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THE DIRICHLET-NEUMANN PROBLEM FOR THE DISSIPATIVE HELMHOLTZ EQUATION IN A 2-D CRACKED DOMAIN WITH THE NEUMANN CONDITION ON CRACKS

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Abstract. The Dirichlet-Neumann problem for the dissipative Helmholtz equation in a connected plane region bounded by closed curves and open arcs (cuts) is studied. The Dirichlet condition is specified on the closed curves, while the Neumann condition is specified on the cuts. The existence of a classical solution is proved by potential theory. The problem is reduced to a Fredholm equation of the second kind, which is uniquely solvable. An integral representation for the solution of the problem is obtained. Our approach holds for both interior and exterior domains.

1. Introduction

The boundary of a 2-D cracked domain includes both closed curves and open arcs (cuts or cracks). The boundary condition is specified on the whole boundary, i.e. on both closed curves and open arcs. Open arcs or cuts model screens, wings, cracks or spits in applied problems. Boundary value problems for PDEs in cracked domains describe different physical processes such as distribution of electric and heat fields, propagation of acoustic waves and scattering by cracks, etc. Stationary waves in isotropic media are described by the Helmholtz equation

$$\Delta u + k^2 u = 0,$$

where Δ is Laplacian. If $\operatorname{Im} k = 0$, then this equation is called propagative. If $\operatorname{Im} k \neq 0$, then this equation is called dissipative, since energy of waves dissipates

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in the media [13]. The skew derivative problem for the propagative Helmholtz equation outside cuts (cracks) in a plane has been studied in [12]. The Dirichlet and Neumann problems for the dissipative Helmholtz equation in cracked domains were studied in [7], [8]. In the present paper we study the mixed problem for the dissipative Helmholtz equation in a cracked domain (interior or exterior), so that the Dirichlet condition is specified on the closed curves, while the Neumann condition is specified on the cuts. We obtain an integral representation for a solution and reduce the problem to the uniquely solvable Fredholm integral equation of the second kind and index zero. The obtained integral equation can be solved numerically by standard codes [9]. The results of the present paper may be helpful in applied inverse problems of determination of crack locations.

2. Formulation of the Problem

By a simple open curve we mean a non-closed smooth arc of finite length without self-intersections [5].

Let γ be a set of curves, which may be closed and open. We say that $\gamma \in C^{2,\lambda}$ (or $\gamma \in C^{1,\lambda}$) if curves γ are of class $C^{2,\lambda}$ (or $C^{1,\lambda}$) with the Hölder exponent $\lambda \in (0,1]$.

In the plane $x = (x_1, x_2) \in R^2$ we consider the multiply connected domain bounded by simple open curves $\Gamma_1^1, \dots, \Gamma_{N_1}^1 \in C^{2,\lambda}$ and simple closed curves $\Gamma_1^2, \dots, \Gamma_{N_2}^2 \in C^{2,\lambda}$, $\lambda \in (0,1]$, so that the curves do not have common points, in particular, endpoints. We will consider both the case of an exterior domain and the case of an interior domain, when the curve Γ_1^2 encloses all others. We put

$$\Gamma^1 = \bigcup_{n=1}^{N_1} \Gamma_n^1, \quad \Gamma^2 = \bigcup_{n=1}^{N_2} \Gamma_n^2, \quad \Gamma = \Gamma^1 \cup \Gamma^2.$$

The connected domain bounded by Γ^2 and containing Γ^1 will be called $\mathcal{D}$, so that $\partial\mathcal{D} = \Gamma^2$, $\Gamma^1 \subset \mathcal{D}$. We assume that each curve Γ_n^j is parametrized by the arc length s: $\Gamma_n^j = \{x : x = x(s) = (x_1(s), x_2(s)), s \in [a_n^j, b_n^j]\}$, $n = 1, \dots, N_j$, $j = 1, 2$, so that

$$a_1^1 < b_1^1 < \cdots < a_{N_1}^1 < b_{N_1}^1 < a_1^2 < b_1^2 < \cdots < a_{N_2}^2 < b_{N_2}^2$$

and the domain $\mathcal{D}$ is to the right when the parameter s increases on Γ_n^2. Therefore points $x \in \Gamma$ and values of the parameter s are in one-to-one correspondence except a_n^2, b_n^2, which correspond to the same point x for $n = 1, \dots, N_2$. Below the sets of the intervals on the Os axis

$$\bigcup_{n=1}^{N_1}[a_n^1, b_n^1], \quad \bigcup_{n=1}^{N_2}[a_n^2, b_n^2], \quad \bigcup_{j=1}^{2}\bigcup_{n=1}^{N_j}[a_n^j, b_n^j]$$

will be denoted by Γ^1, Γ^2 and Γ also.

We put $C^{j,r}(\Gamma_n^2) = \{\mathscr{F}(s) : \mathscr{F}(s) \in C^{j,r}[a_n^2, b_n^2], \mathscr{F}^{(m)}(a_n^2) = \mathscr{F}^{(m)}(b_n^2), m = 0, j\}$, $j = 0, 1$, $r \in [0, 1]$ and

$$C^{j,r}(\Gamma^2) = \bigcap_{n=1}^{N_2} C^{j,r}(\Gamma_n^2).$$

The tangent vector to Γ at the point $x(s)$ we denote by $\tau_x = (\cos\alpha(s), \sin\alpha(s))$, where $\cos\alpha(s) = x_1'(s)$, $\sin\alpha(s) = x_2'(s)$. Let $\mathbf{n}_x = (\sin\alpha(s), -\cos\alpha(s))$ be a normal vector to Γ at $x(s)$. The direction of $\mathbf{n}_x$ is chosen such that it will coincide with the direction of τ_x if $\mathbf{n}_x$ is rotated anticlockwise through an angle of $\pi/2$. So, $\mathbf{n}_x$ is the inward normal to $\mathscr{D}$ on Γ^2.

We consider Γ^1 as a set of cuts. The side of Γ^1 which is on the left, when the parameter s increases will be denoted by $(\Gamma^1)^+$ and the opposite side will be denoted by $(\Gamma^1)^-$.

We say, that the function $u(x)$ belongs to the smoothness class $\mathbf{K}$ if

1) $u \in C^0(\overline{\mathscr{D}\backslash\Gamma^1}) \cap C^2(\mathscr{D}\backslash\Gamma^1)$ and $u(x)$ is continuous at the end-points of the cuts Γ^1,

2) $\nabla u \in C^0(\overline{\mathscr{D}\backslash\Gamma^1}\backslash X)$, where X is a point-set, consisting of the end-points of Γ^1:

$$X = \bigcup_{n=1}^{N_1}(x(a_n^1) \cup x(b_n^1)),$$

3) in the neighbourhood of any point $x(d) \in X$ for some constants $\mathscr{C} > 0$, $\varepsilon > -1$ the inequality holds

$$|\nabla u| \leq \mathscr{C}|x - x(d)|^{\varepsilon}, \tag{1}$$

where $x \to x(d)$ and $d = a_n^1$ or $d = b_n^1$, $n = 1, \ldots, N_1$.

Remark. In the definition of the class $\mathbf{K}$ we consider Γ^1 as a set of cuts in the domain $\mathscr{D}$. According to this definition, $u(x)$ and $\nabla u(x)$ are continuously extensible on cuts $\Gamma^1\backslash X$ from the left and from the right, but their values on $\Gamma^1\backslash X$ from the left and from the right may be different, so that $u(x)$ and $\nabla u(x)$ may have a jump across $\Gamma^1\backslash X$.

Let us formulate the mixed Dirichlet-Neumann problem for the dissipative Helmholtz equation in the domain $\mathscr{D}\backslash\Gamma^1$ (interior or exterior). The Dirichlet condition is specified on the closed curves Γ^2, while the Neumann condition is posed on the cuts Γ^1.

Problem **U**. To find a function $u(x)$ of the class **K** which satisfies the Helmholtz equation

$$u_{x_1x_1}(x) + u_{x_2x_2}(x) + k^2u(x) = 0, \quad x \in \mathscr{D}\backslash\Gamma^1,\ k = \text{const},\ \operatorname{Im} k > 0 \tag{2a}$$

and the boundary conditions

$$\left.\frac{\partial u(x)}{\partial \mathbf{n}_x}\right|_{x(s)\in(\Gamma^1)^+} = F^+(s), \quad \left.\frac{\partial u(x)}{\partial \mathbf{n}_x}\right|_{x(s)\in(\Gamma^1)^-} = F^-(s), \tag{2b}$$

$$u(x)|_{x(s)\in\Gamma^2} = F(s).$$

If $\mathscr{D}$ is an exterior domain, then we add the following conditions at infinity:

$$u = o(|x|^{-1/2}), \quad |\nabla u(x)| = o(|x|^{-1/2}), \quad |x| = \sqrt{x_1^2 + x_2^2} \to \infty. \tag{2c}$$

All conditions of the problem **U** must be satisfied in the classical sense.

The problem **U** includes two particular cases. In the first case $\Gamma^1 = \varnothing$, $\Gamma^2 \neq \varnothing$ and we obtain the Dirichlet problem for the dissipative Helmholtz equation in the domain $\mathscr{D}$ without cuts (this is a particular case of [8] also). In the second case $\Gamma^1 \neq \varnothing$, $\Gamma^2 = \varnothing$ and we obtain the Neumann problem for the dissipative Helmholtz equation outside the cuts Γ^1 on a plane (see [4], [7]).

On the basis of the energy equalities [1, v. IV], [11], we can easily prove the following assertion.

THEOREM 1. *If* $\Gamma \in C^{2,\lambda}$, $\lambda \in (0,1]$, *then the problem* **U** *has at most one solution.*

The theorem holds for both interior and exterior domain $\mathscr{D}$.

3. Integral Equations at the Boundary

Below we assume that

$$F^+(s),\ F^-(s) \in C^{0,\lambda}(\Gamma^1), \quad F(s) \in C^{1,\lambda}(\Gamma^2), \quad \lambda \in (0,1]. \tag{3}$$

If $\mathscr{B}_1(\Gamma^1)$, $\mathscr{B}_2(\Gamma^2)$ are Banach spaces of functions given on Γ^1 and Γ^2, then for functions given on Γ we introduce the Banach space $\mathscr{B}_1(\Gamma^1) \cap \mathscr{B}_2(\Gamma^2)$ with the norm $\|\cdot\|_{\mathscr{B}_1(\Gamma^1)\cap\mathscr{B}_2(\Gamma^2)} = \|\cdot\|_{\mathscr{B}_1(\Gamma^1)} + \|\cdot\|_{\mathscr{B}_2(\Gamma^2)}$.

