $w_1^2+w_2^2+w_3^2=1$, induces the equation $(w_1-w_2)^2+(w_2-w_3)^2+(w_3-w_1)^2=0$. But this is not the case. Hence it follows $\lambda_1w_1^2+\lambda_2w_2^2=0$. This induces $w_1=w_2=0$, which implies $W=(0,0,\pm 1)$. □

Remark. Using the Lagrange's method of indeterminate coefficients, we can see that the critical points of the curvature function $\kappa(W)$ on $S^2(1)$ given by (3.6) are:

$$W=(\pm 1,0,0),(0,\pm 1,0),(0,0,\pm 1),$$

and the critical value where $W=(0,0,\pm 1)$ is $(\lambda_1-\lambda_2)^2/4$, which is neither the maximum nor the minimum of the curvature function κ. We here remark that these calculations need the condition $\lambda_1>\lambda_2$.

Now we consider the $\mathcal{O}(W)$-geometry when $W=(0,0,\pm 1)$. Hereafter we rewrite this W by W_0. To see geometric properties of the $\mathcal{O}(W_0)$-surfaces, namely, the leaves of $\mathcal{O}(W_0)$, we decompose the Levi-Civita connection ∇ of (M,g) into the tangent part D and the normal part Π of the distribution $\mathcal{O}(W_0)$. More pricisely,

$$\nabla_XY=D_XY+\Pi(X,Y)$$

for vector fields $X,Y\in\mathcal{O}(W_0)$. Then the restrictions of D and Π onto each leaf give the Levi-Civita connection on the leaf with respect to the metric induced from g, and the second fundamental form of the leaf, respectively. Now, taking note of the fact that E_1 and E_2 are left invariant base fields of $\mathcal{O}(W_0)$, we can calculate the Levi-Civita connection ∇ of (M,g) as follows:

$$\text{(3.7)}\qquad \nabla_{E_1}E_1=\nabla_{E_2}E_2=0,\quad \nabla_{E_1}E_2=\nabla_{E_2}E_1=-\frac{\lambda_1-\lambda_2}{2}E_3.$$

Consequently it follows $D_{E_i}E_j = 0$ for $i = 1, 2$, and

$$(3.8)\quad \Pi(E_1, E_1) = \Pi(E_2, E_2) = 0, \quad \Pi(E_1, E_2) = \Pi(E_2, E_1) = -\frac{\lambda_1 - \lambda_2}{2} E_3.$$

Summing up the arguments in this section, we have the following theorem.

THEOREM 3.2. *Let (M, g) be the group $E(2)$ with any left invariant metric g which is not the local Euclidean metric, and let (λ_1, λ_2) be the element in $\Lambda(E(2))$ corresponding to g.*

Then, among G-orbital Grassmann geometries of surfaces in M, the $\mathcal{O}(W_0)$-geometry is the only one which has associated surfaces.

Moreover, any $\mathcal{O}(W_0)$-surface S is a minimal flat surface in M such that (i) *it has not a totally geodesic point, and* (ii) *$\kappa(T_pS) = (\lambda_1 - \lambda_2)^2/4 > 0$ for any $p \in S$.*

REMARK. A maximal $\mathcal{O}(W_0)$-surface, a maximal leaf of $\mathcal{O}(W_0)$, is complete since $\mathcal{O}(W_0)$ is a left-invariant distribution on the Lie group M. Moreover, since any left translation is an isometry of (M, g), all the maximal $\mathcal{O}(W_0)$-surfaces are congruent to each other.

4. Grassmann Geometry on the Lie Group $E(1, 1)$

Next we consider the orbital Grassmann geometry of surfaces in $E(1, 1)$ with any left invariant metric g. We denote by (M, g) this Riemannian manifold and we retain the same notations as in the previous section. For example, G is the group of left translations of M, $\mathfrak{l}$ is the Lie algebra of left invariant vector fields on M, κ is the curvature function on the unit sphere $S^2(\mathfrak{l})$, and so on.

