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ON INVARIANT SUBMANIFOLDS OF CONTACT METRIC MANIFOLDS

By

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Abstract. Invariant submanifolds of (κ,μ)-manifolds and (κ,μ)-space forms are studied.

1 Introduction

A differentiable 1-form η on a differentiable manifold M^{2m+1} is called a contact form if $\eta \wedge (d\eta)^m \neq 0$ everywhere on M^{2m+1}, and M^{2m+1} equipped with a contact form is a contact manifold. It is well-known that there exist a unique global vector field ξ, called the characteristic vector field, a $(1,1)$-tensor field φ and a Riemannian metric $\langle\,,\rangle$ satisfying certain relations. The structure $(\eta,\xi,\varphi,\langle\,,\rangle)$ is called a contact metric structure and the manifold M^{2m+1} endowed with such a structure is said to be a contact metric manifold. A contact metric manifold is called a K-contact manifold if the structure vector filed ξ is Killing. A normal contact metric manifold is a Sasakian manifold. A Sasakian manifold is always a K-contact manifold and in dimension three a K-contact manifold is Sasakian. In [3], Blair, Koufogiorgos and Papantoniou introduced the class of contact metric manifolds, in which the structure vector field belongs to the (κ,μ)-nullity distribution. A contact metric manifold belonging to this class is called a (κ,μ)-*manifold.* Characteristic examples of non-Sasakian (κ,μ)-manifolds are the tangent sphere bundles of Riemannian manifolds of constant sectional curvature not equal to one and certain Lie groups [5]. Recently, T. Koufogiorgos introduced the notion of (κ,μ)-space form [11], which contains the well known class of Sasakian space forms for $\kappa = 1$. For more details about contact geometry we refer to [2].

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In this paper we study invariant submanifolds of (κ,μ)-manifolds. The paper is organized as follows. In the section 2 we give a brief account of contact metric manifolds, (κ,μ)-manifolds and (κ,μ)-space forms. Essential details for submanifolds are also given. In the section 3, first we prove that each totally umbilical submanifold of a contact metric manifold tangent to the structure vector field of the ambient manifold is minimal and consequently totally geodesic. Then, we give some basic equations for invariant submanifolds in a (κ,μ)-manifold. As a consequence, every invariant submanifold of a (κ,μ)-manifold becomes a (κ,μ)-manifold. Next, we classify invariant submanifolds in a (κ,μ)-manifold with parallel second fundamental form. Then, using a theorem of D. Blair, we give a classification of invariant submanifolds with parallel second fundamental form in a contact metric manifold whose structure vector field belongs to the κ-nullity distribution. A corollary for invariant submanifolds in a Sasakian manifold is also given. Ricci tensor and scalar curvature for invariant submanifolds in a (κ,μ)-space form are given in the section 4. Using these expressions, we find necessary and sufficient conditions for invariant submanifolds in a (κ,μ)-space form to be totally umbilical and totally geodesic. Then a corollary for invariant submanifold of a Sasakian space form is given. In section 5, we study invariant submanifolds in a (κ,μ)-space form such that the normal connection is trivial. Among other results, it is proved that for an invariant submanifold in a (κ,μ)-space form $\tilde{M}(c)$ with codimension greater than two, the normal connection of the submanifold is trivial provided the submanifold is totally geodesic and $c=1$. As a consequence, we have some corollaries for invariant submanifolds of Sasakian space forms. In the last section, a Simons' type formula for a compact invariant submanifold of a (κ,μ)-space form $\tilde{M}(c)$ is established.

2 (κ,μ)-Contact Manifolds

A $(2m+1)$-dimensional differentiable manifold $\tilde{M}$ is called an almost contact manifold if either its structural group can be reduced to $U(m)\times 1$ or equivalently, there is an almost contact structure $(\tilde{\varphi},\tilde{\xi},\tilde{\eta})$ consisting of a $(1,1)$ tensor field $\tilde{\varphi}$, a vector field $\tilde{\xi}$, and a 1-form $\tilde{\eta}$ satisfying

$$\tilde{\varphi}^2=-I+\tilde{\eta}\otimes\tilde{\xi},\quad \tilde{\eta}(\tilde{\xi})=1,\quad \tilde{\varphi}\tilde{\xi}=0,\quad \tilde{\eta}\circ\tilde{\varphi}=0. \tag{2.1}$$

First and one of the remaining three relations of (2.1) imply the other two relations of (2.1). An almost contact structure $(\tilde{\varphi},\tilde{\xi},\tilde{\eta})$ on $\tilde{M}$ is said to be *normal* if the induced almost complex structure P on the product manifold $\tilde{M}\times\mathbf{R}$ defined by

$$P\left(\tilde{X}, \lambda \frac{d}{dt}\right) = \left(\tilde{\varphi}\tilde{X} - \lambda\tilde{\xi}, \tilde{\eta}(\tilde{X})\frac{d}{dt}\right) \tag{2.2}$$

is integrable, where $\tilde{X}$ is tangent to $\tilde{M}$, t the coordinate of $\mathbf{R}$ and λ a smooth function on $\tilde{M} \times \mathbf{R}$. The condition for being normal is equivalent to vanishing of the torsion tensor $[\tilde{\varphi}, \tilde{\varphi}] + 2d\tilde{\eta} \otimes \tilde{\xi}$, where $[\tilde{\varphi}, \tilde{\varphi}]$ is the Nijenhuis tensor of $\tilde{\varphi}$. Let $\langle\, , \rangle$ be a compatible Riemannian metric with $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta})$, that is,

