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CORRIGENDUM TO: ON THE EXCEPTIONAL SET OF HARDY-LITTLEWOOD'S NUMBERS IN SHORT INTERVALS

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Let $k \geq 2$ be an integer. In the paper [1] we proved a result on the set E_k of the integers which are neither a sum of a prime and a k-power nor a k-power of an integer. Setting $E_k(X) = E_k \cap [1, X]$ and $E_k(X, H) = E_k \cap [X, X+H]$, where X is a sufficiently large parameter and $H = o(X)$, our statement is

Theorem. *Let $k \geq 2$ be a fixed integer and $K = 2^{k-2}$. There exists a (small) positive absolute constant δ such that for $H \geq X^{7/12(1-1/k)+\delta}$*

$$|E_k(X, H)| \ll H^{1-\delta/(5K)}.$$

In fact our proof is not totally correct. There are two corrections to do. The first one is that the level Q of the the Farey dissection has to be fixed equal to $2kY^{1-1/k}$ instead of $4Y^{1-1/k}$ since, in the proof of Lemma 10 of [1], such a condition is needed to estimate, by using the first derivative method, the order of magnitude in a Farey arc of

$$F_k(\alpha) = \sum_{Y/4 \leq m^k \leq Y} e(m^k \alpha),$$

where $Y = X^{7/12+10\delta+\varepsilon}$ and $e(\alpha) = e^{2\pi i \alpha}$. We write here the corrected version of Lemma 10 of [1] whose proof is a slight modification of what Perelli-Zaccagnini [3], eq. (39)–(40), proved.

Lemma 10 of [1]. *Let $(a, q) = 1$, $Q \geq 2kY^{1-1/k}$ and $|\eta| \leq 1/qQ$. Then*

$$\left|F_k\left(\frac{a}{q} + \eta\right)\right| \ll \frac{Y^{1/k-1}}{|\eta|}.$$

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PROOF. Write $m = qt + l$. Hence

$$F_k\left(\frac{a}{q}+\eta\right) = \sum_{l=1}^{q} e\left(l^k\left(\frac{a}{q}+\eta\right)\right) \sum_{t=((Y/4)^{1/k}-l)/q}^{(Y^{1/k}-l)/q} e(f_{k,l}(t)),$$

where, by the Binomial Theorem, we get $f_{k,l}(t) = \eta\sum_{j=1}^{k}\binom{k}{j}(qt)^j l^{k-j}$. It is easy to see that $f'_{k,l}(t) = kq\eta(qt+l)^{k-1}$ and hence $kq|\eta|Y^{1-1/k} \ll |f'_{k,l}(t)| \leq kq|\eta|Y^{1-1/k}$. Now $|f'_{k,l}(t)| \leq 1/2$ follows from $|\eta| \leq 1/qQ$ and $Q \geq 2kY^{1-1/k}$ and hence, by Lemma 4.2 and 4.8 of Titchmarsh [4], Lemma 10 follows. □

This change on Q has no consequences in the rest of the proof of the Theorem.

The second correction concerns the use of Lemma 11 of [1]. Such a lemma implies that our Theorem holds only in the case $k \geq 3$. We restate it here for convenience (we also take this occasion to correct a misprint in its statement).

LEMMA 11 OF [1]. *Let $F(x,y) = x^g y + \sum_{j=0}^{g-1} b_j(y)x^j$ where $g \geq 2$ is a fixed integer and $b_j(y)$ are real-valued functions. Let $\left|\alpha - \frac{a}{q}\right| < 1/q^2$ and $(a,q) = 1$. Then for $T, R, q \leq X$ and for every $\varepsilon > 0$ we have*

$$\sum_{1\leq d\leq T}\left|\sum_{n\leq R} e(\alpha F(n,d))\right| \ll_g TR\left(\frac{1}{q}+\frac{1}{R}+\frac{q}{TR^g}\right)^{1/K} T^{\varepsilon/K}R^{(g-1)\varepsilon/K},$$

where $K = 2^{g-1}$.

Lemma 11 is used in section 4 of [1] to estimate the minor arcs contribution. To this end we choosed $g = k-1$, $k \geq 2$ and $K = 2^{k-2}$. So the second part of equation (19), page 11, in [1], and, consequently the main Theorem, holds only for $k \geq 3$.