We consider the angular potential from [3], [4] for the equation (2a) on Γ^1

$$w_1[\mu](x) = \frac{i}{4}\int_{\Gamma^1} \mu(\sigma) V(x,\sigma)\, d\sigma. \tag{4}$$

The kernel $V(x,\sigma)$ is defined on each curve Γ_n^1 $(n = 1,\dots,N_1)$ by the formula

$$V(x,\sigma) = \int_{a_n^1}^{\sigma} \frac{\partial}{\partial \mathbf{n}_y} \mathscr{H}_0^{(1)}(k|x - y(\xi)|)\, d\xi, \quad \sigma \in [a_n^1, b_n^1],$$

where $\mathscr{H}_0^{(1)}(z)$ is the Hankel function of the first kind [10]:

$$\mathscr{H}_0^{(1)}(z) = \frac{\sqrt{2}\exp(iz - i\pi/4)}{\pi\sqrt{z}} \int_0^{\infty} \exp(-t) t^{-1/2} \left(1 + \frac{it}{2z}\right)^{-1/2} dt,$$

$$y = y(\xi) = (y_1(\xi), y_2(\xi)), \quad |x - y(\xi)| = \sqrt{(x_1 - y_1(\xi))^2 + (x_2 - y_2(\xi))^2}.$$

Below we suppose that $\mu(\sigma)$ belongs to the Banach space $C_q^{\omega}(\Gamma^1)$, $\omega \in (0,1]$, $q \in [0,1)$ and satisfies the following additional conditions

$$\int_{a_n^1}^{b_n^1} \mu(\sigma)\, d\sigma = 0, \quad n = 1,\dots,N_1. \tag{5}$$

We say, that $\mu(s) \in C_q^{\omega}(\Gamma^1)$ if

$$\mu(s) \prod_{n=1}^{N_1} |s - a_n^1|^q |s - b_n^1|^q \in C^{0,\omega}(\Gamma^1),$$

where $C^{0,\omega}(\Gamma^1)$ is a Hölder space with the exponent ω and

$$\|\mu(s)\|_{C_q^{\omega}(\Gamma^1)} = \left\| \mu(s) \prod_{n=1}^{N_1} |s - a_n^1|^q |s - b_n^1|^q \right\|_{C^{0,\omega}(\Gamma^1)}.$$

As shown in [3], [4] for such $\mu(\sigma)$ the angular potential $w_1[\mu](x)$ belongs to the class **K**. In particular, the inequality (1) holds with $\varepsilon = -q$, if $q \in (0,1)$. Moreover, integrating $w_1[\mu](x)$ by parts and using (5) we express the angular potential in terms of a double layer potential

$$w_1[\mu](x) = -\frac{i}{4}\int_{\Gamma^1} \rho(\sigma) \frac{\partial}{\partial \mathbf{n}_y} \mathscr{H}_0^{(1)}(k|x - y(\sigma)|)\, d\sigma, \tag{6}$$

with the density

$$\rho(\sigma) = \int_{a_n^1}^{\sigma} \mu(\xi)\, d\xi, \quad \sigma \in [a_n^1, b_n^1],\ n = 1,\dots,N_1.$$

Consequently, $w_1[\mu](x)$ satisfies both equation (2a) outside Γ^1 and the conditions at infinity (2c).

Let us construct a solution of the problem **U**. This solution can be obtained with the help of potential theory for the Helmholtz equation (2a). We seek a solution of the problem in the following form

$$u[\nu,\mu](x) = v_1[\nu](x) + w[\mu](x), \tag{7}$$

where

$$\begin{gathered} v_1[\nu](x) = \frac{i}{4}\int_{\Gamma^1} \nu(\sigma)\mathscr{H}_0^{(1)}(k|x-y(\sigma)|)\,d\sigma, \\ w[\mu](x) = w_1[\mu](x) + w_2[\mu](x), \\ w_2[\mu](x) = \frac{i}{4}\int_{\Gamma^2} \mu(\sigma)\frac{\partial}{\partial \mathbf{n}_y}\mathscr{H}_0^{(1)}(k|x-y(\sigma)|)\,d\sigma, \end{gathered} \tag{8}$$

and $w_1[\mu](x)$ is given by (4), (6).

By $\int_{\Gamma^j}\cdots d\sigma$ we mean

$$\sum_{n=1}^{N_j}\int_{a_n^j}^{b_n^j}\cdots d\sigma.$$

We will look for $\nu(s)$ in the space $C^{0,\lambda}(\Gamma^1)$, then the single layer potential $v_1[\nu](x)$ belongs to the class **K**, obeys the equation (2a) outside Γ^1 and satisfies the conditions at infinity (2c) in case of an exterior domain $\mathscr{D}$ (see [1, v. IV], [3]).

We will seek $\mu(s)$ from the Banach space $C_q^{\omega}(\Gamma^1)\cap C^{1,\lambda/4}(\Gamma^2)$, $\omega\in(0,1]$, $q\in[0,1)$ with the norm $\|\cdot\|_{C_q^{\omega}(\Gamma^1)\cap C^{1,\lambda/4}(\Gamma^2)} = \|\cdot\|_{C_q^{\omega}(\Gamma^1)} + \|\cdot\|_{C^{1,\lambda/4}(\Gamma^2)}$. Besides, $\mu(s)$ must satisfy conditions (5).

It follows from Theorem 5 in Appendix 1 that for such $\mu(s)$ the double layer potential $w_2[\mu](x)$ belongs to $C^1(\bar{\mathscr{D}})$, and so $w_2[\mu](x)\in\mathbf{K}$. Besides, $w_2[\mu](x)$ obeys the equation (2a) and satisfies the conditions at infinity (2c) if $\mathscr{D}$ is an exterior domain [1, v. IV], [11]. Consequently, for densities $\mu(s)$, $\nu(s)$ described above, the function (7) belongs to the class **K** and satisfies all conditions of the problem **U** except the boundary condition (2b).

To satisfy the boundary condition we put (7) in (2b), use the limit formulas for the angular potential from [3] and arrive at the integral equations for the densities $\mu(s)$, $\nu(s)$:

(9a) $$\pm\frac{1}{2}\nu(s)+\frac{i}{4}\int_{\Gamma^1}\nu(\sigma)\frac{\partial}{\partial\mathbf{n}_x}\mathscr{H}_0^{(1)}(k|x(s)-y(\sigma)|)\,d\sigma$$
$$-\frac{1}{2\pi}\int_{\Gamma^1}\mu(\sigma)\frac{\sin\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|}\,d\sigma+\frac{i}{4}\int_{\Gamma^1}\mu(\sigma)\frac{\partial}{\partial\mathbf{n}_x}V_0(x(s),\sigma)\,d\sigma$$
$$+\frac{i}{4}\int_{\Gamma^2}\mu(\sigma)\frac{\partial}{\partial\mathbf{n}_x}\frac{\partial}{\partial\mathbf{n}_y}\mathscr{H}_0^{(1)}(k|x(s)-y(\sigma)|)\,d\sigma=F^{\pm}(s),\quad s\in\Gamma^1,$$

(9b) $$\frac{i}{4}\int_{\Gamma^1}\nu(\sigma)\mathscr{H}_0^{(1)}(k|x(s)-y(\sigma)|)\,d\sigma+\frac{i}{4}\int_{\Gamma^1}\mu(\sigma)V(x(s),\sigma)\,d\sigma+\frac{1}{2}\mu(s)$$
$$+\frac{i}{4}\int_{\Gamma^2}\mu(\sigma)\frac{\partial}{\partial\mathbf{n}_y}\mathscr{H}_0^{(1)}(k|x(s)-y(\sigma)|)\,d\sigma=F(s),\quad s\in\Gamma^2,$$

where $V(x,\sigma)$ is the kernel of the angular potential (4),

$$V_0(x,\sigma)=\int_{a_n^1}^{\sigma}\frac{\partial}{\partial\mathbf{n}_y}h(k|x-y(\xi)|)\,d\xi,\quad \sigma\in[a_n^1,b_n^1],\ n=1,2,\dots,N_1,$$
$$h(z)=\mathscr{H}_0^{(1)}(z)-\frac{2i}{\pi}\ln\frac{z}{k}.$$

By $\varphi_0(x,y)$ we denote the angle between the vector $\overrightarrow{xy}$ and the direction of the normal $\mathbf{n}_x$. The angle $\varphi_0(x,y)$ is taken to be positive if it is measured anti-clockwise from $\mathbf{n}_x$ and negative if it is measured clockwise from $\mathbf{n}_x$. Besides, $\varphi_0(x,y)$ is continuous in $x,y\in\Gamma$ if $x\neq y$.

Equation (9a) is obtained as $x\to x(s)\in(\Gamma^1)^{\pm}$ and comprises two integral equations. The upper sign denotes the integral equation on $(\Gamma^1)^+$, the lower sign denotes the integral equation on $(\Gamma^1)^-$.

In addition to the integral equations written above we have the conditions (5).

Subtracting the integral equations (9a) we find

(10) $$\nu(s)=(F^+(s)-F^-(s))\in C^{0,\lambda}(\Gamma^1).$$

We note that $\nu(s)$ is found completely and satisfies all required conditions. Hence, the potential $v_1[\nu](x)$ is found completely as well.

We introduce the functions $f_1(s)$ and $f_2(s)$ by the formulae

(11a) $$f_1(s)=\frac{1}{2}(F^+(s)+F^-(s))$$
$$-\frac{i}{4}\int_{\Gamma^1}(F^+(\sigma)-F^-(\sigma))\frac{\partial}{\partial\mathbf{n}_x}\mathscr{H}_0^{(1)}(k|x(s)-y(\sigma)|)\,d\sigma,\quad s\in\Gamma^1,$$

$$\text{(11b)}\quad f_2(s) = F(s) - \frac{i}{4}\int_{\Gamma^1} (F^+(\sigma) - F^-(\sigma))\mathscr{H}_0^{(1)}(k|x(s) - y(\sigma)|)\, d\sigma, \quad s \in \Gamma^2.$$

As shown in [4], if $s \in \Gamma^1$, then $f_1(s) \in C^{0,\lambda}(\Gamma^1)$. Clearly, $f_2(s) \in C^{1,\lambda}(\Gamma^2)$.

Adding the integral equations (9a) and taking into account (9b) we obtain the integral equations for $\mu(s)$ on Γ^1 and on Γ^2

$$\text{(12a)}\quad -\frac{1}{2\pi}\int_{\Gamma^1} \mu(\sigma)\frac{\sin\varphi_0(x(s), y(\sigma))}{|x(s) - y(\sigma)|}\, d\sigma + \frac{i}{4}\int_{\Gamma^1} \mu(\sigma)\frac{\partial}{\partial \mathbf{n}_x} V_0(x(s), \sigma)\, d\sigma$$
$$+ \frac{i}{4}\int_{\Gamma^2} \mu(\sigma)\frac{\partial}{\partial \mathbf{n}_x}\frac{\partial}{\partial \mathbf{n}_y}\mathscr{H}_0^{(1)}(k|x(s) - y(\sigma)|)\, d\sigma = f_1(s), \quad s \in \Gamma^1,$$

$$\text{(12b)}\quad \frac{i}{4}\int_{\Gamma^1} \mu(\sigma) V(x(s), \sigma)\, d\sigma$$
$$+ \frac{1}{2}\mu(s) + \frac{i}{4}\int_{\Gamma^2} \mu(\sigma)\frac{\partial}{\partial \mathbf{n}_y}\mathscr{H}_0^{(1)}(k|x(s) - y(\sigma)|)\, d\sigma = f_2(s), \quad s \in \Gamma^2,$$

where $f_1(s)$, $f_2(s)$ are given in (11).

