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UNIVERSAL SPACES OF NON-SEPARABLE ABSOLUTE BOREL CLASSES

By

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Abstract. In this paper, we show the existence of strongly universal spaces of non-separable Borel class $\alpha \geq 2$. By combining this with the result of Sakai and Yaguchi, we can extend the results concerning absorbing sets due to Bestvina and Mogilski to every non-separable absolute Borel classes.

1. Introduction

Throughout the paper, let τ be an infinite cardinal. All space are metrizable and maps are continuous.

For each space X and for each countable ordinal α, we can define *the additive Borel class* $\Sigma_\alpha(X)$ and *the multiplicative Borel class* $\Pi_\alpha(X)$ in X as follows: Let $\Sigma_0(X)$ be the collection of all open subsets of X, and $\Pi_0(X)$ the one of all closed subsets of X. Suppose that the collections $\Sigma_\zeta(X)$ and $\Pi_\zeta(X)$ have been defined for $\zeta < \alpha$. Define $\Sigma_\alpha(X)$ as the collection of all countable unions $\bigcup_{i \in \mathbf{N}} X_i$ of $X_i \in \bigcup_{\zeta<\alpha} \Pi_\zeta(X)$, and $\Pi_\alpha(X)$ as the one of all countable intersections $\bigcap_{i \in \mathbf{N}} X_i$ of $X_i \in \bigcup_{\zeta<\alpha} \Sigma_\zeta(X)$.

For a countable ordinal α, *the absolute Borel class* $\mathfrak{a}_\alpha(\tau)$ (resp. $\mathfrak{M}_\alpha(\tau)$) is the class of all metrizable spaces X with weight $w(X) \leq \tau$ such that $X \in \Sigma_\alpha(Y)$ (resp. $X \in \Pi_\alpha(Y)$) for an arbitrary metrizable space Y which contains X as a subspace. By the result of [4, CH. III, §35 IV], $X \in \mathfrak{a}_\alpha(\tau)$ $(\alpha \geq 2)$ if and only if $X \in \Sigma_\alpha(E)$ for some completely metrizable space E with $w(E) \leq \tau$, and $X \in \mathfrak{M}_\alpha(\tau)$ $(\alpha \geq 1)$ if and only if $X \in \Pi_\alpha(E)$ for some completely metrizable space E with $w(E) \leq \tau$.

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Note that $\mathfrak{a}_0(\tau) = \emptyset$, $\mathfrak{M}_0(\tau) = \mathfrak{M}_0(\aleph_0)$ is the class of compact metrizable spaces, $\mathfrak{a}_1(\tau)$ is the class of σ-locally compact metrizable spaces with weight $\leq \tau$ (cf. [7]),[1] $\mathfrak{M}_1(\tau)$ is the class of completely metrizable spaces with weight $\leq \tau$, and $\mathfrak{M}_2(\tau)$ is the class of absolute $F_{\sigma\delta}$-spaces which plays an important role in this article.

Let $\ell_2(\tau)$ be the Hilbert space with weight τ and $\ell_2^f(\tau)$ the linear span of the canonical orthonormal basis of $\ell_2(\tau)$. In case $\tau = \aleph_0$, we denote $\ell_2(\aleph_0) = \ell_2$ and $\ell_2^f(\aleph_0) = \ell_2^f$. Let $Q = [-1,1]^{\mathbf{N}}$ be the Hilbert cube. It is well known that ℓ_2 is homeomorphic to ($\approx$) the psuedo-interior $s = (-1,1)^{\mathbf{N}}$ of Q,

$$\ell_2^f \approx \sigma = \{x \in \mathbf{R}^{\mathbf{N}} \mid x(n) = 0 \text{ except for finitely many } n \in \mathbf{N}\} \quad \text{and}$$

$$\ell_2^f \times Q \approx \Sigma = \left\{ (x_i)_{i \in \mathbf{N}} \in Q \,\middle|\, \sup_{i \in \mathbf{N}} |x_i| < 1 \right\},$$

where Σ is called the radial-interior of Q.

In the separable case (i.e., $\tau = \aleph_0$), Bestvina and Mogilski [1] constructed strongly universal spaces for the classes $\mathfrak{a}_\alpha(\aleph_0)$ and $\mathfrak{M}_\alpha(\aleph_0)$ $(\alpha \geq 1)$ as absorbing sets in s (or Q), and characterized them topologically (for the definitions of strong universality and absorbing sets, see Section 2). Using the universality of Σ for the class $\mathfrak{a}_1(\aleph_0)$, they inductively constructed strongly universal spaces for the classes $\mathfrak{a}_\alpha(\aleph_0)$ and $\mathfrak{M}_\alpha(\aleph_0)$ $(\alpha \geq 2)$. In [6], their characterization of strongly universal spaces was extended to non-separable spaces, and it was shown that $\ell_2(\tau) \times \ell_2^f$ is strongly $\mathfrak{M}_1(\tau)$-universal and $\ell_2^f(\tau) \times Q$ is strongly $\mathfrak{a}_1(\tau)$-universal. However, for the classes $\mathfrak{a}_\alpha(\tau)$ and $\mathfrak{M}_\alpha(\tau)$ $(\alpha \geq 2)$, the existence of strongly universal spaces has not been known because separability is used in the proof of [1] (cf. Remark 2 in Section 3).

