

PISOT SUBSTITUTIONS AND THE HAUSDORFF DIMENSION OF BOUNDARIES OF ATOMIC SURFACES

By

De-Jun FENG, Maki FURUKADO, Shunji ITO, Jun WU

Abstract. The atomic surface X_σ from an unimodular Pisot substitution σ usually has the fractal boundary and it generates a self-affine tiling. In this paper, we study the boundary ∂X_σ as the graph directed self-affine fractal and estimate the Hausdorff dimension of the boundary.

0. Introduction

The several properties of self-affine tiles and their boundaries are studied for instance in the articles [26], [15], [3], [16], [9], [17], [18], [4], [27], [1], [24]. In this paper, we treat the sets which have the fractal boundary called atomic surfaces or self-affine tiles based on substitutions.

Let σ be a primitive unimodular Pisot substitution on the free monoid $A^* = \bigcup_{n=0}^{\infty} \{1, 2, \dots, d\}^n$, that is,

- (1) there exists an n such that i occurs in $\sigma^n(j)$ for any pair of letters (i, j) (*primitive*);
- (2) the characteristic polynomial of L_σ is irreducible over \mathbf{Q} and eigenvalues λ_i , $1 \leq i \leq d$ of L_σ satisfy the followings:

$$\lambda_1 > 1 > |\lambda_i|, \quad i = 2, \dots, d \quad (\text{Pisot condition});$$

- (3) $\det L_\sigma = \pm 1$ (unimodular condition).

Let $\omega = (\omega_1, \omega_2, \dots)$ be the fixed point of the substitution σ and $\pi : \mathbf{R}^d \rightarrow \mathcal{P}$ be the projection along the eigenvector with respect to the largest eigenvalue λ_1 of L_σ to the contractive invariant plane \mathcal{P} of L_σ . Let us define the set X_σ by

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$$X_\sigma := \text{the closure of } \left\{ \pi \sum_{k=1}^n e_{\omega_k} \mid n = 1, 2, \dots \right\}$$

where $e_i, i = 1, 2, \dots, d$ are the canonical basis of \mathbf{R}^d . The domain X_σ called the *atomic surface* usually has a fractal boundary. This domain and its boundary are not only interesting from the viewpoint of the fractal geometry, but also ergodic theory, number theory and quasi-crystal theory (see [22], [10], [11], [19], [23], [7]). In this paper, we mainly study the boundary ∂X_σ as the fractals which have graph self-affine in Theorem 2.6 (c.f. [5], [25]) and estimate the Hausdorff dimension of atomic surfaces as follows.

THEOREM 1. *Let σ be a primitive unimodular Pisot substitution with d letters and let X_σ be the atomic surface based on the substitution σ . Then the Hausdorff dimension of the boundary ∂X_σ is estimated by*

$$\dim_H \partial X_\sigma \leq \frac{\log \gamma_1 - \log \lambda_1 - (d - 1) \log |\lambda_d|}{-\log |\lambda_d|}$$

where γ_1 is the largest eigenvalue of the graph matrix M_σ .

Moreover, if the linear map $L_\sigma|_{\mathcal{P}}$ restricted to the contractive invariant plane \mathcal{P} is a similitude, then the Hausdorff dimension of ∂X_σ is given by

$$\dim_H \partial X_\sigma = \frac{(d - 1) \log \gamma_1}{\log \lambda_1}.$$

1. Atomic Surfaces and Their Basic Properties

In this section, we give a survey of the property of the atomic surface which is discussed in [6], [2], [12]. Let \mathcal{A} be an alphabet of d letters $\{1, 2, \dots, d\}$. We denote $\mathcal{A}^* = \bigcup_{n=0}^\infty \mathcal{A}^n$ the free monoid of \mathcal{A} . The substitution σ is a map from \mathcal{A} to \mathcal{A}^* such that $\sigma(i)$ is a non-empty word for any letter i . The substitution σ naturally extends to an endomorphism of the free monoid \mathcal{A}^* by the rule $\sigma(UV) = \sigma(U)\sigma(V)$. Denote $\sigma(i) = W^{(i)}$, where $W^{(i)}$ is a finite word of the length l_i , and we write $W^{(i)} = W_1^{(i)} \dots W_{l_i}^{(i)}$. Denote by $P_k^{(i)}$ the prefix of the length $k - 1$ of $W_k^{(i)}$ (for $k = 1$, this is the empty word), and $S_k^{(i)}$ the suffix of the length $l_i - k$, so that $\sigma(i) = P_k^{(i)} W_k^{(i)} S_k^{(i)}$. For the simplicity, we assume that $W_1^{(1)} = 1$. Under this assumption, the infinite sequence ω given by

$$\omega = \lim_{n \rightarrow \infty} \sigma^n(1)$$

is the fixed point of the substitution σ . There is a natural homomorphism $f : \mathcal{A}^* \rightarrow \mathbf{Z}^d$ obtained by the abelianization of the free monoid \mathcal{A}^* , and we obtain a linear transformation L_σ satisfying the commutative diagram:

$$\begin{array}{ccc} \mathcal{A}^* & \xrightarrow{\sigma} & \mathcal{A}^* \\ f \downarrow & & \downarrow f \\ \mathbf{Z}^d & \xrightarrow{L_\sigma} & \mathbf{Z}^d. \end{array}$$

From now on, we assume that the substitution σ is *primitive*, that is, there exists an n such that i occurs in $\sigma^n(j)$ for any pair of letters (i, j) . It is equivalent to say that the matrix L_σ of σ is primitive. By Perron-Frobenius theorem, L_σ has the largest eigenvalue λ_1 that is positive, simple and strictly bigger in modulus than the other eigenvalues. We denote \mathbf{u}_λ and \mathbf{v}_λ positive eigenvectors associated with λ_1 for L_σ and the transpose of L_σ respectively. Moreover, we assume that the substitution σ satisfies *irreducible Pisot* and *unimodular condition*, that is,

- (1) the characteristic polynomial of L_σ is irreducible over \mathbf{Q} and eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_d$ of L_σ satisfy

$$\lambda_1 > 1 > |\lambda_i|, \quad i = 2, \dots, d \quad (\text{Pisot condition});$$

- (2) the determinant of L_σ is equal to ± 1 (unimodular condition).

Let \mathcal{P} be the plane orthogonal to \mathbf{v}_λ . It is clear that \mathcal{P} is invariant by the linear transformation L_σ . Moreover, the linear transformation L_σ is contractive on \mathcal{P} , that is, there exists a constant $0 < \lambda_0 < 1$ such that

$$d_\mathcal{P}(L_\sigma \mathbf{x}, L_\sigma \mathbf{y}) \leq \lambda_0 d_\mathcal{P}(\mathbf{x}, \mathbf{y}) \quad \text{for } \mathbf{x}, \mathbf{y} \in \mathcal{P}$$

where $d_\mathcal{P}(\cdot, \cdot)$ is the restricted Euclid distance on \mathcal{P} . Let $\pi : \mathbf{R}^d \rightarrow \mathcal{P}$ be the projection along the eigenvector \mathbf{u}_λ .

DEFINITION 1.1. *Let us denote the fixed point $\omega = \lim_{n \rightarrow \infty} \sigma^n(1)$ of σ by*

$$\omega = s_1 s_2 \cdots s_n \cdots,$$

and let us define the set X and X_i , $i = 1, 2, \dots, d$ by

$$X := \text{the closure of } \left\{ \pi \sum_{j=1}^k \mathbf{e}_{s_j} \mid k = 1, 2, \dots \right\},$$

$$X_i := \text{the closure of } \left\{ \pi \sum_{j=1}^k \mathbf{e}_{s_j} \mid s_k = i \text{ for some } k \right\}.$$

The set X is called the atomic surface associated with the substitution σ .

With the notations above, we know the following theorem.

THEOREM 1.2 ([2]). *Let σ be a primitive unimodular Pisot substitution, and X and X_i , $i = 1, 2, \dots, d$ be the atomic surfaces of σ . Then X_i 's satisfy the following relations: for each $i = 1, \dots, d$,*

$$X_i = \sum_{j=1}^d \sum_{\substack{S_k^{(j)} \\ W_k^{(j)}=i, \\ \sigma(j)=P_k^{(j)} W_k^{(j)} S_k^{(j)}}} (L_\sigma X_j - \pi f(S_k^{(j)})) \quad (\text{non-overlap})$$

where $\sum_{j=1}^l A_j$ (non-overlap) means that the Lebesgue measure $|A_j \cap A_k|$ of $A_j \cap A_k$ is equal to zero for each $1 \leq j < k \leq l$.

In [2], we can see implicitly the set equation of X_i , $i = 1, 2, \dots, d$ holds. However, we will give an explicit proof here. For this purpose, we prepare some lemmas and propositions.

LEMMA 1.3. *The set X is bounded. More precisely, we can estimate*

$$\text{diam.} X \leq \frac{2}{1 - \lambda_0} \cdot l \cdot m,$$

where $L_\sigma = (l_{ij})$, $l = \max_{1 \leq j \leq d} \sum_{i=1}^d l_{ij}$, and $m = \max_{1 \leq j \leq d} d_\varphi(\mathbf{0}, \pi(f(j)))$.