$$\langle \tilde{X}, \tilde{Y} \rangle = \langle \tilde{\varphi}\tilde{X}, \tilde{\varphi}\tilde{Y} \rangle + \tilde{\eta}(\tilde{X})\tilde{\eta}(\tilde{Y}) \tag{2.3}$$

or equivalently,

$$\tilde{\Phi}(\tilde{X}, \tilde{Y}) \equiv \langle \tilde{X}, \tilde{\varphi}\tilde{Y} \rangle = -\langle \tilde{\varphi}\tilde{X}, \tilde{Y} \rangle \quad \text{and} \quad \langle \tilde{X}, \tilde{\xi} \rangle = \tilde{\eta}(\tilde{X}) \tag{2.4}$$

for all $\tilde{X}, \tilde{Y} \in T\tilde{M}$. Then, $\tilde{M}$ becomes an almost contact metric manifold equipped with an almost contact metric structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \langle\, , \rangle)$.

A differentiable 1-form $\tilde{\eta}$ on a $(2m+1)$-dimensional differentiable manifold $\tilde{M}$ is called a *contact form* if $\tilde{\eta} \wedge (d\tilde{\eta})^m \neq 0$ everywhere on $\tilde{M}$, and $\tilde{M}$ equipped with a contact form is a *contact manifold.* An almost contact metric structure becomes a *contact metric structure* if $\tilde{\Phi} = d\tilde{\eta}$. A normal contact metric manifold is a *Sasakian manifold.* An almost contact metric manifold is Sasakian if and only if

$$(\tilde{\nabla}_{\tilde{X}}\tilde{\varphi})\tilde{Y} = \langle \tilde{X}, \tilde{Y} \rangle \tilde{\xi} - \tilde{\eta}(\tilde{Y})\tilde{X}, \quad \tilde{X}, \tilde{Y} \in T\tilde{M}, \tag{2.5}$$

where $\tilde{\nabla}$ is Levi-Civita connection, while a contact metric manifold $\tilde{M}$ is Sasakian if and only if the curvature tensor $\tilde{R}$ satisfies

$$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{\xi} = \tilde{\eta}(\tilde{Y})\tilde{X} - \tilde{\eta}(\tilde{X})\tilde{Y}, \quad \tilde{X}, \tilde{Y} \in T\tilde{M}. \tag{2.6}$$

In a contact metric manifold $\tilde{M}$, the $(1,1)$-tensor field $\tilde{h}$ defined by $2\tilde{h} = \mathfrak{L}_{\tilde{\xi}}\tilde{\varphi}$ is symmetric and satisfies

$$\tilde{h}\tilde{\xi} = 0, \quad \tilde{h}\tilde{\varphi} + \tilde{\varphi}\tilde{h} = 0, \quad \tilde{\nabla}_{\tilde{X}}\tilde{\xi} = -\tilde{\varphi}\tilde{X} - \tilde{\varphi}\tilde{h}\tilde{X}, \quad \operatorname{trace}(\tilde{h}) = \operatorname{trace}(\tilde{\varphi}\tilde{h}) = 0. \tag{2.7}$$

The (κ, μ)*-nullity distribution* of a contact metric manifold $\tilde{M}$ is a distribution [3]

$$\begin{aligned} N(\kappa, \mu) : p \to N_p(\kappa, \mu) = \{\tilde{Z} \in T_pM \mid \tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = \kappa(\langle \tilde{Y}, \tilde{Z} \rangle \tilde{X} - \langle \tilde{X}, \tilde{Z} \rangle \tilde{Y}) \\ + \mu(\langle \tilde{Y}, \tilde{Z} \rangle \tilde{h}\tilde{X} - \langle \tilde{X}, \tilde{Z} \rangle \tilde{h}\tilde{Y})\}, \end{aligned}$$

where κ and μ are constants. If $\mu = 0$, the (κ, μ)*-nullity distribution* $N(\kappa, \mu)$ is called the κ*-nullity distribution* $N(\kappa)$. If $\tilde{\xi} \in N(\kappa, \mu)$, that is

$$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{\xi} = \kappa(\tilde{\eta}(\tilde{Y})\tilde{X} - \tilde{\eta}(\tilde{Z})\tilde{Y}) + \mu(\tilde{\eta}(\tilde{Y})\tilde{h}\tilde{X} - \tilde{\eta}(\tilde{Z})\tilde{h}\tilde{Y}), \tag{2.8}$$

then $\tilde{M}$ is called a (κ,μ)-*manifold*. If $\tilde{\xi} \in N(\kappa)$, then

$$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{\xi} = \kappa(\tilde{\eta}(\tilde{Y})\tilde{X} - \tilde{\eta}(\tilde{Z})\tilde{Y}), \tag{2.9}$$

and $\tilde{M}$ will be called an $N(\kappa)$-*contact metric manifold*. In a (κ,μ)-*manifold* the covariant derivatives of $\tilde{\varphi}$ and $\tilde{h}$ satisfy

$$(\tilde{\nabla}_{\tilde{X}}\tilde{\varphi})\tilde{Y} = \langle \tilde{X} + \tilde{h}\tilde{X}, \tilde{Y}\rangle\tilde{\xi} - \tilde{\eta}(\tilde{Y})(\tilde{X} + \tilde{h}\tilde{X}), \tag{2.10}$$

$$\begin{aligned}(\tilde{\nabla}_{\tilde{X}}\tilde{h})\tilde{Y} = {} & \{(1-\kappa)\langle \tilde{X}, \tilde{\varphi}\tilde{Y}\rangle - \langle \tilde{X}, \tilde{\varphi}\tilde{h}\tilde{Y}\rangle\}\tilde{\xi} \\ & - \tilde{\eta}(\tilde{Y})\{(1-\kappa)\tilde{\varphi}\tilde{X} + \tilde{\varphi}\tilde{h}\tilde{X}\} - \mu\tilde{\eta}(\tilde{X})\tilde{\varphi}\tilde{h}\tilde{Y}.\end{aligned} \tag{2.11}$$