It follows from Lemma 4 in Appendix 2 that the sum of integral terms in (12b) belongs to $C^{1,\lambda/4}(\Gamma^2)$ in s for any $\mu(s) \in C_q^\omega(\Gamma^1) \cap C^0(\Gamma^2)$, $\omega \in (0, 1]$, $q \in [0, 1)$. Since $f_2(s) \in C^{1,\lambda}(\Gamma^2)$ in (12b), any solution of the equation (12b) in $C_q^\omega(\Gamma^1) \cap C^0(\Gamma^2)$ automatically belongs to $C_q^\omega(\Gamma^1) \cap C^{1,\lambda/4}(\Gamma^2)$. Consequently, below we will look for a solution $\mu(s)$ of the equations (12) in $C_q^\omega(\Gamma^1) \cap C^0(\Gamma^2)$, $\omega \in (0, 1]$, $q \in [0, 1)$.

Thus, if $\mu(s)$ is a solution of equations (5), (12) from the space $C_q^\omega(\Gamma^1) \cap C^0(\Gamma^2)$, $\omega \in (0, 1]$, $q \in [0, 1)$, then $\mu(s) \in C_q^\omega(\Gamma^1) \cap C^{1,\lambda/4}(\Gamma^2)$ and the potential (7) satisfies all conditions of the problem $\mathbf{U}$.

The following theorem holds.

THEOREM 2. *Let $\Gamma \in C^{2,\lambda}$ and conditions* (3) *hold. If the system of equations* (12), (5) *has a solution $\mu(s)$ from the Banach space $C_q^\omega(\Gamma^1) \cap C^0(\Gamma^2)$, $\omega \in (0, 1]$, $q \in [0, 1)$, then a solution of the problem* $\mathbf{U}$ *is given by* (7), *where $v(s)$ is defined in* (10).

Below we look for $\mu(s)$ in the Banach space $C_q^\omega(\Gamma^1) \cap C^0(\Gamma^2)$.

If $s \in \Gamma^2$, then (12b) is an integral equation of the second kind. If $s \in \Gamma^1$, then (12a) is a singular integral equation [5], [2]. The first term in (12a) is a Cauchy singular integral.

Our further treatment will be aimed to the proof of the solvability of the system (5), (12) in the Banach space $C_q^\omega(\Gamma^1) \cap C^0(\Gamma^2)$. Moreover, we reduce the system (5), (12) to a Fredholm equation of the second kind, which can be easily computed by classical methods.

Equation (12b) on Γ^2 we rewrite in the form

$$\mu(s) + \int_\Gamma \mu(\sigma) A_2(s,\sigma)\, d\sigma = 2f_2(s), \quad s \in \Gamma^2, \tag{13}$$

where

$$A_2(s,\sigma) = \frac{i}{2}(1-\delta(\sigma))V(x(s),\sigma) + \frac{i}{2}\delta(\sigma)\frac{\partial}{\partial \mathbf{n}_y}\mathscr{H}_0^{(1)}(k|x(s)-y(\sigma)|),$$

$$\delta(s) = \begin{cases} 0, & \text{if } s \in \Gamma^1, \\ 1, & \text{if } s \in \Gamma^2, \end{cases}$$

$V(x,\sigma)$ is the kernel of the angular potential (4). According to [3], [4], $A_2(s,\sigma) \in C^0(\Gamma^2 \times \Gamma)$, since $\Gamma \in C^{2,\lambda}$, $\lambda \in (0,1]$.

Remark. Evidently, $f_2(a_n^2) = f_2(b_n^2)$ and $A_2(a_n^2,\sigma) = A_2(b_n^2,\sigma)$ for any $\sigma \in \Gamma$ $(n = 1,\dots,N_2)$. Hence, if $\mu(s)$ is a solution of equation (13) from $C^0\left(\bigcup_{n=1}^{N_2}[a_n^2,b_n^2]\right)$, then, according to the equality (13), $\mu(s)$ automatically satisfies matching conditions $\mu(a_n^2) = \mu(b_n^2)$ for $n = 1,\dots,N_2$ and therefore belongs to $C^0(\Gamma^2)$. This observation is true for equation (12b) also and can be helpful for finding numerical solutions, since we may abandon matching conditions $\mu(a_n^2) = \mu(b_n^2)$ $(n = 1,\dots,N_2)$, which are fulfilled automatically.

It can be easily proved that

$$\frac{\sin\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|} - \frac{1}{\sigma - s} \in C^{0,\lambda}(\Gamma^1 \times \Gamma^1)$$

(see [3], [4] for details). Therefore we can rewrite equation (12a) on Γ^1 in the form

$$\frac{1}{\pi}\int_{\Gamma^1} \mu(\sigma)\frac{d\sigma}{\sigma - s} + \int_\Gamma \mu(\sigma) Y(s,\sigma)\, d\sigma = -2f_1(s), \quad s \in \Gamma^1, \tag{14}$$

where

$$Y(s,\sigma) = \left\{(1-\delta(\sigma))\left[\frac{1}{\pi}\left(\frac{\sin\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|} - \frac{1}{\sigma - s}\right) - \frac{i}{2}\frac{\partial}{\partial \mathbf{n}_x}V_0(x(s),\sigma)\right]\right.$$
$$\left. - \frac{i}{2}\delta(\sigma)\frac{\partial}{\partial \mathbf{n}_x}\frac{\partial}{\partial \mathbf{n}_y}\mathscr{H}_0^{(1)}(k|x(s)-y(\sigma)|)\right\} \in C^{0,p_0}(\Gamma^1 \times \Gamma),$$

$p_0 = \lambda$ if $0 < \lambda < 1$ and $p_0 = 1 - \varepsilon_0$ for any $\varepsilon_0 \in (0,1)$ if $\lambda = 1$ (we took into account Lemma 3 in [4]).

4. The Fredholm Integral Equation and the Solution of the Problem

Inverting the singular integral operator in (14) we arrive at the following integral equation of the second kind [5]:

$$\mu(s) + \frac{1}{Q_1(s)} \int_\Gamma \mu(\sigma) A_0(s,\sigma)\, d\sigma + \frac{1}{Q_1(s)} \sum_{n=0}^{N_1-1} G_n s^n = \frac{1}{Q_1(s)} \Phi_0(s), \quad s \in \Gamma^1, \tag{15}$$

where

$$A_0(s,\sigma) = -\frac{1}{\pi} \int_{\Gamma^1} \frac{Y(\xi,\sigma)}{\xi - s} Q_1(\xi)\, d\xi,$$

$$Q_1(s) = \prod_{n=1}^{N_1} \left| \sqrt{s - a_n^1} \sqrt{b_n^1 - s} \right| \operatorname{sign}(s - a_n^1),$$

$$\Phi_0(s) = \frac{1}{\pi} \int_{\Gamma^1} \frac{2Q_1(\sigma) f_1(\sigma)}{\sigma - s}\, d\sigma,$$

$G_0, \ldots, G_{N_1-1}$ are arbitrary constants.

To derive equations for $G_0, \ldots, G_{N_1-1}$ we substitute $\mu(s)$ from (15) in the conditions (5), then we obtain

$$\int_\Gamma \mu(\sigma) l_n(\sigma)\, d\sigma + \sum_{m=0}^{N_1-1} B_{nm} G_m = H_n, \quad n = 1, \ldots, N_1, \tag{16}$$

where

$$l_n(\sigma) = -\int_{\Gamma_n^1} Q_1^{-1}(s) A_0(s,\sigma)\, ds,$$

$$B_{nm} = -\int_{\Gamma_n^1} Q_1^{-1}(s) s^m\, ds, \tag{17}$$

$$H_n = -\int_{\Gamma_n^1} Q_1^{-1}(s) \Phi_0(s)\, ds.$$

By B we denote the $N_1 \times N_1$ matrix with the elements B_{nm} from (17). As shown in [4], the matrix B is invertible. The elements of the inverse matrix will be called $(B^{-1})_{nm}$. Inverting the matrix B in (16) we express the constants $G_0, \ldots, G_{N_1-1}$ in terms of $\mu(s)$

$$G_n = \sum_{m=1}^{N_1} (B^{-1})_{nm} \left[H_m - \int_\Gamma \mu(\sigma) l_m(\sigma)\, d\sigma \right].$$

We substitute G_n in (15) and obtain the following integral equation for $\mu(s)$ on Γ^1

$$\mu(s) + \frac{1}{Q_1(s)} \int_\Gamma \mu(\sigma) A_1(s,\sigma)\, d\sigma = \frac{1}{Q_1(s)} \Phi_1(s), \quad s \in \Gamma^1, \tag{18}$$

where

$$A_1(s,\sigma) = A_0(s,\sigma) - \sum_{n=0}^{N_1-1} s^n \sum_{m=1}^{N_1} (B^{-1})_{nm} l_m(\sigma),$$

$$\Phi_1(s) = \Phi_0(s) - \sum_{n=0}^{N_1-1} s^n \sum_{m=1}^{N_1} (B^{-1})_{nm} H_m.$$

It can be shown using the properties of singular integrals [2], [5], that $\Phi_0(s)$, $A_0(s,\sigma)$ are Hölder functions if $s \in \Gamma^1$, $\sigma \in \Gamma$. Hence, $\Phi_1(s)$, $A_1(s,\sigma)$ are also Hölder functions if $s \in \Gamma^1$, $\sigma \in \Gamma$. Consequently, any integrable on Γ^1 and continuous on Γ^2 solution of equation (18) belongs to $C^\omega_{1/2}(\Gamma^1)$ with some $\omega \in (0,1]$. Therefore, below we look for $\mu(s)$ on Γ in the space $C^\omega_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ with $\omega \in (0,1]$. It can be easily verified that any solution of equation (18) in this space satisfies both conditions (5) and equation (14). Indeed, if a function $\mu(s) \in C^\omega_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ turns equation (18) to identity, then multiplying this identity by $(s-t)^{-1}$, where $t \in \Gamma^1$, and integrating in s over Γ^1, we obtain identity (14). Integrating identity (18) in s over Γ^1_n $(n = 1,\dots,N_1)$, one can prove that conditions (5) hold.

We put

$$Q(s) = (1-\delta(s))Q_1(s) + \delta(s), \quad s \in \Gamma.$$

Instead of $\mu(s) \in C^\omega_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ we introduce the new unknown function $\mu_*(s) = \mu(s)Q(s) \in C^{0,\omega}(\Gamma^1) \cap C^0(\Gamma^2)$ and rewrite system of equations (13), (18) in the form of one equation

$$\mu_*(s) + \int_\Gamma \mu_*(\sigma) Q^{-1}(\sigma) A(s,\sigma)\, d\sigma = \Phi(s), \quad s \in \Gamma, \tag{19}$$

where

$$A(s,\sigma) = (1-\delta(s))A_1(s,\sigma) + \delta(s)A_2(s,\sigma),$$

$$\Phi(s) = (1-\delta(s))\Phi_1(s) + 2\delta(s) f_2(s).$$

Thus, the system of equations (12), (5) for $\mu(s)$ has been reduced to the equation (19) for the function $\mu_*(s)$. It is clear from our consideration that any solution $\mu_*(s)$ of equation (19) in the space $C^{0,\omega}(\Gamma^1) \cap C^0(\Gamma^2)$ with $\omega \in (0,1]$ produces the solution $\mu(s) = \mu_*(s)/Q(s) \in C^{\omega}_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ of system (12), (5).