In this paper, we characterize $(\ell_2(\tau) \times \ell_2^f)^{\mathbf{N}}$ as a strongly universal space for the class $\mathfrak{M}_2(\tau)$, that is,

PROPOSITION 1.1. *An AR X which is an absolute $F_{\sigma\delta}$-space with $w(X) \leq \tau$ is homeomorpic to $(\ell_2(\tau) \times \ell_2^f)^{\mathbf{N}}$ if and only if X is strongly $\mathfrak{M}_2(\tau)$-universal strong Z_σ-space.*

By the inductive construction, we can obtain strongly universal spaces $\Lambda_\alpha(\tau)$ and $\Omega_\alpha(\tau)$ for the classes $\mathfrak{a}_\alpha(\tau)$ and $\mathfrak{M}_\alpha(\tau)$ $(\alpha \geq 2)$ (for the definitions of the

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[1] A space X is *σ-locally compact* if X is a countable union of locally compact closed subsets. It should be note that X is σ-locally compact if X is a countable union of locally compact subsets. Indeed, let $X = \bigcup_{i \in \mathbf{N}} X_i$, where X_i is locally compact. Then, each X_i is an absolute F_σ-space, hence F_σ in X. Thus, $\mathfrak{a}_1(\tau)$ is equal to the class $\mathfrak{M}_4(\tau)$ in the paper [6].

spaces $\Lambda_\alpha(\tau)$ and $\Omega_\alpha(\tau)$, see Section 3). The following theorem is a main result of this article.

THEOREM 1.2. *For* $\alpha \geq 2$, *an AR* X *with* $w(X) \leq \tau$ *is homeomorpic to* $\Omega_\alpha(\tau)$ (*or* $\Lambda_\alpha(\tau)$) *if and only if* X *is strongly* $\mathfrak{M}_\alpha(\tau)$*-universal* (*or strongly* $\mathfrak{a}_\alpha(\tau)$*-universal*) *and* $X = \bigcup_{i \in \mathbf{N}} X_i$, *where each* X_i *is a strong* Z*-set in* X *and* $X_i \in \mathfrak{M}_\alpha(\tau)$ (*or* $X_i \in \mathfrak{a}_\alpha(\tau)$).

It is also proved in Section 4 that $(\ell_2(\tau) \times \ell_2^f)^{\mathbf{N}} \approx (\ell_2^f(\tau) \times Q)^{\mathbf{N}} \approx \ell_2^f(\tau)^{\mathbf{N}}$.

2. Preliminaries

For each open cover $\mathscr{U}$ of Y, two maps $f, g : X \to Y$ are *$\mathscr{U}$-close* (or f is *$\mathscr{U}$-close* to g) if each $\{f(x), g(x)\}$ is contained in some $U \in \mathscr{U}$. A closed set $A \subset X$ is called a (*strong*) *Z-set* in X provided, for each open cover $\mathscr{U}$ of X, there is a map $f : X \to X$ such that f is $\mathscr{U}$-close to id_X and $f(X) \cap A = \emptyset$ ($\mathrm{cl}\, f(X) \cap A = \emptyset$). When X is an ANR, a closed set A is a Z-set in X if and only if every map $f : \mathbf{I}^k \to X$ ($k \geq 0$) can be approximated by maps $g : \mathbf{I}^k \to X \backslash A$ (i.e., for each open cover $\mathscr{U}$ of X, there is a map $g : \mathbf{I}^k \to X \backslash A$ which is $\mathscr{U}$-close to f). A countable union of (strong) Z-sets in X is called a (*strong*) *Z_σ-set* in X. A space is called a (*strong*) *Z_σ-space* if it is a (strong) Z_σ-set in itself. A *Z-embedding* is an embedding whose image is a Z-set.

A space X is said to be *universal for a class* $\mathscr{C}$ (simply, *$\mathscr{C}$-universal*) if every map $f : C \to X$ of $C \in \mathscr{C}$ is approximated by Z-embeddings. It is said that X is *strongly universal for* $\mathscr{C}$ (simply, *strongly $\mathscr{C}$-universal*) when the following condition is satisfied:

($\mathrm{su}_{\mathscr{C}}$) for each $C \in \mathscr{C}$ and each closed set $D \subset C$, if $f : C \to X$ is a map such that $f|D$ is a Z-embedding, then, for each open cover $\mathscr{U}$ of X, there is a Z-embedding $h : C \to X$ such that $h|D = f|D$ and h is $\mathscr{U}$-close to f.

It should be noted that the condition "$X \in \mathscr{C}$" is not required in the definition of (strong) $\mathscr{C}$-universal.

Let $\mathscr{M}$ be the class of all metrizable spaces. For a class $\mathscr{C} \subset \mathscr{M}$, we denote by $\mathscr{C}(\tau)$ the subclass of $\mathscr{C}$ consisting of all spaces $X \in \mathscr{C}$ with weight $w(X) \leq \tau$. It is said that

- $\mathscr{C}$ is *topological* if $X \in \mathscr{C}$, $X \approx Y \Rightarrow Y \in \mathscr{C}$,
- $\mathscr{C}$ is *closed* (resp. *open*) *hereditary* if $X \in \mathscr{C}$, $A \subset X$ is closed (resp. open) in $X \Rightarrow A \in \mathscr{C}$,
- $\mathscr{C}$ is *additive* if $X = X_1 \cup X_2$ and $X_1, X_2 \in \mathscr{C}$ are closed in $X \Rightarrow X \in \mathscr{C}$.