Moreover, we have

$$\tilde{Q}\tilde{\xi} = 2m\kappa\tilde{\xi}, \quad \tilde{h}^2 = (\kappa - 1)\tilde{\varphi}^2,$$

where $\tilde{Q}$ is Ricci operator. Obviously, $\kappa \leq 1$, equality holds if and only if the manifold is Sasakian. Characteristic examples of non-Sasakian (κ,μ)-manifolds are the tangent sphere bundles of Riemannian manifolds of constant sectional curvature not equal to one and certain Lie groups [5]. For more details we refer to [2], [3] and [11].

The sectional curvature $\tilde{K}(\tilde{X}, \tilde{\varphi}\tilde{X})$ of a plane section spanned by a unit vector $\tilde{X}$ orthogonal to $\tilde{\xi}$ is called a $\tilde{\varphi}$-*sectional curvature*. If the (κ,μ)-manifold $\tilde{M}$ has constant $\tilde{\varphi}$-sectional curvature c then it is called a (κ,μ)-*space form* and is denoted by $\tilde{M}(c)$. The curvature tensor of $\tilde{M}(c)$ is given by [11]

$$\begin{aligned}\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = {} & \frac{c+3}{4}\{\langle \tilde{Y}, \tilde{Z}\rangle\tilde{X} - \langle \tilde{X}, \tilde{Z}\rangle\tilde{Y}\} \\ & + \frac{c-1}{4}\{2\langle \tilde{X}, \tilde{\varphi}\tilde{Y}\rangle\tilde{\varphi}\tilde{Z} + \langle \tilde{X}, \tilde{\varphi}\tilde{Z}\rangle\tilde{\varphi}\tilde{Y} - \langle \tilde{Y}, \tilde{\varphi}\tilde{Z}\rangle\tilde{\varphi}\tilde{X}\} \\ & + \frac{c+3-4\kappa}{4}\{\tilde{\eta}(\tilde{X})\tilde{\eta}(\tilde{Z})\tilde{Y} - \tilde{\eta}(\tilde{Y})\tilde{\eta}(\tilde{Z})\tilde{X} \\ & \qquad + \langle \tilde{X}, \tilde{Z}\rangle\tilde{\eta}(\tilde{Y})\tilde{\xi} - \langle \tilde{Y}, \tilde{Z}\rangle\tilde{\eta}(\tilde{X})\tilde{\xi}\} \\ & + \frac{1}{2}\{\langle \tilde{h}\tilde{Y}, \tilde{Z}\rangle\tilde{h}\tilde{X} - \langle \tilde{h}\tilde{X}, \tilde{Z}\rangle\tilde{h}\tilde{Y} \\ & \qquad + \langle \tilde{\varphi}\tilde{h}\tilde{X}, \tilde{Z}\rangle\tilde{\varphi}\tilde{h}\tilde{Y} - \langle \tilde{\varphi}\tilde{h}\tilde{Y}, \tilde{Z}\rangle\tilde{\varphi}\tilde{h}\tilde{X}\} \\ & + \langle \tilde{\varphi}\tilde{Y}, \tilde{\varphi}\tilde{Z}\rangle\tilde{h}\tilde{X} - \langle \tilde{\varphi}\tilde{X}, \tilde{\varphi}\tilde{Z}\rangle\tilde{h}\tilde{Y} \\ & + \langle \tilde{h}\tilde{X}, \tilde{Z}\rangle\tilde{\varphi}^2\tilde{Y} - \langle \tilde{h}\tilde{Y}, \tilde{Z}\rangle\tilde{\varphi}^2\tilde{X}\end{aligned}$$

$$+\mu\{\tilde{\eta}(\tilde{Y})\tilde{\eta}(\tilde{Z})\tilde{h}\tilde{X}-\tilde{\eta}(\tilde{X})\tilde{\eta}(\tilde{Z})\tilde{h}\tilde{Y}$$
$$+\langle\tilde{h}\tilde{Y},\tilde{Z}\rangle\tilde{\eta}(\tilde{X})\tilde{\xi}-\langle\tilde{h}\tilde{X},\tilde{Z}\rangle\tilde{\eta}(\tilde{Y})\tilde{\xi}\} \tag{2.12}$$

for all $\tilde{X},\tilde{Y},\tilde{Z}\in T\tilde{M}$, where $c+2\kappa=-1=\kappa-\mu$ if $\kappa<1$.

Let M be a submanifold in a manifold $\tilde{M}$ equipped with a Riemannian metric $\langle\,,\rangle$. The Gauss and Weingarten formulae are given respectively by

$$\tilde{\nabla}_X Y=\nabla_X Y+\sigma(X,Y)\quad\text{and}\quad\tilde{\nabla}_X N=-A_N X+\nabla_X^{\perp}N$$

for all $X, Y\in TM$ and $N\in T^{\perp}M$, where $\tilde{\nabla}$, ∇ and $\nabla^{\perp}$ are Riemannian, induced Riemannian and induced normal connections in $\tilde{M}$, M and the normal bundle $T^{\perp}M$ of M respectively, and σ is the second fundamental form related to the shape operator A_N in the direction of N by $\langle\sigma(X,Y),N\rangle=\langle A_N X,Y\rangle$. Moreover, if $\tilde{J}$ is any $(1,1)$-tensor field on $\tilde{M}$, then we have [13]