Now let (λ_1, λ_2) be the pair in $\Lambda(E(1, 1))$ corresponding to g and $\{E_1, E_2, E_3\}$ be the orthonormal basis of $\mathfrak{l}$ given in (2.1). Moreover for $W \in S^2(\mathfrak{l})$, let w_i, $i = 1, 2, 3$, be the i-th coefficients of W with respect to the basis $\{E_1, E_2, E_3\}$. Then, similarly to the case of $E(2)$, the curvature function $\kappa(W)$ is given by

$$(4.9)\qquad \kappa(W) = -\frac{(\lambda_1 - \lambda_2)^2}{4} - \frac{\lambda_1^2 - \lambda_2^2}{2} w_1^2 + \frac{\lambda_1^2 - \lambda_2^2}{2} w_2^2 + \frac{(\lambda_1 - \lambda_2)^2}{2} w_3^2.$$

We here note that though the form of $\kappa(W)$ is the same as in the case of $E(2)$, in this case λ_1 is positive and λ_2 is negative, while in the case of $E(2)$ both λ_1 and λ_2 are positive.

Next for $W \in S^2(\mathfrak{l})$ we consider the G-orbit $\mathcal{O}(W)$ containing the 2-plane $P(W)$, and give the condition for the existence of an $\mathcal{O}(W)$-surface.

PROPOSITION 4.1. *There exists an $\mathcal{O}(W)$-surface if and only if the coefficients w_i of W satisfy the following equations:*

$$w_1^2 + w_2^2 + w_3^2 = 1 \quad \textit{and} \quad \lambda_1 w_1^2 + \lambda_2 w_2^2 = 0. \tag{4.10}$$

PROOF. Similarly to Proposition 3.1, we divide the proof into case (i) that $w_1 = w_2 = w_3 (= \pm 1/\sqrt{3})$ and case (ii) that it doesn't hold, and for each case take the same left invariant vector fields U and V as in the proposition. Then, the condition for the existence of a $\mathcal{O}(W)$-surface is similarly given by the equation $\langle [U, V], W \rangle = 0$.

For case (i), since $\langle [U, V], W \rangle = \pm(\lambda_1 + \lambda_2)/\sqrt{3}$, the above equality occurs only for the case when $\lambda_1 + \lambda_2 = 0$. For case (ii), similarly to Proposition 3.1, the equation holds if and only if $\lambda_1 w_1^2 + \lambda_2 w_2^2 = 0$. These prove our proposition. □

REMARK. Similarly to the case of $E(2)$, the orbit space $\mathcal{S}(G)$ of G-orbits can be identified with the real projective plane over I. Then the set of G-orbits which have the associated surfaces coincides with two projective lines defined by the second equation of (4.10).

Also, using the Lagrange's method of indeterminate coefficients, we can see that the set of critical points of the curvature function $\kappa(W)$ on $S^2(\mathrm{I})$ given by (4.9) is:

$$\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$$

when $\lambda_1 + \lambda_2 \neq 0$, and $\{(0, 0, \pm 1)\} \cup \{(w_1, w_2, w_3); w_1^2 + w_2^2 = 1, w_3 = 0\}$ when $\lambda_1 + \lambda_2 = 0$. Particularly, if we regard $(0, 0, \pm 1)$ as a point in the projective plane, it gives the unique common point on the above two projective lines, and also attains the maximum values $(\lambda_1 - \lambda_2)^2/4$ of $\kappa(W)$.

Now we take a W which satisfies the equations (4.10), and consider the $\mathcal{O}(W)$-geometry for such a W. First, using the formula (2.2), we concretely write down the Levi-Civita connection ∇ of (M, g) by the terms of λ_i's and E_i's:

$$\begin{aligned} \nabla_{E_1} E_2 &= \nabla_{E_2} E_1 = -(\lambda_1 - \lambda_2)/2 E_3, \\ \nabla_{E_1} E_3 &= (\lambda_1 - \lambda_2)/2 E_2, \quad \nabla_{E_2} E_3 = (\lambda_1 - \lambda_2)/2 E_1, \\ \nabla_{E_3} E_1 &= (\lambda_1 + \lambda_2)/2 E_2, \quad \nabla_{E_3} E_2 = -(\lambda_1 + \lambda_2)/2 E_1, \\ \nabla_{E_1} E_1 &= \nabla_{E_2} E_2 = \nabla_{E_3} E_3 = 0. \end{aligned} \tag{4.11}$$