By $\mathscr{C}_\sigma$, we denote the class consisting of all metrizable spaces which can be expressed as countable unions of closed subspaces contained in $\mathscr{C}$. Clearly, if $\mathscr{C}$ is closed hereditary then $\mathscr{C}_\sigma$ is closed and open hereditary.

REMARK 1. The Borel classes $\mathfrak{a}_\alpha(\tau)$ and $\mathfrak{M}_\alpha(\tau)$ $(\alpha \geq 1)$ are closed and open hereditary, additive and topological. For each $\alpha \geq 1$, $\mathfrak{a}_\alpha(\tau)_\sigma = \mathfrak{a}_\alpha(\tau)$ by the definition. We can see that $\mathfrak{M}_\alpha(\tau)_\sigma = \mathfrak{M}_\alpha(\tau)$ for $\alpha \geq 2$ (Remark 2). It should be noted that $\mathfrak{M}_1(\tau) \subsetneqq \mathfrak{M}_1(\tau)_\sigma \subsetneqq \mathfrak{a}_2(\tau)$.

For each space $X \in \mathscr{M}$, we denote by $\mathscr{E}(X)$ the class consisting of all metrizable spaces which is homeomorphic to a closed subset of X. In this paper, the countable product of X is denoted by $X^{\mathbf{N}}$, and $X_f^{\mathbf{N}}$ denotes the weak product of X with a basepoint $* \in X$, that is,

$$X_f^{\mathbf{N}} = \{x \in X^{\mathbf{N}} \mid x(n) = * \text{ except for finitely many } n \in \mathbf{N}\}.$$

Observe that Proposition 2.5 of [1] is valid for a non-separable AR X (cf. [6, footnotes in p. 155]), that is,

PROPOSITION 2.1. *Let X be a non-degenerate AR. Then $X^{\mathbf{N}}$ (resp. $X_f^{\mathbf{N}}$) is strongly $\mathscr{E}(X^{\mathbf{N}})$-universal (resp. strongly $\mathscr{E}(X_f^{\mathbf{N}})$-universal).*

A subset $X \subset M$ is said to be *homotopy dense* if there exists a deformation $h : M \times \mathbf{I} \to M$ such that $h_0 = \mathrm{id}$ and $h_t(M) \subset X$ for $t > 0$.[2] By card A, we denote the cardinality of a set A. Let $D(\tau)$ be a discrete space with card $D(\tau) = \tau$.

LEMMA 2.2. *Let X be an AR with $w(X) = \tau$. Then, the topological classes $\mathscr{E}(X^{\mathbf{N}})$ and $\mathscr{E}(X_f^{\mathbf{N}})$ are additive and closed hereditary. Moreover, they contain $\mathbf{I}^n \times D(\tau)$ as a closed subset for any $n \in \mathbf{N}$.*

PROOF. It was proved that $\mathscr{E}(X_f^{\mathbf{N}})$ is additive and closed hereditary in the proof of [1, Corollary 5.5]. It can be shown by the same way that $\mathscr{E}(X^{\mathbf{N}})$ is additive and closed hereditary. By [5], there exists a complete AR $\tilde{X}$ which contains X as a dense subset. Then, $X_f^{\mathbf{N}}$ is dense in $\tilde{X}^{\mathbf{N}}$. Moreover, $\tilde{X}^{\mathbf{N}}$ is homeomorphic to $\ell_2(\tau)$ by [9]. Since $\ell_2(\tau)$ has a discrete open collection $\mathscr{B}$ with card $\mathscr{B} = \tau$, it follows that $X_f^{\mathbf{N}}$ has a discrete open collection $\mathscr{U}$ with card $\mathscr{U} = \tau$, which is also descrete in $X^{\mathbf{N}}$. Observe that each $U \in \mathscr{U}$ contains an arc. Then, $X_f^{\mathbf{N}}$

[2] It is noted that X is homotopy dense in an ANR M if and only if $M \backslash X$ is locally homotopy negligible in M [8].

contains a copy of $\mathbf{I} \times D(\tau)$ which is closed in $X^{\mathbf{N}}$. Note that $(X_f^{\mathbf{N}})^n \approx X_f^{\mathbf{N}}$ and $(X^{\mathbf{N}})^n \approx X^{\mathbf{N}}$. Therefore, $\mathbf{I}^n \times D(\tau) \in \mathscr{E}(X_f^{\mathbf{N}}) \cap \mathscr{E}(X^{\mathbf{N}})$. □

Lemma 1.3 of [1] is also valid for the non-separable case because separability is not used in the proof. Then, we have the following lemma.