$$(\tilde{\nabla}_X\tilde{J})Y=((\nabla_X J)Y-A_{F_{\tilde{J}}Y}X-t_{\tilde{J}}\sigma(X,Y))$$
$$+((\nabla_X F_{\tilde{J}})Y+\sigma(X,JY)-J^{\perp}\sigma(X,Y)), \tag{2.13}$$
$$(\tilde{\nabla}_X\tilde{J})N=((\nabla_X t_{\tilde{J}})N-A_{J^{\perp}N}X-JA_N X))$$
$$+((\nabla_X J^{\perp})N+\sigma(X,t_{\tilde{J}}N)-F_{\tilde{J}}A_N X)), \tag{2.14}$$

where

$$\tilde{J}X\equiv JX+F_{\tilde{J}}X,\quad X,JX\in TM,\ F_{\tilde{J}}X\in T^{\perp}M,$$
$$\tilde{J}N\equiv t_{\tilde{J}}N+J^{\perp}N,\quad t_{\tilde{J}}N\in TM,\ N,J^{\perp}N\in T^{\perp}M,$$
$$(\nabla_X J)Y\equiv\nabla_X JY-J\nabla_X Y,\quad(\nabla_X F_{\tilde{J}})Y\equiv\nabla_X^{\perp}F_{\tilde{J}}Y-F_{\tilde{J}}\nabla_X Y,$$
$$(\nabla_X t_{\tilde{J}})N\equiv\nabla_X t_{\tilde{J}}N-t_{\tilde{J}}\nabla_X^{\perp}N,\quad(\nabla_X J^{\perp})N\equiv\nabla_X^{\perp}J^{\perp}N-J^{\perp}\nabla_X^{\perp}N.$$

From Gauss and Weingarten formulas, we obtain

$$(\tilde{R}(X,Y)Z)^T=R(X,Y)Z+A_{\sigma(X,Z)}Y-A_{\sigma(Y,Z)}X, \tag{2.15}$$

consequently, the Gauss equation is

$$\tilde{R}(X,Y,Z,W)=R(X,Y,Z,W)-\langle\sigma(X,W),\sigma(Y,Z)\rangle$$
$$+\langle\sigma(X,Z),\sigma(Y,W)\rangle. \tag{2.16}$$

The covariant derivative of σ is defined by

$$(\nabla_X\sigma)(Y,Z)=\nabla_X^{\perp}\sigma(Y,Z)-\sigma(\nabla_X Y,Z)-\sigma(Y,\nabla_X Z). \tag{2.17}$$

Finally for normal vector fields N and V the equation of Ricci-Kühn is

$$\tilde{R}(X, Y, N, V) = R^{\perp}(X, Y, N, V) - \langle [A_N, A_V]X, Y\rangle. \tag{2.18}$$

The mean curvature vector H is expressed by $H = \mathrm{trace}(\sigma)/\dim(M)$. The submanifold M is *totally geodesic* in $\tilde{M}$ if $\sigma = 0$, and *minimal* if $H = 0$. If $\sigma(X, Y) = \langle X, Y\rangle H$ for all $X, Y \in TM$, then M is *totally umbilical.*

3 Invariant Submanifolds

Let $\tilde{M}$ be an almost contact metric manifold with the structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \langle\, , \rangle)$. For a submanifold M of $\tilde{M}$ tangent to $\tilde{\xi}$, we write the orthogonal direct decomposition $TM = \mathscr{D} \oplus \{\xi\}$, where ξ is restriction of $\tilde{\xi}$. Moreover, if the ambient manifold is contact also, then

$$\nabla_{\xi}\xi = 0 \quad \text{and} \quad \sigma(\xi, \xi) = 0. \tag{3.1}$$

Thus, every totally umbilical submanifold M of a contact metric manifold such that $\tilde{\xi} \in TM$ is minimal and consequently totally geodesic. For $H = \langle \xi, \xi\rangle H = \sigma(\xi, \xi) = 0$.

If in a submanifold M of an almost contact metric manifold the structure vector field $\tilde{\xi}$ is tangent to M and $\tilde{\varphi} T_p M \subset T_p M$, then M is called an *invariant submanifold* and inherits an almost contact metric structure $(\varphi, \xi, \eta, \langle\, , \rangle)$ by restriction. Moreover, in view of (2.13) and (2.14), we have

$$(\tilde{\nabla}_X \tilde{\varphi}) Y = (\nabla_X \varphi) Y + \sigma(X, \varphi Y) - \varphi^{\perp}\sigma(X, Y), \tag{3.2}$$

$$(\tilde{\nabla}_X \tilde{\varphi}) N = -A_{\varphi^{\perp} N} X - \varphi A_N X + (\nabla_X \varphi^{\perp}) N. \tag{3.3}$$

For a submanifold M of a contact metric manifold to be invariant, the condition $\tilde{\varphi} T_p M \subset T_p M$ is sufficient. In this case, $\tilde{\xi}$ becomes tangent to M and the induced structure $(\varphi, \xi, \eta, \langle\, , \rangle)$ becomes contact. Moreover, $h = \tilde{h}|_M$, $\sigma(X, \xi) = 0$ and M is minimal [2]. We also have

$$(\tilde{\nabla}_X \tilde{h}) Y = (\nabla_X h) Y + \sigma(X, hY) - h^{\perp}\sigma(X, Y), \tag{3.4}$$

$$(\tilde{\nabla}_X \tilde{h}) N = -A_{h^{\perp} N} X - h A_N X + (\nabla_X h^{\perp}) N. \tag{3.5}$$