PROOF. For any $k > 0$ there exists n such that $l^{(n)} \leq k < l^{(n+1)}$, where $l^{(n)} = |\sigma^n(1)|$ is the length of the word $\sigma^n(1)$. Therefore, there exists j such that

$$\begin{aligned} s_1 \cdots s_k &= \sigma^n(W_1^{(1)}) \cdots \sigma^n(W_{j-1}^{(1)}) t_1 \cdots t_{k'}, \\ t_1 \cdots t_{k'} &< \sigma^n(W_j^{(1)}) \end{aligned}$$

where $u_1 \cdots u_k < v_1 \cdots v_j$ means

$$v_1 \cdots v_j = u_1 \cdots u_k v_{k+1} \cdots v_j.$$

Therefore, we know

$$f(s_1 s_2 \cdots s_k) = f(\sigma^n(W_1^{(1)})) + \cdots + f(\sigma^n(W_{j-1}^{(1)})) + f(t_1 \cdots t_{k'}).$$

On the other hand, we know that

$$d_\varphi(\mathbf{0}, \pi f(\sigma^n(j))) \leq \lambda_0^n d_\varphi(\mathbf{0}, \pi f(j))$$

where $\lambda_0 = \max_{2 \leq i \leq d} (|\lambda_i|)$. Therefore, we have

$$d_\varphi(\mathbf{0}, \pi f(s_1 \cdots s_k)) \leq l \cdot \max_{1 \leq j \leq d} d_\varphi(\mathbf{0}, \pi f(j)) \lambda_0^n + d_\varphi(\mathbf{0}, \pi f(t_1 \cdots t_{k'}))$$

where $l = \max_{1 \leq j \leq d} \sum_{i=1}^d l_{ij}$ and $L_\sigma = (l_{ij})_{1 \leq i, j \leq d}$. Continue the procedure, then we get

$$\text{diam.} X \leq \frac{2}{1 - \lambda_0} \cdot l \cdot \max_{1 \leq j \leq d} d_\varnothing(\mathbf{0}, \pi f(j)). \quad \square$$

LEMMA 1.4. *The following set equation holds: for each $i \in \{1, 2, \dots, d\}$*

$$X_i = \bigcup_{j=1}^d \bigcup_{\substack{S_k^{(j)}: \\ W_k^{(j)}=i \\ \sigma(j)=P_k^{(j)} W_k^{(j)} S_k^{(j)}}} (L_\sigma X_j - \pi f(S_k^{(j)})).$$

PROOF. It is enough to show that

$$L_\sigma^{-1} Y_i = \bigcup_{j=1}^d \bigcup_{\substack{S_k^{(j)}: \\ W_k^{(j)}=i, \\ \sigma(j)=P_k^{(j)} W_k^{(j)} S_k^{(j)}}} (Y_j - L_\sigma^{-1}(\pi f(S_k^{(j)})))$$

where $Y_i = \{\pi f(s_1 \cdots s_k) \mid s_k = i \text{ for some } k\}$. For any k satisfying $s_k = i$, there exist m and t such that

$$s_1 s_2 \cdots s_k = \sigma(s_1 \cdots s_{m-1}) P_t^{(s_m)} W_t^{(s_m)},$$

$$W_t^{(s_m)} = i.$$

Therefore, we have

$$f(s_1 s_2 \cdots s_k) = f(\sigma(s_1 s_2 \cdots s_m)) - f(S_t^{(s_m)}).$$

Thus, the set equation holds. □

LEMMA 1.5. *Let A be a $d \times d$ integer matrix and assume that the characteristic polynomial of A is irreducible, then the eigenvector $\mathbf{u} = {}^t(1, u_1, \dots, u_{d-1})$ of the eigenvalue λ of A is \mathbf{Q} -basis of the field $\mathbf{Q}(\lambda)$, that is,*

- (1) $\mathbf{Q} \cdot 1 + \mathbf{Q} \cdot u_1 + \cdots + \mathbf{Q} \cdot u_{d-1} = \mathbf{Q}(\lambda)$;
- (2) $\{1, u_1, \dots, u_{d-1}\}$ is \mathbf{Q} -independent.

PROOF. Let us denote the simple extension of \mathbf{Q} adjoining λ by $\mathbf{Q}(\lambda)$, then from the irreducibility of the characteristic polynomial of A , we see that $\{1, \lambda, \lambda^2, \dots, \lambda^{d-1}\}$ is the basis of $\mathbf{Q}(\lambda)$, that is,

$$(1) \mathcal{Q} + \mathcal{Q}\lambda + \cdots + \mathcal{Q}\lambda^{d-1} = \mathcal{Q}(\lambda);$$

$$(2) \{1, \lambda, \dots, \lambda^{d-1}\} \text{ is } \mathcal{Q}\text{-independent.}$$

On the other hand, from the definition:

$$A^t[1, u_1, \dots, u_{d-1}] = \lambda^t[1, u_1, \dots, u_{d-1}],$$

we see

$$\lambda = a_{11} + a_{12}u_1 + \cdots + a_{1d}u_{d-1} \quad (1.1)$$

and moreover from the fact that

$$A^k[1, u_1, \dots, u_{d-1}] = \lambda^k[1, u_1, \dots, u_{d-1}],$$

we have

$$\lambda^k = a_{11}^{(k)} + a_{12}^{(k)}u_1 + \cdots + a_{1d}^{(k)}u_{d-1} \quad (1.2)$$

and we see

$$\lambda^k \in \mathcal{Q} + \mathcal{Q}u_1 + \cdots + \mathcal{Q}u_{d-1}.$$

Therefore, we know that

$$(\mathcal{Q}(\lambda) =) \mathcal{Q} + \mathcal{Q}\lambda + \cdots + \mathcal{Q}\lambda^{d-1} \subset \mathcal{Q} + \mathcal{Q}u_1 + \cdots + \mathcal{Q}u_{d-1}.$$

Other direction

$$\mathcal{Q} + \mathcal{Q}\lambda + \cdots + \mathcal{Q}\lambda^{d-1} \supset \mathcal{Q} + \mathcal{Q}u_1 + \cdots + \mathcal{Q}u_{d-1}$$

is easy from the fact that

$$(A - \lambda E)^t[1, u_1, \dots, u_{d-1}] = \mathbf{0}.$$

In fact, $\{1, u_1, \dots, u_{d-1}\}$ is the solution of the linear equation $(A - \lambda E)^t[x_1, \dots, x_d] = \mathbf{0}$, which is the equation with $\mathcal{Q}(\lambda)$ -coefficient, therefore, we see $u_i \in \mathcal{Q}(\lambda)$. And, we have

$$\mathcal{Q} + \mathcal{Q}\lambda + \cdots + \mathcal{Q}\lambda^{d-1} = \mathcal{Q} \cdot 1 + \mathcal{Q} \cdot u_1 + \cdots + \mathcal{Q} \cdot u_{d-1},$$

that is, $\{1, u_1, \dots, u_{d-1}\}$ is the basis of $\mathcal{Q}(\lambda)$. And so, we see $\{1, u_1, \dots, u_{d-1}\}$ is \mathcal{Q} -linearly independent. \square

As the corollary of Lemma 1.5, we have the following.

COROLLARY 1.6. *The closure of $\pi\mathbf{Z}^d = \mathcal{P}$.*

PROPOSITION 1.7. *For the atomic surface X associated with the substitution σ we know the following properties:*

- (1) $\bigcup_{z \in \{\sum_{i=2}^d n_i \pi(e_1 - e_i) \mid n_i \in \mathbb{Z}\}} (X + z) = \mathcal{P}$;
- (2) $\overset{\circ}{X} \neq \emptyset$.

PROOF. For each n let us consider the set of points $l_n = \{\sum_{j=1}^k e_{s_j} \mid 1 \leq k \leq l^{(n)}\}$. We define $Y_n = \pi l_n$ and let us consider the lattice $L_0 := \{\sum_{i=2}^d n_i(e_1 - e_i) \mid n_i \in \mathbb{Z}\}$ on $\mathcal{P}_0 := \{x \in \mathbb{Z}^d \mid \langle x, \overset{\#d}{(1, 1, \dots, 1)} \rangle = 0\}$ where $\langle x, y \rangle$ is the inner product of vectors x and y .

Now define the set of the lattice points by

$$l_n + L_0 = \bigcup_{z \in L_0} (l_n + z).$$

The projection of $l_n + L_0$ by π is denoted by $\bigcup_{z \in L_0} (Y_n + \pi z)$. On the other hand, for any substitution we can see easily the following relation:

$$l_n + L_0 = \{x \in \mathbb{Z}^d \mid \langle x, \overset{\#d}{(1, 1, \dots, 1)} \rangle \geq 0\}.$$

Using the fact that

$$Y_n \subset Y_{n+1},$$

the closure of $\bigcup Y_n = X$,

we know from the boundedness of X and Corollary 1.6,

$$\bigcup_{z \in L_0} (X + \pi z) = \mathcal{P}. \tag{1.3}$$

Using (1.3) and from Baire category theorem, we have $\overset{\circ}{X} = \overset{\circ}{Y} \neq \emptyset$. From Theorem 1.2 and primitivity, we see that

$$\overset{\circ}{X}_i \neq \emptyset \quad \text{for all } i \in \{1, 2, \dots, d\}. \quad \square$$

In order to know that X_i are disjoint each other up to a set of measure 0 (about the sets of measure 0), we would prepare several lemmas. The next result can be found in [2], originally in [21].