Lemma 2.3. *Let M be an ANR and X a homotopy dense subset of M. Suppose that every Z-set in M is a strong Z-set in M. Then, every Z-set in X is a strong Z-set in X.*

Proof. Suppose that $A \subset X$ is a Z-set in X. For each open cover $\mathscr{U}$ of X, we have a collection $\tilde{\mathscr{U}}$ of open sets in M such that $\{U \cap X \mid U \in \tilde{\mathscr{U}}\} = \mathscr{U}$. Then $U = \bigcup \tilde{\mathscr{U}}$ is open in M and X is homotopy dense in U. Let $\mathscr{V}$ be an open cover of U which is a star-refinement of $\tilde{\mathscr{U}}$. Since X is homotopy dense in M, $\mathrm{cl}_M A$ is a Z-set in M, hence a strong Z-set in M. Thus, $\mathrm{cl}_U A = U \cap \mathrm{cl}_M A$ is a strong Z-set in U by Lemma 1.3 of [1]. Hence, there is a map $f : U \to U$ such that f is $\mathscr{V}$-close to id_U and $\mathrm{cl}_U f(U) \cap \mathrm{cl}_U A = \emptyset$. Choose an open refinement $\mathscr{W}$ of $\mathscr{V}$ such that if a map $f' : U \to U$ is $\mathscr{W}$-close to f then $\mathrm{cl}_U f'(U) \cap \mathrm{cl}_U A = \emptyset$. Since X is homotopy dense in U, there exists a map $g : U \to X$ which is $\mathscr{W}$-close to id_U. Then the map $h = g \circ f|_X$ is $\mathscr{U}$-close to id_X. Since $g \circ f$ is $\mathscr{W}$-close to $\mathrm{id}_U \circ f = f$, we have $\mathrm{cl}_U gf(U) \cap \mathrm{cl}_U A = \emptyset$, hence $\mathrm{cl}_X h(X) \cap A = \emptyset$. Therefore, A is a strong Z-set in X. □

Note that every Z-set in $\ell_2(\tau)$ is a strong Z-set [3]. Using the Lemma 2.3, we have the following lemma.

Lemma 2.4. *Let X be a Z_σ-space which is homotopy dense in $\ell_2(\tau)$. Then, $X^{\mathbf{N}}$ and $X_f^{\mathbf{N}}$ are strong Z_σ-spaces which are homotopy dense in $\ell_2(\tau)^{\mathbf{N}}$.*

Proof. Since X is homotopy dense in $\ell_2(\tau)$, X is an AR and $X^{\mathbf{N}}$ is homotopy dense in $\ell_2(\tau)^{\mathbf{N}}$. It is easy to see that $X_f^{\mathbf{N}}$ is homotopy dense in $X^{\mathbf{N}}$. This means that $X_f^{\mathbf{N}}$ is homotopy dense in $\ell_2(\tau)^{\mathbf{N}}$. By Lemma 2.3, every Z-set in $X^{\mathbf{N}}$ (resp. $X_f^{\mathbf{N}}$) is a strong Z-set in $X^{\mathbf{N}}$ (resp. $X_f^{\mathbf{N}}$). Thus, it remains to show that $X^{\mathbf{N}}$ and $X_f^{\mathbf{N}}$ are Z_σ-spaces. It is clear that $X_f^{\mathbf{N}}$ is a Z_σ-space. Since X is a Z_σ-space, we can write that $X = \bigcup_{i \in \mathbf{N}} X_i$, where X_i is a Z-set in X. Then, $X_i \times X^{\mathbf{N}}$ is a Z-set in $X \times X^{\mathbf{N}}$. Hence, $X^{\mathbf{N}} \approx X \times X^{\mathbf{N}} = \bigcup_{i \in \mathbf{N}} X_i \times X^{\mathbf{N}}$ is a Z_σ-space. □

Given a space E, an *E-manifold* is a topological manifold modeled on E, that is, a paracompact Hausdorff space such that each point has an open neighborhood which is homeomorphic to an open set in E.

A *$\mathscr{C}$-absorbing set* in M is a homotopy dense subset $X \subset M$ such that $X \in \mathscr{C}_\sigma$ and X is a strongly $\mathscr{C}$-universal strong Z_σ-space. In [6], Sakai and Yaguchi generalized a characterization of $\mathscr{C}$-absorbing sets by Bestvina and Mogilski [1, Theorem 5.3] to the non-separable case [6, Theorem 3.8]. The following Theorem is an extension of [6, Theorem 3.8]. Note that Proposition 2.1 of [1] are proved without separability.

THEOREM 2.5. *Let $\mathscr{C}$ be a closed hereditary additive topological class of spaces such that $\mathbf{I}^n \times D(\tau) \in \mathscr{C}$ for each $n \in \mathbf{N}$. Suppose that there exists a $\mathscr{C}$-absorbing set Ω in $\ell_2(\tau)$. Then, an AR (or an ANR) X with $w(X) \leq \tau$ is homeomorphic to Ω (or an Ω-manifold) if and only if $X \in \mathscr{C}_\sigma$, X is strongly $\mathscr{C}$-universal and X is a strong Z_σ-space.*

PROOF. This proof is similar to the one of [6, Proposition 4.2]. For the "if" part, just replace "$\mathfrak{M}_i(\tau)$" and "$E_i(\tau)$" by "$\mathscr{C}$" and "Ω". To prove the "only if" part, suppose that X is an Ω-manifold. By Theorem 3.9 (3) of [6], there exists an open embedding $\varphi : X \hookrightarrow \Omega$. Since $\Omega \in \mathscr{C}_\sigma$ and $\mathscr{C}_\sigma$ is open hereditary, we have $X \in \mathscr{C}_\sigma$. By Proposition 2.1 of [1], X is strongly $\mathscr{C}$-universal. Moreover, X is a strong Z_σ-space because so is Ω. □

One should noticed that Theorem 2.5 above means that all $\mathscr{C}$-absorbing sets of $\ell_2(\tau)$ are homeomorphic to each others. Moreover, we can show the topological uniqueness of $\mathscr{C}$-absorbing sets of an $\ell_2(\tau)$-manifold (see the proof of Proposition 4.2 of [6]). Then, the following theorem follows from the classification theorem for $\ell_2(\tau)$-manifold [2, Theorem 6].