As noted above, $\Phi_1(s)$ and $A_1(s,\sigma)$ are Hölder functions if $s \in \Gamma^1$, $\sigma \in \Gamma$. More precisely (see [4], [5]), $\Phi_1(s) \in C^{0,p}(\Gamma^1)$, $p = \min\{1/2, \lambda\}$ and $A_1(s,\sigma)$ belongs to $C^{0,p}(\Gamma^1)$ in s uniformly with respect to $\sigma \in \Gamma$.

We arrive at the following assertion.

LEMMA 1. *Let* $\Gamma \in C^{2,\lambda}$, $\lambda \in (0,1]$, $\Phi(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$, $p = \min\{\lambda, 1/2\}$. *If* $\mu_*(s)$ *from* $C^0(\Gamma)$ *satisfies the equation* (19), *then* $\mu_*(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$.

The condition $\Phi(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$ holds if $f_1(s) \in C^{0,\lambda}(\Gamma^1)$, $f_2(s) \in C^0(\Gamma^2)$.

Hence below we will seek a solution $\mu_*(s)$ of the equation (19) in $C^0(\Gamma)$.

Since $A_1(s,\sigma) \in C^0(\Gamma^1 \times \Gamma)$ and $A_2(s,\sigma) \in C^0(\Gamma^2 \times \Gamma)$, one can verify using the Arzela theorem [6] that the integral operator from (19)

$$\mathbf{A}\mu_* = \int_\Gamma \mu_*(\sigma) Q^{-1}(\sigma) A(s,\sigma)\, d\sigma$$

is a compact operator mapping $C^0(\Gamma)$ into itself. Therefore, (19) is a Fredholm equation of the second kind in the Banach space $C^0(\Gamma)$.

Let us show that homogeneous equation (19) has only a trivial solution in $C^0(\Gamma)$. Then, according to Fredholm's theorems, the inhomogeneous equation (19) has a unique solution in $C^0(\Gamma)$ for any right-hand side in $C^0(\Gamma)$. We will prove this by a contradiction. Let $\mu_*^0(s) \in C^0(\Gamma)$ be a non-trivial solution of the homogeneous equation (19). According to Lemma 1, $\mu_*^0(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$, $p = \min\{\lambda, 1/2\}$. Therefore the function $\mu^0(s) = \mu_*^0(s) Q^{-1}(s) \in C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ converts the homogeneous equations (13), (18) into identities. Using the homogeneous identity (18) we check, that $\mu^0(s)$ satisfies conditions (5). Besides, acting on the homogeneous identity (18) with a singular operator with the kernel $(s-t)^{-1}$ we find that $\mu^0(s)$ satisfies the homogeneous equation (14). Consequently, $\mu^0(s)$ satisfies the homogeneous equations (12). On the basis of Theorem 2, $u[0,\mu^0](x) \equiv w[\mu^0](x)$ is a solution of the homogeneous problem U. According to Theorem 1: $w[\mu^0](x) \equiv 0$, $x \in \mathscr{D}\backslash\Gamma^1$. Using the limit formulas for tangent derivatives of an angular potential [3], we obtain

$$\lim_{x\to x(s)\in(\Gamma^1)^+}\frac{\partial}{\partial\tau_x}w[\mu^0](x)-\lim_{x\to x(s)\in(\Gamma^1)^-}\frac{\partial}{\partial\tau_x}w[\mu^0](x)=\mu^0(s)\equiv 0,\quad s\in\Gamma^1.$$

Hence, $w[\mu^0](x)=w_2[\mu^0](x)\equiv 0$, $x\in\mathscr{D}$, and $\mu^0(s)$ satisfies the following homogeneous equation

$$\frac{1}{2}\mu^0(s)+\frac{i}{4}\int_{\Gamma^2}\mu^0(\sigma)\frac{\partial}{\partial\mathbf{n}_y}\mathscr{H}_0^{(1)}(k|x(s)-y(\sigma)|)\,d\sigma=0,\quad s\in\Gamma^2. \tag{20}$$

Equation (20) has only the trivial solution $\mu^0(s)\equiv 0$ in $C^0(\Gamma^2)$. This is true for both interior and exterior domain $\mathscr{D}$. The detailed proof is presented in the section 5.

Consequently, if $s\in\Gamma$, then $\mu^0(s)\equiv 0$, $\mu_*^0(s)=\mu^0(s)Q^{-1}(s)\equiv 0$ and we arrive at the contradiction to the assumption that $\mu_*^0(s)$ is a non-trivial solution of the homogeneous equation (19). Thus, the homogeneous Fredholm equation (19) has only a trivial solution in $C^0(\Gamma)$.

We have proved the following assertion.

THEOREM 3. *If* $\Gamma\in C^{2,\lambda}$, $\lambda\in(0,1]$, *then* (19) *is a Fredholm equation of the second kind in the space* $C^0(\Gamma)$. *Moreover, equation* (19) *has a unique solution* $\mu_*(s)\in C^0(\Gamma)$ *for any* $\Phi(s)\in C^0(\Gamma)$.

As a consequence of Theorem 3 and Lemma 1 we obtain the corollary.

COROLLARY 1. *If* $\Gamma\in C^{2,\lambda}$, $\lambda\in(0,1]$ *and* $\Phi(s)\in C^{0,p}(\Gamma^1)\cap C^0(\Gamma^2)$, *where* $p=\min\{\lambda,1/2\}$, *then the unique solution of equation* (19) *in* $C^0(\Gamma)$, *ensured by Theorem* 3, *belongs to* $C^{0,p}(\Gamma^1)\cap C^0(\Gamma^2)$.

We recall that $\Phi(s)$ belongs to the class of smoothness required in the corollary if $f_1(s)\in C^{0,\lambda}(\Gamma^1)$, $f_2(s)\in C^0(\Gamma^2)$. As follows from our treatment presented above, if $\mu_*(s)\in C^{0,p}(\Gamma^1)\cap C^0(\Gamma^2)$ is a solution of (19), then $\mu(s)=\mu_*(s)Q^{-1}(s)\in C^p_{1/2}(\Gamma^1)\cap C^0(\Gamma^2)$ is a solution of system (12), (5). We obtain the following statement.

COROLLARY 2. *If* $\Gamma\in C^{2,\lambda}$, $f_1(s)\in C^{0,\lambda}(\Gamma^1)$, $f_2(s)\in C^0(\Gamma^2)$, $\lambda\in(0,1]$, *then the system of equations* (12), (5) *has a solution* $\mu(s)\in C^p_{1/2}(\Gamma^1)\cap C^0(\Gamma^2)$, $p=\min\{1/2,\lambda\}$, *which is expressed by the formula* $\mu(s)=\mu_*(s)Q^{-1}(s)$, *where* $\mu_*(s)\in C^{0,p}(\Gamma^1)\cap C^0(\Gamma^2)$ *is the unique solution of the Fredholm equation* (19) *in* $C^0(\Gamma)$.

REMARK. The solution of system (12), (5) ensured by Corollary 2 is unique in the space $C^{p_o}_{1/2}(\Gamma^1)\cap C^0(\Gamma^2)$ for any $p_o\in(0,p]$. The proof can be given by a contradiction to the assumption that the homogeneous system (12), (5) has a nontrivial solution in this space. The proof is almost the same as the proof of Theorem 3. Consequently, the numerical solution of system (12), (5) can be obtained by the direct numerical inversion of the integral operator from (12), (5). In doing so, Hölder functions can be approximated by continuous piecewise linear functions, which also obey Hölder inequality. The simplification for numerical solving equation (12b) is suggested in the remark to the equation (13) in the section 3.

We remind, that if conditions (3) hold, then $f_1(s) \in C^{0,\lambda}(\Gamma^1)$, $f_2(s) \in C^{1,\lambda}(\Gamma^2) \subset C^0(\Gamma^2)$ and the solution of equations (5), (12) ensured by Corollary 2 belongs to $C^p_{1/2}(\Gamma^1) \cap C^{1,\lambda/4}(\Gamma^2)$. On the basis of Corollary 2 and Theorem 2 we arrive at the final result.

THEOREM 4. *If* $\Gamma \in C^{2,\lambda}$ *and conditions* (3) *hold, then the solution of the problem* **U** *exists and is given by* (7), *where* $\nu(s)$ *is defined in* (10) *and* $\mu(s)$ *is a solution of equations* (12), (5) *from* $C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$, $p = \min\{1/2, \lambda\}$, *ensured by Corollary* 2. *More precisely,* $\mu(s) \in C^p_{1/2}(\Gamma^1) \cap C^{1,\lambda/4}(\Gamma^2)$.

It can be checked directly that the solution of the problem **U** satisfies condition (1) with $\varepsilon = -1/2$. Explicit expressions for singularities of the solution gradient at the end-points of the cuts Γ^1 can be easily obtained with the help of formulas presented in [4].

Theorem 4 ensures existence of a classical solution of the problem **U** when $\Gamma \in C^{2,\lambda}$ and conditions (3) hold. The uniqueness of the classical solution follows from Theorem 1. On the basis of our consideration we suggest the following scheme for solving the problem **U**. First, we find the unique solution $\mu_*(s)$ of the Fredholm equation (19) from $C^0(\Gamma)$. This solution automatically belongs to $C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$, $p = \min\{\lambda, 1/2\}$. Second, we construct the solution of equations (12), (5) from $C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ by the formula $\mu(s) = \mu_*(s)Q^{-1}(s)$. This solution automatically belongs to $C^p_{1/2}(\Gamma^1) \cap C^{1,\lambda/4}(\Gamma^2)$. Finally, substituting $\nu(s)$ from (10) and $\mu(s)$ in (7) we obtain the solution of the problem **U**.

Modern methods for numerical analysis of integral equations with singular integrals are presented in [9].

5. Analysis of Equation (20)

The Fredholm equation (20) is well-known in classical mathematical physics. We arrive at (20) when solving the homogeneous Dirichlet problem for the Helmholtz equation (2a) in the domain $\mathscr{D}$ by the double layer potential.

Our aim is to prove the following assertion.

PROPOSITION. *If* $\Gamma^2 \in C^{2,\lambda}$, $\lambda \in (0,1]$, *then there is only the trivial solution of the homogeneous Fredholm equation* (20) *in* $C^0(\Gamma^2)$.