To see geometric properties of the $\mathcal{O}(W)$-surfaces, namly, the leaves of the distribution $\mathcal{O}(W)$, we next see the Gauss curvature κ_D and the second fundamental form Π of leaves. Let $W=(w_1,w_2,w_3)$. Then, by the relations (4.10) between w_i's, the formula (4.9) of $\kappa(W)$ is rewritten in the term of w_1 as follows:

$$\kappa(W)=\frac{(\lambda_1-\lambda_2)^2}{4}+2\lambda_1(\lambda_2-\lambda_1)w_1^2. \tag{4.12}$$

Now we divide the case into two according as w_1 is zero or not.

First suppose $w_1\neq 0$, and set $X=(w_2,-w_1,0)$ and $Y=(w_3,0,-w_1)$. Then the pair $\{X,Y\}$ is a basis of $\mathcal{O}(W)$. By the fomulas (4.11), it follows

$$\begin{aligned}
&\nabla_X X=w_1w_2(\lambda_1-\lambda_2)E_3,\quad \nabla_Y Y=-w_1w_3\lambda_1E_2,\\
&\nabla_X Y=\frac{\lambda_1-\lambda_2}{2}w_1(-w_2E_2+w_3E_3+w_1E_1),\\
&\nabla_Y X=\frac{w_1}{2}\{(\lambda_1-\lambda_2)w_3E_3-(\lambda_1+\lambda_2)w_2E_2-(\lambda_1+\lambda_2)w_1E_1\},
\end{aligned}$$

and moreover by taking the normal components of these,

$$\begin{aligned}
&\Pi(X,X)=w_1w_2w_3(\lambda_1-\lambda_2)W,\quad \Pi(Y,Y)=-w_1w_2w_3\lambda_1W,\\
&\Pi(X,Y)=\frac{\lambda_1-\lambda_2}{2}w_1(1-2w_2^2)W=\frac{\lambda_1-\lambda_2}{2}w_1\left(1+\frac{2\lambda_1}{\lambda_2}w_1^2\right)W,\\
&\Pi(Y,X)=\frac{w_1}{2}\{(\lambda_1-\lambda_2)w_3^2-(\lambda_1+\lambda_2)w_2^2-(\lambda_1+\lambda_2)w_1^2\}W\\
&\qquad\quad=\frac{\lambda_1-\lambda_2}{2}w_1\left(1+\frac{2\lambda_1}{\lambda_2}w_1^2\right)W
\end{aligned}\tag{4.13}$$

where the second equations in the last two equations are obtained by using (4.10). Also, by the first equation of (4.10), it follows

$$\langle X,X\rangle\langle Y,Y\rangle-\langle X,Y\rangle^2=(w_1^2+w_2^2)(w_1^2+w_3^2)-w_2^2w_3^2=w_1^2. \tag{4.14}$$

Then, by (4.12), (4.13), (4.14) and the Gauss equation of leaves, the Gauss curvature $\kappa_D(W)$ is given as follows:

$$\begin{aligned}
\kappa_D(W)&=\kappa(W)-\frac{\langle\Pi(X,Y),\Pi(X,Y)\rangle-\langle\Pi(X,X),\Pi(Y,Y)\rangle}{\langle X,X\rangle\langle Y,Y\rangle-\langle X,Y\rangle^2}\\
&=\frac{(\lambda_1-\lambda_2)^2}{4}+2\lambda_1(\lambda_2-\lambda_1)w_1^2-\frac{(\lambda_1-\lambda_2)^2(1-2w_2^2)^2}{4}\\
&\quad-\lambda_1(\lambda_1-\lambda_2)w_2^2w_3^2\\
&=-\lambda_1(\lambda_1-\lambda_2)w_1^2<0
\end{aligned}\tag{4.15}$$

where the equations (4.10) are again used. Next, to calculate the mean curvature H of leaves, we make an orthonormal basis $\{\hat{X}, \hat{Y}\}$ from $\{X, Y\}$ by the Schmidt orthonormalization method. Namely let

$$\hat{X} = X/\|X\|, \quad \hat{Y} = (Y - \langle Y, \hat{X} \rangle \hat{X})/\|Y - \langle Y, \hat{X} \rangle \hat{X}\|.$$

Then, using the equations (4.10), we can see that

$$\text{(4.16)} \quad \langle H, W \rangle = \langle \Pi(\hat{X}, \hat{X}) + \Pi(\hat{Y}, \hat{Y}), W \rangle = -\frac{w_2 w_3}{w_1}(\lambda_1 w_1^2 + \lambda_2 w_2^2) = 0.$$

Hence the leaves of $\mathcal{O}(W)$ are minimal.