LEMMA 1.8. *Let M be a primitive matrix with the largest eigenvalue λ . Suppose that \mathbf{v} is a positive vector such that $M\mathbf{v} \geq \lambda\mathbf{v}$. Then the inequality is an equality and \mathbf{v} is the eigenvector with respect to λ .*

Hereafter, we will note $|K|$ the measure of the set K .

LEMMA 1.9. *The vector of volumes $(|X_i|)_{1 \leq i \leq d}$ satisfies the following inequality:*

$$L_\sigma^{-1}(|X_1|, \dots, |X_d|) \geq \lambda_1(|X_1|, \dots, |X_d|).$$

PROOF. From the form of X_i in the equation of Lemma 1.4, we see

$$|L_\sigma^{-1} X_i| \leq \sum_{j=1}^d (L_\sigma)_{ij} |X_j|.$$

Since the determinant of L_σ^{-1} restricted to \mathcal{P} is λ_1 , we know that $|L_\sigma^{-1} X_i| = \lambda_1 |X_i|$. Hence we arrive at the conclusion. \square

From the Lemma 1.8, Lemma 1.9 and the fact that $|X_j| > 0$, we obtain the proof of Theorem 1.2.

REMARK. We don't know whether

$$X = \sum_{j=1}^d X_j \quad (\text{non-overlap})$$

and we see in [2] that $X = \bigcup_{j=1}^d X_j$ is non-overlap if σ satisfies the coincidence condition.

COROLLARY 1.10. *The relation that $X = \text{the closure of } \overset{\circ}{X}$ holds.*

PROOF. Moreover by rewriting Theorem 1.2, for any $n > 0$ we have

$$X = \sum_{i=1}^d \sum_{j=1}^d \sum_{\substack{S_{n,k}^{(j)} \\ W_{n,k}^{(j)}=i \\ \sigma^n(j)=P_{n,k}^{(j)} W_{n,k}^{(j)} S_{n,k}^{(j)}}} (L_\sigma^n X_j - \pi f(S_{n,k}^{(j)})).$$

For any $x \in X$ and $\delta > 0$, let $B_x(\delta)$ be the ball with the center x and the radius δ on \mathcal{P} , then by the above rewritten formula, there exist n and $S_{n,k}^{(j)}$ such that

$$B_x(\delta) \supset L_\sigma^n X_j - \pi f(S_{n,k}^{(j)}) \quad \text{and} \quad L_\sigma^n \overset{\circ}{X}_j \neq \emptyset.$$

This means that the relation that $X = \text{the closure of } \overset{\circ}{X}$ holds. \square

2. Structure of Boundary and Mauldin-Williams Graph

We say that the point $(x, i^*) \in \mathbf{Z}^d \times \{1, 2, \dots, d\}$ is an element of the stepped surface \mathcal{P} if $\langle x, v_\lambda \rangle \geq 0$ and $\langle x - e_i, v_\lambda \rangle < 0$. Put all of the elements of the stepped surface \mathcal{P} by \mathcal{S} .

LEMMA 2.1. *If a pair $(x, i^*) \neq (y, j^*)$ are the elements of S , then the element (z, k^*) given by*

$$(z, k^*) := \begin{cases} (x - y, i^*) & \text{if } \langle x - y, v_\lambda \rangle \geq 0 \\ (y - x, j^*) & \text{if } \langle y - x, v_\lambda \rangle > 0 \end{cases}$$

is also an element of S .

The proof is easy.

Let us define the map $\varphi : S \times S \rightarrow S \times S$ as follows:

$$\varphi((x, i^*), (y, j^*)) = ((\mathbf{0}, l^*), (z, k^*))$$

where (z, k^*) is given as Lemma 2.1 and l^* is given by

$$l^* = \begin{cases} j^* & \text{if } \langle x - y, v_\lambda \rangle \geq 0 \\ i^* & \text{if } \langle y - x, v_\lambda \rangle > 0 \end{cases}$$

$$\left(z = \begin{cases} x - y & \text{if } \langle x - y, v_\lambda \rangle \geq 0 \\ y - x & \text{if } \langle y - x, v_\lambda \rangle > 0 \end{cases} \right).$$

LEMMA 2.2. *Let us define the operator σ^* on S by*

$$\sigma^* : (x, i^*) := \sum_{j \in \{1, \dots, d\}} \sum_{\substack{S_k^{(j)}: \\ W_k^{(j)} = i, \\ \sigma(j) = P_k^{(j)} W_k^{(j)} S_k^{(j)}}} (L_\sigma^{-1}x + L_\sigma^{-1}f(S_k^{(j)}), j^*).$$

Then all of the elements in $\sigma^(x, i^*)$ are also the elements of S .*

The proof can be found in [2].

Let us consider the set V_0 of the pair of elements such that

$$V_0 = \{((\mathbf{0}, i^*), (x, j^*)) \mid (x, j^*) \in S, \|\pi x\| < 2D\}$$

where $\|x\|$ be the length of the vector x and D be the diameter of X estimated in Lemma 1.3. Then, we see that the cardinarity of V_0 is finite. Let us define the set of the pair $V^{(i)}$ such that

$$V^{(i)} := \{\varphi((x, j^*), (y, k^*)) \mid (x, j^*), (y, k^*) \in \sigma^*(\mathbf{0}, i^*), \|\pi(x - y)\| < 2D\},$$

and $V_0^{(0)} := \bigcup_{i=1, 2, \dots, d} V^{(i)}$, then $V_0^{(0)} \subset V_0$.

Let us define the arrow from the point $((\mathbf{0}, i^*), (w, j^*)) \in V_0$ by the following manner: for each pair $((\mathbf{0}, i^*), (w, j^*))$ let us pick up the pair such that $(x, k^*) \in$

$\sigma^*(\mathbf{0}, l^*)$, $(y, l^*) \in \sigma^*(w, j^*)$ and if $\|\pi(x - y)\| < 2D$, we give the arrow from $((\mathbf{0}, l^*), (w, j^*))$ to $\varphi((x, k^*), (y, l^*))$.

Let us define the graph $G_0 = (V_1, E, i, t)$ by the following manner:

1st step: let us consider the arrows starting from the vertex $u \in V_0^{(0)}$. If we can not find the arrow from u , then the vertex u is cancelled; If we can find the arrow e from u to v we denote $i(e) = u$, $t(e) = v$ and if moreover the vertex is new, that is, $v \in V_0 \setminus V_0^{(0)}$, then we call v the first generation of u . We denote the set of the first generation from $V_0^{(0)}$ by $V_0^{(1)}$.

2nd step: let us consider the arrow starting from the vertex of the first generation $v \in V_0^{(1)}$. If we cannot find any arrows from v , then we cancel the vertex $v \in V_0^{(1)}$ and the arrow e such that $t(e) = v$; if we can find the arrow e' from v to w and the terminal $t(e')$ is new, that is, $\omega = t(e') \in V_0 \setminus (V_0^{(0)} \cup V_0^{(1)})$, then we call the terminal ω the 2nd generator of u and denote $V_0^{(2)}$.

kth step: if we can not find any arrows from the vertex v_k , we cancelled the vertex v_k and the arrow e such that $t(e) = v_k$. And by the cancellation of the arrow e if $v_{k-1} = i(e)$ has no arrow e' such that $v_{k-1} = i(e')$ then the vertex v_{k-1} and the arrow e'' such that $t(e'') = v_{k-1}$ are also cancelled and so on. From the finiteness of the cardinality of V_0 , we can stop this procedure. We denote the final step by q .

Now we get the graph with vertices $V_1 = \bigcup_{j=1}^q V_0^{(j)}$ and each vertex u has the arrow e such that $u = i(e)$.

We denote the graph by $G_B = (V_1, E, i, t)$ and call the graph of the boundary of the atomic surface. For the simplicity, we denote the vertex $((\mathbf{0}, l^*), (x, j^*))$ by (i, j, x) .