We will give a proof by a contradiction. Let $\mu^0(s)$ be a nontrivial solution of equation (20) in $C^0(\Gamma^2)$. So, $\mu^0(s)$ transforms equation (20) into an identity. It follows from Lemma 4 in Appendix 2 that the integral term in this identity belongs to $C^{1,\lambda/4}(\Gamma^2)$. It follows from identity (20) that $\mu^0(s) \in C^{1,\lambda/4}(\Gamma^2)$. Consider the double layer potential $w_2[\mu^0](x) = \frac{i}{4}\int_{\Gamma^2} \mu^0(\sigma)\frac{\partial \mathscr{H}_0^{(1)}(k|x-y(\sigma)|)}{\partial \mathbf{n}_y}\,d\sigma$. According to Theorem 5 in Appendix 1, $w_2[\mu^0](x) \in C^1(\overline{\mathscr{D}}) \cap C^1(\overline{R^2\backslash\mathscr{D}})$. Moreover, the potential $w_2[\mu^0](x)$ satisfies the Helmholtz equation $(w_2)_{x_1x_1} + (w_2)_{x_2x_2} + k^2 w_2 = 0$, $x \in \mathscr{D}$ and the homogeneous Dirichlet boundary condition on Γ^2: $\lim_{\substack{x\to x^0\in\Gamma^2\\ x\in\mathscr{D}}} w_2[\mu^0](x) = 0$, which follows from identity (20). Besides, $w_2[\mu^0](x)$ satisfies conditions at infinity (2c) if $\mathscr{D}$ is an exterior domain. So, $w_2[\mu^0](x)$ is a solution of the particular case of the problem **U** when $\Gamma^1 = \varnothing$. According to Theorem 1, this problem has only the trivial solution $w_2[\mu^0](x) \equiv 0$ in $\mathscr{D}$. Let $\mathscr{D}_n$ be a domain bounded by the curve Γ_n^2 and such that $\mathscr{D} \not\subset \mathscr{D}_n$, $n = 1,\dots,N_2$. Therefore $\mathscr{D}_2,\dots,\mathscr{D}_{N_2}$ are interior domains, while $\mathscr{D}_1$ is interior if $\mathscr{D}$ is exterior, and $\mathscr{D}_1$ is exterior if $\mathscr{D}$ is interior. We consider Γ^2 as double-sided curves. We denote that side of Γ^2 which is on the left when the parameter s increases on Γ^2 by $(\Gamma^2)^+$, the opposite side will be denoted by $(\Gamma^2)^-$. Set

$$w_2[\mu^0](x) = w_{21}[\mu^0](x) + w_{22}[\mu^0](x) \tag{21}$$

where

$$w_{21}[\mu^0](x) = -\frac{1}{2\pi}\int_{\Gamma^2} \mu^0(\sigma)\frac{\partial \ln|x-y(\sigma)|}{\partial \mathbf{n}_y}\,d\sigma,$$

$$w_{22}[\mu^0](x) = \frac{i}{4}\int_{\Gamma^2} \mu^0(\sigma)\frac{\partial h(k|x-y(\sigma)|)}{\partial \mathbf{n}_y}\,d\sigma.$$

According to Lemma 2 in Appendix 1,

$$\left.\frac{\partial w_{22}[\mu^0](x)}{\partial \mathbf{n}_x}\right|_{(\Gamma^2)^+} = \left.\frac{\partial w_{22}[\mu^0](x)}{\partial \mathbf{n}_x}\right|_{(\Gamma^2)^-}. \tag{22}$$

It follows from (27) in Theorem 5 (Appendix 1) and from Lemma 3 in Appendix 1 that

(23) $$\left.\frac{\partial w_{21}[\mu^0](x)}{\partial \mathbf{n}_x}\right|_{(\Gamma^2)^+} = \left.\frac{\partial v_{21}[(\mu^0)'](x)}{\partial \tau_x}\right|_{(\Gamma^2)^+},$$

(24) $$\left.\frac{\partial w_{21}[\mu^0](x)}{\partial \mathbf{n}_x}\right|_{(\Gamma^2)^-} = \left.\frac{\partial v_{21}[(\mu^0)'](x)}{\partial \tau_x}\right|_{(\Gamma^2)^-}$$

where $v_{21}[(\mu^0)'](x) = -\frac{1}{2\pi}\int_{\Gamma^2}\{\mu^0(\sigma)\}' \ln|x - y(\sigma)|\, d\sigma$. It is shown in Lemma 3 (Appendix 1) that

(25) $$\left.\frac{\partial v_{21}[(\mu^0)'](x)}{\partial \tau_x}\right|_{(\Gamma^2)^+} = \left.\frac{\partial v_{21}[(\mu^0)'](x)}{\partial \tau_x}\right|_{(\Gamma^2)^-}.$$

According to (21)–(25), $\left.\frac{\partial w_2[\mu^0](x)}{\partial \mathbf{n}_x}\right|_{(\Gamma^2)^+} = \left.\frac{\partial w_2[\mu^0](x)}{\partial \mathbf{n}_x}\right|_{(\Gamma^2)^-}$. Since $w_2[\mu^0](x) \equiv 0$ in $\mathscr{D}$, we have $\left.\frac{\partial w_2[\mu^0](x)}{\partial \mathbf{n}_x}\right|_{(\Gamma^2)^+} = 0$. Therefore, the function

$$w_2[\mu^0](x) \in C^1(\overline{\mathscr{D}_n}) \cap C^2(\mathscr{D}_n)$$

satisfies the Helmholtz equation

(26) $$\Delta w_2 + k^2 w_2 = 0, \quad k = \text{const}, \ \text{Im}\, k > 0$$

in the domain $\mathscr{D}_n$ $(n = 1, \dots, N_2)$ and obeys the homogeneous Neumann boundary condition $\left.\frac{\partial w_2[\mu^0](x)}{\partial \mathbf{n}_x}\right|_{(\Gamma_n^2)^+} = 0$, $n = 1, \dots, N_2$. In addition, if $n = 1$ and if $\mathscr{D}_1$ is an exterior domain ($\mathscr{D}$ is interior) then $w_2[\mu^0](x)$ satisfies conditions at infinity (2c). So, $w_2[\mu^0](x)$ obeys the homogeneous Neumann problem for dissipative Helmholtz equation (26) in the domain $\mathscr{D}_n$. Using the method of energy equalities [1, v. IV], [11], we multiply (26) by $\overline{w_2}$ (the complex conjugate function to w_2) and integrate by parts in $\mathscr{D}_n$. Using condition at infinity (2c) for $n = 1$ if $\mathscr{D}_1$ is an exterior domain, we obtain the identity $\|\nabla w_2\|^2_{L_2(\mathscr{D}_n)} - k^2\|w_2\|^2_{L_2(\mathscr{D}_n)} = \int_{(\Gamma_n^2)^+} \overline{w_2}\frac{\partial w_2}{\partial \mathbf{n}_x}\, ds = 0$, $n = 1, \dots, N_2$. If $\text{Re}\, k \neq 0$ then taking the imaginary part in this identity and remembering that $\text{Im}\, k > 0$, we obtain $\|w_2\|^2_{L_2(\mathscr{D}_n)} = 0$; $n = 1, \dots, N_2$. The same result follows from the identity if $\text{Re}\, k = 0$ since $\text{Im}\, k > 0$. So in any case, $w_2[\mu^0](x) \equiv 0$ in $\mathscr{D}_n$ $(n = 1, \dots, N_2)$. Therefore, $w_2[\mu^0](x) \equiv 0$ in $R^2 \backslash \Gamma^2$. Using the jump formula for the limit values of the double layer potential [1], [11], we obtain $w_2[\mu^0](x)|_{(\Gamma^2)^+} - w_2[\mu^0](x)|_{(\Gamma^2)^-} = -\mu^0(s) \equiv 0$. Hence $\mu^0(s) \equiv 0$ on Γ^2 and we arrive at a contradiction to the assumption that $\mu^0(s)$ is a nontrivial solution of equation (20). Thus, equation (20) has only the trivial solution in $C^0(\Gamma^2)$. The proof is completed.

As a consequence of Proposition and the Fredholm alternative, we obtain

COROLLARY 3. *If* $\Gamma^2 \in C^{2,\lambda}$, $\lambda \in (0,1]$ *then nonhomogeneous Fredholm equation* (20) *is uniquely solvable in* $C^0(\Gamma^2)$ *for any right-hand side from* $C^0(\Gamma^2)$.

Appendix 1

Let us study smoothness properties of the double layer potential with a differentiable density. Let G be an interior simply connected domain bounded by a simple closed curve ∂G of the class $C^{1,\lambda}$, $\lambda \in (0,1]$. The boundary ∂G is parametrized by the arc length s counted from some fixed point on ∂G:

$$\partial G = \{y : y = y(s) = (y_1(s), y_2(s)) \in R^2, s \in [a,b]; y_1(s), y_2(s) \in C^{1,\lambda}[a,b]; \\ y(a) = y(b), y'(a) = y'(b)\}.$$

It is assumed that the domain G is situated on the left when the parameter s increases on ∂G. Introduce at a point $y \in \partial G$ the tangent vector τ_y showing the increment direction of s and the vector $\mathbf{n}_y$ of the outward normal to G. Then $\tau_y = (\cos\alpha(s), \sin\alpha(s))$, $\mathbf{n}_y = (\sin\alpha(s), -\cos\alpha(s))$, $\cos\alpha(s) = y_1'(s)$, $\sin\alpha(s) = y_2'(s)$. Note that the curvilinear integral of the first kind $\int_{\partial G} \cdots d\sigma$ coincides with the integral $\int_a^b \cdots d\sigma$. Let

$$\mu(s) \in C^{1,\lambda}(\partial G) = \{\mu(s) \in C^{1,\lambda}[a,b], \mu(a) = \mu(b), \mu'(a) = \mu'(b)\}.$$

Introduce the designations:

$$W[\mu](x) = \frac{i}{4}\int_{\partial G} \mu(\sigma) \frac{\partial \mathcal{H}_0^{(1)}(k|x - y(\sigma)|)}{\partial \mathbf{n}_y}\, d\sigma = W_1[\mu](x) + W_2[\mu](x)$$

is the double layer potential for the Helmholtz equation,

$$W_1[\mu](x) = -\frac{1}{2\pi}\int_{\partial G} \mu(\sigma) \frac{\partial \ln|x - y(\sigma)|}{\partial \mathbf{n}_y}\, d\sigma$$

is the double layer potential for the Laplace equation,

$$W_2[\mu](x) = \frac{i}{4}\int_{\partial G} \mu(\sigma) \frac{\partial h(k|x - y(\sigma)|)}{\partial \mathbf{n}_y}\, d\sigma,$$

where $h(z) = \mathcal{H}_0^{(1)}(z) - \frac{2i}{\pi}\ln\frac{z}{k}$, $h(z) = C^1[0,+\infty)$. We will prove the following result.