THEOREM 2.6. *Under the assumption of Theorem 2.5, two Ω-manifolds are homeomorphic to each others if and only if they have the same homotopy type.*

PROOF. Let X and Y be Ω-manifolds which have the same homotopy type. By Theorem 3.9 (4) of [6], there exist $\ell_2(\tau)$-manifolds $\tilde{X}$ and $\tilde{Y}$ in which X and Y can be embedded as $\mathscr{C}$-absorbing sets, respectively. Since X and Y are homotopy dense in $\tilde{X}$ and $\tilde{Y}$ respectively, $\tilde{X}$ and $\tilde{Y}$ have the same homotopy type. By the classification theorem of $\ell_2(\tau)$-manifolds [2, Theorem 6], we have $\tilde{X} \approx \tilde{Y}$. Hence, Y also can be embedded into $\tilde{X}$ as a $\mathscr{C}$-absorbing set. From uniqueness of $\mathscr{C}$-absorbing sets, it follows that X and Y are homeomorphic. □

3. Existence of Absorbing Sets in $\ell_2(\tau)$

First, we will show the following lemmas.

LEMMA 3.1. *For each F_σ-subset X of $\ell_2(\tau)$, there exists a closed embedding $\varphi : \ell_2(\tau) \hookrightarrow \ell_2(\tau) \times \ell_2$ such that $\varphi^{-1}(\ell_2(\tau) \times \ell_2^f) = X$.*

PROOF. As a special case of Lemma 3.3 of [11], we have a map $f : \ell_2(\tau) \to \ell_2^f$ such that $f^{-1}(\ell_2^f) = X$. Now, we define a map $\varphi : \ell_2(\tau) \to \ell_2(\tau) \times \ell_2$ by $\varphi(x) = (x, f(x))$. Then, φ is a closed embedding such that $\varphi^{-1}(\ell_2(\tau) \times \ell_2^f) = X$. □

LEMMA 3.2. *For each $F_{\sigma\delta}$-subset X of $\ell_2(\tau)$, there exists a closed embedding $\varphi : \ell_2(\tau) \hookrightarrow (\ell_2(\tau) \times \ell_2)^{\mathbf{N}}$ such that $\varphi^{-1}((\ell_2(\tau) \times \ell_2^f)^{\mathbf{N}}) = X$.*

PROOF. Suppose $X = \bigcap_{i \in \mathbf{N}} X_i$, where each X_i $(i \in \mathbf{N})$ is F_σ in $\ell_2(\tau)$. By Lemma 3.1, there exist closed embeddings $\varphi_i : \ell_2(\tau) \to \ell_2(\tau) \times \ell_2$, $i \in \mathbf{N}$, such that $\varphi_i^{-1}(\ell_2(\tau) \times \ell_2^f) = X_i$. Define a map $\varphi : \ell_2(\tau) \to (\ell_2(\tau) \times \ell_2)^{\mathbf{N}}$ by $\varphi(x) = (\varphi_i(x))_{i \in \mathbf{N}}$. Then, φ is a closed embedding and $\varphi^{-1}((\ell_2(\tau) \times \ell_2^f)^{\mathbf{N}}) = X$. □

LEMMA 3.3. *Let A_i be an F_σ-subset of a space X_i for each $i \in \mathbf{N}$. Then, the subset $\prod_{i \in \mathbf{N}} A_i$ of the product space $\prod_{i \in \mathbf{N}} X_i$ is $F_{\sigma\delta}$ in $\prod_{i \in \mathbf{N}} X_i$.*

PROOF. Let $A_i = \bigcup_{j \in \mathbf{N}} F_{ij}$ where each F_{ij} is a closed subset of X_i. Then, it follows that

$$\begin{aligned}
\prod_{i \in \mathbf{N}} A_i &= \bigcap_{i \in \mathbf{N}} \prod_{n=1}^{i-1} X_n \times A_i \times \prod_{n=i+1}^{\infty} X_n \\
&= \bigcap_{i \in \mathbf{N}} \prod_{n=1}^{i-1} X_n \times \bigcup_{j \in \mathbf{N}} F_{ij} \times \prod_{n=i+1}^{\infty} X_n \\
&= \bigcap_{i \in \mathbf{N}} \bigcup_{j \in \mathbf{N}} \prod_{n=1}^{i-1} X_n \times F_{ij} \times \prod_{n=i+1}^{\infty} X_n \\
&\subset \prod_{i \in \mathbf{N}} X_i.
\end{aligned}$$

For every $i, j \in \mathbf{N}$,

$$\prod_{n=1}^{i-1} X_n \times F_{ij} \times \prod_{n=i+1}^{\infty} X_n$$

is closed in $\prod_{i\in\mathbf{N}} X_i$. Therefore, $\prod_{i\in\mathbf{N}} A_i$ is $F_{\sigma\delta}$ in $\prod_{i\in\mathbf{N}} X_i$. □

LEMMA 3.4. $\mathscr{E}((\ell_2(\tau)\times\ell_2^f)^{\mathbf{N}}) = \mathfrak{M}_2(\tau)$