Now, we prove the following

PROPOSITION 3.1. *Let M be a $(2n+1)$-dimensional invariant submanifold of a (κ, μ)-manifold. Then, we have*

$$(\nabla_X \varphi) Y = \langle X + hX, Y\rangle \xi - \eta(Y)(X + hX), \tag{3.6}$$

$$\varphi^{\perp}\sigma(X,Y)=\sigma(X,\varphi Y)=\sigma(\varphi X,Y), \tag{3.7}$$

$$A_{\varphi^{\perp}N}=\varphi A_N=-A_N\varphi, \tag{3.8}$$

$$(\tilde{\nabla}_X\tilde{\varphi})N=0, \tag{3.9}$$

$$\begin{aligned}(\nabla_X h)Y=&\{(1-\kappa)\langle X,\varphi Y\rangle-\langle X,\varphi hY\rangle\}\xi\\ &-\eta(Y)\{(1-\kappa)\varphi X+\varphi hX\}-\mu\eta(X)\varphi hY,\end{aligned} \tag{3.10}$$

$$\varphi^{\perp}h(X,Y)=\sigma(X,hY)=\sigma(hX,Y), \tag{3.11}$$

$$A_{h^{\perp}N}=hA_N=A_Nh, \tag{3.12}$$

$$(\tilde{\nabla}_X\tilde{h})N=0, \tag{3.13}$$

$$Q\xi=2n\kappa\xi,\quad h^2=(\kappa-1)\varphi^2,$$

where Q is Ricci operator on the invariant submanifold.

Proof. From (2.10), we have

$$(\tilde{\nabla}_X\tilde{\varphi})Y=\langle X+hX,Y\rangle\xi-\eta(Y)(X+hX). \tag{3.14}$$

Equating tangential and normal parts of right hand sides of (3.2) and (3.14), we get (3.6) and (3.7). Equation (3.8) is equivalent to (3.7). From (2.10) we have (3.9). Similarly, we can prove (3.10)–(3.13). Using $\sigma(X,\xi)=0$ in (2.15), we get

$$\tilde{R}(X,Y)\xi=R(X,Y)\xi,$$

which in view of (2.8), gives

$$R(X,Y)\xi=\kappa(\eta(Y)X-\eta(Z)Y)+\mu(\eta(Y)hX-\eta(Z)hY).$$

This completes the proof. □

In view of the previous discussion in this section, we can state the following

Theorem 3.2. *An invariant submanifold of a (κ,μ)-manifold is a (κ,μ)-manifold.*

We recall the following Lemma for later uses.

Lemma 3.3 [7]. *Let M be an invariant submanifold of a contact metric manifold. Then, we have*

$$\varphi A_N = -A_N \varphi, \quad A_N \xi = 0, \tag{3.15}$$

$$A_N = A_N h \quad \textit{if and only if} \quad (\nabla_X A_N)\xi = 0. \tag{3.16}$$

Now, we prove the following

THEOREM 3.4. *Let M be an invariant submanifold in a (κ, μ)-manifold. If $\nabla \sigma = 0$, then either $\kappa = 0$ or M is totally geodesic.*

PROOF. For any submanifold in a Riemannian manifold, first we note that

$$\langle (\nabla_X \sigma)(Y, Z), N \rangle = \langle (\nabla_X A_N) Y - A_{\nabla^{\perp} N} Y, Z \rangle.$$

Thus taking in to account $\nabla \sigma = 0$ and $A_N \xi = 0$, the above equation gives

$$(\nabla_X A_N)\xi = 0,$$

which in view of (3.16) implies that $A_N h = 0$. Thus we have

$$(1 - \kappa) A_N X = (\kappa - 1) A_N \varphi^2 X = A_N h^2 X = A_N X,$$

which provides

$$\kappa A_N = 0.$$

Hence, either $\kappa = 0$ or the invariant submanifold is totally geodesic. □

The above theorem provides the following

COROLLARY 3.5 [9]. *An invariant submanifold of a Sasakian manifold is totally geodesic, provided the second fundamental form of immersion is covariantly constant.*

Now, we recall the following

THEOREM 3.6 [1]. *Let M be a $(2n+1)$-dimensional manifold endowed with a contact metric structure $(\varphi, \xi, \eta, \langle\, , \rangle)$ such that*

$$R(X, Y)\xi = 0, \quad X, Y \in TM,$$

where R is the Riemann curvature tensor. Then, M is locally isometric to $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$.

In view of Theorem 3.4 and Theorem 3.6, we have the following

THEOREM 3.7. *Let $\tilde{M}$ be a contact metric manifold with its structure vector field belonging to the κ-nullity distribution. Let M be a $(2n+1)$-dimensional invariant submanifold in $\tilde{M}$, whose second fundamental form is covariantly constant, then either M is totally geodesic or M is locally isometric to $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$.*

We close this section by proving the following

PROPOSITION 3.8. *Let M be a $(2n+1)$-dimensional invariant submanifold in a (κ, μ)-manifold $\tilde{M}$. Then*

$$\operatorname{trace}(hA^2) = 0, \tag{3.17}$$

$$(\operatorname{trace}(hA))^2 \le 2n(1-\kappa)\operatorname{trace}(A^2). \tag{3.18}$$

PROOF. Since $h^2 = (\kappa - 1)\varphi^2$, therefore h may be represented by

$$h = \begin{pmatrix} aI_n & & 0 \\ & -aI_n & \\ 0 & & 0 \end{pmatrix},$$

where $a = (1-\kappa)^{1/2}$. Since (3.12) holds true, we may take the same orthogonal matrix to orthogonalize A. Therefore, from (3.15), A can be represented as

$$A = \begin{pmatrix} a_1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & a_n & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & -a_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & -a_n & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Thus, we have

$$\operatorname{trace}(hA) = 2(1-\kappa)^{1/2}(a_1 + \cdots + a_n) \quad \text{and} \quad \operatorname{trace}(hA^2) = 0.$$