THEOREM 5. *Let* $\partial G \in C^{1,\lambda}$, $\lambda \in (0,1]$, $\mu(s) \in C^{1,\lambda}(\partial G)$, *then*

$$W[\mu](x) \in C^1(\overline{G}) \cap C^1(\overline{R^2 \backslash G}).$$

PROOF. It will be proved in Lemma 2 (see below) that $W_2[\mu](x) \in C^1(R^2)$ for any density $\mu(s) \in C^0(\partial G) = \{\mu(s) \in C^0[a,b], \mu(a) = \mu(b)\}$. Thus, to finish the proof of the theorem, we need to demonstrate that

$$W_1[\mu](x) \in C^1(\overline{G}) \cap C^1(\overline{R^2 \backslash G}).$$

According to [11, §31.2], if $\mu(s) \in C^0(\partial G)$ and $\partial G \in C^{1,\lambda}$ then

$$W_1[\mu](x) \in C^0(\overline{G}) \cap C^0(\overline{R^2 \backslash G}).$$

Thereby, it remains to prove that $\nabla W_1[\mu](x) \in C^0(\overline{G}) \cap C^0(\overline{R^2 \backslash G})$. Let $x \notin \partial G$, then

$$\frac{\partial W_1[\mu](x)}{\partial x_j} = -\frac{1}{2\pi} \int_{\partial G} \mu(\sigma) \frac{\partial}{\partial x_j} \frac{\partial \ln|x - y(\sigma)|}{\partial \mathbf{n}_y} \, d\sigma, \quad j = 1, 2.$$

Transform the densities in the integrals for $\dfrac{\partial W_1}{\partial x_1}, \dfrac{\partial W_1}{\partial x_2}$ taking into account that $\mathbf{n}_y = (y_2'(\sigma), -y_1'(\sigma))$:

$$\begin{aligned}
\frac{\partial}{\partial x_1} \frac{\partial \ln|x-y|}{\partial \mathbf{n}_y} &= y_2' \frac{\partial}{\partial x_1} \frac{\partial \ln|x-y|}{\partial y_1} - y_1' \frac{\partial}{\partial x_1} \frac{\partial \ln|x-y|}{\partial y_2} \\
&= -y_2' \frac{\partial}{\partial y_1} \frac{\partial \ln|x-y|}{\partial y_1} + y_1' \frac{\partial}{\partial y_1} \frac{\partial \ln|x-y|}{\partial y_2} \\
&= y_2' \frac{\partial}{\partial y_2} \frac{\partial \ln|x-y|}{\partial y_2} - y_1' \frac{\partial}{\partial y_1} \frac{\partial \ln|x-y|}{\partial x_2} \\
&= -y_2' \frac{\partial}{\partial x_2} \frac{\partial \ln|x-y|}{\partial y_2} - y_1' \frac{\partial}{\partial x_2} \frac{\partial \ln|x-y|}{\partial y_1} \\
&= -\frac{\partial}{\partial x_2} \frac{\partial \ln|x-y(\sigma)|}{\partial \sigma},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial x_2} \frac{\partial \ln|x-y|}{\partial \mathbf{n}_y} &= y_2' \frac{\partial}{\partial x_2} \frac{\partial \ln|x-y|}{\partial y_1} - y_1' \frac{\partial}{\partial x_2} \frac{\partial \ln|x-y|}{\partial y_2} \\
&= -y_2' \frac{\partial}{\partial y_2} \frac{\partial \ln|x-y|}{\partial y_1} + y_1' \frac{\partial}{\partial y_2} \frac{\partial \ln|x-y|}{\partial y_2} \\
&= y_2' \frac{\partial}{\partial y_2} \frac{\partial \ln|x-y|}{\partial x_1} - y_1' \frac{\partial}{\partial y_1} \frac{\partial \ln|x-y|}{\partial y_1}
\end{aligned}$$

$$= y_2' \frac{\partial}{\partial x_1} \frac{\partial \ln|x-y|}{\partial y_2} + y_1' \frac{\partial}{\partial x_1} \frac{\partial \ln|x-y|}{\partial y_1}$$

$$= \frac{\partial}{\partial x_1} \frac{\partial \ln|x-y(\sigma)|}{\partial \sigma},$$

where the Laplace equation

$$\frac{\partial}{\partial y_1} \frac{\partial \ln|x-y|}{\partial y_1} = -\frac{\partial}{\partial y_2} \frac{\partial \ln|x-y|}{\partial y_2}, \quad x \neq y$$

and the relation

$$\frac{\partial \ln|x-y|}{\partial x_j} = -\frac{\partial \ln|x-y|}{\partial y_j}, \quad j=1,2$$

are used. Consequently, integrating by parts, we obtain for $x \notin \partial G$

$$\frac{\partial W_1[\mu](x)}{\partial x_j} = \frac{-(-1)^j}{2\pi} \int_{\partial G} \mu(\sigma) \frac{\partial}{\partial x_{3-j}} \frac{\partial \ln|x-y(\sigma)|}{\partial \sigma} \, d\sigma$$

$$= \frac{(-1)^j}{2\pi} \int_{\partial G} \mu'(\sigma) \frac{\partial \ln|x-y(\sigma)|}{\partial x_{3-j}} \, d\sigma = -(-1)^j \frac{\partial V_1[\mu'](x)}{\partial x_{3-j}}, \quad j=1,2,$$

where $\mu'(s) \in C^{0,\lambda}(\partial G) = \{\mu'(s) \in C^{0,\lambda}[a,b], \mu'(a) = \mu'(b)\}$. By

$$V_1[\mu'](x) = -\frac{1}{2\pi} \int_{\partial G} \mu'(\sigma) \ln|x-y(\sigma)| \, d\sigma$$

we denote the single layer potential with the density $\mu'(\sigma)$. It follows from Lemma 3 (see below) that $\nabla V_1[\mu'](x) \in C^0(\overline{G}) \cap C^0(\overline{R^2 \backslash G})$. Since

$$\frac{\partial W_1[\mu](x)}{\partial x_j} = -(-1)^j \frac{\partial V_1[\mu'](x)}{\partial x_{3-j}}, \quad j=1,2, \tag{27}$$

we obtain $\nabla W_1[\mu](x) \in C^0(\overline{G}) \cap C^0(\overline{R^2 \backslash G})$. The proof is completed.

LEMMA 2. *If* $\partial G \in C^{1,\lambda}$, $\lambda \in (0,1]$, $\mu(s) \in C^0(\partial G)$, *then* $W_2[\mu](x) \in C^1(R^2)$. *Besides,* $\nabla W_2[\mu](x)|_{\partial G}$ *can be calculated by differentiation under the integral.*

PROOF. Set $(x \neq y)$

$$\cos \psi(x,y) = \frac{x_1 - y_1}{|x-y|} = -|x-y|'_{y_1} = |x-y|'_{x_1},$$

$$\sin \psi(x,y) = \frac{x_2 - y_2}{|x-y|} = -|x-y|'_{y_2} = |x-y|'_{x_2}.$$

Then for $x \neq y$

$$\nabla_y h(k|x-y|) = -kh'(k|x-y|)(\cos\psi(x,y), \sin\psi(x,y)).$$

Since $h'(z) \in C^0[0,+\infty)$ and $h'(z) = O(z\ln z)$, then $\dfrac{\partial h(k|x-y|)}{\partial \mathbf{n}_y}$ is continuous in $(x,y) \in R^2 \times \partial G$. We obtain from the theorem on a continuity of a proper integral in parameter that $W_2[\mu](x) \in C^0(R^2)$. It is easy to verify that

$$\nabla_x \cos\psi(x,y) = -\frac{\sin\psi(-\sin\psi, \cos\psi)}{|x-y|}, \quad \nabla_x \sin\psi(x,y) = \frac{\cos\psi(-\sin\psi, \cos\psi)}{|x-y|}.$$

Hence,

$$\frac{\partial \nabla_y h(k|x-y|)}{\partial x_1} = -k^2 h''(k|x-y|)\cos\psi(x,y)(\cos\psi(x,y), \sin\psi(x,y)) + \frac{kh'(k|x-y|)\sin\psi(x,y)(-\sin\psi(x,y), \cos\psi(x,y))}{|x-y|},$$

$$\frac{\partial \nabla_y h(k|x-y|)}{\partial x_2} = -k^2 h''(k|x-y|)\sin\psi(x,y)(\cos\psi(x,y), \sin\psi(x,y)) - \frac{kh'(k|x-y|)\cos\psi(x,y)(-\sin\psi(x,y), \cos\psi(x,y))}{|x-y|}.$$

Taking into account (see [3, 4]) that $h'(z) = -\frac{i}{\pi} z\ln z + zh_1(z)$, $h_1(z) \in C^1[0,+\infty)$, $h''(z) = -\frac{i}{\pi}\ln z + h_2(z)$, $h_2(z) \in C^1[0,+\infty)$, we obtain

$$\begin{aligned}\frac{\partial \nabla_y h(k|x-y|)}{\partial x_1} = \frac{ik^2(1,0)\ln(k|x-y|)}{\pi} &- k^2[h_2(k|x-y|)\cos\psi(x,y)(\cos\psi(x,y), \sin\psi(x,y)) \\ &+ h_1(k|x-y|)\sin\psi(x,y)(\sin\psi(x,y), -\cos\psi(x,y))],\end{aligned}$$

$$\begin{aligned}\frac{\partial \nabla_y h(k|x-y|)}{\partial x_2} = \frac{ik^2(0,1)\ln(k|x-y|)}{\pi} &- k^2[h_2(k|x-y|)\sin\psi(x,y)(\cos\psi(x,y), \sin\psi(x,y)) \\ &+ h_1(k|x-y|)\cos\psi(x,y)(-\sin\psi(x,y), \cos\psi(x,y))].\end{aligned}$$

Thus,

$$\frac{\partial W_2[\mu](x)}{\partial x_1} = -\frac{k^2}{4\pi}\int_{\partial G} \mu(\sigma)\sin\alpha(\sigma)\ln(k|x-y(\sigma)|)\,d\sigma - \frac{ik^2}{4}\int_{\partial G} \mu(\sigma)[h_2(k|x-y|)\cos\psi(x,y)(\cos\psi(x,y)\sin\alpha$$

$$- \sin \psi(x, y) \cos \alpha) + h_1(k|x - y|) \sin \psi(x, y)$$

$$\times (\sin \psi(x, y) \sin \alpha + \cos \psi(x, y) \cos \alpha)] \, d\sigma,$$

$$\frac{\partial W_2[\mu](x)}{\partial x_2} = \frac{k^2}{4\pi} \int_{\partial G} \mu(\sigma) \cos \alpha(\sigma) \ln(k|x - y(\sigma)|) \, d\sigma$$

$$- \frac{ik^2}{4} \int_{\partial G} \mu(\sigma)[h_2(k|x - y|) \sin \psi(x, y)(\cos \psi(x, y) \sin \alpha$$

$$- \sin \psi(x, y) \cos \alpha) + h_1(k|x - y|) \cos \psi(x, y)$$

$$\times (-\sin \psi(x, y) \sin \alpha - \cos \psi(x, y) \cos \alpha)] \, d\sigma.$$

Here the arguments are omitted: $y = y(\sigma)$, $\alpha = \alpha(\sigma)$. The first terms in the formulae for $\frac{\partial W_2}{\partial x_1}$, $\frac{\partial W_2}{\partial x_2}$ are the single layer potentials with continuous on ∂G densities. They are continuous in x on the whole plane (see [11, §31.2]). Let G_0 be a bounded open domain and $\partial G \subset G_0$. If $x \in G_0$ then the integrands in the second integrals in $\frac{\partial W_2}{\partial x_1}$ and in $\frac{\partial W_2}{\partial x_2}$ are uniformly bounded, that is their absolute values can be majorized by a constant uniformly in $x \in G_0$, $y \in \partial G$. If $x \notin \partial G$ and $x \to x^0 \in \partial G$ then the integrands in these integrals tend to their direct values on ∂G for all $y \in \partial G$ except $y = x^0$ (that is the limit exists 'nearly everywhere'). Applying the theorem on proceeding to limit under the Lebesgue integral [6, ch. V, §5.5], [1, v. V, §54], we obtain that for $x \notin \partial G$, $x \to x^0 \in \partial G$ the second integrals in $\frac{\partial W_2}{\partial x_1}$, $\frac{\partial W_2}{\partial x_2}$ tend to their direct values on ∂G. Hence, $\frac{\partial W_2}{\partial x_1}, \frac{\partial W_2}{\partial x_2} \in C^0(R^2)$. Moreover, these derivatives can be calculated under the integral for all $x \in R^2$. The proof is completed.