PROOF. We have $\mathfrak{M}_2(\tau) \subset \mathscr{E}((\ell_2(\tau)\times\ell_2^f)^{\mathbf{N}})$ by Lemma 3.2. To see $\mathscr{E}((\ell_2(\tau)\times\ell_2^f)^{\mathbf{N}}) \subset \mathfrak{M}_2(\tau)$, it suffices to show that $(\ell_2(\tau)\times\ell_2)^{\mathbf{N}} \in \mathfrak{M}_2(\tau)$ because $\mathfrak{M}_2(\tau)$ is closed hereditary. Since ℓ_2^f is F_σ in ℓ_2, $\ell_2(\tau)\times\ell_2^f$ is F_σ in $\ell_2(\tau)\times\ell_2$. Then, $(\ell_2(\tau)\times\ell_2^f)^{\mathbf{N}}$ is $F_{\sigma\delta}$ in $(\ell_2(\tau)\times\ell_2)^{\mathbf{N}}$ by Lemma 3.3. This means $(\ell_2(\tau)\times\ell_2^f)^{\mathbf{N}} \in \mathfrak{M}_2(\tau)$. □

By combining Proposition 2.1 and Lemmas 2.4, 3.4, we have the following:

PROPOSITION 3.5. *The space* $(\ell_2(\tau)\times\ell_2^f)^{\mathbf{N}}$ *is an* $\mathfrak{M}_2(\tau)$*-absorbing set in* $(\ell_2(\tau)\times\ell_2)^{\mathbf{N}}$. □

Now, for each countable ordinal $\alpha \geq 2$, we shall construct an $\mathfrak{M}_\alpha(\tau)$-absorbing set $\Omega_\alpha(\tau)$ and an $\mathfrak{a}_\alpha(\tau)$-absorbing set $\Lambda_\alpha(\tau)$ in $\ell_2(\tau)$. This construction is the same way as [1], where "s" ($\approx \ell_2$) is just replaced by "$\ell_2(\tau)$". Take any homeomorphisms $\varphi : (\ell_2(\tau)\times\ell_2)^{\mathbf{N}} \to \ell_2(\tau)$ and $\psi : \ell_2(\tau)^{\mathbf{N}} \to \ell_2(\tau)$. First, define $\Omega_2(\tau) = \varphi((\ell_2(\tau)\times\ell_2^f)^{\mathbf{N}})$. Suppose that $\Omega_\alpha(\tau) \subset \ell_2(\tau)$ has been defined. Then, we define

$$\Lambda_\alpha(\tau) = \psi((\ell_2(\tau)\backslash\Omega_\alpha(\tau))_f^{\mathbf{N}}) \subset \ell_2(\tau)$$

Suppose that $\Lambda_\zeta(\tau) \subset \ell_2(\tau)$ have been defined for $2 \leq \zeta < \alpha$. In case $\alpha = \beta+1$, let

$$\Omega_\alpha(\tau) = \psi(\Lambda_\beta(\tau)^{\mathbf{N}}) \subset \ell_2(\tau).$$

When α is a limit ordinal, we define

$$\Omega_\alpha(\tau) = h_\alpha\left(\prod\nolimits_{2\leq\zeta<\alpha} \Lambda_\zeta(\tau)^{\mathbf{N}}\right) \subset \ell_2(\tau)$$

where $h_\alpha : \prod_{2\leq\zeta<\alpha} \ell_2(\tau)^{\mathbf{N}} \to \ell_2(\tau)$ is a homeomorphism.

The following is easily proved by the induction on $\alpha \geq 2$.

- $\Omega_\alpha(\tau) \in \mathfrak{M}_\alpha(\tau)$ and $\Lambda_\alpha(\tau) \in \mathfrak{a}_\alpha(\tau)$.
- $\Omega_\alpha(\tau)$ and $\Lambda_\alpha(\tau)$ are homotopy dense in $\ell_2(\tau)$.
- $\Omega_\alpha(\tau)$ and $\Lambda_\alpha(\tau)$ are strong Z_σ-space.

The following lemma is the non-separable version of [1, Lemma 6.3], where "s" and "Q" are replaced by "$\ell_2(\tau)$". The proof is basically same as [1, Lemma 6.3].

LEMMA 3.6. *Let $\alpha \geq 2$ be a countable ordinal. Suppose $X \in \mathfrak{M}_\alpha(\tau)$ (resp. $X \in \mathfrak{a}_\alpha(\tau)$) is embedded into $\ell_2(\tau)$. Then there is a closed embedding $\varphi_\alpha : \ell_2(\tau) \hookrightarrow \ell_2(\tau)$ such that $\varphi_\alpha^{-1}(\Omega_\alpha(\tau)) = X$ (resp. ${\varphi_\alpha}^{-1}(\Lambda_\alpha(\tau)) = X$).*

PROOF. Lemma 3.2 means the case of $\Omega_2(\tau)$. Similarly to [1, Lemma 6.3], other cases are shown by induction on α. □

Now, we have the following non-separable version of [1, Proposition 6.4]. This is an answer for Problem 5 in [6]. The proof is same as [1, Proposition 6.4].