Next suppose $w_1 = 0$. In this case $w_2 = 0$ by (4.10). Set $X = (1, 0, 0)$ and $Y = (0, 1, 0)$. We note that the pair $\{X, Y\}$ is an orthonormal basis of $\mathcal{O}(W)$. Then, by (4.11) and (4.12), it follows

$$\nabla_X X = \nabla_Y Y = 0, \quad \nabla_X Y = \nabla_Y X = -(\lambda_1 - \lambda_2)/2E_3,$$

$$\text{(4.17)} \quad \Pi(X, X) = \Pi(Y, Y) = 0, \quad \Pi(X, Y) = \Pi(Y, X) = -(\lambda_1 - \lambda_2)/2E_3,$$

and moreover

$$\text{(4.18)} \qquad \kappa(W) = (\lambda_1 - \lambda_2)^2/4.$$

Hence by (4.17) it follows $H = 0$, and by (4.18) and the Gauss equation, it follows $\kappa_D(W) = 0$. These imply that the leaves of $\mathcal{O}(W)$ are flat minimal surfaces.

Summing up the arguments in this section, we have the following theorem.

THEOREM 4.2. *Let (M, g) be the group $E(1, 1)$ with any left invariant metric g and (λ_1, λ_2) the element in $\Lambda(E(1, 1))$ corresponding to g. Moreover take $W \in S^2(\mathfrak{l})$ and let w_i, $i = 1, 2, 3$, be the coefficients of W with respect to the orthonormal basis $\{E_1, E_2, E_3\}$ of $\mathfrak{l}$.*

Then, an $\mathcal{O}(W)$-geometry has an $\mathcal{O}(W)$-surface if and only if the coefficients w_i of W satisfy the equations (4.10). Under this condition, the curvature function $\kappa(W)$ moves the values between the positive maximum $(\lambda_1 - \lambda_2)^2/4$ and the minimum $(\lambda_1 + (3 + 2\sqrt{2})\lambda_2)(\lambda_1 + (3 - 2\sqrt{2})\lambda_2)/4$.

Moreover, for such a $\mathcal{O}(W)$-geometry, the $\mathcal{O}(W)$-surfaces are minimal surfaces in M with the constant nonpositive curvature $-\lambda_1(\lambda_1 - \lambda_2)w_1^2$, where $0 \leq w_1^2 \leq \lambda_2/(\lambda_2 - \lambda_1)$. In particular they are flat if and only if $w_1 = 0$, thus, $w_2 = 0$, equivalently $W = (0, 0, \pm 1)$.

Also, an $\mathcal{O}(W)$-geometry has the totally geodesic $\mathcal{O}(W)$-surfaces if and only if $\lambda_1 + \lambda_2 = 0$ and the coefficient w_3 is equal to 0.

PROOF. The first claim is the one of Proposition 4.1. From (4.10), we can see that $0 \leq w_1^2 \leq \lambda_2/(\lambda_2 - \lambda_1)$. Then, by (4.12), we can calculate the bounds of $K(W)$ for such W's.

The second claim has been done in the arguments of this section. We prove the third claim. We first note that if it is the case when $w_1 = 0$, the $\mathcal{O}(W)$-surfaces are not totally geodesic and in this case $w_3 \neq 0$ by (4.10). Hence we may consider the case when $w_1 \neq 0$. In this case it follows $w_2 \neq 0$ by (4.10). We now recall the formulas (4.13) for the second fundamental form of $\mathcal{O}(W)$-surfaces. In the formulas, it holds $w_3 = 0$ if and only if $\Pi(X, X) = 0$. Also, if $w_3 = 0$, it holds $\Pi(Y, Y) = 0$ and then it holds $\Pi(Y, X) = 0$ if and only if $\lambda_1 + \lambda_2 = 0$. These prove the third claim. $\square$

REMARK. As described in the last remark of the previous section, a maximal $\mathcal{O}(W)$-surface, if it exists, is complete, and then all the maximal $\mathcal{O}(W)$-surfaces are congruent to each other.