The existence of the arrow from (i, p, x_0) to (j, q, x_1) means that on the notation:

$$\sigma^*(\mathbf{0}, l^*) = \sum_{l \in \{1, \dots, d\}} \sum_{\substack{S_k^{(l)}: \\ \sigma(l) = P_k^{(l)} \cdot i \cdot S_k^{(l)}}} (-L_\sigma^{-1}(f(S_k^{(l)})), l^*) \quad (2.4)$$

$$\sigma^*(x_0, p^*) = \sum_{m \in \{1, \dots, d\}} \sum_{\substack{S_{k'}^{(m)}: \\ \sigma(m) = P_{k'}^{(m)} \cdot p \cdot S_{k'}^{(m)}}} (-L_\sigma^{-1}(f(S_{k'}^{(m)})), m^*) + L_\sigma^{-1}(x_0), \quad (2.5)$$

there exist l, k, m and k' such that (j, q, x_1) is given explicitly by

$$\mathbf{x}_1 = \begin{cases} L_\sigma^{-1}(f(S_k^{(l)}) - f(S_{k'}^{(m)}) + \mathbf{x}_0) & \text{if } \langle L_\sigma^{-1}(f(S_k^{(l)}) - f(S_{k'}^{(m)}) + \mathbf{x}_0), \mathbf{v}_\lambda \rangle \geq 0 \\ -L_\sigma^{-1}(f(S_k^{(l)}) - f(S_{k'}^{(m)}) + \mathbf{x}_0) & \text{if } \langle L_\sigma^{-1}(f(S_k^{(l)}) - f(S_{k'}^{(m)}) + \mathbf{x}_0), \mathbf{v}_\lambda \rangle < 0 \end{cases} \quad (2.6)$$

$$(j, q) = \begin{cases} (l, m) & \text{if } \langle L_\sigma^{-1}(f(S_k^{(l)}) - f(S_{k'}^{(m)}) + \mathbf{x}_0), \mathbf{v}_\lambda \rangle \geq 0 \\ (m, l) & \text{if } \langle L_\sigma^{-1}(f(S_k^{(l)}) - f(S_{k'}^{(m)}) + \mathbf{x}_0), \mathbf{v}_\lambda \rangle < 0 \end{cases}. \quad (2.7)$$

PROPOSITION 2.3. For each vertex $(i, j, \mathbf{x}) \in V_1$ we know $X_i \cap (X_j + \pi\mathbf{x}) \neq \emptyset$ and $|X_i \cap (X_j + \pi\mathbf{x})| = 0$.

PROOF. Suppose that $X_i \cap (X_j + \pi\mathbf{x}) = \emptyset$, then from the compactness of X_i and X_j we see

$$d_\varphi(X_i, (X_j + \pi\mathbf{x})) > 0,$$

where $d_\varphi(A, B) := \inf\{d_\varphi(x, y) \mid x \in A, y \in B\}$, and so we have

$$d_\varphi(L_\sigma^{-1}X_i, L_\sigma^{-1}(X_j + \pi\mathbf{x})) \geq \lambda_0^{-1}d_\varphi(X_i, (X_j + \pi\mathbf{x})).$$

From the set equation given by Theorem 1.2 and the relation (2.4) and (2.5), we know that

$$L_\sigma^{-1}X_i \supset X_l - L_\sigma^{-1}\pi f(S_k^{(l)}) \quad \text{for } (l, k) \text{ satisfying } W_k^{(l)} = i$$

$$L_\sigma^{-1}(X_p + \mathbf{x}_0) \supset X_m - L_\sigma^{-1}\pi f(S_{k'}^{(m)}) + L_\sigma^{-1}(\mathbf{x}_0) \quad \text{for } (m, k') \text{ satisfying } W_{k'}^{(m)} = p.$$

Moreover, from the fact that the vertex (i_1, j_1, \mathbf{x}_1) from $(i, j, \mathbf{x}) \in V_1$ is given by (2.6) and (2.7), in particular (i_1, j_1) is chosen as (l, m) or (m, l) on the notation (2.4), (2.5). Therefore we see

$$d_\varphi(X_{i_1}, X_{j_1} + \pi\mathbf{x}_1) \geq d_\varphi(L_\sigma^{-1}X_i, L_\sigma^{-1}(X_j + \pi\mathbf{x})),$$

that is,

$$d_\varphi(X_{i_1}, X_{j_1} + \pi\mathbf{x}_1) \geq \lambda_0^{-1}d_\varphi(X_i, X_j + \pi\mathbf{x}).$$

Continuing this procedure, we have

$$d_\varphi(X_{i_n}, X_{j_n} + \pi\mathbf{x}_n) \geq \lambda_0^{-n}d_\varphi(X_i, X_j + \pi\mathbf{x}).$$

On the other hand, from the definition of V_0 and Lemma 1.3, we know

$$d_\varphi(X_p, X_q + \pi\mathbf{x}) < 3D \quad \text{for all } (p, q, \mathbf{x}) \in V_0.$$

Therefore, we see that

$$d_{\varphi}(X_i, X_j + \pi x) = 0.$$

This contradicts to $d_{\varphi}(X_i, X_j + \pi x) > 0$. From the definition of $(i, j, x) \in V_1$, there exist $k, n, (y, l^*)$ and $(w, m^*) \in {}^*\sigma^n(\mathbf{0}, k^*)$ such that

$$(i, j, x) = \varphi((y, l^*), (w, m^*))$$

where we denote ${}^*\sigma^n$ instead of $(\sigma^*)^n$. Therefore, from the non-overlapping property in Theorem 1.2, we have

$$|X_i \cap (X_j + \pi x)| = 0. \quad \square$$

PROPOSITION 2.4. *For each vertices $(i, j, x) \in V_1$, we see*

$$\partial X_i \supset X_i \cap (X_j + \pi x).$$

PROOF. Assume that

$$\partial X_i \not\supset X_i \cap (X_j + \pi x).$$

Then, we see that

$$(X_j + \pi x) \cap \overset{\circ}{X}_i \neq \emptyset.$$

Therefore, there exist $a \in \overset{\circ}{X}_i$ and an open ball $B_{\delta}(a)$ with the center a and the radius δ such that

$$a \in X_j + \pi x \quad \text{and} \quad B_{\delta}(a) \subset \overset{\circ}{X}_i.$$

Since the closure of $\overset{\circ}{X}_j$ is equal to X_j , we know $B_{\delta}(a) \cap (\overset{\circ}{X}_j + \pi x) \neq \emptyset$, and thus there exists $B_{\delta'}(b)$ such that

$$B_{\delta'}(b) \subset B_{\delta}(a) \cap (\overset{\circ}{X}_j + \pi x).$$

Therefore,

$$|B_{\delta}(a) \cap (\overset{\circ}{X}_j + \pi x)| > 0.$$

From Proposition 2.3 this contradicts to

$$|X_i \cap (X_j + \pi x)| = 0. \quad \square$$

PROPOSITION 2.5. *For each $j \in \{1, \dots, d\}$, there exist n and W_0 such that $\sigma^n(j) = Y \cdot 1 \cdot W_0$ and satisfying the following form:*

$$\begin{aligned} & \partial(X_j - \pi L_\sigma^{-n}(f(W_0))) \\ = & \sum_{\substack{k, W: \\ \sigma^n(k)=Y^i \cdot 1 \cdot W \text{ if } k \neq j \\ \text{or } \sigma^n(k)=Y^n \cdot 1 \cdot W \text{ and } W \neq W_0 \text{ if } k=j}} ((X_j - \pi L_\sigma^{-n}f(W_0)) \cap (X_k - \pi L_\sigma^{-n}(f(W)))) \quad (2.8) \end{aligned}$$

and

$$\varphi((j, f(W_0)), (k, f(W))) \in V_1 \quad \text{if } (X_j - \pi L_\sigma^{-n}f(W_0)) \cap (X_k - \pi L_\sigma^{-n}(f(W))) \neq \emptyset.$$

In particular, we have

$$\partial X_j = \sum_{\substack{k, W: \\ \sigma^n(k)=Y^i \cdot 1 \cdot W \text{ if } k \neq j \\ \text{or } \sigma^n(k)=Y^n \cdot 1 \cdot W \text{ and } W \neq W_0 \text{ if } k=j}} (X_j \cap (X_k - \pi(L_\sigma^{-n}(f(W)) - f(W_0)))). \quad (2.9)$$

PROOF. From Theorem 1.2, we know

$$L_\sigma^{-n} X_1 = \sum_{j=1}^d \sum_{\substack{W: \\ \sigma^n(j)=Y \cdot 1 \cdot W}} (-\pi L_\sigma^{-n}(f(W)) + X_j).$$

For the fixed j and the sufficient large n , we can find a ball V contained $L_\sigma^{-n} X_1$ and W_0 such that the ball V contains $X_j - \pi L_\sigma^{-n}f(W_0)$ and W_0 satisfies $\sigma^n(j) = Y \cdot 1 \cdot W_0$. Therefore, we see that

$$\begin{aligned} & \partial(X_j - \pi L_\sigma^{-n}(f(W_0))) \\ = & \sum_{\substack{k, W: \\ \sigma^n(k)=Y^i \cdot 1 \cdot W \text{ if } k \neq j \\ \text{or } \sigma^n(k)=Y^n \cdot 1 \cdot W \text{ and } W \neq W_0 \text{ if } k=j}} (X_j - \pi L_\sigma^{-n}f(W_0)) \cap (X_k - \pi L_\sigma^{-n}(f(W))). \quad (2.10) \end{aligned}$$

In the formula (2.8), if $(X_j - \pi L_\sigma^{-n}f(W_0)) \cap (X_k - \pi L_\sigma^{-n}(f(W))) \neq \emptyset$, then

$$\varphi((j, L_\sigma^{-n}f(W_0)), (k, L_\sigma^{-n}f(W))) \in V_1. \quad \square$$

For each arrow $e_{u,v} \in E$ let us define the transformation $T_{u,v} : \mathcal{P} \rightarrow \mathcal{P}$ by

$$T_{u,v} \mathbf{x} = L_\sigma \mathbf{x} + \pi \mathbf{f}_{u,v} \quad (2.11)$$

where $u = (i, p, \mathbf{x}_0)$ and $v = (j, q, \mathbf{x}_1)$ given by (2.6) and (2.7), and $\pi \mathbf{f}_{u,v}$ is given by

$$\pi f_{u,v} = \begin{cases} -\pi f(S_{k'}^{(m)}) + \mathbf{x}_0 & \text{if } \langle L_\sigma^{-1}(f(S_k^{(l)}) - f(S_{k'}^{(m)}) + \mathbf{x}_0), \mathbf{v}_\lambda \rangle \geq 0 \\ \pi f(S_k^{(l)}) & \text{if } \langle L_\sigma^{-1}(f(S_k^{(l)}) - f(S_{k'}^{(m)}) + \mathbf{x}_0), \mathbf{v}_\lambda \rangle < 0 \end{cases}.$$