PROPOSITION 3.7. *For a countable ordinal $\alpha \geq 2$, the space $\Omega_\alpha(\tau)$ is $\mathfrak{M}_\alpha(\tau)$-absorbing in $\ell_2(\tau)$ and $\Lambda_\alpha(\tau)$ is $\mathfrak{a}_\alpha(\tau)$-absorbing in $\ell_2(\tau)$.* □

REMARK 2. Recall that $\Omega_\alpha(\tau) \in \mathfrak{M}_\alpha(\tau)$ for $\alpha \geq 2$. By Proposition 3.5 of [6], $\Omega_\alpha(\tau)$ is strongly $\mathfrak{M}_\alpha(\tau)_\sigma$-universal, which means $\mathfrak{M}_\alpha(\tau)_\sigma = \mathfrak{M}_\alpha(\tau)$. Moreover, we have $\Omega_\alpha(\tau)_f^{\mathbf{N}} \approx \Omega_\alpha(\tau)$ $(\approx \Omega_\alpha(\tau)^{\mathbf{N}})$ by Theorem 3.8 below because $\Omega_\alpha(\tau)_f^{\mathbf{N}} \in \mathfrak{M}_\alpha(\tau)$ can be embedded into $\ell_2(\tau)$ as an $\mathfrak{M}_\alpha(\tau)$-absorbing set.

By combining Proposition 3.7 and Theorem 2.5, we have the following non-separable version of [1, Theorem 6.5].

THEOREM 3.8. *For $\alpha \geq 2$, an AR X with $w(X) \leq \tau$ is homeomorpic to $\Omega_\alpha(\tau)$ (or $\Lambda_\alpha(\tau)$) if and only if X is strongly $\mathfrak{M}_\alpha(\tau)$-universal (or strongly $\mathfrak{a}_\alpha(\tau)$-universal) and $X = \bigcup_{i \in \mathbf{N}} X_i$, where each X_i is a strong Z-set in X and $X_i \in \mathfrak{M}_\alpha(\tau)$ (or $X_i \in \mathfrak{a}_\alpha(\tau)$).* □

THEOREM 3.9. *For $\alpha \geq 2$, an ANR X with $w(X) \leq \tau$ is an $\Omega_\alpha(\tau)$-manifold (or a $\Lambda_\alpha(\tau)$-manifold) if and only if X is strongly $\mathfrak{M}_\alpha(\tau)$-universal (or strongly $\mathfrak{a}_\alpha(\tau)$-universal) and $X = \bigcup_{i \in \mathbf{N}} X_i$, where each X_i is a strong Z-set in X and $X_i \in \mathfrak{M}_\alpha(\tau)$ (or $X_i \in \mathfrak{a}_\alpha(\tau)$).* □

REMARK 3. We have defined $\Omega_2(\aleph_0)$ as $(\ell_2 \times \ell_2^f)^{\mathbf{N}}$. On the other hand, in [1], Ω_2 was defined as $\Sigma^{\mathbf{N}} \approx (\ell_2^f \times Q)^{\mathbf{N}}$. In this connection, we shall show that $(\ell_2^f(\tau) \times Q)^{\mathbf{N}} \approx \Omega_2(\tau)$ in Section 4.

Remark 4. It was shown in [6] that $\ell_2^f(\tau) \times Q$ can be embedded into $\ell_2(\tau)$ as an $\mathfrak{a}_1(\tau)$-absorbing set.[3] Thus, as a generalization of $\Lambda_1 = \Sigma \approx \ell_2^f \times Q$ in [1], $\Lambda_1(\tau)$ should be defined as $\ell_2^f(\tau) \times Q$. In [1], Λ_1 is the first step of the inductive construction of Ω_α and Λ_α. Then, it seems that $\Lambda_1(\tau)$ can be used as the first step in the construction. Thus, it is natural to ask whether Lemma 3.6 is valid for $\Lambda_1(\tau)$ and $X \in \mathfrak{a}_1(\tau)$ or not. In other words, we have the following question.

Question. For each F_σ-subset X of $\ell_2(\tau)$, does there exist a closed embedding $\varphi : \ell_2(\tau) \hookrightarrow \ell_2(\tau) \times Q$ such that $\varphi^{-1}(\ell_2^f(\tau) \times Q) = X$?

However, even if this question is affirmative, we cannot obtain Lemma 3.6 for $\alpha \geq 2$ from this directly. Because there exists an absolute $F_{\sigma\delta}$-space which cannot be expressed as a countable intersection of absolute F_σ-spaces (e.g., the space $\ell_2(\tau)$ for any $\tau > \aleph_0$).

4. Consistency with the Separable Case

In this section, we shall show $(\ell_2(\tau) \times \ell_2^f)^{\mathbf{N}} \approx (\ell_2^f(\tau) \times Q)^{\mathbf{N}} \approx \ell_2^f(\tau)^{\mathbf{N}}$ (cf. Remark 1 in the previous section). Recall that $\mathfrak{M}_2(\tau) = \mathscr{E}((\ell_2(\tau) \times \ell_2^f)^{\mathbf{N}})$ by Lemma 3.4.

Lemma 4.1. $\mathfrak{M}_2(\tau) = \mathscr{E}(\ell_2^f(\tau)^{\mathbf{N}})$.