Hence,

$$\begin{aligned} (\operatorname{trace}(hA))^2 &= 4(1-\kappa)(a_1 + \cdots + a_n)^2 \\ &\le 4n(1-\kappa)(a_1^2 + \cdots + a_n^2) \\ &= 2n(1-\kappa)\operatorname{trace}(A^2), \end{aligned} \tag{3.19}$$

which completes the proof. $\square$

4 Ricci Tensor and Scalar Curvature

In a (κ,μ)-space form $\tilde{M}(c)$, from (2.12), we obtain

$$\tilde{R}(\tilde{X},\tilde{\xi})\tilde{\xi} = -\kappa\tilde{\varphi}^2 X + \mu\tilde{h}X.$$

Consequently, if $\kappa = 0 \neq \mu$, then $\tilde{h}$ is determined completely in terms of the Riemann curvature.

In view of (2.12) and (2.16) we are able to state the following

PROPOSITION 4.1. *In a $(2n+1)$-dimensional invariant submanifold in a (κ,μ)-space form $\tilde{M}(c)$, the Ricci tensor and the scalar curvature are given respectively by*

$$\begin{aligned} S(X,Y) = {} & \frac{1}{2}((n+1)c + 3(n-1) + 2\kappa)\langle X, Y\rangle \\ & - \frac{1}{2}\{(n+1)c + 3(n-1) - 2(2n-1)\kappa\}\eta(X)\eta(Y) \\ & + (\mu + 2n - 2)\langle hX, Y\rangle - \sum_{i=1}^{2n+1}\langle\sigma(e_i, X), \sigma(Y, e_i)\rangle, \end{aligned} \tag{4.1}$$

$$r = n((n+1)c + 3(n-1)) + 4n\kappa - \|\sigma\|^2, \tag{4.2}$$

where

$$\|\sigma\|^2 = \sum_{i,j=1}^{2n+1}\langle\sigma(e_i, e_j), \sigma(e_i, e_j)\rangle.$$

The equations (4.1) and (4.2) give the following

THEOREM 4.2. *For a $(2n+1)$-dimensional invariant submanifold in a (κ,μ)-space form $\tilde{M}(c)$, the following statements are equivalent:*

1. *M is totally umbilical,*
2. *M is totally geodesic,*
3. *Ricci tensor is given by*

$$\begin{aligned} S(X,Y) = {} & (\mu + 2n - 2)\langle hX, Y\rangle + \frac{1}{2}\{(n+1)c + 3(n-1) + 2\kappa\}\langle X, Y\rangle \\ & - \frac{1}{2}\{(n+1)c + 3(n-1) - 2(2n-1)\kappa\}\eta(X)\eta(Y), \end{aligned} \tag{4.3}$$

4. *Scalar curvature is given by*

$$r = n((n+1)c + 3(n-1)) + 4n\kappa. \tag{4.4}$$

As an immediate consequence, we have the following

Corollary 4.3. *For a $(2n+1)$-dimensional invariant submanifold in a Sasakian space form $\tilde{M}(c)$, the following statements are equivalent:*

1. *M is totally umbilical,*
2. *M is totally geodesic,*
3. *Ricci tensor is given by*

$$S(X,Y) = \frac{1}{2}((n+1)c + 3n - 1)\langle X, Y\rangle - \frac{1}{2}(n+1)(c-1)\eta(X)\eta(Y), \tag{4.5}$$

4. *Scalar curvature is given by*

$$r = n((n+1)c + 3n + 1). \tag{4.6}$$

In particular, a 3-dimensional totally geodesic invariant submanifold M in a Sasakian space form $\tilde{M}(c)$ has Ricci tensor $S = (c+1)g + (1-c)\eta \otimes \eta$.

Remark 4.4. *A Sasakian manifold M is η-Einstein if its Ricci tensor satisfies*

$$S(X,Y) = a\langle X, Y\rangle + b\eta(X)\eta(Y), \quad X, Y \in TM,$$

where a and b are some constants [10]. *A 3-dimensional Sasakian manifold is known to be η-Einstein* [4] *and its Ricci curvature is given by $S = \left(\frac{r}{2} - 1\right)g + \left(3 - \frac{r}{2}\right)\eta \otimes \eta$. Thus from* (4.5), *we see that M is η-Einstein.*

5 Invariant Submanifolds with Trivial Normal Connection

In this section, we assume that for an invariant submanifold M in a (κ,μ)-space form $\tilde{M}(c)$, the normal connection is trivial. Then, for a unit vector $N \in T^{\perp}M$ and $X, Y \in TM$, from (2.12) we get

$$2\tilde{R}(X, \varphi Y, N, \tilde{\varphi}N) = (1-c)\langle \varphi X, \varphi Y\rangle. \tag{5.1}$$

On the other hand, from the equation of Ricci-Kühn, we also have

$$\begin{aligned} \tilde{R}(X, \varphi Y, N, \tilde{\varphi}N) &= \langle \sigma(A_N X, \varphi Y), \tilde{\varphi}N\rangle - \langle \sigma(X, A_N\varphi Y), \tilde{\varphi}N\rangle \\ &= \langle A_N X, A_{\tilde{\varphi}N}\varphi Y\rangle - \langle A_{\tilde{\varphi}N}X, A_N\varphi Y\rangle \\ &= \langle A_N X, \varphi A_N \varphi Y\rangle + \langle \varphi A_N X, \varphi A_N Y\rangle \\ &= 2\langle \varphi A_N X, \varphi A_N Y\rangle = 2\langle A_N X, A_N Y\rangle, \end{aligned} \tag{5.2}$$

where (3.8) and (3.15) are used. Thus, we obtain

$$(1-c)\langle\varphi X,\varphi Y\rangle = 4\langle A_N X, A_N Y\rangle. \tag{5.3}$$