Then the transformation $T_{u,v}$ on \mathcal{P} is a contractive map. Therefore, we have the list of compact sets $(\mathcal{X}_u)_{u \in V_1}$ uniquely satisfying $\mathcal{X}_u = \bigcup T_{u,v}(K_v)$ (see [20]). On the other hand, for each vertex (i, p, \mathbf{x}_0) from Proposition 2.3, we know $X_i \cap (X_p + \pi \mathbf{x}_0) \neq \emptyset$ and each X_i and $X_p + \mathbf{x}_0$ are decomposed by Theorem 1.2,

$$X_i = \sum_{l=1}^d \sum_{\substack{S_k^{(l)}: \\ W_k^{(l)}=i, \\ \sigma(l)=P_k^{(l)} W_k^{(l)} S_k^{(l)}}} (L_\sigma X_l - \pi f(S_k^{(l)})),$$

$$X_p + \pi \mathbf{x}_0 = \sum_{m=1}^d \sum_{\substack{S_{k'}^{(m)}: \\ W_{k'}^{(m)}=p, \\ \sigma(m)=P_{k'}^{(m)} W_{k'}^{(m)} S_{k'}^{(m)}}} (L_\sigma X_m - \pi(f(S_{k'}^{(m)}) - \mathbf{x}_0)).$$

Therefore, we have

$$X_i \cap (X_p + \pi \mathbf{x}_0) = \sum_{\substack{S_k^{(l)}, S_{k'}^{(m)}: \\ (W_k^{(l)}, W_{k'}^{(m)})=(i,p)}} (L_\sigma(X_l) - \pi f(S_k^{(l)})) \cap (L_\sigma(X_m) - \pi(f(S_{k'}^{(m)}) - \mathbf{x}_0)).$$

Using (2.6), (2.7) and $\pi f_{u,v}$ we have

$$\begin{aligned} X_i \cap (X_p + \pi \mathbf{x}_0) &= \bigcup_{\substack{v:v=(j,q,\mathbf{x}_1) \in V_1, \\ e \in E_{u,v}}} L_\sigma(X_j \cap (X_q + \pi L(\mathbf{x}_1))) + \pi f_{u,v} \\ &= \bigcup_{\substack{v:v=(j,q,\mathbf{x}_1) \in V_1, \\ e \in E_{u,v}}} T_{u,v}(X_j \cap (X_q + \pi \mathbf{x}_1)). \end{aligned} \quad \square$$

Therefore, we have the following theorem.

THEOREM 2.6. *Let $G_B = (V_1, E, i, t)$ be the graph from the substitution σ and let $T_{u,v} : \mathcal{P} \rightarrow \mathcal{P}$ be the transformation given by (2.11). Then, the list of compact sets $(\mathcal{X}_u)_{u \in V_1}$ satisfying*

$$\mathcal{K}_u = \bigcup_{\substack{v \in V_1, \\ e \in E_{u,v}}} T_{u,v}(\mathcal{K}_v)$$

is given by

$$\mathcal{K}_u = X_i \cap (X_j + \pi \mathbf{x})$$

where $u = (i, j, \mathbf{x}) \in V_1$.

3. Hausdorff Dimension of Boundaries

In this section, we discuss the Hausdorff dimension of the boundary of atomic surfaces.

THEOREM 3.1. *Let σ be a primitive unimodular Pisot substitution with d letters. Let X be the atomic surface with respect to σ . Then the Hausdorff dimension of ∂X is estimated by*

$$\dim_H \partial X \leq \dim_B \partial X \leq \frac{\log \gamma_1 - \log \lambda_1 - (d - 1) \log |\lambda_d|}{-\log |\lambda_d|}$$

where $\dim_B \partial X$ is the Box dimension of ∂X and γ_1 is the largest eigenvalue of the matrix of the graph G_B .

PROOF. By Proposition 2.5, the boundary ∂X is constructed by the sets $(X_i \cap (X_j + \pi \mathbf{x}))$, $(i, j, \mathbf{x}) \in V_1$. For any $\varepsilon > 0$, each set $X_i \cap (X_j + \pi \mathbf{x})$ can be covered by $c(\gamma_1 + \varepsilon)^n$ pieces parallelograms $L_\sigma^n(\pi \mathcal{U})$ from the unit square \mathcal{U} and the parallelogram $L_\sigma^n(\pi \mathcal{U})$ is covered at most $c' \left(\frac{|\lambda_2|}{|\lambda_d|} \cdot \frac{|\lambda_3|}{|\lambda_d|} \cdot \dots \cdot \frac{|\lambda_{d-1}|}{|\lambda_d|} \right)^n$ pieces of the cube whose length of the edge is $|\lambda_d|^n$. Therefore, the Box dimension of $X_i \cap (X_j + \pi \mathbf{x})$ can be estimated by

$$\begin{aligned} \dim_B(X_i \cap (X_j + \pi \mathbf{x})) &\leq \lim_{n \rightarrow \infty} \frac{\log c(\gamma_1 + \varepsilon)^n + \log c'(\lambda_1 |\lambda_d|^{d-1})^{-n}}{-\log |\lambda_d|^n} \\ &= \frac{\log(\gamma_1 + \varepsilon) - \log \lambda_1 - (d - 1) \log |\lambda_d|}{-\log |\lambda_d|} \end{aligned}$$

for any $\varepsilon > 0$. Therefore, by Proposition 2.5, we see

$$\dim_H \partial X \leq \dim_B \partial X \leq \frac{\log \gamma_1 - \log \lambda_1 - (d - 1) \log |\lambda_d|}{-\log |\lambda_d|}. \quad \square$$

If we know the explicit values γ_1 , λ_1 and λ_d , we see probably that $\dim_H \partial X < d - 1$. But we have no idea to say

$$\frac{\log \gamma_1 - \log \lambda_1 - (d - 1) \log |\lambda_d|}{-\log |\lambda_d|} < d - 1.$$

Therefore, we give the next theorem (c.f. [14]).

THEOREM 3.2. *Under the same assumption for σ as in Theorem 3.1, we have*

$$\dim_H \partial X < d - 1.$$

PROOF. From the set equations in Theorem 1.2 of $\{X_i\}_{i=1,2,\dots,d}$ and $\mathring{X}_1 \neq \emptyset$, for the sufficient large n_0 there exist $j_0 \in \{1, \dots, d\}$ and k_0 such that

$$\begin{aligned} \sigma^{n_0}(j_0) &= P_{n_0, k_0}^{(j_0)} \cdot 1 \cdot S_{n_0, k_0}^{(j_0)}, \\ L_\sigma^{(n_0)} X_{j_0} - \pi f(S_{n_0, k_0}^{(j_0)}) &\subset \mathring{X}_1. \end{aligned}$$

This means

$$\left\{ \begin{aligned} \partial X_1 &\subset \sum_{j=1}^d \sum_{\substack{(j, S_{n,k}^{(j)}) \neq (j_0, S_{n_0, k_0}^{(j_0)}), \\ \sigma^{n_0}(j) = P_{n_0, k}^{(j)} \cdot 1 \cdot S_{n_0, k}^{(j)}}} (L_\sigma^{n_0}(\partial X_j) - \pi f(S_{n_0, k}^{(j)})) \\ \partial X_i &\subset \sum_{j=1}^d \sum_{\substack{S_{n,k}^{(j)}, \\ \sigma^{n_0}(j) = P_{n,k}^{(j)} \cdot i \cdot S_{n,k}^{(j)}}} (L_\sigma^{n_0}(\partial X_j) - \pi f(S_{n,k}^{(j)})) \end{aligned} \right. \quad (3.12)$$

From the above properties, we say that we can cover ∂X_1 by

at most $L_\sigma^{n_0}(1, 1)$ pieces of $L_\sigma^{n_0}(\partial X_1)$

...

at most $L_\sigma^{n_0}(j_0, 1) - 1$ pieces of $L_\sigma^{n_0}(\partial X_{j_0})$

...

at most $L_\sigma^{n_0}(d, 1)$ pieces of $L_\sigma^{n_0}(\partial X_d)$

and on the definition of the matrix

$$D = \begin{bmatrix} L_\sigma^{n_0}(1, 1) & \cdots & L_\sigma^{n_0}(1, d) \\ \cdots & \cdots & \cdots \\ L_\sigma^{n_0}(j_0, 1) - 1 & \cdots & L_\sigma^{n_0}(j_0, d) \\ \cdots & \cdots & \cdots \\ L_\sigma^{n_0}(d, 1) & \cdots & L_\sigma^{n_0}(d, d) \end{bmatrix},$$

we see that $D < L_\sigma^{n_0}$ and D is primitive for sufficient large n_0 . Therefore, we know that the largest eigenvalue μ of D is strictly smaller than $\lambda_1^{n_0}$. The boundary ∂X_1 can be covered by at most c^p -pieces of parallelogram $\pi L_\sigma^{pn_0}(\mathcal{U})$ for any $\mu < \nu < \lambda_1^{n_0}$. By analogous discussion in Theorem 3.1, we see that the boundary ∂X_1 is covered by at most $c^p(\lambda_1|\lambda_d|^{d-1})^{-pn_0}$ pieces of cubes with the length $|\lambda_d|^{pn_0}$. Therefore, the α -dimensional Hausdorff measure $\mathcal{H}^\alpha(\partial X_1)$ can be estimated by

$$\mathcal{H}^\alpha(\partial X_1) \leq \lim_{p \rightarrow \infty} \nu^p \frac{1}{(\lambda_1|\lambda_d|^{d-1})^{pn_0}} (|\lambda_d|^{pn_0})^\alpha.$$