REMARK. The solution of the equations (4.10) can be parametrized by the coefficient w_1 as follows:

$$\left(w_1, \pm\sqrt{-\frac{\lambda_1}{\lambda_2}}w_1, \pm\sqrt{1 - \frac{(\lambda_2 - \lambda_1)}{\lambda_2}w_1^2}\right),$$

where the signs of the 2nd and the 3rd parts can be taken independently. Since $\mathcal{O}(W) = \mathcal{O}(-W)$, we may here assume that $w_1 \geq 0$. Then the above parametrization gives that of all the G-orbital $\mathcal{O}(W)$-geometries which have the associated surfaces. This parametization is devided into 4 series by the difference of the signs of the 2nd and the 3rd parts. In each series if two geometries are different, their associated surfaces are not congruent to each other. Because they have different Gauss curvatures.

Also, these 4 series look very similar from geometric properties of the associated surfaces. In fact, consider the linear transformation φ of $\mathfrak{l}$ defined by the equation $\varphi(w_1, w_2, w_3) = (w_1, -w_2, -w_3)$. Then it is an isometric automorphism of $\mathfrak{l}$ and at the same time changes the signs of the 2nd and the 3rd components of the above parametrization. So, if in place of $E(1,1)$, we consider its universal covering, φ induces an isometric automorphism on it and consequently it holds $\varphi(\mathcal{O}(W)) = \mathcal{O}(\varphi(W))$. Hence the $\mathcal{O}(W)$-geometry and the $\mathcal{O}(\varphi(W))$-geometry on the universal covering are equivalent to each other. But we don't know a relationship between 2 series obtained by changing the either sign of the 2nd and the 3rd parts. In this situation, the linear isometry ϕ of $\mathfrak{l}$ defined by the equation $\phi(w_1, w_2, w_3) = (w_1, -w_2, w_3)$ is not an automorphism of $\mathfrak{l}$.

REMARK. We last remark about a relation between the Grassmann geometries of $E(2)$ and $E(1,1)$. Consider the complexification $\mathfrak{l}^{\mathbf{C}}$ of the Lie algebra $\mathfrak{l}$ of $E(1,1)$ and put

$$F_1 = \sqrt{-1}E_1, \quad F_2 = E_2, \quad F_3 = \sqrt{-1}E_3.$$

Then these F_i generate a real form of $\mathfrak{l}^{\mathbf{C}}$ and satisfy that

$$[F_1, F_2] = 0, \quad [F_2, F_3] = \lambda_1 F_1, \quad [F_3, F_1] = (-\lambda_2)F_2,$$

where $-\lambda_2 \geq \lambda_1 > 0$. Hence the real form is isomorphic to the Lie algebra $E(2)$. From this fact and the results of Theorem 4.2, it may be expected that there exist somewhat relations between two $\mathcal{O}(W)$-geometries of $E(2)$ and $E(1,1)$ when $W = (0, 0, \pm 1)$, and between the local Euclidean geometry of $E(2)$ and the geometry of $E(1,1)$ when $\lambda_1 + \lambda_2 = 0$ and $w_3 = 0$.

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References

[1] M. P. do Carmo, Differential Geometry of Curves and Surfaces, Prentice-Hall Inc., New Jersey, 1976.
[2] J. Inoguchi, K. Kuwabara, and H. Naitoh, Grassmann geometry on the 3-dimensional Heisenberg group, Hokkaido Math. J., to appear.
[3] J. Milnor, Curvatures of left invariant metrics on Lie groups, Advances in Math. **21** (1976), 293–329.
[4] V. Patrangenaru, Classifying 3 and 4 dimensional homogeneous Riemannian manifolds by Cartan triples, Pacific J. of Math. **173-2** (1996), 511–532.

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