Proof. To see $\mathscr{E}((\ell_2(\tau) \times \ell_2^f)^{\mathbf{N}}) \subset \mathscr{E}(\ell_2^f(\tau)^{\mathbf{N}})$, it suffices to prove that $(\ell_2(\tau) \times \ell_2^f)^{\mathbf{N}} \in \mathscr{E}(\ell_2^f(\tau)^{\mathbf{N}})$. Let $J(\tau)$ be the hedgehog with weight τ, that is, the cone over the canonical orthonormal basis of $\ell_2(\tau)$ with the vertex $0 \in \ell_2(\tau)$. Then, $J(\tau)^{\mathbf{N}} \approx \ell_2(\tau)$ by Theorem 5.1 of [9] (cf. [10]). Since $J(\tau)$ is a closed subset of the space $\ell_2^f(\tau)$, we have a closed embedding of $\ell_2(\tau)$ into $\ell_2^f(\tau)^{\mathbf{N}}$. On the other hand, ℓ_2^f can be embedded into $\ell_2^f(\tau)$ as a closed set. Then, we have $\ell_2(\tau) \times \ell_2^f \in \mathscr{E}(\ell_2^f(\tau)^{\mathbf{N}} \times \ell_2^f(\tau))$. Hence,

$$(\ell_2(\tau) \times \ell_2^f)^{\mathbf{N}} \in \mathscr{E}((\ell_2^f(\tau)^{\mathbf{N}} \times \ell_2^f(\tau))^{\mathbf{N}}) = \mathscr{E}(\ell_2^f(\tau)^{\mathbf{N}}).$$

To see $\mathscr{E}(\ell_2^f(\tau)^{\mathbf{N}}) \subset \mathfrak{M}_2(\tau)$, observe that $\ell_2^f(\tau)$ is an F_σ-subspace of $\ell_2(\tau)$. By Lemma 3.3, we have that $\ell_2^f(\tau)^{\mathbf{N}}$ is $F_{\sigma\delta}$ in $\ell_2(\tau)^{\mathbf{N}}$, and $\mathscr{E}(\ell_2^f(\tau)^{\mathbf{N}}) \subset \mathfrak{M}_2(\tau)$. □

Proposition 4.2. *Suppose that E and F are Z_σ-spaces which are homotopy*

[3] See the footnote 1.

dense in $\ell_2(\tau)$. *If there exist closed embeddings* $f : E^{\mathbf{N}} \hookrightarrow F^{\mathbf{N}}$ *and* $g : F^{\mathbf{N}} \hookrightarrow E^{\mathbf{N}}$ (*resp.* $f : E_f^{\mathbf{N}} \hookrightarrow F_f^{\mathbf{N}}$ *and* $g : F_f^{\mathbf{N}} \hookrightarrow E_f^{\mathbf{N}}$), *then* $E^{\mathbf{N}} \approx F^{\mathbf{N}}$ (*resp.* $E_f^{\mathbf{N}} \approx F_f^{\mathbf{N}}$).

Proof. Because of similarity, we shall only prove $E^{\mathbf{N}} \approx F^{\mathbf{N}}$. By the assumption, we have $\mathscr{C}(E^{\mathbf{N}}) = \mathscr{C}(F^{\mathbf{N}})$, which is an additive closed hereditary topological class such that $\mathbf{I}^n \times D(\tau) \in \mathscr{C}(E^{\mathbf{N}})$ for all $n \in \mathbf{N}$ by Lemma 2.2. By Proposition 2.1 and Lemma 2.4, $E^{\mathbf{N}}$ and $F^{\mathbf{N}}$ are strongly $\mathscr{C}(E^{\mathbf{N}})$-universal strong Z_σ-spaces which are homotopy dense in $\ell_2(\tau)^{\mathbf{N}}$. Then, $E^{\mathbf{N}}$ and $F^{\mathbf{N}}$ are $\mathscr{C}(E^{\mathbf{N}})$-absorbing in $\ell_2(\tau)^{\mathbf{N}}$. Hence, $E^{\mathbf{N}} \approx F^{\mathbf{N}}$ by the topological uniqueness of absorbing sets. □

Then, we have the following.

Proposition 4.3. $(\ell_2(\tau) \times \ell_2^f)^{\mathbf{N}} \approx \ell_2^f(\tau)^{\mathbf{N}} \approx (\ell_2^f(\tau) \times Q)^{\mathbf{N}}$.

Proof. Since $\mathscr{C}((\ell_2(\tau) \times \ell_2^f)^{\mathbf{N}}) = \mathfrak{M}_2(\tau) = \mathscr{C}(\ell_2^f(\tau)^{\mathbf{N}})$, we have $(\ell_2(\tau) \times \ell_2^f)^{\mathbf{N}} \approx \ell_2^f(\tau)^{\mathbf{N}}$ by Proposition 4.2. Now, we show $(\ell_2^f(\tau) \times Q)^{\mathbf{N}} \approx \ell_2^f(\tau)^{\mathbf{N}}$. Note that $\ell_2 \times Q \approx \ell_2$ and $\ell_2^f(\tau) \times \mathbf{R} \approx \ell_2^f(\tau)$. Then it follows that

$$\begin{aligned}(\ell_2^f(\tau) \times Q)^{\mathbf{N}} &\approx (\ell_2^f(\tau) \times \mathbf{R} \times Q)^{\mathbf{N}} \approx \ell_2^f(\tau)^{\mathbf{N}} \times \ell_2 \times Q \\ &\approx \ell_2^f(\tau)^{\mathbf{N}} \times \ell_2 \approx (\ell_2^f(\tau) \times \mathbf{R})^{\mathbf{N}} \approx \ell_2^f(\tau)^{\mathbf{N}}.\end{aligned}$$ □

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