Let us assume that $\nu = \lambda_1^{n_0-x}$ for some $0 < x < 1$. Then the Hausdorff measure is estimated by

$$\mathcal{H}^\alpha(\partial X_1) \leq \lim_{p \rightarrow \infty} (\lambda_1^{(x-1)}|\lambda_d^{\alpha-(d-1)}|)^{pn_0},$$

we can choose $\alpha_0 > 0$ such that

$$\alpha_0 < d - 1 \quad \text{and} \quad \lambda_1^{(x-1)}\lambda_d^{\alpha_0-(d-1)} < 1,$$

and so we know that $\mathcal{H}^{\alpha_0}(\partial X_1) = 0$. Therefore we have

$$\dim_H(\partial X_1) \leq \alpha_0 < d - 1.$$

By analogous discussion, we see

$$\dim_H(\partial X_i) < d - 1$$

and so we get

$$\dim_H(\partial X) < d - 1. \quad \square$$

From now on, we will assume that the linear transformation L_σ on \mathcal{P} is a similitude. In two cases (i) $d = 2$ (ii) $d = 3$ and L_σ is the complex Pisot matrix, we know that the linear transformation is the similitude on \mathcal{P} .

Let the list $\{X_1 \dots, X_d\}$ of compact sets be the atomic surfaces, then we had known the sets satisfy the equation in Theorem 1.2. Therefore, we can get the graph $G_\sigma = \{\mathcal{V}, E, i, t\}$ which is constructed by $\mathcal{V} = \{1, \dots, d\}$, $e_{ij} \in E$ if there exists $j \in \{1, \dots, d\}$ such that $\sigma(i) = P_k^{(j)} \cdot i \cdot S_k^{(j)}$. And for each $e_{ij} \in E$ let us define the contracting transformation $T_{ij} : \mathcal{P} \rightarrow \mathcal{P}$ by

$$T_{ij}(x) = L_\sigma x - \pi f(S_k^{(j)})$$

which is the similitude with some contractive constant $0 < s < 1$. Then we see that $\{\mathcal{V}, E, i, t, \{T_{ij}\}\}$ is a Mauldin-Williams graph and that $\{X_i \mid i = 1, 2, \dots, d\}$ is

the graph construction set. Moreover, the graph satisfies the locally finite condition, that is, there exists a constant $H > 0$ such that for any $1 > r > 0$ and any $x \in \mathcal{P}$

$$\# \left\{ (i_1 i_2 \cdots i_l) \left| \begin{array}{l} e_{i_j, i_{j+1}} \in E, \quad 1 \leq j \leq l-1, \\ tr \leq t^l \leq r, \\ T_{i_1 i_2} \circ T_{i_2 i_3} \circ \cdots \circ T_{i_{l-1} i_l}(X_{i_l}) \cap B_x(r) \neq \emptyset \end{array} \right. \right\} < H, \quad (*)$$

since the sets $\overset{\circ}{X}_j, j = 1, 2, \dots, d$ satisfy the open set condition.

Therefore, we have the following lemma.

LEMMA 3.3. *Let $G_B = (V_1, E, i, t, \{T_{u,v}\})$ be a Mauldin-William graph in Theorem 2.6. Then the graph satisfies the locally finite condition.*

PROOF. From the locally finite condition of $G_B = \{V, E, i, t, \{T_{ij}\}\}$, we see that

$$\begin{aligned} & \# \left\{ (u_1, u_2, \dots, u_n) \left| \begin{array}{l} e_{u_i, u_{i+1}} \in E, \quad tr < t^n < r, \\ T_{u_1 u_2} T_{u_2 u_3} \cdots T_{u_{n-1} u_n}(X_p \cap (X_q + \pi y)) \cap B_x(r) \neq \emptyset \end{array} \right. \right\} \\ & < C_H^2 = \frac{H(H-1)}{2}. \end{aligned}$$

Using Lemma 3.3 and Theorem 1 in [20], we have the following theorem.

THEOREM 3.4. *Let σ be the primitive unimodular Pisot substitution. Let us assume that the linear transformation L_σ on the invariant surface \mathcal{P} is a similitude. Then the Hausdorff dimension of ∂X is given by*

$$\dim_H \partial X = \frac{(d-1) \log \gamma_1}{\log \lambda_1}$$

where γ_1 is the largest eigenvalue of the matrix of the graph G_B .

4. Examples

In this section, we propose some examples of atomic surfaces.

EXAMPLE 4.1. *Let σ be the following substitution:*

$$\sigma : \begin{array}{l} 1 \rightarrow 112 \\ 2 \rightarrow 21 \end{array} .$$

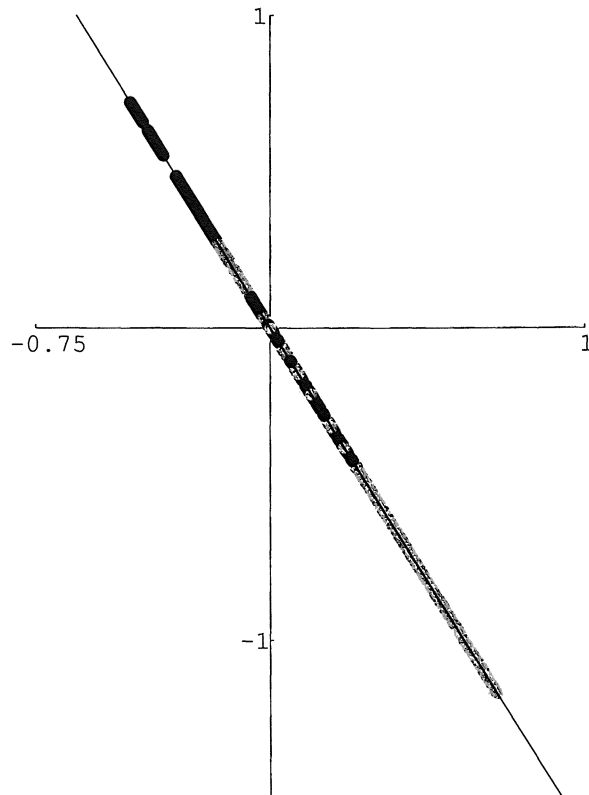


Figure 1: the atomic surface $X = \bigcup_{i=1,2} X_i$ in Example 4.1.

This substitution is a simple example which is not invertible. Therefore, the atomic surface is not an interval (see [6]). In this example, the graph G_B of the boundary of the atomic surface is given by the following form (see Figure 2):

The matrix M_σ of the graph G_B is given by

$$M_\sigma = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and the characteristic polynomial of M_σ is given by

$$x^2(x^2 - 2x - 1)(x - 1)^2$$

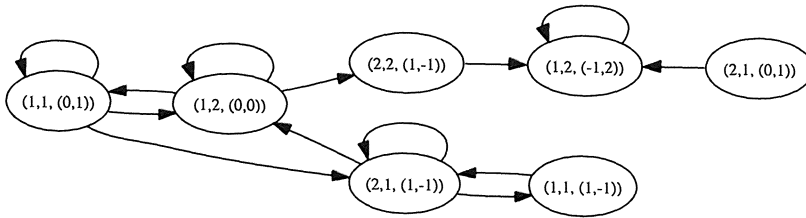


Figure 2: the graph G_B from the substitution: $1 \mapsto 112, 2 \mapsto 21$.

where the largest eigenvalue of M_σ comes from $x^2 - 2x - 1$. And so by using Theorem 3.4, the Hausdorff dimension of the boundary of the atomic surface is given by

$$\dim_H \partial X = \frac{\log \gamma_1}{\log \lambda_1} = \frac{\log 2.41421}{\log 2.61803} = 0.915785 \dots$$

where γ_1 and λ_1 are the largest eigenvalues of the graph matrix M_σ and L_σ respectively.

EXAMPLE 4.2. Let us consider the substitution called Rauzy substitution [22]:

$$\begin{aligned} 1 &\rightarrow 12 \\ \sigma : 2 &\rightarrow 13 \\ 3 &\rightarrow 1. \end{aligned}$$

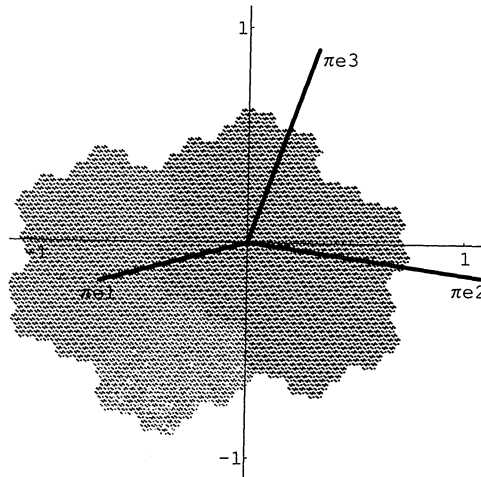


Figure 3: the atomic surface $X = \bigcup_{i=1,2,3} X_i$ in Example 4.2.