Moreover, if $U, V \in T^{\perp}M$ are mutually perpendicular unit vectors, then $(U+V)/\sqrt{2}$ is another unit vector. Thus, in view of (5.3), we obtain

$$\langle A_U A_V X, Y\rangle + \langle A_V A_U X, Y\rangle = 0,$$

which gives

$$A_U A_V = -A_V A_U. \tag{5.4}$$

Now, in view of (5.3), we are able to state the following

THEOREM 5.1. *For an invariant submanifold M in a (κ,μ)-space form $\tilde{M}(c)$ with trivial normal connection, we have $c \le 1$ with equality condition if and only if M is totally geodesic.*

When the codimension of the invariant submanifold is greater than two, we have a stronger result in the form of following

THEOREM 5.2. *Let M be a $(2n+1)$-dimensional invariant submanifold in a (κ,μ)-space form $\tilde{M}(c)$ with codimension greater than two. Then the following statements are equivalent:*

(i) *the normal connection of M is trivial,*

(ii) *M is totally geodesic and $c = 1$.*

PROOF. In view of (2.12) and (2.18), it is easy to see that **(ii)** implies **(i)**. Let the normal connection be trivial and M be not totally geodesic. Consider a φ-basis $\{e_1, e_2, \dots, e_{2n}, \xi\}$ for T_pM with $e_{n+i} = \varphi e_i$, $i = 1, \dots, n$. If $A_U e_i = 0$ for some unit vector $U \in T^{\perp}M$, then from (5.3), M is totally geodesic. So $A_U e_i \neq 0$ for any N and e_i. From (5.3), it follows that $A_U e_1, \dots, A_U e_{2n}$ are linearly independent. Using (5.4) in Ricci-Kühn equation, for mutually orthogonal unit vectors $U, V \in T^{\perp}M$ we obtain

$$\tilde{R}(X, Y, U, V) = 2\langle A_U X, A_V Y\rangle,$$

while in view of (2.12), we get

$$2\tilde{R}(X, Y, U, V) = (1-c)\langle X, \varphi Y\rangle\langle\varphi^{\perp} U, V\rangle.$$

From the above two equations, we have

$$(1-c)\langle X, \varphi Y\rangle\langle\varphi^{\perp} U, V\rangle = 4\langle A_U X, A_V Y\rangle.$$

If codimension is greater then two, we may take V orthogonal to U and $\varphi^{\perp}U$; thus the above equation gives

$$\langle A_U X, A_V Y\rangle = 0, \quad X, Y \in TM.$$

By assumption, $A_V e_i \neq 0$, $i = 1, \dots, 2n$. Therefore, $A_U e_i$ are orthogonal to $A_V e_j$, $i, j = 1, \dots, 2n$. Thus, $A_U e_1, \dots, A_U e_{2n}$, $A_V e_1, \dots, A_V e_{2n}$ are linearly independent, which is a contradiction. Therefore, M must be totally geodesic and hence $c = 1$. □

Theorem 5.1 and Theorem 5.2 provides the following two Corollaries.

COROLLARY 5.3 [10]. *For an invariant submanifold M in a Sasakian space form $\tilde{M}(c)$ with trivial normal connection, we have $c \leq 1$ with equality condition if and only if M is totally geodesic.*

COROLLARY 5.4 [10]. *Let M be an invariant submanifold in a Sasakian space form $\tilde{M}(c)$ with codimension greater than two. Then the following statements are equivalent*:

(i) *the normal connection of M is trivial,*

(ii) *M is totally geodesic and $c = 1$.*

6 Simons' Type Formula

Let M be a $(2n+1)$-dimensional invariant submanifold of a $(2m+1)$-dimensional (κ, μ)-space form $\tilde{M}(c)$. We choose a local field of orthonormal frames $e_1, \dots, e_{2m+1}$ such that, restricted to M, $e_1, \dots, e_n$, $e_{n+1} = \varphi e_1, \dots, e_{2n} = \varphi e_n$ are tangent to $\mathscr{D}$ and $e_{2n+1} = \xi$. We use the following convention on range of indices:

$$1 \leq i, j, k, \dots \leq 2n+1,$$

$$2n+2 \leq \alpha, \beta, \gamma \cdots \leq 2m+1.$$

We put

$$\sigma(e_i, e_j) = \sum \sigma_{ij}^{\alpha} e_{\alpha},$$

$$(\nabla_{e_k}\sigma)(e_i, e_j) = \sum \sigma_{ijk}^{\alpha} e_{\alpha},$$

$$R_{jkl}^{i} = \langle R(e_k, e_l)e_j, e_i\rangle,$$

where R is the curvature tensor of M.