LEMMA 3. *Let* $\partial G \in C^{1,\lambda}$, $\lambda \in (0, 1]$,

$$v(s) \in C^{0,\lambda}(\partial G) = \{v(s) \in C^{0,\lambda}[a, b], v(a) = v(b)\}.$$

Then $V_1[v](x) = -\frac{1}{2\pi} \int_{\partial G} v(\sigma) \ln|x - y(\sigma)| \, d\sigma \in C^1(\overline{G}) \cap C^1(\overline{R^2 \backslash G})$. *Besides,* $\left.\frac{\partial V_1(x)}{\partial \tau_x}\right|_{x \in (\partial G)^+} = \left.\frac{\partial V_1(x)}{\partial \tau_x}\right|_{x \in (\partial G)^-}$, *where* ∂G *is considered as a double-sided curve,* $(\partial G)^+$ *is the side of* ∂G, *which is on the left, when parameter* s *increases, while* $(\partial G)^-$ *is another side.*

Proof. The functions $(V_1)_{x_1}$ and $(-V_1)_{x_2}$ are defined for $x \in R^2\backslash\partial G$ and satisfy the Cauchy-Riemann relations $(V_1)_{x_1x_2} = -(-V_1)_{x_2x_1}$, $(V_1)_{x_1x_1} = (-V_1)_{x_2x_2}$ since $\Delta V = 0$ for $x \in R^2\backslash\partial G$. Let $z = x_1 + ix_2$, $t = y_1 + iy_2$, consider the analytic complex function

$$\begin{aligned}\Phi(z) &= (V_1)_{x_1} - i(V_1)_{x_2}\\ &= -\frac{1}{2\pi}\int_{\partial G} \nu(\sigma)\left[\frac{\cos\psi(x, y(\sigma))}{|x-y|} - i\frac{\sin\psi(x, y(\sigma))}{|x-y|}\right] d\sigma\\ &= -\frac{1}{2\pi}\int_{\partial G} \nu(\sigma)\frac{e^{-i\alpha(\sigma)}}{|x-y|e^{i\psi(x,y)}}\,dt = -\frac{1}{2\pi}\int_{\partial G} \nu_1(\sigma)\frac{dt}{z-t},\end{aligned}$$

where $\nu_1(\sigma) = \nu(\sigma)e^{-i\alpha(\sigma)} \in C^{0,\lambda}(\partial G)$, $dt = t'(\sigma)\,d\sigma = e^{i\alpha(\sigma)}\,d\sigma$ since $y_1'(\sigma) = \cos\alpha(\sigma)$, $y_2'(\sigma) = \sin\alpha(\sigma)$. Denote $\nu_2(t) = \nu_2(t(\sigma)) = \nu_1(\sigma)$. Consider an arbitrary non-closed arc γ_1 which is a part of ∂G. If $x(a) = x(b) \notin \gamma_1 = \{y \in \gamma_1 : y = y(s),\ s \in [c,d]\}$, then $\nu_1(\sigma) \in C^{0,\lambda}[c,d]$, that is

$$\begin{aligned}|\nu_2(t_1(\sigma_1)) - \nu_2(t_2(\sigma_2))| &= |\nu_1(\sigma_1) - \nu_1(\sigma_2)| \le c|\sigma_1 - \sigma_2|^\lambda\\ &\le c_1|t_1(\sigma_1) - t_2(\sigma_2)|^\lambda, \quad \forall\sigma_1, \sigma_2 \in [c,d].\end{aligned}$$

Here we used the fact (see [3, lemma 1]) that

$$\frac{|\sigma_1 - \sigma_2|}{|y(\sigma_1) - y(\sigma_2)|} \in C^0([c,d]\times[c,d]),$$

consequently,

$$\frac{|\sigma_1 - \sigma_2|}{|y(\sigma_1) - y(\sigma_2)|} < \text{const},$$

therefore

$$|\sigma_1 - \sigma_2| < \text{const}|t_1(\sigma_1) - t_2(\sigma_2)|.$$

Hence, $\nu_2(t) \in C^{0,\lambda}(\gamma_1)$,

$$\Phi(z) = -\frac{1}{2\pi}\int_{\partial G} \nu_2(t)\frac{dt}{z-t}.$$

If $x(a) = x(b)$ is an interior point of $\gamma_1 = \{x \in \gamma_1 : x = x(s), s \in [a,c] \cup [d,b]\}$, then introduce a new parametrization on γ_1:

$$\gamma_1 = \{\tilde{y} \in \gamma_1 : \tilde{y} = \tilde{y}(\xi), \xi \in [d, b + c - a]\},$$

where $\tilde{y}(\xi) = y(\xi)$, $\tilde{v}_1(\xi) = v_1(\xi)$ if $\xi \in [d,b]$; $\tilde{y}(\xi) = y(\xi - (b-a))$, $\tilde{v}_1(\xi) = v_1(\xi - (b-a))$ if $\xi \in [b, b+c-a]$. Obviously, $\tilde{v}_1(\xi) \in C^{0,\lambda}[d,b]$, $\tilde{v}_1(\xi) \in C^{0,\lambda}[b, b+c-a]$ and $\tilde{v}_1(\xi) \in C^0[d, b+c-a]$. It is known from [5, §5.1] that $\tilde{v}_1(\xi) \in C^{0,\lambda}[d, b+c-a]$. Similarly, $\tilde{x}(\xi) \in C^{1,\lambda}[d, b+c-a]$. It can be shown by repeating the above arguments for the function $\tilde{v}_2(t) = \tilde{v}_2(t(\xi)) = \tilde{v}_1(\xi)$ in the new parametrization ξ that $\tilde{v}_2(t) \in C^{0,\lambda}(\gamma_1)$. Then

$$\Phi(z) = -\frac{1}{2\pi}\int_{\gamma_1} \frac{\tilde{v}_2(t)\,dt}{z-t} - \frac{1}{2\pi}\int_{\partial G\backslash\gamma_1} \frac{v_2(t)\,dt}{z-t}.$$

In any case, the Cauchy integral $\Phi(z)$ has a Hölder density on γ_1. It is known from [5, § 15] that the function $\Phi(z)$ is continuously extensible on γ_1 from the left and from the right (except, maybe, the end-points of γ_1). Since γ_1 is an arbitrary non-closed arc of ∂G, the function $\Phi(z)$ is continuously extensible on ∂G from the left and from the right at all points. Thereby, $\nabla V_1 \in C^0(\bar{G}) \cap C^0(\overline{R^2\backslash G})$. Since $V_1[\mu](x) \in C^0(R^2)$ (see [11, ch. V, §31.2]), we obtain that $V_1[\mu](x) \in C^1(\bar{G}) \cap C^1(\overline{R^2\backslash G})$. Consider ∂G a double-sided curve. The limiting values of functions on $(\gamma_1)^+$ and on $(\gamma_1)^-$ will be denoted by superscripts $+$ and $-$ respectively. Using the expression for function $\Phi(z)$ and Sokhotsky's formulae from [5], we obtain

$$\left(\frac{\partial V_1(x)}{\partial \tau_x}\right)^{\pm}\Bigg|_{x(s)\in\gamma_1} = \mathrm{Re}[e^{i\alpha(s)}\Phi^{\pm}(t_0)]|_{t_0\in\gamma_1} = \mathrm{Re}\left[\pm\frac{v(s)}{2}i + e^{i\alpha(s)}\Phi(t_0)\right]\Bigg|_{t_0\in\gamma_1}$$
$$= \mathrm{Re}(e^{i\alpha(s)}\Phi(t_0))|_{t_0\in\gamma_1},$$

where $t_0 = x_1 + ix_2$, $\Phi(t_0)$ is the direct value of the function $\Phi(z)$ at the point t_0 on γ_1. Hence, $\left.\frac{\partial V_1(x)}{\partial \tau_x}\right|_{x\in(\gamma_1)^+} = \left.\frac{\partial V_1(x)}{\partial \tau_x}\right|_{x\in(\gamma_1)^-} = [\mathrm{Re}(e^{i\alpha(s)}\Phi(t_0))]|_{t_0\in\gamma_1}$. Since γ_1 is an arbitrary open arc contained in ∂G, we obtain $\left.\frac{\partial V_1(x)}{\partial \tau_x}\right|_{x\in(\partial G)^+} = \left.\frac{\partial V_1(x)}{\partial \tau_x}\right|_{x\in(\partial G)^-}$. The proof is completed.

Appendix 2

Let us study smoothness of the direct value of the double layer potential on the curve.