To save the result for the case $k = 2$ we need to insert in the body of the proof of the Theorem a minor arc estimate of the following kind:

$$\int_{-2/H}^{2/H}\left|F_2\left(\frac{a}{q}+\eta\right)\right|^2 d\eta \ll P^{-1/2} = Y^{2/k-1}P^{-1/2K} \quad \text{for } P < q \leq Q \text{ and } k = 2, \quad (1)$$

where, as in [1], $H = QP$, $Q = 2kY^{1-1/k} = 4Y^{1/2}$ and P is essentially equal to Y^δ (in fact the definition of P depends on the existence of the exceptional zero $\tilde{\beta}$, see section 1, page 3, of [1]).

The estimate (1) can be obtained following the line of section 5 of Perelli-Pintz [2].

By Gallagher's Lemma (see, *e.g.*, Lemma 7 of [1]), we get that

$$\int_{-2/H}^{2/H} \left| F_2\left(\frac{a}{q}+\eta\right)\right|^2 d\eta \ll H^{-2} \int_{Y/10}^{Y} \left| \sum_{m^2 \in I(x)} e\left(m^2 \frac{a}{q}\right)\right|^2 dx,$$

where $I(x) = [x, x+H/4]$. Writing $d = m-n$ we obtain

$$\left| \sum_{m^2 \in I(x)} e\left(m^2 \frac{a}{q}\right)\right|^2 = 2\,\mathrm{Re} \sum_{\substack{0 \le d \le H/Y^{1/2} \\ 2d \not\equiv 0 \pmod q}} e\left(d^2 \frac{a}{q}\right) \sum_{n^2, (n+d)^2 \in I(x)} e\left(n \frac{2da}{q}\right)$$

$$+ \sum_{m^2 \in I(x)} \sum_{\substack{n^2 \in I(x) \\ 2m \equiv 2n \pmod q}} 1 = S_1 + S_2, \tag{2}$$

say. Now we have

$$\int_{Y/10}^{Y} S_1\, dx \ll \sum_{\substack{0 \le d \le H/Y^{1/2} \\ 2d \not\equiv 0 \pmod q}} |\Sigma(d)| + \frac{H^2}{Y},$$

where

$$\Sigma(d) = \sum_{Y_1 \le n \le Y_2} e\left(n \frac{2da}{q}\right) \max\left(0; \frac{H}{4} - d^2 - 2nd\right)$$

and

$$Y_1 = \left(\frac{Y}{10} + \frac{H}{4}\right)^{1/2}, \quad Y_2 = \left(Y - \frac{H}{4}\right)^{1/2} - d.$$

Denoting by $\|a\|$ the distance of a from the nearest integer and using the fact

$$\Sigma(d, y) = \sum_{Y_1 \le n \le y} e\left(n \frac{2da}{q}\right) \ll \left\| \frac{2da}{q} \right\|^{-1},$$

we have, by partial summation, that

$$\Sigma(d) \ll H \left\| \frac{2da}{q} \right\|^{-1}.$$

Using now Lemma 2.2 of Vaughan [5], we obtain

$$\int_{Y/10}^{Y} S_1\, dx \ll \frac{H^2}{Y^{1/2}} \log X + Hq \log X + \frac{H^2}{Y}. \tag{3}$$

Moreover we have

$$\int_{Y/10}^{Y} S_2\,dx \ll \int_{Y/10}^{Y} \frac{H}{x^{1/2}}\left(\frac{H}{qx^{1/2}}+1\right)dx \ll HY^{1/2}+\frac{H^2}{q}. \tag{4}$$

Recalling $P < q \leq Q$, $H = QP$, $Q = 2kY^{1-1/k} = 4Y^{1/2}$, we get from (2)–(4) that

$$\int_{-2/H}^{2/H} \left| F_2\left(\frac{a}{q}+\eta\right)\right|^2 d\eta \ll Y^{-1/2}\log X + P^{-1}\log X + P^{-1} \ll P^{-1}\log X. \tag{5}$$

Hence, recalling $k = 2$ and $K = 2^{k-2}$, from (5) we obtain that (1) holds.

Now we have that the second part of equation (19), page 11, in [1] holds in the case $k = 2$ too. Inserting it in the rest of the proof of the Theorem we get that it holds for every $k \geq 2$.

References

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