Let Δ be the Laplace operator acting on $C^\infty(M)$. Then we have the following (see, (3.12) in [6]).

$$\frac{1}{2}\Delta\|\sigma\|^2 = \sum(\sigma_{ijk}^\alpha)^2 + \sum \sigma_{ij}^\alpha \Delta \sigma_{ij}^\alpha. \tag{6.1}$$

The equation of Codazzi implies that

$$\sigma_{ijk} = \sigma_{ikj}. \tag{6.2}$$

Since M is minimal, from (2.21) in [6] and (6.2) we obtain

$$\sum \sigma_{ij}^\alpha \Delta \sigma_{ij}^\alpha = \sum \sigma_{ij}^\alpha \sigma_{km}^\alpha R_{ijk}^m + \sigma_{ij}^\alpha \sigma_{mi}^\alpha R_{kjk}^m - \sigma_{ij}^\alpha \sigma_{ki}^\beta R_{\beta jk}^\alpha. \tag{6.3}$$

Moreover by using the Ricci-Kühn equation, we see that the right side of (6.3) is equal to the following;

$$\begin{aligned}\sum \sigma_{ij}^\alpha \sigma_{km}^\alpha \tilde{R}_{ijk}^m &+ \sigma_{ij}^\alpha \sigma_{mi}^\alpha \tilde{R}_{kjk}^m - \sigma_{ij}^\alpha \sigma_{ki}^\beta \tilde{R}_{\beta jk}^\alpha \\ &+ \sum_{\lambda,\mu} \operatorname{trace}(A_{e_\lambda}A_{e_\mu} - A_{e_\mu}A_{e_\lambda})^2 - \sum_{\lambda,\mu}(\operatorname{trace} A_{e_\lambda}A_{e_\mu})^2,\end{aligned} \tag{6.4}$$

where $\tilde{R}$ is the curvature tensor of $\tilde{M}(c)$.

In view of (3.7) and (3.8), we observe that the shape operator of invariant submanifolds in contact metric manifolds has similar properties as that of Kaehler submanifolds in [12]. Hence by applying Proposition 3.1, Lemma 3.4 and (6.10) in [12], we have

$$\begin{aligned}&\sum_{\lambda,\mu} \operatorname{trace}(A_{e_\lambda}A_{e_\mu} - A_{e_\mu}A_{e_\lambda})^2 - \sum_{\lambda,\mu}(\operatorname{trace} A_{e_\lambda}A_{e_\mu})^2 \\ &\qquad \geq -\|\sigma\|^4 - \frac{1}{2}\|\sigma\|^4 = -\frac{3}{2}\|\sigma\|^4.\end{aligned} \tag{6.5}$$

Also by a straightforward computation, we get

$$\sum \sigma_{ij}^\alpha \sigma_{km}^\alpha \tilde{R}_{ijk}^m = c\|\sigma\|^2, \tag{6.6}$$

$$\sum \sigma_{ij}^\alpha \sigma_{mi}^\alpha \tilde{R}_{kjk}^m = \left\{\frac{(c+3)(2n-1)}{4} + \frac{3(c-1)}{4} + \kappa\right\}\|\sigma\|^2, \tag{6.7}$$

$$\sum \sigma_{ij}^\alpha \sigma_{ki}^\beta \tilde{R}_{\beta jk}^\alpha = \frac{c-1}{2}\|\sigma\|^2, \tag{6.8}$$

and hence

$$\sum(\sigma_{ij}^{\alpha}\sigma_{km}^{\alpha}\tilde{R}_{ijk}^{m}+\sigma_{ij}^{\alpha}\sigma_{mi}^{\alpha}\tilde{R}_{kjk}^{m}-\sigma_{ij}^{\alpha}\sigma_{ki}^{\beta}\tilde{R}_{\beta jk}^{\alpha})$$
$$=\left\{\frac{c(n+2)+3n}{2}+\kappa-1\right\}\|\sigma\|^{2}. \tag{6.9}$$

Combining (6.1), (6.4), (6.5) and (6.9), we obtain

$$\frac{1}{2}\Delta\|\sigma\|^{2}\geq\|\nabla\sigma\|^{2}+\left\{-\frac{3}{2}\|\sigma\|^{2}+\frac{c(n+2)+3n}{2}+\kappa-1\right\}\|\sigma\|^{2}. \tag{6.10}$$

Now, we assume that M is compact. Then by applying Green's theorem, we have

$$\int_{M}\left\{\frac{3}{2}\|\sigma\|^{2}-\frac{c(n+2)+3n}{2}-\kappa+1\right\}\|\sigma\|^{2}\,dv_{M}\geq\int_{M}\|\nabla\sigma\|^{2}\,dv_{M}. \tag{6.11}$$

Theorem 3.4 and (6.11) yield us the following.

THEOREM 6.1. *Let M be a compact $(2n+1)$-dimensional invariant submanifold in a (κ,μ)-space form $\tilde{M}(c)$. Then either $\kappa=0$ and $\|\sigma\|^{2}=\frac{c(n+2)+3n}{3}-\frac{2}{3}$, or M is totally geodesic, or at some point $p\in M$, we have*

$$\|\sigma\|^{2}(p)>\frac{c(n+2)+3n}{3}+\frac{2\kappa}{3}-\frac{2}{3}.$$

PROOF. If $\|\sigma\|^{2}\leq\frac{c(n+2)+3n}{3}+\frac{2\kappa}{3}-\frac{2}{3}$ at every point of M^{2n+1}, from (6.11) we obtain that $\|\sigma\|^{2}=\frac{c(n+2)+3n}{3}+\frac{2\kappa}{3}-\frac{2}{3}$ and $\nabla\sigma=0$, or $\sigma=0$ on M. By applying Theorem 3.4, we can prove the statement. □

REMARK 6.2. We have the following remarks. **(a)** μ does not appear in (6.10). **(b)** In case of $\kappa=1$, our Theorem 6.1 becomes Theorem 4.1 of Endo [8] or Theorem 2.1 of Kon [10]. But in case of $\kappa\neq 1$ and $n\neq 1$, Theorem 6.1 does not coincide with Theorem 4.1 of Endo [8].

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