LEMMA 4. *Let γ be a simple closed curve of class $C^{2,\lambda}$, $\lambda \in (0,1]$, parametrized by the arc length s: $\gamma = \{x : x = x(s), s \in [a,b]\}$, and*

$$\mu(s) \in C^0(\gamma) = \{\mu(s) \in C^0[a,b], \mu(a) = \mu(b)\}.$$

Let

$$I(s) = \frac{i}{4}\int_\gamma \mu(\sigma)\frac{\partial \mathscr{H}_0^{(1)}(k|x(s)-y(\sigma)|)}{\partial \mathbf{n}_y}\,d\sigma$$

be the direct value of the double layer potential on γ. *Then*

$$I(s) \in C^{1,\lambda/4}(\gamma) = \{I(s) \in C^{1,\lambda/4}[a,b], I(a) = I(b), I'(a) = I'(b)\}.$$

PROOF. Let $I(s) = I_1(s) + I_2(s)$, where

$$I_1(s) = -\frac{1}{2\pi}\int_\gamma \mu(\sigma)\frac{\partial \ln|x(s)-y(\sigma)|}{\partial \mathbf{n}_y}\,d\sigma,$$

$$I_2(s) = \frac{i}{4}\int_\gamma \mu(\sigma)\frac{\partial h(k|x(s)-y(\sigma)|)}{\partial \mathbf{n}_y}\,d\sigma, \quad h(z) = \mathscr{H}_0^{(1)}(z) - \frac{2i}{\pi}\ln\frac{z}{k}.$$

1) Consider $I_1(s)$ and prove that $I_1(s) \in C^{1,\lambda/4}[a,b]$. Taking into account that $\mathbf{n}_y = (y_2'(\sigma), -y_1'(\sigma))$, we find

$$\frac{\partial \ln|x(s)-y(\sigma)|}{\partial \mathbf{n}_y} = \frac{T(s,\sigma)}{g(s,\sigma)}, \quad g(s,\sigma) = \frac{|x(s)-y(\sigma)|^2}{(s-\sigma)^2},$$

$$T(s,\sigma) = \frac{[x_2(s)-y_2(\sigma)]y_1'(\sigma) - [x_1(s)-y_1(\sigma)]y_2'(\sigma)}{(s-\sigma)^2}.$$

Note that $y(\sigma)$ is a point on Γ corresponding to $s=\sigma$. So, we may put $x(\sigma) = y(\sigma)$. For $j=1,2$ we have [3, §3]

$$x_j(s) - x_j(\sigma) = (s-\sigma)Z_j^1(s,\sigma) = -x_j'(\sigma)(\sigma - s) + (\sigma - s)^2 Z_j^2(\sigma, s),$$

where

$$Z_j^1(s,\sigma) = \int_0^1 x_j'(\sigma + \xi(s-\sigma))\,d\xi \in C^{1,\lambda}([a,b]\times[a,b]),$$

$$Z_j^2(\sigma,s) = \int_0^1 \xi x_j''(s + \xi(\sigma - s))\,d\xi \in C^{0,\lambda}([a,b]\times[a,b]).$$

Note that the function

$$g(s,\sigma) = \frac{|x(s)-x(\sigma)|^2}{(s-\sigma)^2} = \{[Z_1^1(s,\sigma)]^2 + [Z_2^1(s,\sigma)]^2\} \in C^{1,\lambda}([a,b]\times[a,b])$$

does not equal zero anywhere on Γ and $g(s,s) = 1$, therefore

$$\frac{1}{g(s,\sigma)} \in C^1([a,b]\times[a,b]).$$

Further,

$$\frac{\partial}{\partial s}\frac{1}{g(s,\sigma)} = \frac{\partial}{\partial s}\frac{(s-\sigma)^2}{|x(s)-x(\sigma)|^2} = -\frac{g'_s(s,\sigma)}{g^2(s,\sigma)}$$

$$= -2\frac{Z_1^1(s,\sigma)[Z_1^1(s,\sigma)]'_s + [Z_2^1(s,\sigma)]'_s Z_2^1(s,\sigma)}{g^2(s,\sigma)} \in C^{0,\lambda}([a,b]\times[a,b]).$$

Consequently, $\frac{1}{g(s,\sigma)} \in C^{1,\lambda}([a,b]\times[a,b])$. Similarly,

$$T(s,\sigma) = \frac{[x_2(s)-x_2(\sigma)]x_1'(\sigma) - [x_1(s)-x_1(\sigma)]x_2'(\sigma)}{(s-\sigma)^2}$$

$$= [Z_2^2(\sigma,s)x_1'(\sigma) - Z_1^2(\sigma,s)x_2'(\sigma)] \in C^{0,\lambda}([a,b]\times[a,b]).$$

Consider $\frac{\partial T(s,\sigma)}{\partial s} = J_1(s,\sigma) - 2J_2(s,\sigma)$, where

$$J_1(s,\sigma) = \frac{x_2'(s)x_1'(\sigma) - x_1'(s)x_2'(\sigma)}{(s-\sigma)^2}$$

$$= \frac{[x_2'(s)-x_2'(\sigma)]x_1'(\sigma) - [x_1'(s)-x_1'(\sigma)]x_2'(\sigma)}{(s-\sigma)^2}$$

$$= \frac{1}{s-\sigma}\left\{x_1'(\sigma)\int_0^1 x_2''[s+\xi(\sigma-s)]\,d\xi - x_2'(\sigma)\int_0^1 x_1''[s+\xi(\sigma-s)]\,d\xi\right\};$$

$$J_2(s,\sigma) = \frac{[x_2(s)-x_2(\sigma)]x_1'(\sigma) - [x_1(s)-x_1(\sigma)]x_2'(\sigma)}{(s-\sigma)^3}$$

$$= \frac{1}{s-\sigma}\left\{x_1'(\sigma)\int_0^1 \xi x_2''[s+\xi(\sigma-s)]\,d\xi - x_2'(\sigma)\int_0^1 \xi x_1''[s+\xi(\sigma-s)]\,d\xi\right\}.$$

Then

$$\frac{\partial T(s,\sigma)}{\partial s} = \frac{1}{s-\sigma}\left\{x_1'(\sigma)\int_0^1 (1-2\xi)x_2''[s+\xi(\sigma-s)]\,d\xi\right.$$

$$\left. - x_2'(\sigma)\int_0^1 (1-2\xi)x_1''[s+\xi(\sigma-s)]\,d\xi\right\} = \frac{K(s,\sigma)}{s-\sigma},$$

where $K(s,\sigma) \in C^{0,\lambda}([a,b]\times[a,b])$ and $K(s,s) = 0$. According to [5, §5.7], the following representation holds:

$$\frac{\partial T(s,\sigma)}{\partial s} = \frac{K^*(s,\sigma)}{|s-\sigma|^{1-\lambda/4}},$$

$K^*(s,\sigma) \in C^{0,3\lambda/4}([a,b]\times[a,b])$. Using properties of Hölder functions [5], we obtain the representation

$$\frac{\partial}{\partial s}\frac{\partial \ln|x(s)-y(\sigma)|}{\partial \mathbf{n}_y} = \frac{1}{g(s,\sigma)}\frac{\partial T(s,\sigma)}{\partial s} + T(s,\sigma)\frac{\partial}{\partial s}\frac{1}{g(s,\sigma)}$$
$$= \frac{K_1(s,\sigma)}{|s-\sigma|^{1-\lambda/4}} + K_2(s,\sigma),$$

where $K_1(s,\sigma)\in C^{0,3\lambda/4}([a,b]\times[a,b])$, $K_2(s,\sigma)\in C^{0,\lambda}([a,b]\times[a,b])$. By formal differentiation under the integral, we find

$$\frac{dI_1(s)}{ds} = -\frac{1}{2\pi}\int_\gamma \mu(\sigma)\frac{\partial}{\partial s}\frac{\partial \ln|x(s)-y(\sigma)|}{\partial \mathbf{n}_y}\,d\sigma$$
$$= -\frac{1}{2\pi}\int_\gamma \mu(\sigma)\frac{K_1(s,\sigma)}{|s-\sigma|^{1-\lambda/4}}\,d\sigma - \frac{1}{2\pi}\int_\gamma \mu(\sigma)K_2(s,\sigma)\,d\sigma.$$

The validity of differentiation under the integral can be proved in the same way as at the end of §1.6 in [11] (Fubini theorem on change of integration order is used). Taking into account the obtained representation for $\frac{dI_1(s)}{ds}$ and applying results of [5, §51.1], we obtain that $\frac{dI_1(s)}{ds}\in C^{0,\lambda/4}[a,b]$.

2) Consider $I_2(s)$. It follows from lemma 3.3 in [3] that

$$\frac{\partial}{\partial s}\frac{\partial h(k|x(s)-y(\sigma)|)}{\partial \mathbf{n}_y} \in C^{0,\lambda/4}([a,b]\times[a,b]).$$

This kernel is continuous, hence differentiation under the integral is valid (by the theorem on differentiation of proper integral with respect to a parameter):

$$\frac{dI_2(s)}{ds} = \frac{i}{4}\int_\gamma \mu(\sigma)\frac{\partial}{\partial s}\frac{\partial h(k|x(s)-y(\sigma)|)}{\partial \mathbf{n}_y}\,d\sigma, \quad \frac{dI_2(s)}{ds}\in C^{0,\lambda/4}[a,b].$$

It follows from the points 1) and 2) that $I(s)\in C^{1,\lambda/4}[a,b]$ and

$$\frac{dI(s)}{ds} = \frac{i}{4}\int_\gamma \mu(\sigma)\frac{\partial}{\partial s}\frac{\partial \mathscr{H}_0^{(1)}(k|x(s)-y(\sigma)|)}{\partial \mathbf{n}_y}\,d\sigma.$$

Note that $\frac{\partial}{\partial s} = \frac{\partial}{\partial \tau_x}$ due to the parametrization chosen. It is easy to verify that the kernels in the integrals $I(s)$, $I'(s)$ depend on $x_j^{(m)}(s)$, $j=1,2$; $m=0,1,2$. The values of these functions at $s=a$ and at $s=b$ are equal. Therefore, $I(a)=I(b)$, $I'(a)=I'(b)$. Thus, $I(s)\in C^{1,\lambda/4}(\gamma)$. That accomplishes the proof.

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References

[1] Smirnov V. I., A course of higher mathematics. Vol. IV, V. Pergamon Press, Oxford, 1964.
[2] Gakhov F. D., Boundary value problems. Pergamon Press, Oxford; Addison-Wesley, Reading, Mass., 1966.
[3] Krutitskii P. A., Dirichlet problem for the Helmholtz equation outside cuts in a plane. Comp. Maths. Math. Phys., 1994, v. 34, No. 8/9, pp. 1073–1090. (Originally published in Russian in Zh. Vych. Mat. Mat. Fiz., 1994, v. 34, No. 8/9, p. 1237–1258).
[4] Krutitskii P. A., Neumann problem for the Helmholtz equation outside cuts in a plane. Comp. Maths. Math. Phys., 1994, v. 34, No. 11, pp. 1421–1431. (Originally published in Russian in Zh. Vych. Mat. Mat. Fiz., 1994, v. 34, No. 11, p. 1652–1665).
[5] Muskhelishvili N. I., Singular integral equations. Noordhoff, Groningen, 1953, reprinted 1972. (3rd Russian edition: Nauka, Moscow, 1968.)
[6] Kolmogorov A. N., Fomin S. V., Elements of the theory of functions and functional analysis. Dover Pubns, N.Y., 1999.
[7] Krutitskii P. A., The Neumann problem for the 2-D Helmholtz equation in a domain, bounded by closed and open curves. Int. J. Maths. Math. Sci., 1998, v. 21, No. 2, pp. 209–216.
[8] Krutitskii P. A., The Dirichlet problem for the dissipative Helmholtz equation in a plane domain bounded by closed and open curves. Hiroshima Math. J., 1998, v. 28, No. 1, p. 149–168.
[9] Lifanov I. K., Singular integral equations and discrete vortices. VSP, Zeist, 1996.
[10] Nikiforov A. F., Uvarov V. B., Special functions of mathematical physics. Birkhäuser, Basel, 1988. (2-nd Russian edition: Nauka, Moscow, 1984).
[11] Vladimirov V. S., Equations of Mathematical Physics. Marcel Dekker, N.Y., 1971. (4-th Russian edition: Nauka, Moscow, 1981).
[12] Krutitskii P. A., The skew derivative problem for the Helmholtz equation outside cuts in a plane. Rendiconti di Mathematica (Roma), Serie VII, 1999, Vol. 19, p. 367–384.
[13] Colton D., Kress R., Integral equation methods in scattering theory. Wiley, N.Y., 1983.