The Hausdorff dimension had been calculated in [10]. In our method, the graph G_B of the boundary of the atomic surface is given by the following form (see Figure 4):

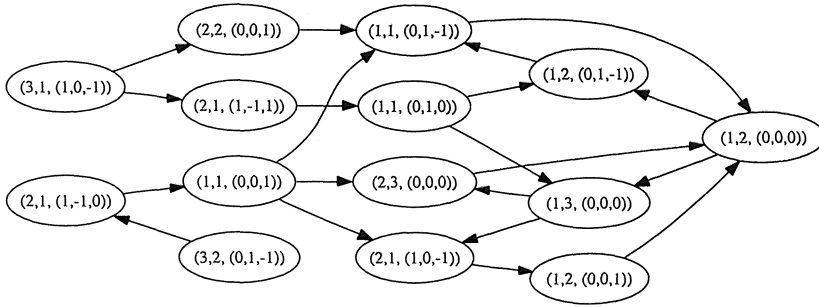


Figure 4: the graph G_B from Rauzy substitution: $1 \mapsto 12$, $2 \mapsto 13$, $3 \mapsto 1$.

The matrix M_σ of the graph G_B is given by

$$M_\sigma = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the characteristic polynomial of M_σ is given by

$$x^3(x^4 - 2x - 1).$$

Therefore, the Hausdorff dimension of ∂X_σ is calculated by

$$\dim_H \partial X = \frac{2 \log \gamma_1}{\log \lambda_1} = \frac{2 \log 1.39534}{\log 1.83929} = 1.09337 \dots$$

where γ_1 and λ_1 are the largest eigenvalues of the graph matrix M_σ and L_σ respectively.

EXAMPLE 4.3. *Let us consider the following substitution:*

$$\begin{aligned} 1 &\rightarrow 12 \\ \sigma : 2 &\rightarrow 31. \\ 3 &\rightarrow 1 \end{aligned}$$

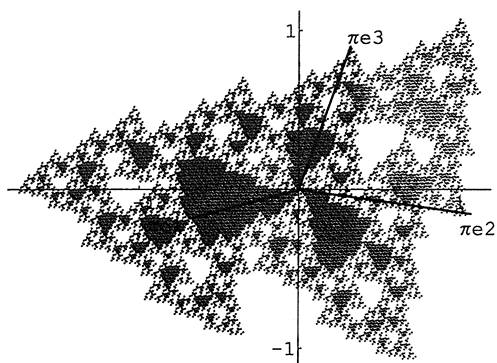


Figure 5: the atomic surface $X = \bigcup_{i=1,2,3} X_i$ in Example 4.3.

The matrix L_σ of the substitution is same as one of Rauzy substitution. But the shape of the atomic surface is perfectly different. The graph G_B of the boundary of the atomic surface is given by the following form (see Figure 6): The characteristic polynomial of M_σ is given by

$$(x^6 - x^5 - x^4 - x^2 + x - 1)(x^2 + x + 1)^2 x^{15} (x - 1)^2.$$

Therefore, the Hausdorff dimension of ∂X_σ is calculated by

$$\dim_H \partial X = \frac{2 \log \gamma_1}{\log \lambda_1} = \frac{2 \log 1.72629}{\log 1.83929} = 1.7919 \dots$$

where γ_1 and λ_1 are the largest eigenvalues of the graph matrix M_σ and L_σ respectively.

EXAMPLE 4.4. *Let us consider the substitution:*

$$\begin{aligned} 1 &\rightarrow 112 \\ \sigma : 2 &\rightarrow 13 \\ 3 &\rightarrow 1. \end{aligned}$$

This substitution is an example of a class of Pisot substitutions:

$$\begin{aligned} \sigma_{k_1, k_2} : \\ 1 &\rightarrow \overbrace{11 \dots 12}^{\#k_1} \\ 2 &\rightarrow \overbrace{11 \dots 13}^{\#k_2} \\ 3 &\rightarrow 1 \end{aligned}$$

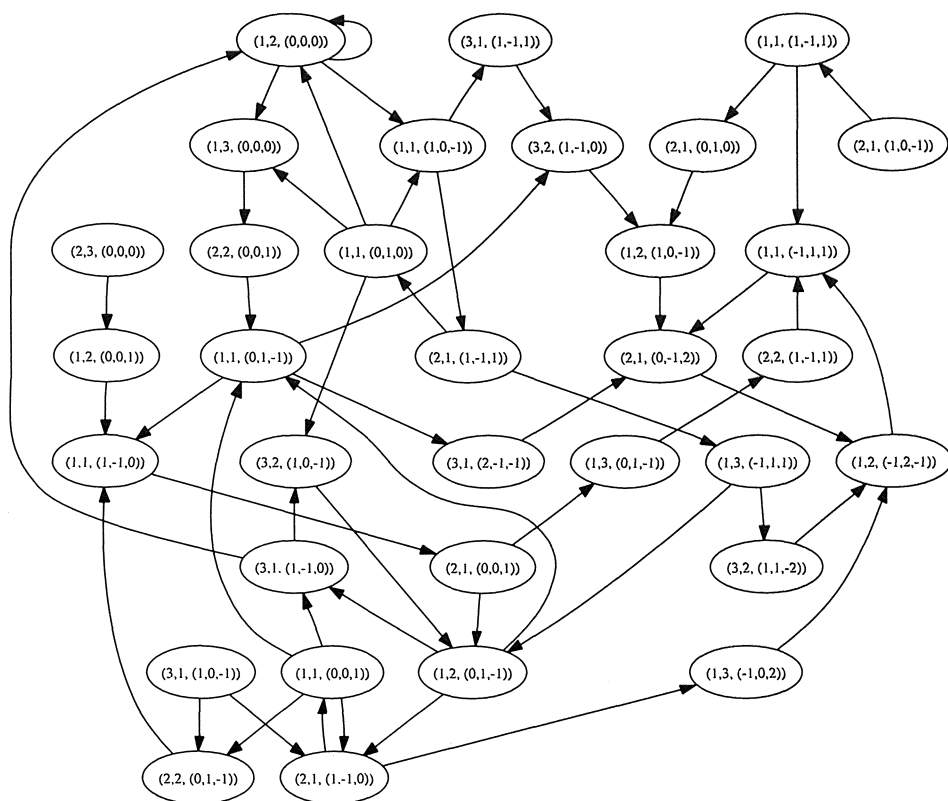


Figure 6: the graph G_B from the substitution: $1 \mapsto 12$, $1 \mapsto 31$, $1 \mapsto 1$.

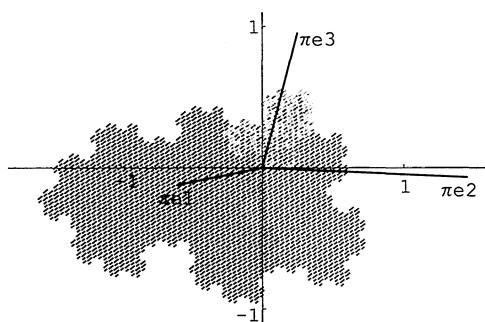


Figure 7: the atomic surface $X = \bigcup_{i=1,2,3} X_i$ in Example 4.4.

which is related to Pisot β -expansions (see [13]). The graph G_B of the boundary of the atomic surface is given by the following form (see Figure 8):

The matrix M_σ of the graph G_B is given by

$$M_\sigma = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the characteristic polynomial of M_σ is given by

$$x^5(x^4 - x^2 - 3x - 1).$$

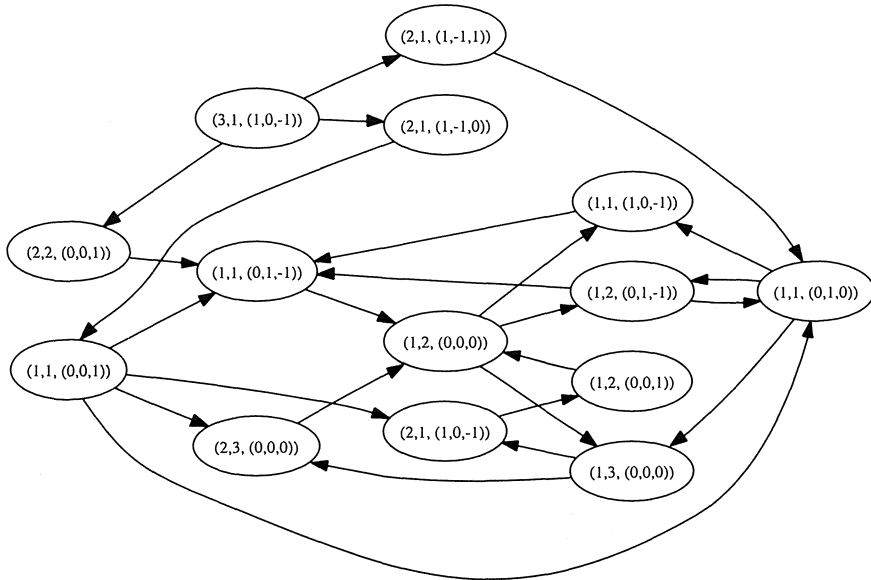


Figure 8: the graph G_B from the β -substitution: $1 \mapsto 112$, $2 \mapsto 13$, $3 \mapsto 1$.

Therefore, the Hausdorff dimension of ∂X_σ is calculated by

$$\dim_H \partial X = \frac{2 \log \gamma_1}{\log \lambda_1} = \frac{2 \log 1.74553}{\log 2.54682} = 1.19177 \dots$$

where γ_1 and λ_1 are the largest eigenvalues of the graph matrix M_σ and L_σ respectively.

EXAMPLE 4.5. *Let us consider the substitution:*

$$\begin{aligned} 1 &\rightarrow 13 \\ \sigma : 2 &\rightarrow 1 \ . \\ 3 &\rightarrow 32 \end{aligned}$$

This substitution is coming from Example 4 in [8] ($L_\sigma = M^2$).

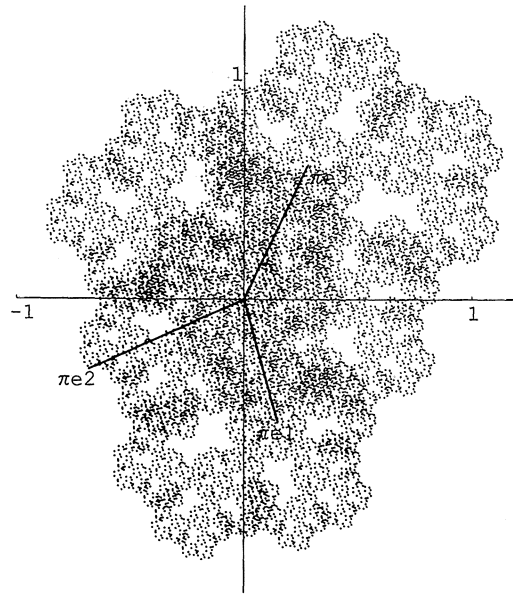


Figure 9: the atomic surface $X = \bigcup_{i=1,2,3} X_i$ in Example 4.5.

This example is that the atomic surface is not simply connected. The characteristic polynomial of M_σ is given by

$$\begin{aligned} &x^{32}(x^{13} - x^{12} - x^{10} + x^9 - 2x^8 - 4x^7 - 2x^5 - 4x^4 + x^3 - 4x^2 - 1) \\ &\times (x^5 - 2x^3 + x - 1)(x^4 + x^3 + x^2 + x + 1)(x - 1) \end{aligned}$$

and the largest eigenvalue of M_σ is coming from the polynomial $(x^{13} - x^{12} - x^{10} + x^9 - 2x^8 - 4x^7 - 2x^5 - 4x^4 + x^3 - 4x^2 - 1)$. Therefore, the Hausdorff dimension of ∂X_σ is calculated by

$$\dim_H \partial X = \frac{2 \log \gamma_1}{\log \lambda_1} = \frac{2 \log 1.72864}{\log 1.75478} = 1.94643 \dots$$

where γ_1 and λ_1 are the largest eigenvalues of the graph matrix M_σ and L_σ respectively.

EXAMPLE 4.6. *Let us consider the substitution:*

$$\begin{aligned} 1 &\rightarrow 12123 \\ \sigma : 2 &\rightarrow 1 \\ 3 &\rightarrow 12. \end{aligned}$$

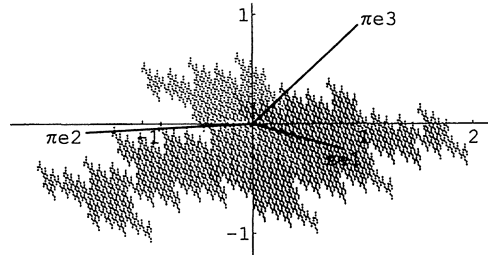


Figure 10: the atomic surface $X = \bigcup_{i=1,2,3} X_i$ in Example 4.6.

This substitution is coming from $\sigma_1 \circ \sigma_2$ for σ_m Example 1 in [10].

This is an example such that the boundary of the atomic surface is not double point free. The graph G_B of the boundary of the atomic surface is given the following form (see Figure 11);

The matrix M_σ of the graph G_B is given by

$$M_\sigma = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The characteristic polynomial of M_σ is given by

$$x^{13}(x^3 - 3x^2 + 2x - 1)(x - 1).$$

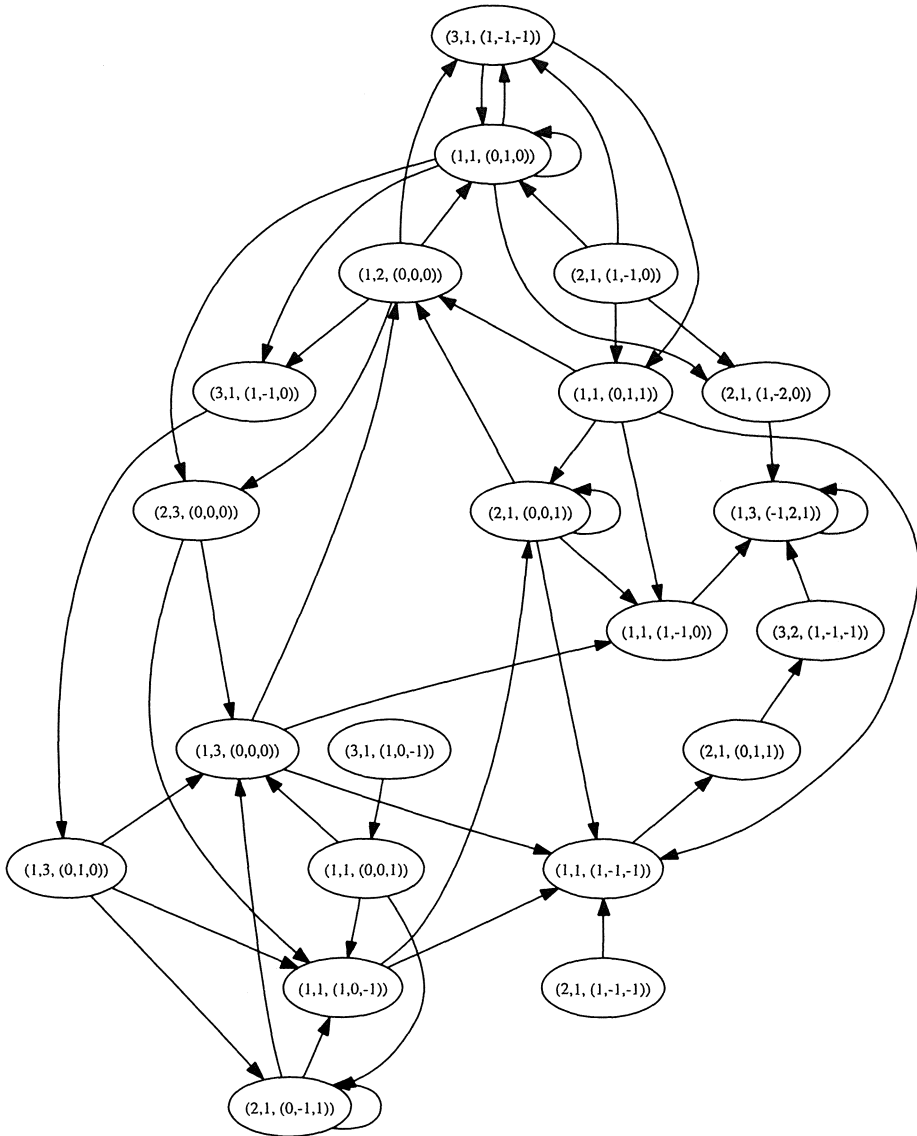


Figure 11: the graph G_B from the substitution: $1 \mapsto 12123$, $2 \mapsto 1$, $3 \mapsto 12$.

Therefore, the Hausdorff dimension of ∂X_σ is calculated by

$$\dim_H \partial X = \frac{2 \log \gamma_1}{\log \lambda_1} = \frac{2 \log 2.32472}{\log 3.0796} = 1.5$$

where γ_1 and λ_1 are the largest eigenvalues of the graph matrix M_σ and L_σ respectively.

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De-Jun Feng

Department of Mathematical Science

Tsinghua University

Beijing, 100084, People's Republic of China

E-mail: dfeng@math.tsinghua.edu.cn

Maki Furukado

Faculty of Business Administration

Yokohama National University

Tokiwadai, Hodogaya-ku Yokohama, 240-8501, Japan

E-mail: furukado@ynu.ac.jp

Shunji Ito

Graduate School of Natural Science and Technology

Kanazawa Universtiy, Kakuma-machi, Kanazawa

Ishikawa, 920-1192, Japan

E-mail: ito@t.kanazawa-u.ac.jp

Jun Wu

Department of Mathematics

Huazhong University of Science and Technology,

Wuhan, Hubei, 430074, People's Republic of China

E-mail: wujunyu@public.wh.hb.cn